On regularity criteria for the \(n\)-dimensional Navier–Stokes equations in terms of the pressure

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Abstract

We study the Cauchy problem for the \(n\)-dimensional Navier–Stokes equations \((n \geq 3)\), and prove some regularity criteria involving the integrability of the pressure or the pressure gradient for weak solutions in the Morrey, Besov and multiplier spaces.

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1. Introduction

We study regularity criteria for the Cauchy problem for the Navier–Stokes equations in \(\mathbb{R}^n\) \((n \geq 3)\):

\[
\begin{align*}
& u_t + u \cdot \nabla u + \nabla P = \Delta u \quad \text{in} \ (0, T) \times \mathbb{R}^n, \\
& \text{div} \ u = 0 \quad \text{in} \ (0, T) \times \mathbb{R}^n, \\
& u|_{t=0} = u_0(x) \quad \text{in} \ \mathbb{R}^n,
\end{align*}
\]

where \(u \in \mathbb{R}^n\) is the velocity, \(P\) is the pressure, and \(u_0\) with \(\text{div} \ u_0 = 0\) is the initial velocity.

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The global well-posedness for (1.1)–(1.3) was studied first by Leray and Hopf. They proved in their pioneering work [14,18] that a weak solution $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ to (1.1)–(1.3) for $u_0 \in L^2$ exists globally in time. However, the uniqueness and regularity of weak solutions are still a very challenging open problem. On the other hand, due to the well-known fact that the Leray–Hopf solution coincides with the smooth solution as long as the latter exists, it is conjectured that the Leray–Hopf solution is regular. This was first shown by Serrin [24] under certain additional hypotheses; also by Ohyama [21] independently.

**Theorem 1.1.** Suppose $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ is a weak solution of the problem (1.1)–(1.3), and assume in addition that $u \in L^s(0,T;L^r)$ where $2/s + n/r \leq 1$. Then $u$ is smooth.

We should point out that in [24] the case $2/s + n/r < 1$ was treated, while the case $2/s + n/r = 1$ was dealt with by Fabes, Jones and Riviére [12], Giga [13], Sohr [25], and von Wahl [26]. The corresponding local regularity result of Serrin was extended to the limit case by Struwe [28] and Takahashi [30,31]. In the case $r = n = 3$, $s = \infty$ a smallness condition was required at first and removed recently by Escauriaza, Seregin and Šverák [11,23]. For further references, see [2], for example.

Similar regularity criteria involving either the pressure $p$, or combinations of $u$ and $p$ have been studied by a lot of authors, see for example Beirão da Veiga [4–6], Chae and Lee [9], or Berselli and Galdi [2], and the references cited therein. Note that a rough dimensional analysis as in Caffarelli, Kohn and Nirenberg [7] predicts that weak solutions to (1.1)–(1.3) are regular, provided either the condition

$$\nabla P \in L^s(0,T;L^r) \quad \text{with} \quad \frac{2}{s} + \frac{n}{r} \leq 3,$$

or the condition

$$P \in L^s(0,T;L^{r^*}) \quad \text{with} \quad \frac{2}{s} + \frac{n}{r^*} \leq 2 \quad (1.5)$$

is satisfied. Indeed, Berselli and Galdi [2] obtained the following conditional regularity concerning the pressure, some technical improvements of which have been obtained recently by Zhou [33].

**Theorem 1.2.** Suppose $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ is a weak solution of the problem (1.1)–(1.3) with associated pressure $P$. Assume that either (i) the pressure satisfies (1.5) with $1 \leq s < \infty$, or (ii) the pressure gradient satisfies (1.4) with $1 \leq s \leq n$. Then $u$ is smooth.

It was also remarked in [2, Remark 1.3] that regularity would still hold in the limit case $P \in L^\infty(0,T;L^{n/2})$, provided the corresponding norm is sufficiently small.

The restriction on $s$ in the second part of Theorem 1.2 was removed by Zhou [34,35] when $n \leq 4$. Recently, Struwe [29] extended Zhou’s result to arbitrary dimensions $n \geq 3$. More precisely, he proved

**Theorem 1.3.** Let $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ be a weak solution of the problem (1.1)–(1.3) with associated pressure $P$. Assume $\nabla P$ satisfies condition (1.4) with $n/3 < r < \infty$, $2/3 < s < \infty$. Then $u$ is smooth.
Very recently, the following conditional regularity involving the pressure in Lorentz spaces was obtained by Cai, Fan and Zhai [8].

**Theorem 1.4.** Let \( u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \) be a weak solution to (1.1)–(1.3) with associated pressure \( P \). Also assume that one of the following conditions is satisfied:

(i) \( P \in L^s(0, T; L_r^{r^*}) \) with \( \frac{2}{s} + \frac{n}{r^*} = 2 \), with \( 1 \leq s < \infty \). \hspace{1cm} (1.6)

(ii) \( P \in L^\infty(0, T; L_n^{n/2}) \), provided the corresponding norm is sufficiently small. \hspace{1cm} (1.7)

(iii) \( \nabla P \in L^s(0, T; L_r^r) \) with \( \frac{2}{s} + \frac{n}{r} = 3 \), \( \frac{n}{3} < r < \infty \), \( \frac{2}{3} < s < \infty \). \hspace{1cm} (1.8)

Then \( u \) is smooth.

Here \( L^{p,q} \) is the standard Lorentz space in \( \mathbb{R}^n \), see [27,32] for example, and \( L_r^{r,\infty} \equiv L_r \) is the weak space.

The aim of this paper is to extend Theorems 1.2–1.4 to the Morrey, Besov and the multiplier spaces. We point out here that the pointwise multipliers between different spaces of differentiable functions have been studied by Maz’ya and co-workers [19,20]. They are useful tools for stating minimal regularity requirements on the coefficients of partial differential operators for proving regularity or uniqueness of solutions. The main result of this paper reads:

**Theorem 1.5.** Assume \( u_0 \in L^2 \cap L^\theta \) for some \( \theta > \max\{n, 4\} \) and \( \text{div} u_0 = 0 \) in \( \mathbb{R}^n \). Let \( u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \) be a weak solution of the Navier–Stokes system (1.1)–(1.3) with associated pressure \( P \). Assume that one of the following conditions is satisfied:

(1) \( P \in L^s(0, T; \dot{M}_r^{r*}) \) with \( \frac{2}{s} + \frac{n}{r^*} = 2 \), with \( 1 \leq s < \infty \), \( r^* \geq \hat{q} \). \hspace{1cm} (1.9)

(2) \( P \in L^\infty(0, T; \dot{M}_n^{n/2}, \hat{q}) \), provided \( \|P\|_{L^\infty(0, T; \dot{M}_n^{n/2}, \hat{q})} \) is sufficiently small and \( \frac{n}{2} \geq \hat{q} \). \hspace{1cm} (1.10)

(3) \( \nabla P \in L^s(0, T; \dot{M}_r^{r,q}) \) with \( \frac{2}{s} + \frac{n}{r} = 3 \), \( \frac{n}{3} < r < \infty \), \( \frac{2}{3} < s < \infty \), \( r \geq q \). \hspace{1cm} (1.11)

(4) \( P \in L^s(0, T; \dot{B}_r^{0,\sigma}) \) with \( n = 3 \), \( \frac{2}{s} + \frac{3}{r} = 2 \), \( \frac{3}{2} < r \leq \infty \), \( \sigma \leq \frac{2r}{3} \). \hspace{1cm} (1.12)

(5) \( P \in L^{\frac{2}{r},\infty}(0, T; \dot{X}_r^r) \) for any \( r \in (0, 1] \). \hspace{1cm} (1.13)

(6) \( P \in L^{\frac{2}{1+\alpha},\infty}(0, T; \dot{Y}_1^{1+\alpha}) \) for any \( \alpha \in [0, 1) \). \hspace{1cm} (1.14)

Then \( u \) is smooth.

