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β-Perfect Graphs

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The class of β-perfect graphs is introduced. We draw a number of parallels between these graphs and perfect graphs. We also introduce some special classes of β-perfect graphs. Finally, we show that the greedy algorithm can be used to colour a graph G with no even chordless cycles using at most $2(\chi(G)-1)$ colours.

1. β-Perfect Graphs

Contrary to our usual practice, we feel obliged to begin this paper with a few definitions. So, let $G=(V(G), E(G))$ be a graph without loops or multiple edges (for this and other definitions see Berge [1]). We denote the chromatic number of G by $\chi(G)$. We let $\omega(G)$ be the size of the largest clique in G. Clearly, $\chi(G)\geqslant\omega(G)$. For a vertex x of G we let $d(x)$ be the degree of x in G. We let δ_G be the minimum degree of a vertex in G and we let Δ_G be the maximum degree in G. We let $\beta(G)=\max\{\delta_{G'}+1 \mid G'$ is an induced subgraph of $G\}$. Now, we can order the vertices of G arbitrarily and then colour them greedily. Thus $\chi(G)\leqslant\Delta_G+1$. Actually, we can do better. We order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. (See [9] for results on this order; in [4] this order is discussed in relation to perfect graphs). Colouring greedily on this order shows that $\chi(G)\leqslant\beta(G)$ (as was proven in [9] and [14]).

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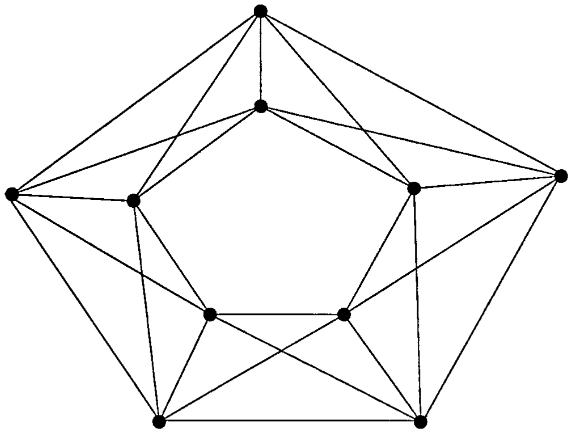

FIGURE 1

A graph G is *perfect* (Berge [2]) if for each induced subgraph H of G, $\chi(H)=\omega(H)$. Berge conjectured, and Lovász [7] proved, that G is perfect if and only if $\bar{G}$ is. Berge conjectured, and no one has proved, that G is perfect if and only if neither G nor $\bar{G}$ contains a chordless cycle on $2k+1$ vertices for $k \geqslant 2$.

We call a graph, G, β-perfect if for each induced subgraph H of G, $\chi(H)=\beta(H)$. Clearly, G can be β-perfect even though $\bar{G}$ is not (consider the chordless cycle on four vertices). However, a β-perfect graph can contain no chordless cycle on $2k$ vertices, $k \geqslant 2$ (since $\beta(C_{2k})=3$ and $\chi(C_{2k})=2$). In fact, we shall show that G and $\bar{G}$ are β-perfect if and only if neither G nor $\bar{G}$ contains a chordless cycle of length $2k$ for $k \geqslant 2$. This is an interesting analogue of Berge's conjecture. We prove this result in Section 3.

Now, although every β-perfect graph has no even induced cycle, some graphs with no induced even cycles are not β-perfect. (One such graph is depicted in Fig. 1. It is obtained from an induced cycle of length 5 by substituting an edge for each vertex. We can obtain similar examples by substituting an edge for each vertex of any odd cycle of length at least 5. For a related result, see [15], Theorem 6.) It would be of interest to determine exactly which graphs are β-perfect. Markossian and Karpetjan [8] and Meyniel [11] showed independently that if G is a graph in which every cycle on $2k+1$ vertices for $k \geqslant 2$ has at least two chords then G is perfect. In Section 4, we show that if G is a graph in which every cycle on $2k$ vertices for $k \geqslant 2$ has at least two chords then G is β-perfect. We shall also prove a more general analogue of their result, to wit:

If G is a graph which has no chordless cycle on $2k$ vertices (for $k \geqslant 2$) and no cycle on $2k$ vertices (for $k \geqslant 2$) with precisely one chord and such that this chord forms a triangle with two edges of the cycle (we shall call such a cycle *short-chorded*) then G is β-perfect.

