EQUIVARIANT EMBEDDINGS OF PRINCIPAL $\mathbb{Z}_n$-BUNDLES INTO COMPLEX VECTOR BUNDLES

Jesper Michael MØLLER
Mathematical Institute, University of Copenhagen, Universitetsparken 5, DK-2100 København Ø, Denmark

Received 3 December 1982
Revised 7 March 1983

Let $\pi : E \rightarrow X$ be a principal $\mathbb{Z}_n$-bundle and $p : V \rightarrow X$ an $m$-dimensional complex vector bundle over, say, a connected CW-complex $X$. An equivariant embedding of $\pi$ into $p$ is an embedding $h : E \rightarrow V$ commuting with projections such that $h(e \cdot z) = zh(e)$ for all $e \in E$ and $z \in \mathbb{Z}_n \subseteq \mathbb{S}^1 \subseteq \mathbb{C}$. We compute the primary obstruction $c \in H^{2m}(X; \mathbb{Z})$ to embedding $\pi$ equivariantly into $p$. If $\dim X \leq 2m$, then $c = 0$ if and only if $\pi$ admits an equivariant embedding into $p$. If $\dim X > 2m$ and $\pi$ embeds equivariantly into $p$, then $c = 0$. Other embedding criteria exist in case $p$ is the trivial $m$-plane bundle $e^m$. We use these criteria for a discussion of the classification of the equivalence classes of principal $\mathbb{Z}_n$-bundles that admit equivariant embeddings into $e^m$. Finally, we offer an example of a principal $\mathbb{Z}_n$-bundle that admit an ordinary but not an equivariant embedding into $e^m$.

AMS Subj. Class. (1980): Primary 57M12; Secondary 57S17
principal $\mathbb{Z}_n$-bundle Chern class universal bundle complex vector bundle obstruction equivariantly polynomial principal $\mathbb{Z}_n$-bundle equivariant embedding lens space braid group

1. Introduction

For an arbitrary finite group $G$, the paper [5] contains an existence theorem for equivariant embeddings of principal $G$-bundles into $G$-vector-bundles with trivial action on the base space. The purpose of this paper is to investigate the special case where $G = \mathbb{Z}_n$ is the cyclic group of order $n \geq 2$ acting on complex vector bundles by scalar multiplication. In comparison to the general case, this special case requires much less effort and more information is obtainable.

Throughout this paper, let $X$ be a connected topological space. Let $\pi : E \rightarrow X$ be a (numerable) principal $\mathbb{Z}_n$-bundle, and let $p : V \rightarrow X$ be an $m$-dimensional complex vector bundle over $X$. Since $\mathbb{Z}_n$ is a subgroup of the topological group $\mathbb{S}^1 \subseteq \mathbb{C}$ of complex units, any complex vector bundle is a $\mathbb{Z}_n$-vector-bundle by scalar multiplication. Thus an equivariant embedding of $\pi$ into $p$ is a map $h : E \rightarrow V$ which commutes with projections,

Diagram:

\[ E \xrightarrow{h} V \xrightarrow{p} X \]
maps $E$ homeomorphically onto its image $h(E)$ in $V$ and satisfies $h(e \cdot z) = z h(e)$ for all $e \in E$ and all $z \in \mathbb{Z}_n \subset S^1 \subset \mathbb{C}$. As usual, we consider the $\mathbb{Z}_n$-action on $V$ as a left action.

The paper consists of two more sections. In Section 2 we offer a simple criterion for the existence of equivariant embeddings of $\pi$ into $p$. This leads in an easy way to the result, also contained in [5], that when $X$ is a $CW$-complex of dimension $< 2m$, then an equivariant embedding always exists. But moreover, we compute, when the dimension of $X$ is $2m$, the obstruction in $H^{3m}(X; \mathbb{Z})$ to embedding $\pi$ equivariantly into $p$.

In Section 3 we discuss equivariant embeddings of principal $\mathbb{Z}_n$-bundles into trivial vector bundles. In particular, we consider equivariant embeddings into trivial line bundles. As an application of the derived results we construct a principal $\mathbb{Z}_n$-bundle that cannot be embedded equivariantly but nevertheless admits an ordinary embedding into the trivial line bundle.