The notation appeared in Theorem 1.5 will be given at the end of this section.

**Remark 1.1.** Since \( L^p \subset L^{p,\infty} \subset \dot{M}_{p,q} (p > q) \), Theorem 1.5 generalizes Theorems 1.2–1.4.
Remark 1.2. When taking \( s = 1, r = \sigma = \infty \) in (1.12), we get \( P \in L^1(0, T; \dot{B}^0_{\infty, \infty}(\mathbb{R}^3)) \), which gives an interesting generalization of the condition (1.5) with \( n = 3 \) and \( s = 1 \) (i.e. \( P \in L^1(0, T; L^\infty(\mathbb{R}^3)) \)) because \( L^\infty \subset BMO \subset \dot{B}^0_{\infty, \infty} \).

Roughly speaking, the proof of Theorem 1.5 is based on derivation of an a priori estimate and an application of Theorem 1.1. The a priori estimate is the key step in the proof and is obtained by exploiting the features of the Morrey, Besov and multiplier spaces, employing delicately (a priori) estimates and interpolation inequalities, and applying our important Lemma 1.2 below which is proved by using techniques from harmonic analysis.

Finally, we introduce the function spaces and the notation used throughout this paper. Let \( 1 < q \leq p < \infty \), we define the homogeneous Morrey space in \( \mathbb{R}^n \) by

\[
\dot{M}_{p,q} = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n) \mid \|f\|_{\dot{M}_{p,q}} = \sup_{R > 0} \sup_{x \in \mathbb{R}^n} R^n \left( \frac{1}{p} - \frac{1}{q} \right) \left( \int_{B(x,R)} |f(y)|^q \, dy \right)^{1/q} < \infty \right\}.
\]

Let \( 1 \leq p' \leq q' < \infty \), we define the homogeneous space \( \dot{N}_{p',q'} \) by

\[
\dot{N}_{p',q'} = \left\{ f \in L^{q'}(\mathbb{R}^n) \mid f = \sum_{k \in \mathbb{N}} g_k, \quad \text{where } (g_k) \subset L^{q'}_{\text{comp}}(\mathbb{R}^n) \quad \text{and} \quad \sum_{k \in \mathbb{N}} d_k^{n \left( \frac{1}{p'} - \frac{1}{q'} \right)} \left\| g_k \right\|_{L^{q'}} < \infty, \quad \text{where for any } k, \, d_k = \text{diam}(\text{supp } g_k) < \infty \right\},
\]

where \( L^{q'}_{\text{comp}}(\mathbb{R}^n) \) is the space of all \( L^{q'} \)-functions in \( \mathbb{R}^n \) with compact support.

Remark 1.3. \( \dot{N}_{p',q'} \) is a Banach space when it is equipped with the norm

\[
\|f\|_{\dot{N}_{p',q'}} = \inf \left\{ \sum_{k \in \mathbb{N}} d_k^{n \left( \frac{1}{p'} - \frac{1}{q'} \right)} \left\| g_k \right\|_{L^{q'}} \right\},
\]

where the infimum is taken over all possible decompositions.

We have the following important properties on the spaces \( \dot{M}_{p,q} \) and \( \dot{N}_{p',q'} \), which will be frequently used in the proof of Theorem 1.5.

Lemma 1.1. (See [10,17].) Let \( 1 < p' \leq q' < \infty \), and \( p, q \) satisfy \( \frac{1}{p} + \frac{1}{p'} = 1 \), \( \frac{1}{q} + \frac{1}{q'} = 1 \). Then, \( \dot{M}_{p,q}(\mathbb{R}^n) \) is the dual space of \( \dot{N}_{p',q'}(\mathbb{R}^n) \).

Lemma 1.2. Let \( 1 < p' \leq q' < 2 \), \( m \geq 2 \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Denote \( \alpha = -\frac{n}{2} + \frac{n}{p} + \frac{n}{m} \in (0, 1] \). Then there exists a constant \( C > 0 \), such that for any \( u \in L^m(\mathbb{R}^n) \) and \( v \in \dot{H}^\alpha(\mathbb{R}^n) \),

\[
\|uv\|_{\dot{N}_{p',q'}} \leq C \|u\|_{L^m} \|v\|_{\dot{H}^\alpha}, \tag{1.15}
\]

which will be denoted by \( L^m(\mathbb{R}^n) \times \dot{H}^\alpha(\mathbb{R}^n) \hookrightarrow \dot{N}_{p',q'}(\mathbb{R}^n) \) in what follows.

Proof. The case \( m = 2 \) is proved in [10,17]. We will prove the lemma for the case \( m > 2 \) in Section 3.  \( \Box \)
Remark 1.4. Lemma 1.2 is of independent interest and could have applications in the theory of harmonic analysis and partial differential equations.

By $BMO$ we denote the space of functions with bounded mean oscillations, i.e.,

$$BMO := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^3), \sup_{x,R} \frac{1}{|B_R|} \int_{B_R(x)} |f(y) - \overline{f_{B_R(x)}}| \, dy < \infty \right\},$$

where $\overline{f_{B_R(x)}}$ is the average of $f$ over $B_R(x) := \{ y \in \mathbb{R}^3 \mid |x - y| < R \}$ (cf. Stein [27]). $\dot{B}_{p,q}^s$ denotes the homogeneous Besov space, see [27,32] for details on $BMO$ and $\dot{B}_{p,q}^s$.

By a multiplier acting from one functional space, $S_1$, into another, $S_2$, we mean a function which defines a bounded linear mapping of $S_1$ into $S_2$ by pointwise multiplication. Thus, with any pair of spaces $S_1, S_2$ we associate a third, the space of multipliers $M(S_1, S_2)$.