In Section 2 we shall make a few more introductory remarks about β-perfect graphs, even cycles, and how they relate to perfect graphs.

2. Even Holes and β-Perfect Graphs

By a *hole*, we mean an induced subgraph isomorphic to a chordless cycle on at least four vertices. C_k denotes a hole on k vertices. C_k is *odd* if k is odd and *even* if k is even. If we say that G *contains* a cycle with one chord or a shortchorded even cycle, we mean that these structures appear as induced subgraphs in G. If G is β-perfect then G contains no even holes but there are graphs with no even holes which are not β-perfect. Nevertheless, we will show that if G has no even hole then for each induced subgraph G' of G, $\chi(G') \geqslant \beta(G')/2+1$. (Bruce Shepherd [12] has asked if there is an analogous result linking the clique number to the chromatic number in graphs with no odd holes.)

First, however, we consider the class of graphs which are both perfect and β-perfect. Clearly, any such graph must contain no holes. The graphs with no holes are called *chordal* or *triangulated* (see [5]), and we denote this class by T. Rose [13] showed that in any chordal graph there is a vertex of minimum degree whose neighbourhood is a clique; such vertices are called *simplicial*. It follows that for such graphs $\beta(G)=\omega(G)=\chi(G)$ and in fact $\beta(H)=\omega(H)=\chi(H)$ for any induced subgraph H. Thus, every triangulated graph is both perfect and β-perfect. So, the graphs which are both perfect and β-perfect are precisely the triangulated graphs.

We now show that if G contains no even hole then $\chi(G) \geqslant \beta(G)/2+1$. We note first that G has no even hole if and only if every 2-colourable induced subgraph of G is acyclic. Let G be a graph with no even hole and consider a colouring of G with $\chi(G)$ colours. If $\chi(G)=1$ then the result is trivial. So, we can assume $\chi(G) \geqslant 2$. Let G' be any induced subgraph of G and for $1 \leqslant i \leqslant \chi(G)$ let S_i be the set of vertices of G' with colour i. Clearly, $|E(G')| = \sum_{i<j} |E(S_i \cup S_j)|$. Now, by our earlier remark, $S_i \cup S_j$ is acyclic. Thus, $|E(G')| \leqslant \sum_{i<j} (|S_i|+|S_j|-1)$. It follows that $|E(G')| \leqslant (\chi(G)-1)\,|V(G')| - \binom{\chi(G)}{2}$. Thus, $|E(G')| < (\chi(G)-1)\,|V(G')|$. So there is some vertex of G' with degree less than $2(\chi(G)-1)$. Therefore, $\beta(G) \leqslant 2(\chi(G)-1)$, as required.

We note that this bound can be achieved for $\chi(G)=2$ (any tree) or $\chi(G)=3$ (see Fig. 2). Also, the situation for arbitrary graphs is markedly different. The complete bipartite graph $K_{n,n}$ has $\beta(K_{n,n})/\chi(K_{n,n})=n/2$. It

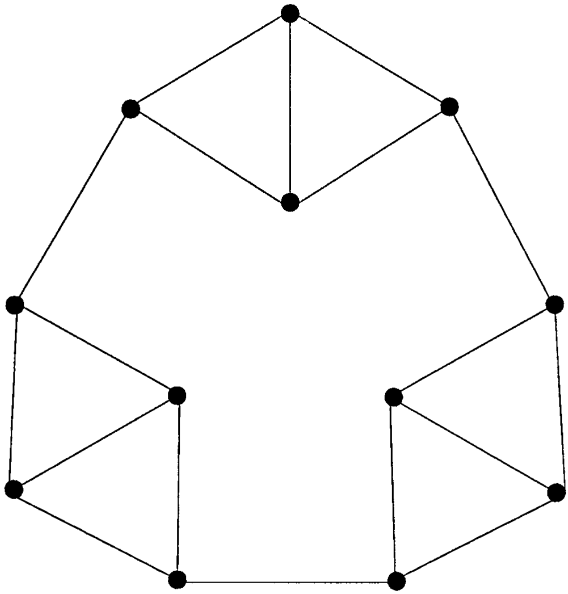

FIGURE 2

would be of interest to find a polynomial time recognition algorithm for the class of graphs which contain no even holes.