Several of the statements contained in this paper are also valid if we let $\pi : E \to X$ be a principal $S^1$-bundle.

2. The existence theorem

Let $\pi[\mathbb{C}] : E[\mathbb{C}] \to X$ denote the complex line bundle associated to the principal $\mathbb{Z}_n$-bundle $\pi : E \to X$ with respect to the usual $\mathbb{Z}_n$-action on $\mathbb{C}$ by scalar multiplication. The elements of the total space $E[\mathbb{C}] = E \times \mathbb{C}/\mathbb{Z}_n$ are orbits of the form $(e, z) \mathbb{Z}_n$, where $(e, z) \in E \times \mathbb{C}$. Let $\text{Hom}(\pi[\mathbb{C}], p)$ denote the complex homomorphism vector bundle as defined e.g. in Husemoller ([6], pp. 65–67).

In the following Lemma we reformulate the embedding problem as a section problem.

Lemma 2.1. There is a bijective correspondence between the set of equivariant embeddings of $\pi$ into $p$ and the set of nowhere zero sections of $\text{Hom}(\pi[\mathbb{C}], p)$.

Proof. According to [6, Example 6.8, p. 67], there is a bijective correspondence between the set of nowhere zero sections of $\text{Hom}(\pi[\mathbb{C}], p)$ and the set of nowhere zero vector bundle morphisms of $\pi[\mathbb{C}]$ into $p$. Hence we must establish a one-to-one correspondance between the set of vector bundle morphisms of rank 1 everywhere and the set of equivariant embeddings of $\pi$ into $p$.

Let $u : E[\mathbb{C}] \to V$ be a vector bundle morphism of rank 1 everywhere, and let $h_u : E \to V$ be the map defined by

$$h_u(e) = u((e, 1) \mathbb{Z}_n).$$
for all $e \in E$. Then clearly $h_u$ commutes with projections and $h_u(e \cdot z) = zh_u(e)$ for all $z \in \mathbb{Z}_n$. Moreover, since $u$ is of constant rank 1, $h_u$ is an embedding. Consequently, $h_u$ is an equivariant embedding of $\pi$ into $p$.

Consider next an equivariant embedding $h : E \to V$ of $\pi$ into $p$. Let $u_h : E[\mathbb{C}] \to V$ be the map defined by

$$u_h((e, z)\mathbb{Z}_n) = zh(e),$$

for all $(e, z) \in E \times \mathbb{C}$. Then $u_h$ is a well defined vector bundle morphism of $\pi[\mathbb{C}]$ into $p$ of constant rank 1.

Finally, it is easily seen that $u \to h_u$ and $h \to u_h$ are maps that are inverse to each other. Therefore the lemma is now proved.

Now follows the main result in this section. Similar results with $n = 2$ and $p$ a real vector bundle can be found in [2] and [9]. See also [3, Theorems 7.1 and 7.4].

Let $c_1(\pi) \in H^2(X; \mathbb{Z})$ denote the first Chern class of the line bundle $\pi[\mathbb{C}]$, and let $c_i(p) \in H^{2i}(X; \mathbb{Z})$ denote the $i$'th Chern class of $p$.

**Theorem 2.2.** Assume that $X$ is a CW-complex of dimension $k \geq 1$. If $k < 2m$, then any principal $\mathbb{Z}_n$-bundle $\pi : E \to X$ can be embedded equivariantly into the complex vector bundle $p : V \to X$ of dimension $m$. If $k = 2m$, then

$$0 = \sum_{i=0}^{m} (-1)^i c_1(\pi)^m-i \cup c_i(p),$$

is a necessary and sufficient condition that there exists an equivariant embedding of $\pi$ into $p$. If $k > 2m$, then (*) is a necessary condition.

