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Perron Theorem in the monotone iteration method for traveling waves in delayed reaction–diffusion equations

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Abstract

In this paper we revisit the existence of traveling waves for delayed reaction–diffusion equations by the monotone iteration method. We show that Perron Theorem on existence of bounded solution provides a rigorous and constructive framework to find traveling wave solutions of reaction–diffusion systems with time delay. The method is tried out on two classical examples with delay: the predator–prey and Belousov–Zhabotinskii models.

MSC: 35K55; 35R10

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1. Introduction

We shall be concerned with the existence of traveling waves solutions for the delayed reaction–diffusion system

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(u_t). \tag{1}$$

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Due to their important applications in population dynamics and biological models, see e.g. [3–9,12,14–16,20,23,24,27,28,30–34,36,37], equations such as (1) have evolved from the simple one-dimensional scalar reaction–diffusion equation to systems that include a delay in time for more realistic modeling. Traveling waves, although are a classical topic in applied mathematics, remain a driving force in the study of (1). We refer the reader to [10,30] which contain many surveys on methods used to study traveling waves in parabolic differential equations. Note that when delay is introduced, most methods would fail if they are not modified appropriately (see e.g. [11,34]).

Recently, Wu and Zou [32,36] have extended the method of monotone iterations to deal with the existence of traveling wave fronts for delayed equations (1) (see also [14,15,20,21]). This produces a monotone sequence of positive functions that converges to a traveling wave front, which is an increasing positive solution of an equation of the form

$$D\phi''(\xi) - c\phi'(\xi) + f_c(\phi_\xi) = 0, \quad \xi \in \mathbb{R}. \tag{2}$$

This method is contingent on the construction of a pair of upper and lower solutions, which satisfy the inequality versions of (2). It turns out that this issue is closely related to the existence and uniqueness of smooth and bounded solutions of nonhomogeneous equations of the form

$$Dx''(t) - cx'(t) - \beta x(t) + g(t) = 0, \quad \text{for all } t \in \mathbb{R}, \tag{3}$$

where the constants $c, \beta > 0$. In differential equations results of this type are known as Perron Theorem, see [1,13,25,26] and for interesting applications see [1,2] and the references therein.

We would like to point out that the smoothness required for upper and lower solutions of (2) obstruct the search for bounded solutions of (3). One faces the following dilemma, see e.g. [20, 32,36]. An excessive relaxation of the smoothness of upper and lower solutions of (2) simplifies their finding, but would not generate the sought monotone iteration scheme. The main reason is due to the failure of Perron Theorem for this class of weak solutions of (3).

In the light of the above remark, the question of existence of traveling waves front in predator–prey or Belousov–Zhabotinskii models with delay, as addressed in [20], remains open.

The purpose of this paper is twofold: First to set up a rigorous framework for the monotone iteration method and then apply it to the predator–prey and Belousov–Zhabotinskii models with delay.

We now briefly outline the plan of this paper. In the next section, to recall the main concepts and tools, we discuss a modified version of Perron Theorem for C^1-solutions of (3). Next we show how crucial it is to the monotone iteration method. Remarks and counterexamples are used to explain the pitfalls of non-smooth upper solutions. In Theorem 11 one finds a rigorous framework to construct fail-safe upper and lower solutions which are then used for the delayed predator–prey and Belousov–Zhabotinskii equations, see Theorems 15, 17.

To conclude we would like to emphasise that Perron Theorem dictates C^1-smoothness which makes the search for upper and lower solutions much harder than C^0-smooth solutions as in [20], and of course easier than C^2-solutions, which seems to be impossible for the above mentioned equations.

2. The monotone iteration scheme

In this section we introduce a modified version of Perron Theorem which is central to a rigorous framework for the monotone iteration scheme. We show that this framework provides a clean procedure for constructing upper and lower solutions of reaction–diffusion equation with delay.

2.1. Bounded solutions of nonhomogeneous equations

The theory of bounded solutions of nonhomogeneous equations is a classical topic of the theory of ordinary differential equations, which can be found in [1,13,19,25] and the references therein.

Let us first consider the equation

$$u''(t) + \alpha u'(t) + \beta u(t) + f(t) = 0, \quad t \in \mathbb{R},\ u(t) \in \mathbb{R}, \tag{4}$$

where f is a function that is continuous and bounded on $\mathbb{R} \setminus \{0\}$ and has the right and left limits at $x = 0$, $f(0^+)$ and $f(0^-)$; we always assume that α and β are real numbers with $\beta < 0$ so that the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0$$

has two distinct real roots of opposite signs $\lambda_1 < 0$ and $\lambda_2 > 0$. To our knowledge, all available results on classical solutions deal with bounded and continuous forcing terms f on the entire real line (see e.g. [1,13,19,25]). As we allow jump discontinuities in f in (4), we need to modify the concept of solutions as well as the conditions for their existence and boundedness. We first need a definition.

Definition 1. Suppose that f is a bounded and continuous function on $\mathbb{R} \setminus \{0\}$ and both $f(0^+)$ and $f(0^-)$ exist. Then, a function u defined on $\mathbb{R}$ is said to be a generalized solution of (4) if

(1) u and u' are bounded and continuous on $\mathbb{R}$,
(2) u'' exists and is continuous on $\mathbb{R} \setminus \{0\}$, and both $u''(0^-)$ and $u''(0^+)$ exist.

Below is a version of Perron Theorem for generalized solutions with discontinuous f.