Let $r, \alpha \in (0,1]$ and denote $\dot{X}_r := M(\dot{H}^r, L^2)$ and $\dot{Y}_{1+\alpha} := M(\dot{H}^{\alpha}, L^{2n/(n+2)})$. The space $\dot{X}_r$ has been characterized by Maz’ya [19] in terms of Sobolev capacities. $\dot{X}_r$ has been used in the study of the Navier–Stokes equations in [17] where it is pointed out that

$$\dot{M}_{n/r,q} \subset \dot{X}_r, \quad \frac{n}{r} > q \geq 2,$$

follows easily from Lemma 1.2 when $m = 2$. Thus, similarly one has

$$\dot{M}_{n/(1+\alpha),q} \subset \dot{Y}_{1+\alpha}, \quad \frac{n}{1+\alpha} > q \geq 2.$$

Similarly to [2,8,29,33,35], the following inequalities will be used in our proof:

$$\| P \|_{L^q} \leq C \| u \|_{L^{2q}}, \quad 1 < q < \infty, \quad (1.16)$$

$$\| \nabla P \|_{L^q} \leq C \| \nabla u \| \| u \|_{L^q}, \quad 1 < q < \infty, \quad (1.17)$$

$$\| P \|_{\dot{M}^{r,*}_q} \leq C \| \nabla p \|_{\dot{M}^{r,q}_q} \text{ with } r \leq n, \ r^{n/r} = \hat{q}/q, \quad (1.18)$$

and the Gagliardo–Nirenberg inequalities:

$$\| u \|_{\dot{H}^\alpha} \leq C \| u \|_{L^2}^{1-\alpha} \| \nabla u \|_{L^2}^{\alpha}, \quad \alpha \in (0,1], \quad (1.19)$$

$$\| u \|_{L^p} \leq C \| u \|_{L^2}^{1+\frac{\alpha}{2}} \| \nabla u \|_{L^2}^{\frac{\alpha}{2} - \frac{\alpha}{p}}, \quad 2 \leq p < \frac{2n}{n-2}. \quad (1.20)$$

The inequalities (1.16) and (1.17) can be easily verified by applying the Calderón–Zygmund inequality to the following equation for the pressure obtained by using (1.1) and (1.2),

$$-\Delta P = \text{div}(u \cdot \nabla u) = \sum_{i,j=1}^n \partial_i \partial_j (u^i u^j),$$

while the estimate (1.18) can be found in [16, Lemma 4.1].
Throughout this paper $C$ will denote a generic positive constant which can vary from line to line. For simplicity, we shall use $\int$ to denote $\int_{\mathbb{R}^n}$ or $\int_{\mathbb{R}^3}$. We also use the following abbreviations:

\[
L^p \equiv L^p(\mathbb{R}^n), \quad \dot{H}^r \equiv \dot{H}^r(\mathbb{R}^n),
\]

\[
\dot{M}_{p,q} \equiv \dot{M}_{p,q}(\mathbb{R}^n), \quad \dot{N}_{p',q'} \equiv \dot{N}_{p',q'}(\mathbb{R}^n).
\]

In the next section we prove Theorem 1.5, and in Section 3 we give the proof of Lemma 1.2.

2. Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5. To this end, we collect some preliminary results, due to Kato [15] and Giga [13].

Proposition 2.1. The following properties hold:

1. Suppose that $u_0 \in L^\theta$ for some $\theta > \max\{n, 4\}$ and $\text{div} \, u_0 = 0$ in $\mathbb{R}^n$. Then there are $T_0 > 0$ and a unique solution of (1.1)–(1.3) on $[0, T_0)$ such that

\[
u(\tau) \in BC([0, T_0), L^\theta) \cap L^s(0, T_0; L^r), \quad t^{1/s} \, u \in BC([0, T_0), L^r)
\]  

(2.1)

for $2/s + n/r = n/\theta$, $r > n$, where $BC$ denotes the class of bounded and continuous functions.

2. Let $(0, T^*)$ be the maximal interval, such that $u$ solves the problem (1.1)–(1.3) in $C((0, T^*); L^\theta)$ with $\theta > n$. Then

\[
\|u(\tau)\|_{L^\theta} \geq C (T^* - \tau)^{\frac{n-\theta}{2\theta}}
\]  

(2.2)

for some positive constant $C$ independent of $T^*$ and $\theta$.

3. Let $u$ be a solution of (1.1)–(1.3) on $(0, T_0)$ in the function class (2.1). Assume that $u_0 \in L^2$, then $u$ is also a weak solution, that is,

\[
u \in L^\infty(0, T_0; L^2) \cap L^2(0, T_0; H^1)
\]

and $u$ satisfies the energy inequality

\[
\|u(t)\|^2_{L^2} + 2 \int_0^t \|\nabla u(\tau)\|^2_{L^2} \, d\tau \leq \|u_0\|^2_{L^2} \quad \text{for all} \; t \in [0, T_0].
\]

4. Let $u$ be a weak solution satisfying $u \in L^s(0, T; L^r)$ for some $r > n$, where $2/s + n/r \leq 1$. Then, $u \in C^\infty(\mathbb{R}^n \times (0, T))$.

Proof of Theorem 1.5. By virtue of Proposition 2.1, the weak solution $u$ is smooth in some time-interval $(0, T_0)$. In particular, $(u, p) \in C^\infty(\mathbb{R}^n \times (0, T_0))$ and $u$ is in the class (2.1). Thus, for any $T > 0$ we assume that $u$ is a smooth solution to (1.1)–(1.3) on $(0, T) \times \mathbb{R}^n$ and will establish a priori bounds that will allow to extend $u$ for all time.

(I) We first show that Theorem 1.5 holds under one of the conditions (1.9)–(1.11).
Multiplying Eq. (1.1) with $|u|^\theta$ for some number $\theta > \max\{n, 4\}$ in $L^2$, integrating then by parts and using (1.2), we obtain the well-known identity (see [3,22])

$$
\frac{1}{\theta} \frac{d}{dt} \int |u|^\theta \, dx + \int |\nabla u|^2 |u|^\theta \, dx + \frac{4(\theta - 2)}{\theta^2} \int |\nabla |u|^{\theta/2}|^2 \, dx \\
= - \int u |u|^\theta \nabla P \, dx := I(t), \quad t \in (0, T),
$$

(2.3)

where $I(t)$ can be easily bounded as follows, employing integration by parts and (1.2):

$$
I(t) \leqslant (\theta - 2) \int |P||u|^\theta |\nabla |u|| \, dx \\
\leqslant \frac{2(\theta - 2)}{\theta} \left( \int |P|^2 |u|^\theta \, dx \right)^{1/2} \left( \int |\nabla |u|^{\theta/2}|^2 \, dx \right)^{1/2}.
$$

(2.4)