**Proof.** According to Lemma 2.1, it suffices to compute the Euler class $e \in H^{2m}(X; \mathbb{Z})$ of the vector bundle $\text{Hom}(\pi[\mathbb{C}], p)$, since the theorem then will follow from obstruction theory. Using the vector bundle equivalence $\text{Hom}(\pi[\mathbb{C}], p) = \pi[\mathbb{C}] \otimes p$, where $\pi[\mathbb{C}]$ is the conjugate bundle [6, Definition 7.6, p. 691] to $\pi[\mathbb{C}]$, we find that

$$e = c_m(\text{Hom}(\pi[\mathbb{C}], p))$$

$$= c_m(\pi[\mathbb{C}] \otimes p)$$

$$= \sum_{i=0}^{m} c_i(\pi)[\mathbb{C}]^{m-i} \cup c_i(p)$$

$$= \sum_{i=0}^{m} (-1)^{m-i}c_1(\pi)^{m-i} \cup c_i(p)$$

$$= \pm \sum_{i=0}^{m} (-1)^i c_1(\pi)^{m-i} \cup c_i(p).$$
In this context it should be noted that the first Chern class $c_1(\pi) \in H^2(X; \mathbb{Z})$ of a principal $\mathbb{Z}_n$-bundle $\pi : E \to X$ always satisfies the equation

$$nc_1(\pi)^j = 0,$$

for any power $j > 0$. This follows from the fact that $H^{2j}(B\mathbb{Z}_n; \mathbb{Z}) = \mathbb{Z}$, where $B\mathbb{Z}_n$ is the classifying space for numerable principal $\mathbb{Z}_n$-bundles. Alternatively, one could (for $j = 1$) refer to [7], where it is proved that $\pi[C]$ is an element of order $n$ in the group of isomorphism classes of line bundles over $X$.

When $n = 2$, there is an equivariant embedding if and only if there is an ordinary embedding of $\pi$ into $p$; cf. [3, p. 37 below Remark 7.3] and [9, Lemma 2.1]. Hence we have

**Corollary 1.3.** Let $X$ and $k$ be as in Theorem 1.2 and let $\pi : E \to X$ be a 2-fold covering over $X$. If $k < 2m$, then any 2-fold covering over $X$ embeds into $p$. If $k = 2m$, then (*1) is a necessary and sufficient condition that there exists an embedding of $\pi$ into $p$. If $k > 2m$, then (*) is a necessary condition.

### 3. Equivariant embeddings into trivial vector bundles

In this Section we discuss the special case where the vector bundle $p$ in question is the trivial complex $m$-plane bundle $\epsilon_m^n(X)$ over $X$.

For $1 \leq m \leq \infty$ and $2 \leq n < \infty$, let

$$\mu^m(n) : S^{2m-1} \to L^{2m-1}(n)$$

be the canonical principal $\mathbb{Z}_n$-bundle over the lens space $L^{2m-1}(n)$ as defined e.g. in Whitehead [10, Example 3, p. 91]. Clearly, $\mu^m(n)$ admits an equivariant embedding into $\epsilon_m^n(L^{2m-1}(n))$. On the other hand one has

**Example 3.1.** For $1 < m < \infty$, $c_1(\mu^m(n))^{m-1}$ is the generator of $H^{2m-2}(L^{2m-1}(n); \mathbb{Z}) \cong \mathbb{Z}$. Hence by Theorem 2.2, $\mu^m(n)$ cannot be embedded equivariantly into $\epsilon_{m-1}^m(L^{2m-1}(n))$. Similarly, $\mu^\infty(n)$ cannot be embedded equivariantly into any (finite dimensional) trivial vector bundle over $L^\infty(n)$.

The equivariant embedding $S^{2m-1} \to L^{2m-1}(n) \times C^m$ of $\mu^m(n)$ into $\epsilon_m^n(L^{2m-1}(n))$ is in fact a universal model for equivariant embeddings of principal $\mathbb{Z}_n$-bundles into trivial vector bundles. This statement is made precise in