3. β-PERFECT GRAPHS WITH β-PERFECT COMPLEMENTS

In this section, we show that both G and $\overline{G}$ are β-perfect if and only if neither G nor $\overline{G}$ contains an even hole. We have already seen that G and $\overline{G}$ can both be β-perfect only if neither contains an even hole. To show the converse we need the following structure theorem.

THEOREM 1. *G and $\overline{G}$ both contain no even holes if and only if either*

(i) *the vertices of G can be partitioned into a clique C and a stable set S, or*

(ii) *the vertices of G can be partitioned into a clique C, a stable set S and a hole H of length 5, such that each vertex of H is adjacent to all of C and none of S.*

Proof. It is easy to see that if G satisfies (i) or (ii) then neither G nor $\overline{G}$ contains an even hole. It remains to prove the converse.

So let G be a graph such that neither G nor $\overline{G}$ contains an even hole. We show first that if G contains a C_5 then it satisfies (ii).

If G contains a C_5 let H be a C_5 of G. We note that any vertex x of $G-H$ is adjacent to either none of H or all of H, as otherwise $H+x$ or $\overline{H+x}$ contains a C_4. Let C be the set of vertices which are adjacent to all of H. Let S be the set of vertices which are adjacent to none of H. If some $x, y \in C$ are not adjacent then $x+y+H$ contains a C_4. If some $x, y \in S$ are adjacent then $x+y+H$ contains a $\overline{C_4}$. Thus, we know that C is a clique and S is a stable set, as required.

Now, consider the case when G contains no C_5. Let C be a maximum clique of G chosen in addition so that $\sum_{x \in C} d(x)$ is maximized. We claim that $G-C$ is a stable set.

Otherwise, there is some edge xy in $G-C$. If there are $w, z \in C$ such that $xw, yz \in E(G)$, $xz, yw \notin E(G)$ then $\{x, w, z, y\}$ is a C_4 in G, a contradiction. Thus, we can assume that x is adjacent to every vertex of C which is adjacent to y. Now if there are $w, z \in C$ such that $xw, xz \notin E(G)$ then $\{w, x, z, y\}$ is a $\overline{C_4}$ in G, a contradiction. Thus x misses at most one vertex of C. Since C is maximum, x misses exactly one vertex of C call this vertex z. Now, $C-z+x$ is also a clique. Also x sees y but z does not. Thus, by our choice of C, there is some vertex w which sees z but not x. Since C is maximum, there is some vertex z' of C which misses w. Now, w sees y as otherwise $\{w, z, x, y\}$ is a $\overline{C_4}$. Thus, y misses z' as otherwise $\{w, z, z', y\}$ is a C_4. But then, $\{w, z, z', x, y\}$ is a C_5 in G, a contradiction. ▮

COROLLARY. *If neither G nor $\bar{G}$ contain an even hole then G is β-perfect.*

Proof. Let G be a graph such that neither G nor $\bar{G}$ contains on even hole. Then if G' is an induced subgraph of G, G' satisfies (i) or (ii). Now, if G' satisfies (i) then every vertex of minimum degree in G' is simplicial, so $\chi(G) \geqslant \delta_{G'}+1$. If G' satisfies (ii) then $\chi(G') = |C|+3$ and each vertex of H has degree at most $|C|+2$ so $\chi(G) \geqslant \delta_{G'}+1$. Thus, $\chi(G) \geqslant \beta(G)$. The result follows. ▮

COROLLARY. *Both G and $\bar{G}$ are β-perfect if and only if both G and $\bar{G}$ contain no even holes.* (*Note that this is equivalent to requiring that neither G nor $\bar{G}$ contains a C_4.*)

We note that graphs which satisfy (i) *are precisely $T \cap \bar{T}$. These are studied in* [5] *where they are called* split *graphs.*

Determining if either G or $\bar{G}$ contains a C_4 can clearly be done in polynomial time.