For simplicity, denote $v = |u|^\theta/2$. Then we have by (2.3) and (2.4) that

$$
\frac{d}{dt} \int v^2 \, dx + \frac{1}{C} \int |\nabla u|^2 |u|^\theta \, dx + \frac{1}{C} \int |\nabla v|^2 \, dx \\
\leqslant C \int |P||u|^\theta/2 - 1 |\nabla v| \, dx \\
\leqslant C \int |P||v|^{1-2/\theta} |\nabla v| \, dx \\
\leqslant \epsilon \int |\nabla v|^2 \, dx + C\epsilon^{-1} \int |P|^2 v^{2(\theta - 2)/\theta} \, dx \quad (\forall 0 < \epsilon < 1).
$$

(2.5)

(1) Assume that (1.9) holds. When $r^* > n$, the last term of (2.5) can be bounded as follows, using (1.15) with $m = 2$ and $\alpha = n/r^*$, (1.16), and (1.19) with $\alpha = n/r^*$,

$$
\int |P|^2 v^{2(\theta - 2)/\theta} \, dx \leqslant C \|P\|_{L^{\theta/2}} \|P v^{\alpha/\theta}\|_{L^2} \|v\|_{\dot{H}^{n/r^*}} \\
\leqslant C \|P\|_{L^{\theta/2}} \|u\|_{L^{\theta/2}} \|v\|_{L^2}^{\alpha/\theta} \|v\|_{\dot{H}^{n/r^*}} \\
\leqslant C \|P\|_{L^{\theta/2}} \|v\|_{L^2} \|v\|_{\dot{H}^{n/r^*}} \\
\leqslant C \|P\|_{L^{\theta/2}} \|v\|_{L^2}^{2 - n/r^*} \|\nabla v\|_{L^2}^{n/r^*} \\
\leqslant \epsilon \|\nabla v\|_{L^2}^2 + C\epsilon^{-1} \|P\|_{L^{\theta/2}} \|v\|_{L^2}^2 \\
= \epsilon \|\nabla v\|_{L^2}^2 + C\epsilon^{-1} \|P\|_{L^{\theta/2}} \|v\|_{L^2}^2 \quad (\forall 0 < \epsilon < 1).
$$

(2.6)

When $r^* = n$, the last term of (2.5) can be easily estimated as follows,
\[
\int |P|^2 v^{\frac{2(\theta - 2)}{\theta}} \, dx \leq C \|P\|_{M_{\tilde{r}, \tilde{q}}} \left\| P v^{\frac{\theta - 4}{\theta}} \right\|_{L^2} \|v\|_{\dot{H}^1} \leq C \|P\|_{M_{\tilde{r}, \tilde{q}}} \|v\|_{L^2} \|\nabla v\|_{L^2} \\
\leq \epsilon \|\nabla v\|_{L^2}^2 + C \epsilon^{-1} \|P\|_{M_{\tilde{r}, \tilde{q}}}^2 \|v\|_{L^2}^2 \text{ for any } \epsilon \in (0, 1).
\]  

(2.7)

When \( n/2 < r^* < n \), the last term of (2.5) can be bounded again as follows, using (1.15) with \( m = 2^* = 2n/(n - 2) \) and \( \alpha = n/r^* - 1 \in (0, 1) \), (1.16), and (1.19) with \( \alpha = n/r^* - 1 \),

\[
\int |P|^2 v^{\frac{2(\theta - 2)}{\theta}} \, dx \leq C \|P\|_{M_{\tilde{r}, \tilde{q}}} \left\| P v^{\frac{\theta - 4}{\theta}} \right\|_{L^{2n}} \|v\|_{\dot{H}^1} \\
\leq C \|P\|_{M_{\tilde{r}, \tilde{q}}} \|v\|_{L^{2n}} \|v\|_{\dot{H}^1} \\
\leq C \|P\|_{M_{\tilde{r}, \tilde{q}}} \|v\|_{L^{2n}} \|v\|_{\dot{H}^1} \\
\leq \epsilon \|\nabla v\|_{L^2}^2 + C \|P\|_{M_{\tilde{r}, \tilde{q}}}^2 \|v\|_{L^2}^2 \text{ for } \forall \epsilon \in (0 < 1).
\]  

(2.8)

Now, inserting (2.6), or (2.7), or (2.8) into (2.5), taking \( \epsilon \) small and applying Gronwall’s inequality, we conclude \( u \in L^{\infty}(0, T; L^{\tilde{q}}) \subset L^{\tilde{q}}(0, T; L^{\tilde{q}}) \) for some \( \tilde{q} \) satisfying \( 2/\tilde{q} + n/\tilde{q} = 1 \). Thus, we may again invoke Proposition 2.1 to conclude that \( u \) extends smoothly to \([0, T]\).

(2) Assume that (1.10) holds. In this case, one can easily bound the last term of (2.5) as follows:

\[
\int |P|^2 v^{\frac{2(\theta - 2)}{\theta}} \, dx \leq C \|P\|_{M_{\tilde{r}, \tilde{q}}} \left\| P v^{\frac{\theta - 4}{\theta}} \right\|_{L^{2n}} \|v\|_{\dot{H}^1} \\
\leq C \|P\|_{M_{\tilde{r}, \tilde{q}}} \|v\|_{L^{2n}} \|v\|_{\dot{H}^1} \\
\leq C \|P\|_{M_{\tilde{r}, \tilde{q}}} \|v\|_{L^2}^2.
\]

Inserting the above inequality into (2.5) and recalling the smallness of \( \|P\|_{M_{\tilde{r}, \tilde{q}}} \), we find that \( u \in L^{\infty}(0, T; L^{\tilde{q}}) \subset L^{\tilde{q}}(0, T; L^{\tilde{q}}) \) for some \( \tilde{q} \) satisfying \( 2/\tilde{q} + n/\tilde{q} = 1 \), which together with Proposition 2.1 immediately implies that \( u \) extends smoothly to \([0, T]\).

(3) Assume that (1.11) holds. Without loss of generality, we may assume that \( r \geq n \). Indeed, by (1.18) for \( r < n \), we see that

\[
\|P\|_{L^1(0, T; M_{\tilde{r}, \tilde{q}})} \leq C \|\nabla P\|_{L^1(0, T; M_{\tilde{r}, \tilde{q}})}
\]

with \( 1/r^* = 1/r - 1/n \). Therefore the case \( r < n \) is already covered by (1.9) and (1.10).