**Theorem 3.2.** The following three statements are equivalent:

1. $\pi$ admits an equivariant embedding into $\epsilon_m^n(x)$.
2. The $m$-fold Whitney sum $\pi[C] \oplus \cdots \oplus \pi[C]$ has a nowhere zero section.
3. $\pi$ is equivalent to the pull-back of $\mu^m(n)$ along some map of $X$ into $L^{2m-1}(n)$.
Proof. Since \( \text{Hom}(\pi[C], e^n_\mathbb{C}(X)) \cong \tilde{\pi}[C] \otimes e^n_\mathbb{C}(X) \cong \tilde{\pi}[C] \otimes \cdots \otimes \tilde{\pi}[C] \), the equivalence of (1) and (2) follows from Lemma 2.1. Clearly, (3) implies (1). To see that (1) implies (3), assume that \( h : E \to X \times \mathbb{C}^m \) is an equivariant embedding of \( \pi \) into \( e^n_\mathbb{C}(X) \). Then actually \( h(E) \subseteq X \times (\mathbb{C}^m - \{0\}) \). Let \( \phi : E \to S^{2m-1} \) be the composite map \( \phi = P \circ pr_2 \circ h \), where \( pr_2 : X \times (\mathbb{C}^m - \{0\}) \to \mathbb{C}^m - \{0\} \) is projection on the second factor, and \( P : \mathbb{C}^m - \{0\} \to S^{2m-1} \) is radial projection. Then \( \phi \) is a \( Z_n \)-map, and hence \( \phi \) induces a map \( f : X \to L^{2m-1}(n) \) between the base spaces such that \( \pi \cong f^*(\mu^m(n)) \). □

Remark 3.3. Cf. [4, Example 4.3]. Let \( [\cdot, \cdot] \) denote sets of free homotopy classes of maps, and let \( P^m \mathbb{Z}_n(X) \) denote the set of equivalence classes of principal \( Z_n \)-bundles over \( X \) that admit equivariant embeddings into \( e^n_\mathbb{C}(X) \). According to Theorem 3.2 and covering space theory, there is an exact sequence

\[
\{\ast\} \to [X, S^{2m-1}] \to [X, L^{2m-1}(n)] \to P^m \mathbb{Z}_n(X) \to \{\ast\}
\]

of pointed sets. Thus the surjective map

\[
[X, L^{2m-1}(n)] \to P^m \mathbb{Z}_n(X),
\]

defined by pull-back of \( \mu^m(n) \), is bijective if and only if every map of \( X \) into \( S^{2m-1} \) is nullhomotopic. Hence \( \mu^m(n) \) is not a universal bundle in the sense of Husemoller [6, Definition 10.5, p. 52] for the functor \( P^m \mathbb{Z}_n \). By methods similar to those of [8], one can even show that no principal \( Z_n \)-bundle is universal for \( P^m \mathbb{Z}_n \).

We now make a further specialization assuming that \( m = 1 \). A principal \( Z_n \)-bundle \( \pi : E \to X \) that admits an equivariant embedding into \( e^1_\mathbb{C}(x) \) is said to be an equivariantly polynomial principal \( Z_n \)-bundle. Thus the equivariantly polynomial principal \( Z_n \)-bundles are the representatives of the elements of \( P^1 \mathbb{Z}_n(X) \). For the general definition of polynomial coverings, the reader is referred to [3].

Example 3.4. Let \( a : X \to S^1 \subseteq \mathbb{C} \) be a map and let \( P_a : X \times \mathbb{C} \to \mathbb{C} \) be the simple Weierstrass polynomial ([3], Definition 2.1) defined by

\[
P_a(x, z) = z^n - a(x),
\]

for all \( x \in X \) and \( z \in \mathbb{C} \). It is readily seen that the polynomial covering \( \pi_a \) associated to \( P_a \) admits the structure of a principal \( Z_n \)-bundle such that \( \pi_a \) becomes an equivariantly polynomial principal \( Z_n \)-bundle. Moreover, Theorem 3.2 implies, that for any equivariantly polynomial principal \( Z_n \)-bundle \( \pi \), there is a map \( a : X \to S^1 \) such that \( \pi \cong \pi_a \).