4. Two Classes of β-Perfect Graphs

In this section, we discuss graphs which contain no even holes and no even cycles with one chord. This class of graphs turns out to have a very simple characterization. We also discuss those graphs which contain no even holes and no short-chorded even cycles. (Note that the first class is strictly contained in the second). We show that these graphs are β-perfect. We shall need the following lemma in the proof of both results.

LEMMA 2. *Let G be a graph with no even holes and no short-chorded even cycles. Then every block of G is either a clique or triangle-free.*

Proof. Let G be a graph with no even holes and no short-chorded even cycles. Let B be a block of G which contains a triangle. Let C be a maximum clique of B. If $C \neq B$ then for each x in $B - C$ we can find two vertex disjoint paths from x to C. By choosing x and the paths P_1, P_2 so that the total length of the paths is minimized, we can ensure that $V(P_1) \cup V(P_2)$ induces a chordless cycle H. Now, if H is a triangle then H consists of an edge of C and x. Since C is maximal, we can choose a vertex z of C which x misses and then $z + H$ is a short-chorded even cycle. If H is not a triangle then choose some $z \in C - H$. Clearly, $z + H$ contains an even hole or a short-chorded even cycle. These contradictions imply that $C = B$, as required. ▮

THEOREM 3. *A graph G contains no even hole and no even cycle with one chord if and only if every block of G is either a clique or an odd hole.*

Proof. Clearly if every block of G is either a clique or an odd hole then every even cycle in G has at least two chords. Thus, we need only show that if G is a graph with no even hole and no even cycle with one chord then every block of G is either a clique or an odd hole. By Lemma 2, we need only show that each triangle-free block of G is an odd hole or an edge. Let B be a triangle-free block of G which is not an edge. Since B contains a cycle, B must contain an odd hole.

We can choose an odd hole H such that no $x \in B - H$ sees more than one vertex of H; we simply choose the minimum length odd hole in B. Assume $B \neq H$.

Now choose $y \in B - H$ and vertex disjoint paths P_1 and P_2 from y to H so that $|P_1| + |P_2|$ is minimized. Let u be the endpoint of P_1 on H and let v be the endpoint of P_2 on H. Let P_3 be the path of H between u and v with the same parity as $P_1 + P_2$. Then, $P_1 + P_2 + P_3$ is an even cycle and $P_1 + P_2 + P_3$ has no chords unless uv is an edge of G in which case $P_1 + P_2 + P_3$ may have one chord. This is a contradiction. Thus $B = H$, as required. ▮

In order to prove that graphs with no even holes and no short-chorded even cycles are β-perfect we shall need the following result which is of some independent interest.

THEOREM. *Let G be a graph with no triangle and no even hole. Let x be a vertex of G. Then either x sees all of G or there is some y of $G-x$ not adjacent to x such that y has degree at most 2 in G.*

Proof. Assume the theorem is not true and consider a counterexample G with as few vertices as possible. We note that no vertex z of G sees all of $G-z$ or otherwise G is a star centered at z. Now, let x be a vertex of G such that there is no vertex of $G-x$ non-adjacent to x which has degree at most 2. We show first that:

(1) G is two-connected.

To prove this we assume G is not two-connected and derive a contradiction. To do so, we will need to consider the *block-tree* of G. Its vertices are the blocks and cutpoints of G. A cutpoint is adjacent precisely to those blocks which contain it. We note that all the leaves of this tree are blocks of G containing exactly one cutpoint. If x is not in all the blocks of G then there is some leaf B_1 of this tree not containing x. If x is in all the blocks of G then the block tree of G has x as its unique non-leaf node. Then since x does not see all of G there is a block B_1 of G which has x as its unique cutpoint and is not an edge. In either case if G is not two-connected we arrive at a contradiction by considering B_1. If B_1 is an edge then the vertex of B_1 which is not a cutpoint of G has degree 1 in G and misses x. If B_1 is not an edge then consider x_1 the unique cutpoint of B_1. By minimality there is a vertex y of B_1-x_1 which misses x_1 and has degree at most two in B_1. Clearly, y has degree at most two in G, a contradiction. Next, we note that:

(2) There is an odd hole C through x such that no vertex of $G-C$ sees more than one vertex of C.