Now, following the calculations in [29] and taking \( \theta > 2r - 2 \), we utilize (2.4) and Young’s inequality to deduce that

\[
I(t) \leq C \left( \int |\nabla P||u|^\theta \, dx \right)^{1/2} \left( \int |P|^2 |u|^\theta \, dx \right)^{1/4} \left( \int |\nabla v|^2 \, dx \right)^{1/4} \\
\leq \epsilon \int |\nabla v|^2 \, dx + C \epsilon^{-1} \left( \int |\nabla P||u|^\theta \, dx \right)^{2/3} \left( \int |P|^2 |u|^\theta \, dx \right)^{1/3}.
\]  

(2.9)
If we apply Hölder’s inequality and use (1.16) with \( q = \frac{\theta+2}{2} \), we get

\[
\int |P|^{2} |u|^{\theta-2} \, dx \leq \|P\|^{2}_{L^{\frac{\theta+2}{2}}} \|u\|^{\frac{\theta-2}{\theta+2}}_{L^{\theta+2}} \leq C \|u\|^{\frac{\theta+2}{\theta+2}}_{L^{\theta+2}} = C \|v\|^{2(\theta+2)/\theta}_{L^{2(\theta+2)/\theta}}. \tag{2.10}
\]

Moreover, we use (1.8) with \( m = \frac{2(\theta+2)}{\theta-2} \) and \( \alpha = \frac{n}{m} - \frac{n}{r} = \frac{n}{r} - \frac{2n}{\theta+2} \in (0, 1) \) to infer that

\[
\int |
abla P| |u|^{\theta-1} \, dx \leq C \|\nabla P\|^{\theta/2}_{\dot{M}_{r,q}} \|v\|^{\theta/2}_{\tilde{H}^{\alpha}} \leq C \|\nabla P\|^{\theta/2}_{\dot{M}_{r,q}} \|v\|^{\theta/2}_{L^{2(\theta+2)/\theta}} \|v\|^{2}_{\dot{H}^{\alpha}}. \tag{2.11}
\]

Now, substituting (2.10) and (2.11) into (2.9), utilizing Young’s inequality, (1.19) and (1.20), we find that

\[
I(t) \leq \epsilon \int |\nabla v|^{2} \, dx + C \epsilon^{-1} \|\nabla P\|^{2/3}_{\dot{M}_{r,q}} \|v\|^{4/3}_{\dot{H}^{\alpha}} \|v\|^{2/3}_{L^{2(\theta+2)/\theta}} \leq 2 \epsilon \int |\nabla v|^{2} \, dx + C \epsilon^{-1} \|\nabla P\|^{2}_{\dot{M}_{r,q}} \|v\|^{2}_{\dot{H}^{\alpha}}.
\]

Inserting the above estimate into (2.5), taking \( \epsilon \) small, and applying Gronwall’s inequality, one obtains \( u \in L^{\infty}(0, T; L^{\theta}) \subset L^{\tilde{\theta}}(0, T; L^{\theta}) \) for some \( \tilde{\theta} \) satisfying \( 2/\tilde{\theta} + n/\theta = 1 \), from which and Proposition 2.1 the smoothness of \( u \) follows immediately.

(II) We next prove that Theorem 1.5 holds under the condition (1.12).

We apply (2.5) with \( \theta = 4 \) and \( \epsilon \) being small to get

\[
\frac{d}{dt} \int v^{2} \, dx + \frac{1}{C} \int |\nabla v|^{2} \, dx + \frac{1}{C} \int |\nabla u|^{2} \, dx \leq C \int P^{2} v \, dx =: J(t), \tag{2.12}
\]

where \( v = |u|^{2} \).

Using the Littlewood–Paley decomposition (1.18) (see, e.g., [27,32]), we decompose \( P \) as follows

\[
P = \sum_{j=\infty}^{\infty} \Delta_{j} P = \sum_{j<-N} \Delta_{j} P + \sum_{j=-N}^{N} \Delta_{j} P + \sum_{j>N} \Delta_{j} P,
\]

where \( N \) is a positive integer to be chosen later. Substituting this decomposition into \( J(t) \), we obtain

\[
J(t) = \sum_{j<-N} \int P v \Delta_{j} P \, dx + \sum_{j=-N}^{N} \int P v \Delta_{j} P \, dx + \sum_{j>N} \int P v \Delta_{j} P \, dx =: J_{1}(t) + J_{2}(t) + J_{3}(t).
\]

Next, we estimate each \( J_{i} \) \( (i = 1, 2, 3) \). First, recalling

\[
\|\Delta_{j} f\|_{L^{q}} \leq C 2^{3j (\frac{1}{p} - \frac{1}{q})} \|\Delta_{j} f\|_{L^{p}}, \quad 1 \leq p \leq q \leq \infty, \tag{2.13}
\]
with $C$ being a positive constant independent of $f$ and $j$, we apply Hölder’s inequality and (1.16) to infer that

$$J_1(t) \leq \|P\|_{L^2} \|v\|_{L^2} \sum_{j < -N} \|\Delta_j P\|_{L^\infty}$$

$$\leq C \|P\|_{L^2} \|v\|_{L^2} \sum_{j < -N} 2^{2j} \|\Delta_j P\|_{L^2}$$

$$\leq C 2^{-\frac{3}{2}N} \|P\|_{L^2}^2 \|v\|_{L^2}^2$$

$$\leq C 2^{-\frac{3}{2}N} \|v\|_{L^2}^3.$$

For $J_2(t)$, we see that by Hölder’s inequality and (1.16),

$$J_2(t) \leq \|P\|_{L^{2r'}} \|v\|_{L^{2r'}} \sum_{j = -N}^{N} \|\Delta_j P\|_{L^{r'}}$$

$$\leq C N^{\frac{2r-3}{2r'}} \|P\|_{L^{2r'}} \|v\|_{L^{2r'}} \|P\|_{\dot{B}^{0}_{r, \frac{2r}{2r'}}}$$

$$\leq C N^{\frac{2r-3}{2r'}} \|v\|_{L^{2r'}}^2 \|P\|_{\dot{B}^{0}_{r, \frac{2r}{2r'}}}$$

$$\leq C N^{\frac{2r-3}{2r'}} \|v\|_{L^{2r'}}^2 \|\nabla v\|_{L^2}^{\frac{3}{r'}} \|P\|_{\dot{B}^{0}_{r, \frac{2r}{2r'}}},$$

where $r' = r/(r - 1)$ denotes the conjugate exponent of $r$, and we have used Gagliardo–Nirenberg’s inequality (1.20) with $n = 3$ and $p = 2r'$ in the last step.

Finally, for $J_3(t)$ we make use of (2.13), (1.16) and (1.17) to deduce that

$$J_3(t) \leq \|P\|_{L^3} \|v\|_{L^3} \sum_{j > N} \|\Delta_j P\|_{L^3}$$

$$\leq C \|P\|_{L^3} \|v\|_{L^3} \sum_{j > N} 2^{j/2} \|\Delta_j P\|_{L^2}$$

$$\leq C \|P\|_{L^3} \|v\|_{L^3} \left( \sum_{j > N} 2^{-j} \right)^{1/2} \left( \sum_{j > N} 2^{2j} \|\Delta_j P\|_{L^2}^2 \right)^{1/2}$$

$$\leq C 2^{-N/2} \|P\|_{L^3} \|v\|_{L^3} \|\nabla P\|_{L^2}$$

$$\leq C 2^{-N/2} \|v\|_{L^3}^2 \|\nabla u|u|\|_{L^2}$$

$$\leq C 2^{-N/2} \|v\|_{L^2} \|\nabla v\|_{L^2} \|\nabla u|u|\|_{L^2}$$

$$\leq C 2^{-N/2} \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 + C 2^{-N/2} \|v\|_{L^2} \|\nabla u|u|\|_{L^2}^2.$$

