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Short note on Hilbert's inequality

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Abstract Considering five different parameters, we obtain some new Hilbert-type integral inequalities for functions $f(x)$, $g(x)$ in $L^2[0,\infty)$. Then, we extract from our results some special cases which have been proved before.

MATHEMATICS SUBJECT CLASSIFICATION: 26Dxx; 26D07; 26D10

1. Introduction

We study advanced variants of the classical integral Hilbert-type inequality [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)\ g(x)}{x+y}\ dx\ dy \leqslant \frac{\pi}{\sin(\pi/k)} \left(\int_0^\infty f^k(x)dx\right)^{1/k} \times \left(\int_0^\infty g^{k'}(x)dx\right)^{1/k'}, \tag{1}$$

unless $f(x) \equiv 0$ or $g(x) \equiv 0$, where $k > 1$, $k' = k/(k-1)$. Inequality (1) would be false for some $f(x)$, $g(x)$ if $\pi\ cosec(\pi/k)$ were replaced by any smaller number see [1]. Inequality (1) with its improvements has played a fundamental role in the development of many mathematical branches see for instance [2–4]. We centre our attention on the case when $k = k' = 2$ in (1), which takes the following form:

$$\int_0^\infty \int_0^\infty \frac{f(x)\ g(x)}{x+y}\ dx\ dy \leqslant \pi\left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right)^{1/2}, \quad f(x),\ g(x) \in L^2[0,\infty). \tag{2}$$

Inequality (2) has many generalizations concerning the denominator of the left-hand side see for example [5,6,2,3,7].

Our main goal is to obtain new generalizations of Hilbert-type inequality (2). In the following section, we state the main result of this paper of which many special cases can be obtained.

2. Main results and discussion

In this section, we state and discuss our main theorem together with its special cases.

For three different parameters r, t, $\lambda \in (0,1]$, we have the following general result.

Theorem 2.1. *Suppose that $0 < a < b$ and $0 < r, t, \lambda \leqslant 1$. Then for functions $f(x)$, $g(x) \in L^2[0,\infty)$ the following Hilbert-type inequality holds:*

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$$\int_a^b \int_a^b \frac{f(x)\, g(y)}{(x^r + y^t)^{\lambda}}\, dx\, dy \leqslant \left[\int_a^b \frac{x^{\alpha}}{t}\left(\beta(p,q) - 2h_{r,t,\lambda}(1)\ \Psi(a,b,r,t,\lambda)\ x^{\frac{r(t-r)\lambda}{4t}}\right) f^2(x) dx \times \int_a^b \frac{y^{\alpha'}}{r}\left(\beta(p',q') - 2h_{r,t,\lambda}(1)\ \Psi'(a,b,r,t,\lambda)\ y^{\frac{t(r-t)\lambda}{4r}}\right) g^2(y) dy\right]^{1/2}, \tag{3}$$

where $\alpha = 1 - \left(\frac{4rt+t^2-r^2}{4t}\right)\lambda$, $\alpha' = 1 - \left(\frac{4rt+r^2-t^2}{4r}\right)\lambda$, $\beta(\theta, \phi)$ *is the* $\beta -$*function with* $p = \frac{(r+t)\lambda}{4t}$, $q = \lambda - \frac{(r+t)\lambda}{4t}$, $p' = \frac{(r+t)\lambda}{4r}$, $q' = \lambda - \frac{(r+t)\lambda}{4r}$, $\Psi(a,b,r,t,\lambda) = \left(\frac{a^{(r+t)\lambda}}{b^{(3t-r)\lambda}}\right)^{1/8}$, $\Psi'(a,b,r,t,\lambda) = \left(\frac{a^{(r+t)\lambda}}{b^{(3r-t)\lambda}}\right)^{1/8}$, *and* $h_{r,t,\lambda}(\zeta) = \zeta^{-\frac{(r+t)\lambda}{4t}} \int_0^{\zeta} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{1-\frac{(r+t)\lambda}{4t}} du.$

As a special case of Theorem 2.1 when $t = r$, we have the following corollary:

Corollary 2.2. *Suppose that* $0 < a < b$ *and* $0 < t, \lambda \leqslant 1$. *Then for functions* $f(x), g(x) \in L^2[0,\infty)$ *the following Hilbert-type inequality holds:*

$$\int_a^b \int_a^b \frac{f(x)\, g(y)}{(x^t + y^t)^{\lambda}}\, dx\, dy \leqslant \frac{\beta\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)}{t}\left(1 - \left(\frac{a}{b}\right)^{t\ \lambda/4}\right) \times \left(\int_a^b x^{1-t\ \lambda} f^2(x)\ dx \int_a^b x^{1-t\ \lambda} g^2(x)\ dx\right)^{1/2}.$$