To see this, simply consider a minimum length cycle C through x, obviously it is induced and hence it must be an odd hole. If some vertex y sees vertices a and b of C let P_1 be the path of C from a to b containing x and let P_2 be the other path of C from a to b. Since $y \cup P_2$ does not contain a triangle and is not a C_4, the cycle formed by $y \cup P_1$ is shorter than C, a contradiction.

Now, let C have vertices $\{v_0, v_1, v_2, ..., v_k\}$ and edges $v_i v_{i+1}$ for i between 0 and k (addition modulo $k+1$) where $v_0 = x$. Let S_i be the set of vertices of $G-C$ which see v_i and let $S = \bigcup_{i=0}^{k} S_i$. We note that, as G is triangle-free, each S_i is a stable set. Furthermore, by our choice of C, the S_i are disjoint. Note that:

(3) If $|i-j| > 1$ (modulo $k+1$) then there is no component H of $G-C-S$ containing vertices y and z such that y sees a vertex u_i of S_i and z sees a vertex u_j of S_j.

Otherwise consider an induced path P_3 from u_i to u_j in $H \cup \{u_i, u_j\}$. Consider the two paths P_1 and P_2 from v_i to v_j in C. Clearly, $P_1 + P_3$ and $P_2 + P_3$ are induced cycles of different parity in G, a contradiction. Similarly, we can show:

(4) S is a stable set.

Furthermore:

(5) There is no component H of $G - C - S$ such that for some i, every edge with exactly one endpoint in H has its other endpoint in S_i.

Otherwise, consider F the graph induced by $H + v_i + S_i$. By induction, there is a vertex y of $F - v_i$ non-adjacent to v_i which has degree at most two in F. Clearly, y is in H and hence is non-adjacent to x and has degree at most 2 in G, a contradiction.

Now, by (3) and (5) we can partition $G - C - S$ up into subgraphs $H_0, \ldots, H_k$ such that

$H_i = \cup \{H \mid H$ is a component of $G - C - S$ and there is an edge between H and S_i, and an edge between H and S_{i+1} but there is no edge between H and $S - S_i - S_{i+1}\}$.

We note that:

(6) If $u_i \in S_i$, $u_{i+1} \in S_{i+1}$ and u_i and u_{i+1} see vertices x and y respectively in some component H of H_i then any induced u_i to u_{i+1} path in $H \cup \{u_i, u_{i+1}\}$ has even length.

This is because for any u_i to u_{i+1} induced path P in $H \cup \{u_i, u_{i+1}\}$, we have that $P \cup \{v_i, v_{i+1}\}$ induces a hole.

We can conclude from (6) that:

(7) There is no vertex a of $S_i \cup S_{i+1}$ which sees vertices in two components of H_i.

To see this, assume the contrary and let $a \in S_i \cup S_{i+1}$ see vertices in two components F_1 and F_2 of H_i. By symmetry we can assume that $a \in S_i$. Let b_1 and b_2 be vertices of S_{i+1} such that b_1 sees some element of F_1 and b_2 sees some element of F_2. Let P_1 be an induced a to b_1 path in $F_1 \cup \{a, b_1\}$ and let P_2 be an induced a to b_2 path in $F_2 \cup \{a, b_2\}$. Now, by (6), if $b_1 = b_2$ then $P_1 \cup P_2$ induces an even hole, a contradiction. If $b_1 \neq b_2$ then, again by (6), $P_1 \cup P_2 \cup \{v_{i+1}\}$ induces an even hole, a contradiction.

Using (7), we can show that:

(8) For any i and any component F of H_i, either there is no vertex in S_i which sees both a vertex of F and vertex of $G - F - v_i$ or there is no vertex in S_{i+1} which sees both a vertex of $G - F - v_{i+1}$ and a vertex of F.