Substituting the above estimates for $J_i$ ($i = 1, 2, 3$) into (2.12), we obtain
\[
\frac{d}{dt} \int v^2 \, dx + \frac{1}{C} \int |\nabla v|^2 \, dx + \frac{1}{C} \int |\nabla u|^2 |u|^2 \, dx \\
\leq C2^{-3N/2} \|v\|_{L^2}^3 + CN^{2r-\frac{3}{2r}} \|v\|_{L^2}^{2-3/r} \|\nabla v\|_{L^2}^{3/r} \|P\|_{L^{r, \frac{N}{r}}}^0 \\
+ C2^{-N/2} \|v\|_{L^2} \|\nabla v\|_{L^2}^2 + C2^{-N/2} \|v\|_{L^2} \|\nabla u\|_{L^2}^2. 
\]

(2.14)

Now, we choose \( N \) in (2.14) so that \( 2^{-N/2} \|v\|_{L^2} \leq 1/(2C) \), i.e.,

\[
N \geq 2 \log^+ \left( \frac{C^2 \|v\|_{L^2}}{\log 2} \right) + 2,
\]

to conclude

\[
\frac{d}{dt} \int v^2 \, dx + \frac{1}{C} \int |\nabla v|^2 \, dx \leq C \|P\|_{L^{r, \frac{N}{r}}}^F \|v\|_{L^2}^2 \log(e + \|v\|_{L^2}) + C,
\]

which, by applying Gronwall’s inequality and (1.12), implies

\[
\|v\|_{L^\infty(0,T;L^2)} + \|v\|_{L^2(0,T;H^1)} \leq C,
\]

and thus, \( u \in L^\infty(0,T;L^4) \subset L^2(0,T;L^4) \). Hence, \( u \) is smooth by Proposition 2.1.

(III) We now prove that Theorem 1.5 holds under (1.13) or (1.14).

(1) We first assume that (1.13) holds. In this case, we use Hölder’s inequality, (1.16) and (1.19) to bound the last term on the right-hand side of (2.5) as follows

\[
\int |P|^2 v^{\frac{2(\theta-2)}{\theta}} \, dx \leq \|Pv\|_{L^2} \|Pv^{\frac{\theta-4}{\theta}}\|_{L^2} \\
\leq \|P\|_{\dot{H}^\theta} \|v\|_{\dot{H}^\theta} \|P\|_{L^0} \|v^{\frac{\theta-4}{\theta}}\|_{L^{2/\theta}} \\
\leq C \|P\|_{\dot{H}^\theta} \|v\|_{\dot{H}^\theta} \|u^2\|_{L^0} \|v\|_{L^2}^{(\theta-4)/\theta} \\
\leq C \|P\|_{\dot{H}^\theta} \|v\|_{\dot{H}^\theta} \|v\|_{L^2}^{2} \\
\leq C \|P\|_{\dot{H}^\theta} \|v\|_{L^2}^2 \|v\|_{\dot{H}^1} \\\n\leq \epsilon \|\nabla v\|_{L^2}^2 + C\epsilon^{-1} \|P\|_{\dot{H}^\theta} \|v\|_{L^2}^2. 
\]

Inserting the above estimates into (2.5) and taking \( \epsilon \) appropriately small, we apply Gronwall’s inequality to give \( u \in L^\infty(0,T;L^\theta) \subset L^\theta(0,T;L^\theta) \) for some \( \theta \) satisfying \( 2/\theta + n/\theta = 1 \). Hence, \( u \) is smooth due to Proposition 2.1.

(2) Assume that (1.14) holds. In this case, the last term on the right-hand side of (2.5) can be still bounded as follows, using Hölder’s inequality, (1.16) and (1.19),

\[
\int |P|^2 v^{\frac{2(\theta-2)}{\theta}} \, dx \leq \|Pv\|_{L^2}^{2n} \|Pv^{\frac{\theta-4}{\theta}}\|_{L^2} \\
\leq \|P\|_{\dot{H}^\theta} \|v\|_{\dot{H}^\theta} \|P\|_{L^2} \|v^{\frac{\theta-4}{\theta}}\|_{L^{2/\theta}} \\
\leq C \|P\|_{\dot{H}^\theta} \|v\|_{\dot{H}^\theta} \|v\|_{L^2}^{2} \\
\leq C \|P\|_{\dot{H}^\theta} \|v\|_{L^2}^2 \|v\|_{\dot{H}^1} \\\n\leq \epsilon \|\nabla v\|_{L^2}^2 + C\epsilon^{-1} \|P\|_{\dot{H}^\theta} \|v\|_{L^2}^2.
\]
Therefore, we obtain an atomic decomposition for
\[ uv \]
and
\[ 2974 \]

\[ \|v\|_{\dot{H}^{\alpha}}^{1}\sim\sum_{\epsilon,j,k}\|4^{j\alpha}|c_{\epsilon,j,k}|^{2}\|^{1/2}_{L^{2}} \]

\[ \|v\|_{L^{p}}\sim\sum_{\epsilon}\left(\sum_{j,k}|c_{\epsilon,j,k}|^{2}2^{nj}\mathbf{1}_{Q_{j,k}}(x)\right)^{1/2}_{L^{p}}, \quad 1 < p < \infty. \]

From now on, we drop the index \( \epsilon \) for the sake of simplicity, and use the following abbreviation:

\[ uv = u \sum_{Q \in \mathcal{Q}} C_{Q} \psi_{Q}, \]

We substitute the above estimate into (2.5), take \( \epsilon \) suitably small and apply Gronwall’s inequality to conclude that \( u \in L^{\infty}(0, T; L^{\tilde{\theta}}) \subset L^{\tilde{\theta}}(0, T; L^{\tilde{\theta}}) \) for some \( \tilde{\theta} \) satisfying \( 2/\tilde{\theta} + n/\tilde{\theta} = 1 \). Therefore, \( u \) can be extend smoothly to \([0, T]\) by Proposition 2.1. The proof is complete. \( \square \)

3. Proof of Lemma 1.2

We will use a wavelet decomposition but a different reordering and an argument due to Aguirre, Escobedo, Peral and Tchamitchian [1] to prove Lemma 1.2. Our proof is an adaptation and modification of an argument in [10,17] to our case.