Another special case of Theorem 2.1 is when $t = r = 1$, this leads to the following corollary (which has been proved in [5]):

Corollary 2.3. *Let* $0 < a < b$ *and* $0 < t \leqslant 1$, $f(x)$, $g(x) \in L^2[0,\infty)$. *Then*

$$\int_a^b \int_a^b \frac{f(x)\, g(y)}{(x+y)^{\lambda}} dx\, dy \leqslant \beta\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(1 - \left(\frac{a}{b}\right)^{\lambda/4}\right) \times \left(\int_a^b x^{1-\lambda} f^2(x)\, dx \int_a^b x^{1-\lambda} g^2(x)\, dx\right)^{1/2}.$$

Before proving Theorem 2.1, let us state and prove the following two lemmas.

Lemma 2.4. *For parameters* r, t, λ *where* $0 < t, \lambda \leqslant 1$, *define* $h_{r,t,\lambda}(\zeta)$ *as*

$$h_{r,t,\lambda}(\zeta) := \zeta^{-\frac{(r+t)\lambda}{4t}} \int_0^{\zeta} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{1-\frac{(r+t)\lambda}{4t}} du, \quad \zeta \in (0, 1]. \tag{4}$$

Then $h_{r,t,\lambda}(\zeta)$ *is strictly decreasing, i.e.,* $h_{r,t,\lambda}(\zeta) \geqslant h_{r,t,\lambda}(1)$. *The equality holds when* $\zeta = 1$.

Proof. For $\zeta \in (0, 1]$, we have

$$\frac{d}{d\zeta} h_{r,t,\lambda}(\zeta) = \frac{\zeta^{-1}}{(1+\zeta)^{\lambda}} - \zeta^{-1-\frac{(r+t)\lambda}{4t}} \int_0^{\zeta} \frac{1}{(1+u)^{\lambda}}\ du^{\left(\frac{(r+t)\lambda}{4t}\right)}.$$

Integrating by parts gives

$$\frac{d}{d\zeta} h_{r,t,\lambda}(\zeta) = -\lambda\ \zeta^{-1-\frac{(r+t)\lambda}{4t}} \int_0^{\zeta} \frac{1}{(1+u)^{1+\lambda}}\ u^{\left(\frac{(r+t)\lambda}{4t}\right)}\ du < 0.$$

Therefore, $h_{r,t,\lambda}(\zeta)$ is strictly decreasing on $(0, 1]$. Hence $h_{r,t,\lambda}(\zeta) \geqslant h_{r,t,\lambda}(1)$. This completes the proof. □

In the light of Lemma 2.4, one can think of the following lemma:

Lemma 2.5. *For* $0 < a < b$ *and* $r, t, \lambda \in (0,1]$, *define*

$$w_{r,t,\lambda}(a,b,x) := \int_a^b \frac{1}{(x^r + y^t)^{\lambda}} \left(\frac{x}{y}\right)^{1-\frac{(r+t)\lambda}{4}} dy, \quad x \in [a,b], \tag{5}$$

and

$$w_{r,t,\lambda}(a,b,y) := \int_a^b \frac{1}{(x^r + y^t)^{\lambda}} \left(\frac{y}{x}\right)^{1-\frac{(r+t)\lambda}{4}} dx, \quad y \in [a,b]. \tag{6}$$

Then, the following inequalities hold under the condition that $\frac{a^t}{x^r}, \frac{x^r}{b^t} \in (0, 1]$

$$w_{r,t,\lambda}(a,b,x) \leqslant \frac{x^{\alpha}}{t}\left(\beta(p,q) - 2h(1)\left(\Psi(a,b,r,t,\lambda)\ x^{\frac{r(t-r)\lambda}{4t}}\right)\right), \tag{7}$$

and

$$w_{r,t,\lambda}(a,b,y) \leqslant \frac{y^{\alpha'}}{r}\left(\beta(p',q') - 2h(1)\left(\Psi'(a,b,r,t,\lambda)\ y^{\frac{t(r-t)\lambda}{4r}}\right)\right), \tag{8}$$

where $\alpha, \alpha', p, q, p', q', \Psi(a,b,r,t,\lambda), \Psi'(a,b,r,t,\lambda)$ *and* $h(.)$ *are as defined in Theorem* 2.1.