Below we list some criteria for \( \pi : E \to X \) to be an equivariantly polynomial principal \( Z_n \)-bundle. Let \( \partial : \pi_1(X, x_0) \to Z_n \) be the boundary operator in the homotopy sequence for \( \pi \), and let \( \delta_n : Z \to Z_n \) be reduction modulo \( n \) of the integers. Clearly, \( \delta_n \) can be identified with the boundary operator in the homotopy sequence for \( \mu^1(n) : S^1 \to L^1(n) \).
Proposition 3.5. For any principal $Z_n$-bundle $\pi : E \to X$ over a connected CW-complex $X$ with base point $x_0 \in X$, the following four statements are equivalent:

1. $\pi$ is an equivariantly polynomial principal $Z_n$-bundle.
2. $\pi[\mathcal{C}] = \varepsilon(X)$.
3. $c_1(\pi) = 0$.
4. There is a homomorphism $\partial' : \pi_1(X, x_0) \to \mathbb{Z}$ such that $\partial_n \circ \partial' = \partial$.

With the aid of this proposition we obtain an algebraic classification of $P^1 Z_n(X)$.

Proposition 3.6. For any CW-complex $X$, there is a bijective correspondence between the set $P^1 Z_n(X)$ and the group $H^1(X; \mathbb{Z}) \otimes \mathbb{Z}_n$.

Proof. The short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_n \to 0$$

of coefficient groups induces an exact sequence of cohomology groups

$$H^1(X; \mathbb{Z}) \to H^1(X; \mathbb{Z}) \to H^1(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}) \to \cdots,$$

in which $\lambda : H^1(X; \mathbb{Z}) \to H^1(X; \mathbb{Z}_n)$ is reduction modulo $n$ of the integer cohomology classes. According to the equivalence of the first and the fourth statement of Proposition 3.5, $P^1 Z_n(X)$ is in one-to-one correspondence with the image $\text{im} \lambda$. But by exactness, $\text{im} \lambda = H^1(X; \mathbb{Z})/nH^1(X; \mathbb{Z}) = H^1(X; \mathbb{Z}) \otimes \mathbb{Z}_n$. □

It follows from Proposition 3.6 that all equivariantly polynomial principal $Z_n$-bundles over $X$ are trivial, i.e. $P^1 Z_n(X) = \{\ast\}$, if and only if $H^1(X; \mathbb{Z})$ is divisible by $n$. The other extreme, i.e. every principal $Z_n$-bundle over $X$ admits an equivariant embedding into $\varepsilon(X)$, occurs if and only if $\lambda : H^1(X; \mathbb{Z}) \to H^1(X; Z_n)$ is an epimorphism.

We conclude this paper with an example of a principal $Z_n$-bundle that is polynomial but not equivariantly polynomial.

Let $\Sigma_n$ denote the symmetric group on $n$ letters and let $F_n(C)$ denote the configuration space for $n$ ordered points in $C$ with the usual $\Sigma_n$-action as considered in [1] and [4]. The $n$-cycle $(1, 2, 3, \ldots, n) \in \Sigma_n$ generates a cyclic subgroup isomorphic to $Z_n$. By restricting the $\Sigma_n$-action to this cyclic subgroup $Z_n \subset \Sigma_n$, we get a principal $Z_n$-bundle $\nu^1(n) : F_n(C) \to F_n(C)/\Sigma_n$. Obviously, $\nu^1(n)$ is a polynomial covering, but on the other hand one has

Proposition 3.7. For $n > 2$, the principal $Z_n$-bundle $\nu^1(n)$ does not admit an equivariant embedding into the trivial complex line bundle.

Proof. The identity map $1$ on $F_n(C)$ together with the canonical projection $p : F_n(C)/Z_n \to F_n(C)/\Sigma_n$ define a map between fibrations
Consider the induced map between the associated homotopy sequences

\[ \begin{array}{c}
1 \rightarrow \pi_1(F_n(\mathbb{C})) \rightarrow \pi_1(F_n(\mathbb{C})/\mathbb{Z}_n) \rightarrow \mathbb{Z}_n \rightarrow 1 \\
\downarrow \rho \downarrow \tau_n \\
1 \rightarrow \pi_1(F_n(\mathbb{C})) \rightarrow \pi_1(F_n(\mathbb{C})/\Sigma_n) \rightarrow \Sigma_n \rightarrow 1.
\end{array} \]