Let \( u \in L^{m}(\mathbb{R}^{n}) (m > 2) \) and \( v \in \dot{H}^{\alpha}(\mathbb{R}^{n}) \). We use a wavelet decomposition for \( v \) in order to obtain an atomic decomposition for \( uv \). Let \( \{\psi_{\epsilon}\}_{1 < \epsilon < 2^{n}} \) be a set of regular, compactly supported mother wavelets. Denote

\[ \mathcal{Q} = \{ Q = Q_{j,k} = 2^{-j}(k + [0, 1]^{n}), \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^{n} \} \]

be the collection of dyadic cubes of \( \mathbb{R}^{n} \);

\[ \psi_{\epsilon,q} : x \mapsto 2^{n/2}\psi_{\epsilon}(2^{j}x - k) \]

be the wavelet adjusted to the cube \( Q = Q_{j,k} \in \mathcal{Q} \);

\[ C_{Q} = C_{\epsilon,j,k} = (v, \psi_{\epsilon,q}) \]

be the wavelet coefficient of \( v \) associated to the wavelet \( \psi_{\epsilon,q} \).

Thus, we have the following decomposition for \( v \):

\[ v = \sum_{1 < \epsilon < 2^{n}} \sum_{Q \in \mathcal{Q}} C_{Q} \psi_{\epsilon,q} = \sum_{1 < \epsilon < 2^{n}} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^{n}} C_{\epsilon,j,k} 2^{nj/2} \psi_{\epsilon}(2^{j}x - k). \]

We recall that

\[ \|v\|_{\dot{H}^{\alpha}} \sim \left(\sum_{\epsilon,j,k}4^{j\alpha}|c_{\epsilon,j,k}|^{2}\right)^{1/2} \sim \sum_{\epsilon}\left(\sum_{j,k}4^{j\alpha}|c_{\epsilon,j,k}|^{2}2^{nj}\mathbf{1}_{Q_{j,k}}(x)\right)^{1/2}_{L^{2}} \]

and

\[ \|v\|_{L^{p}} \sim \sum_{\epsilon}\left(\sum_{j,k}|c_{\epsilon,j,k}|^{2}2^{nj}\mathbf{1}_{Q_{j,k}}(x)\right)^{1/2}_{L^{p}}, \quad 1 < p < \infty. \]
where $\psi$ denotes one of the mother wavelets. We define
\[
Av(x) := \left( \sum_{Q \in \mathcal{Q}} 4^{j}|C_Q|^2 \frac{1_{Q(x)}}{|Q|} \right)^{1/2} = \left( \sum_{j,k} 4^{j(\alpha+n/2)}|C_{j,k}|^2 1_{Q_{j,k}(x)} \right)^{1/2}, \quad x \in \mathbb{R}^n.
\]

It is easy to see that if $v \in \dot{H}^\alpha(\mathbb{R}^n)$, then $Av \in L^2(\mathbb{R}^n)$. We divide the proof of Lemma 1.2 into four steps.

**Step 1.** Letting $N \in \mathbb{Z}$, we put $E_N = \{ x \in \mathbb{R}^n \mid Av(x) > 2^N \}$. It is easy to observe that $E_N + 1 \subset \frac{1}{13}E_N$, $|E_N| \leq 4^{-N} \| Av \|^2_{L^2}$ and
\[
\sum_{N \in \mathbb{Z}} 2^{2N} |E_N| \leq 2 \| Av \|^2_{L^2}.
\]

Next, we decompose $E_N$ into dyadic cubes. Notice that $Av$ can be rewritten in the following form
\[
Av(x) = \left( \sum_{Q \in \mathcal{Q}, x \in Q} 4^{j(\alpha+n/2)}|C_Q|^2 \right)^{1/2} = \sup_{Q \in \mathcal{Q}, x \in Q} L_Q,
\]

where $L_Q = (\sum_{Q' \subset Q} 4^{j(\alpha+n/2)}|C_{Q'}|^2)^{1/2}$.

Let $x \in E_N$, then there exists $Q(x) \in \mathcal{Q}$, such that $x \in Q(x)$ and $L_{Q(x)} > 2^N$. We have $Q(x) \subset E_N$. So
\[
E_N = \bigcup_{Q \in \mathcal{C}_N} Q, \quad \text{where } \mathcal{C}_N = \bigcup_{x \in E_N} \mathcal{C}_N(x), \quad \mathcal{C}_N(x) = \{ Q \in \mathcal{Q} \mid x \in Q, L_Q > 2^N \}.
\]

Noting that the collections $\mathcal{C}_N$ are at most numerable and $\mathcal{C}_{N+1} \subset \mathcal{C}_N$, for any $Q \in \mathcal{C}_N$, we have $|Q| \leq |E_N| < \infty$. Hence, there exists $H_N = \{ Q_{N,\ell} \}_{\ell \in \mathbb{N}} \subset \mathcal{C}_N$, a sequence of maximal dyadic cubes which forms a partition of $E_N$. We notice that
\[
\{ Q \in \mathcal{Q} \mid C_Q \neq 0 \} \subset \bigcup_{N \in \mathbb{N}} \mathcal{C}_N = \bigcup_{N \in \mathbb{Z}} \mathcal{C}_N \setminus \mathcal{C}_{N+1}.
\]

Thus, we put
\[
\Delta_N = \mathcal{C}_N \setminus \mathcal{C}_{N+1} = \{ Q \in \mathcal{C}_N \mid Q \notin \mathcal{C}_{N+1} \},
\]
\[
\Delta_{N,\ell} = \{ Q \in \Delta_N \mid Q \subset Q_{N,\ell} \}, \quad F(N) = \{ \ell \in \mathbb{N} \mid \Delta_{N,\ell} \neq \emptyset \}.
\]

Observing that $\{ \Delta_{N,\ell} \}_{\ell \in F(N)}$ is a partition of $\Delta_N$ and $\{ \Delta_N \}_{N \in \mathbb{Z}}$ itself is a partition of $\bigcup_{N \in \mathbb{Z}} \mathcal{C}_N$, we obtain the corresponding decomposition for $uv$
\[
uv = \sum_{N \in \mathbb{Z}} \sum_{\ell \in F(N)} u v_{N,\ell},
\]

where $v_{N,\ell} = \sum_{Q \in \Delta_{N,\ell}} C_Q \psi_Q$. This is the atomic decomposition we want.
Step 2. Let \( N \in \mathbb{Z}, \ell \in F(N) \), then for all \( x \in \mathbb{R}^n \), we have
\[
Av_{N, \ell}(x) = \left( \sum_{Q \in \Delta_{N, \ell}} 4^{j} |C_Q|^2 \frac{1_Q(x)}{|Q|} \right)^{1/2} \leq C2^N 1_{Q_N, \ell}(x). \tag{3.1}
\]

In fact, the left-hand side of (3.1) is supported in \( Q_N, \ell \). Let \( x \in \mathbb{R}^n \). If \( x \notin E_{N+1} \), then we get \( Av_{N, \ell}(x) \leq Av(x) \leq 2^{N+1} \) immediately.