Proof. Putting $u = \frac{y^t}{x^r}$ in (5) gives

$$w_{r,t,\lambda}(a,b,x) = \frac{x^{\alpha}}{t}\left(\int_0^{\infty} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{1-\frac{(r+t)\lambda}{4t}} du - \int_0^{a^t/x^r} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{1-\frac{(r+t)\lambda}{4t}} du - \int_{b^t/x^r}^{\infty} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{1-\frac{(r+t)\lambda}{4t}} du\right),$$

where $\alpha = 1 - r\lambda - \frac{t\lambda}{4} + \frac{r^2\lambda}{4t}$. Use the definition of the Beta function $\left(\beta(\theta, \phi) = \int_0^{\infty} \frac{z^{\theta-1}}{(1+z)^{\theta+\phi}}\ dz\right)$ in the first integral and the substitution $u = \frac{1}{v}$ in the third integral to have

$$w_{r,t,\lambda}(a,b,x) = \frac{x^{\alpha}}{t}\left(\beta(p,q) - \int_0^{a^t/x^r} \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{1-\frac{(r+t)\lambda}{4t}} du - \int_0^{x^r/b^t} \frac{1}{(1+v)^{\lambda}} \left(\frac{1}{v}\right)^{1-\frac{(3t-r)\lambda}{4t}} dv\right), \tag{9}$$

where $p = \frac{(r+t)\lambda}{4t}$, and $q = \lambda - \frac{(r+t)\lambda}{4t}$. Now applying Lemma 2.4 to the second and third terms in (9) leads to

$$w_{r,t,\lambda}(a,b,x) = \frac{x^{\alpha}}{t}\left(\beta(p,q) - \left(\left(\frac{a^t}{x^r}\right)^{\frac{(r+t)\lambda}{4t}} h_{r,t,\lambda}\left(\frac{a^t}{x^r}\right) + \left(\frac{x^r}{b^t}\right)^{\frac{(3t-r)\lambda}{4t}} h_{r,t,\lambda}\left(\frac{x^r}{b^t}\right)\right)\right)$$
$$\leqslant \frac{x^{\alpha}}{t}\left(\beta(p,q) - 2\ h_{r,t,\lambda}(1)\left(\Psi(a,b,r,t,\lambda)\ x^{\frac{(t-r)r\lambda}{4t}}\right)\right),$$

which is (7). Similarly, we can prove (8). This completes the proof. □

3. Proving the main result

Proof of Theorem 2.1. By Cauchy's inequality, we can estimate the left-hand side of (3) as follows

$$\int_a^b \int_a^b \frac{f(x)\ g(y)}{(x^r + y^t)^{\lambda}}\ dx\ dy$$
$$= \int_a^b \int_a^b \frac{f(x)}{(x^r + y^t)^{\lambda/2}} \left(\frac{x}{y}\right)^{\frac{\left(1-\frac{(r+t)\lambda}{4}\right)}{2}} \frac{g(y)}{(x^r + y^t)^{\lambda/2}} \left(\frac{y}{x}\right)^{\frac{\left(1-\frac{(r+t)\lambda}{4}\right)}{2}} dx\ dy$$
$$\leqslant \left[\int_a^b \int_a^b \frac{f^2(x)}{(x^r + y^t)^{\lambda}} \left(\frac{x}{y}\right)^{\left(1-\frac{(r+t)\lambda}{4}\right)} dx\ dy \times \int_a^b \int_a^b \frac{g^2(y)}{(x^r + y^t)^{\lambda}} \left(\frac{y}{x}\right)^{\left(1-\frac{(r+t)\lambda}{4}\right)} dx\ dy\right]^{1/2}$$
$$= \left[\int_a^b w_{r,t,\lambda}(a,b,x) f^2(x)\ dx \int_a^b w_{r,t,\lambda}(a,b,y) g^2(y)\ dy\right]^{1/2}, \quad (10)$$

where $w_{r,t,\lambda}(a,b,x)$ and $w_{r,t,\lambda}(a,b,y)$ are as defined in (5) and (6) respectively. Applying Lemma 2.5 to inequality (10) yields (3) as required. This compelets the proof. □

References

[1] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge University Press, 1934.

[2] Y. Bicheng, L. Debnath, On a new generalization of Hardy–Hilbert's inequality and its applications, Journal of Mathematical Analysis and Applications 233 (1999) 484–497.

[3] K. Jichang, L. Debnath, On new generalization of Hilbert's inequality and their applications, Journal of Mathematical Analysis and Applications 245 (2000) 248–265.

[4] G. Mingzhe, On Hilbert's inequality and its applications, Journal of Mathematical Analysis and Applications 212 (1997) 316–323.

[5] Y. Bicheng, Note on Hilbert's integral inequality, Journal of Mathematical Analysis and Applications 220 (1998) 778–785.

[6] Y. Bicheng, On new generalizations of Hilbert's inequality, Journal of Mathematical Analysis and Applications 248 (2000) 29–40.

[7] H. Ke, On Hilbert's inequality, Chinese Annals of Mathematics, Series B 13 (1992) 35–39.