Otherwise, let u_i be a vertex of S_i which sees a vertex of $G-F-v_i$ and a vertex of F and let u_{i+1} be a vertex of S_{i+1} which sees a vertex of $G-F-v_{i+1}$ and a vertex of F. By (7), u_i sees a vertex of H_{i-1} and u_{i+1} sees a vertex of H_{i+1}. Now, let P_1 be an induced path in $F \cup \{u_i, u_{i+1}\}$ from u_i to u_{i+1}. Let P_2 be a v_{i-1} to u_i path in $\{v_{i-1}, u_i\} \cup S_{i-1} \cup H_{i-1}$. Let P_3 be a u_{i+1} to v_{i+2} path in $\{u_{i+1}, v_{i+2}\} \cup H_{i+1} \cup S_{i+2}$. By (6), P_1 has even length and both P_2 and P_3 have odd length. It follows that $P_1 \cup P_2 \cup P_3 \cup C - \{v_i, v_{i+1}\}$ is an even hole in G, a contradiction.

Now, recall that $x = v_0$. We claim that:

(9) One of H_1 or H_2, must be non-empty.

Otherwise, if $S_2 \neq \emptyset$ each vertex of S_2 would have degree one in G while if $S_2 = \emptyset$ then v_2 would have degree 2 in G. In either case, there is clearly a vertex of G which is not adjacent to x but has degree at most two in G, a contradiction.

To complete our proof consider some component F of H_j where j is 1 or 2. (We know such an F exists by (9)). Let G_1 be the graph induced by $F \cup \{v_j, v_{j+1}\} \cup \{x \mid S_j \cup S_{j+1}$ and x sees some vertex of $F\}$. By (8), either there is no vertex of $S_j \cap G_1$ which sees a vertex of $G-F-v_j$ or there is no vertex of $S_{j+1} \cap G_1$ which sees a vertex of $G-F-v_{j+1}$. In the first case set $k = j+1$ and in the second set $k = j$. By the minimality of G, there is a vertex y of $G_1 - v_k$ non-adjacent to v_k which has degree at most 2 in G_1. By our choice of k, y also has degree at most 2 in G. It follows that y is a vertex of $G-x$ non-adjacent to x and of degree at most 2, a contradiction. ∎

We shall use Lemma 2 and Theorem 4 to prove the following.

Theorem 5. *Any graph with no even holes and no even short-chorded cycle is β-perfect.*

Proof. Let G be a graph with no even holes and no even short-chorded cycle. Clearly it is enough to show that $\chi(G') \geqslant \delta_{G'} + 1$ for any induced subgraph G' of G with $|G'| \geqslant 2$. Now, let B be a block which is a leaf of the block tree of G'. Let x be the unique vertex of B in some other block of G' (choose x arbitrarily if $G = B$).

By Lemma 2, B is a clique or triangle-free. If B is a clique let y be any vertex of $B-x$. Clearly $\delta_{G'} + 1 \leqslant d_{G'}(y) + 1 \leqslant |B| \leqslant \chi(G')$. If B is not a clique then B is a two-connected graph containing no triangles and no even holes. Thus, B contains an odd hole H and so x is not adjacent to all of B. By Theorem 4, $B-x$ contains a vertex y not adjacent to x such that y has degree at most two in B. Now, $\delta_{G'} + 1 \leqslant d_{G'}(y) + 1 \leqslant 3 \leqslant \chi(H) \leqslant \chi(G')$. It follows that $\chi(G') \geqslant \delta_{G'} + 1$ for each induced subgraph G', as required. ∎

5. Remarks

It would be of interest to determine the complexity of deciding whether or not a graph contains an even hole. We note that Bienstock [3] has shown that determining if a graph contains an even hole through a specified vertex is NP-complete. Reed (unpublished) has shown that we can decide whether or not a planar graph contains an even hole in polynomial time. McDiarmid, Reed, Schrijver, and Shepherd [10] have shown that we can determine if two specified vertices of a planar graph lie on a hole in polynomial time. It would be of interest to determine the complexity of deciding whether or not there is an even hole through a specified vertex of a planar graph. It would also be of interest to determine the complexity of deciding if a given graph is β-perfect. This problem is in co-NP for if a graph is not β-perfect then it contains a subgraph whose chromatic is less than or equal to its minimum degree. To show that the graph is not β-perfect, we merely exhibit an optimal colouring of such an H.

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