Hence, assume \( x \in E_{N+1} \). In this case there exists \( Q(x) \in H_{N+1} \), such that \( x \in Q(x) \). Let \( \overline{Q(x)} \) denote its dyadic father, then we have \( L_{\overline{Q(x)}} \leq 2^{N+1} \). Let \( Q \in \Delta_{N, \ell} \) such that \( x \in Q \), then \( Q \cap \overline{Q(x)} \neq \emptyset \). Thus, either \( Q \subseteq \overline{Q(x)} \) which is impossible because of \( Q \notin C_{N+1} \), or \( \overline{Q(x)} \subset Q \) which implies \( L_Q \leq L_{\overline{Q(x)}} \). Therefore, we deduce
\[
Av_{N, \ell}(x) = \sup_{Q \in \Delta_{N, \ell}, x \in Q} \left( \sum_{Q' \in \Delta_{N, \ell}, Q \subset Q'} 4^{j(\alpha+n/2)} |C_{Q'}|^2 \right)^{1/2}
\leq \sup_{Q \in \Delta_{N, \ell}, x \in Q} L_Q
\leq \sup_{Q \in \Delta_{N, \ell}, x \in Q} L_{\overline{Q}}
\leq 2^{N+1},
\]
which gives (3.1).

Step 3. As \( \psi \) is compactly supported, there exists \( M > 0 \) such that \( \text{supp} \psi \subset [-M, M]^n \). Then, one has \( \text{supp} \psi_Q \subset \overline{Q} = \overline{Q}_{j,k} = 2^{-j} (k + [-M, M]^n) \), and for any \( N \in \mathbb{Z} \) and \( \ell \in F(N) \), \( \text{supp} v_{N, \ell} \subset \overline{Q}_{N, \ell} \).

Let \( s \in (1, \infty) \), then \( v_{N, \ell} \in L^s \). Indeed, one has
\[
\| v_{N, \ell} \|_{L^s} \sim \left\| \sum_{Q \in \Delta_{N, \ell}} |C_Q|^2 2^{jn} 1_Q(x) \right\|_{L^s}^{1/2}.
\]
Let \( Q \in \Delta_{N, \ell} \), then \( Q \subset Q_{N, \ell} \). Hence \( j(Q_{N, \ell}) \leq j(Q) \). We obtain after a straightforward calculation that
\[
\| v_{N, \ell} \|_{L^s} \leq C |Q_{N, \ell}|^{\alpha/n} \left\| \sum_{Q \in \Delta_{N, \ell}} 4^{j(\alpha+n/2)} |C_Q|^2 1_Q(x) \right\|_{L^s}^{1/2}
\leq C d_{Q_N, \ell}^{\alpha/2} 2^N \| 1_{Q_N, \ell} \|_{L^s}
\leq C d_{Q_N, \ell}^{\alpha+n/s} 2^N < \infty,
\]
which implies \( v_{N, \ell} \in L^s(\mathbb{R}^n) \), where for \( Q \in \mathcal{Q} \), \( d_Q = \text{diam}(Q) \sim 2^{-j} \).
Step 4. Next, we majorize the series

\[ \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} d_Q^n \left( \frac{1}{p'} - \frac{1}{q'} \right) \| u v_{N, \ell} \|_{L^{q'}}. \]

where, for simplicity, we have denoted \( Q_{N, \ell} \) by \( \tilde{Q} \).

Take \( s, t, \sigma \) such that \( \frac{1}{q'} = \frac{1}{\sigma} + \frac{1}{s}, \frac{1}{\sigma} = \frac{1}{m} + \frac{1}{t} \) and \( 1 < q' < 2 \). Then, we have by Hölder’s inequality that

\[ \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} d_Q^n \left( \frac{1}{p'} - \frac{1}{q'} \right) \tilde{Q} \| u v_{N, \ell} \|_{L^{q'}} \leq C \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} d_Q^n \left( \frac{1}{p'} - \frac{1}{q'} \right) \| u \|_{L^\sigma} \| v_{N, \ell} \|_{L^s} \]

\[ \leq C \left( \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} \| u \|_{L^\sigma}^{2n \left( \frac{1}{m} + \frac{1}{t} \right)} \| v_{N, \ell} \|_{L^s} \right)^{\frac{t+2}{2}} \left( \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} d_Q^n 2^{2N} \right)^{\frac{t-2}{2}} \]

\[ \leq C \left( \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} \| u \|_{L^\sigma}^{2n \left( \frac{1}{m} + \frac{1}{t} \right)} \| v_{N, \ell} \|_{L^s} \right)^{\frac{t+2}{2}} \left( \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} d_Q^n 2^{2N} \right)^{\frac{t-2}{2}} \]

\[ = C \| u \|_{L^m} \left( \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} 1 \tilde{Q}^{2 \left( \frac{1}{m} + \frac{1}{t} \right)} \right)^{\frac{1}{\sigma}} \left( \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} d_Q^n 2^{2N} \right)^{\frac{t-2}{2}}, \quad (3.2) \]

where the last term on the right-hand side of (3.2) can be bounded as follows, using the property of \( Av \),

\[ \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} d_Q^n 2^{2N} = C \sum_{N \in \mathbb{Z}} \sum_{\ell \in \mathbb{F}(N)} |Q_{N, \ell}| 2^{2N} \]

\[ \leq C \sum_{N \in \mathbb{Z}} |E_N| 2^{2N} \]

\[ \leq C \| Av \|_{L^2}^2. \quad (3.3) \]

To derive bounds for the second term on the right-hand side of (3.2), we can use an argument of maximal functions (see [17]) that allows us to replace \( \tilde{Q} = Q_{N, \ell} \) by \( Q \), and deduce that
\[
\left\| \sum_{N \in \mathbb{Z}} \sum_{\ell \in F(N)} 1_{Q_2}^{2N} \right\|_{L^t} \leq C \left\| \sum_{N \in \mathbb{Z}} \sum_{\ell \in F(N)} 1_{Q_2}^{2N} \right\|_{L^t} \\
= C \left( \sum_{N \in \mathbb{Z}} \sum_{\ell \in F(N)} |Q_{N,\ell}|^{2N} \right)^{1/t} \\
\leq C \left( \sum_{N \in \mathbb{Z}} |E_N|^{2N} \right)^{1/t} \\
\leq C \|A v\|^{2/t}_{L^2}. \tag{3.4}
\]

Inserting (3.3) and (3.4) into (3.2), we conclude

\[
\sum_{N \in \mathbb{Z}} \sum_{\ell \in F(N)} d_{\tilde{Q}}^{n(p')^{-\frac{1}{p'}}} \, \|u v_{N,\ell}\|_{L^{p'}} \leq C \|u\|_{L^m} \|A v\|_{L^2} \leq C \|u\|_{L^m} \|v\|_{\dot{H}^{\alpha}}.
\]

which completes the proof.

**Remark 3.1.** Similarly, we can show the following imbedding

\[
L^{m_1}(\mathbb{R}^n) \times W^{\alpha,m_2}(\mathbb{R}^n) \hookrightarrow \dot{N}_{p',q'}(\mathbb{R}^n)
\]

with \(-n/p' = -n/m_1 + \alpha - n/m_2\).

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**References**