In the lower short sequence, \( \pi_1(F_n(\mathbb{C})/\mathbb{Z}_n) \) is the braid group \( B(n) \) on \( n \) strings. Let \( \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \) be the standard generators of \( B(n) \) as presented e.g. in [1, Theorem 1.8, p. 18]. Then the boundary operator \( \tau_n : B(n) \rightarrow \Sigma_n \) takes \( \sigma_i \) to the 2-cycle \((i, i + 1) \in \Sigma_n, 1 \leq i \leq n - 1\). By the 5-lemma, \( \rho_* \) is a monomorphism. Hence we can identify \( \pi_1(F_n(\mathbb{C})/\mathbb{Z}_n) \) with the subgroup \( \tau_n^{-1}(\mathbb{Z}_n) \) of \( B(n) \). The boundary operator \( \partial \) in the homotopy sequence for \( \nu^1(n) \) is thereby identified with the restriction \( \tau_n|\tau_n^{-1}(\mathbb{Z}_n) \) of \( \tau_n \) to \( \tau_n^{-1}(\mathbb{Z}_n) \).

Let \( \Lambda \in B(n) \) be the word

\[ \Lambda = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})(\sigma_1 \sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)(\sigma_1). \]

By [1, Lemma 2.5.1, p. 76] we have the identity

\[ \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 = \Delta (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) \Delta^{-1}, \]

so, since \( (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n \) generates the center of \( B(n) \) [1, Corollary 1.8.4, p. 28], it follows that

\[ (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1)^n \Delta = (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) \Delta^{-1} = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n. \]

Now assume that \( \delta' : \tau_n^{-1}(\mathbb{Z}_n) \rightarrow \mathbb{Z} \) is a homomorphism such that \( \partial_n \circ \delta' = \tau_n \mid \tau_n^{-1}(\mathbb{Z}_n) \).

By Proposition 3.5, we are through if we can bring this assumption to a contradiction.

Since \( \tau_n(\sigma_1 \sigma_2 \cdots \sigma_{n-1}) = (1, 2, \ldots, n) \) and \( \tau_n(\sigma_{n-1} \sigma_{n-2} \cdots, \sigma_1) = (1, 2, \ldots, n)^{-1} \), the words \( \sigma_1 \sigma_2 \cdots \sigma_{n-1} \) and \( \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 \) are elements of \( \tau_n^{-1}(\mathbb{Z}_n) \). From the equations

\[ n \delta'(\sigma_1 \sigma_2 \cdots \sigma_{n-1}) = \delta'(\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n \]

\[ = \delta'(\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1)^n \]

\[ = n \delta'(\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1), \]
we infer that $\partial'(\sigma_1\sigma_2\cdots\sigma_{n-1}) = \partial'(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$. Hence

$$(1, 2, \ldots, n) = \tau_n(\sigma_1\sigma_2\cdots\sigma_{n-1})$$

$$= \partial_n \circ \partial'(\sigma_1\sigma_2\cdots\sigma_{n-1})$$

$$= \partial_n \circ \partial'(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$$

$$= \tau_n(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$$

$$= (1, 2, \ldots, n)^{-1}.$$ 

For $n > 2$, this is indeed a contradiction. $\square$

To clarify the role of $\nu^1(n)$, note that the principal $\Bbb{Z}_n$-bundle $\pi : E \to X$ admits an ordinary embedding into $\varepsilon \Bbb{C}(X)$ if and only if $\pi$ is equivalent to the pull-back of $\nu^1(n)$ along some map of $X$ into $F_n(\Bbb{C})/\Bbb{Z}_n$. Similarly, let $\nu^m(n) : F_n(\Bbb{C}^m) \to F_n(\Bbb{C}^m)/\Bbb{Z}_n$ be the principal $\Bbb{Z}_n$-bundle defined by an obvious generalization of $\nu^1(n)$. Then $\pi : E \to X$ admits an ordinary embedding into $\varepsilon \Bbb{C}(X)$ if and only if $\pi$ is equivalent to the pull-back of $\nu^m(n)$ along some map of $X$ into $F_n(\Bbb{C}^m)/\Bbb{Z}_n$.

References