Lemma 2. *Consider Eq.* (4) *with $\beta < 0$, and assume that*

(1) *f is a bounded and continuous function on $\mathbb{R} \setminus \{0\}$ and both $f(0^+)$ and $f(0^-)$ exist,*
(2) *Eq.* (4) *holds in the classical sense for all t except possibly at $t = 0$.*

Then, Eq. (4) *has a unique generalized solution u given by*

$$u(t) = Gf(t) := \frac{1}{\lambda_1 - \lambda_2}\left(\int_{-\infty}^{t} e^{\lambda_1(t-s)} f(s)\,ds + \int_{t}^{+\infty} e^{\lambda_2(t-s)} f(s)\,ds\right). \tag{5}$$

Proof. Observe that function Gf, which is defined by (5), exists and is bounded. A simple computation shows that Gf is in fact continuously differentiable on $\mathbb{R}$, and $(Gf)''$ exists and is continuous on the whole $\mathbb{R}$ with a possible exception at $t = 0$ at which both $(Gf)''(0^+)$ and $(Gf)''(0^-)$ exist. It is important to observe that while Gf and $(Gf)'$ are bounded on $\mathbb{R}$, Gf is a particular solution of (4) on each of the two disjoint intervals $(-\infty, 0)$ and $(0, +\infty)$. Obviously, $v = u - Gf$ is then a classical solution of the homogeneous equation associated with (4) on $(0, \infty)$, and hence

$$u(t) - Gf(t) = ae^{\lambda_1 t} + be^{\lambda_2 t}, \quad \text{for } t > 0,$$

where a, b are constants. From the boundedness of u and Gf on $(0, \infty)$ it follows that $b = 0$, i.e.

$$u(t) - Gf(t) = ae^{\lambda_1 t}, \quad \text{for } t > 0.$$

Similarly the classical solution $u - Gf$ on $(-\infty, 0)$ is of the form

$$u(t) - Gf(t) = de^{\lambda_2 t}, \quad \text{for } t < 0.$$

Combining both behaviors we have

$$u(t) - Gf(t) = \begin{cases} ae^{\lambda_1 t}, & t > 0, \\ de^{\lambda_2 t}, & t \leqslant 0. \end{cases} \tag{6}$$

Use the fact that although $u(t) - Gf$ has no second derivative at $t = 0$, it is still continuously differentiable on the real line, and in particular at $t = 0$, to deduce the interface conditions

$$\begin{cases} a = d, \\ \lambda_1 a = \lambda_2 d. \end{cases}$$

Since $0 \neq \lambda_1 \neq \lambda_2 \neq 0$, we must have $a = d = 0$, that is $u - Gf = 0$. This completes the proof of the lemma. □

Remark 3. Lemma 2 shows that the continuity of u' plays a crucial role in the uniqueness of the solution. The failure of the continuity even at a single point, would lead to a non-uniqueness of bounded solutions. We now illustrate this important fact by a simple and yet explicit example.

Example 4. Consider bounded solutions for the equation

$$y''(t) - y(t) = 0 \quad \text{where } t \in \mathbb{R}. \tag{7}$$

Since $f = 0$, by Lemma 2 the only bounded continuously differentiable solution (that is, generalized solution) would be 0. However, if we allow for a non-differentiable function to be a solution, then the continuous function

$$y(t) := \begin{cases} e^{-t}, & t > 0, \\ e^{t}, & t \leqslant 0, \end{cases} \tag{8}$$

satisfies (7) for all $t \in \mathbb{R} \setminus \{0\}$. Obviously, it is a non-zero bounded and continuous function on $\mathbb{R}$, but not differentiable at $t = 0$.

Remark 5. The function y in (8) can also serve as a counterexample to the identity (see the proof of [20, Lemma 3.5])

$$\frac{d}{\lambda_2-\lambda_1}\left(\int_{-\infty}^{t} e^{\lambda_1(t-s)}\varphi(s)\,ds+\int_{t}^{\infty} e^{\lambda_2(t-s)}\varphi(s)\,ds\right)$$
$$=y(t)+\frac{1}{\lambda_2-\lambda_1}\sum_{j=k+1}^{m} e^{\lambda_1(t-T_j)}\big(y'(T_j+)-y'(T-)\big)$$
$$-\sum_{j=1}^{k} e^{\lambda_2(t-T_j)}\big(y'(T_j+)-y'(T-)\big), \tag{9}$$

where y is a piecewise C^2 solution of

$$dy''(t)-cy'(t)-\beta y(t)=\varphi(t)\quad \text{on } \mathbb{R}\setminus\{T_0,\dots,T_m\}.$$

In the particular case when $d=1$, $\lambda_1=-1$, $\lambda_2=1$, $c=0$, $\beta=1$, $m=0$, $T_0=0$, the expression (8) defines a solution to $y''(t)-y(t)=0$ for all $t\neq 0$. Since $\varphi=0$, the left-hand side of (9) is zero while the right-hand side is not

$$0=y(t)+\frac{1}{2}\sum_{j=k+1}^{m} e^{-t}\big(y'(0^+)-y'(0^-)\big)-\frac{1}{2}\sum_{j=1}^{k} e^{t}\big(y'(0^+)-y'(0^-)\big),$$
$$0=y(t)+\frac{1}{2}\sum_{j=k+1}^{m} e^{-t}(-2)-\frac{1}{2}\sum_{j=1}^{k} e^{t}(-2),$$
$$0=y(t)-e^{-t}+e^{t}\neq 0.$$

We hope that the argument for a continuous y' is by now clear and self-evident.

2.2. Delayed reaction–diffusion equations

Consider the following system of reaction–diffusion equations with time delay

$$\frac{\partial u(x,t)}{\partial t}=D\frac{\partial^2 u(x,t)}{\partial x^2}+f(u_t), \tag{10}$$

where $t\geqslant 0$, $x\in\mathbb{R}$, $u(x,t)\in\mathbb{R}^n$, $D=\operatorname{diag}(d_1,\dots,d_n)$ with $d_i>0$ for all $i=1,\dots,n$, $f: C([-\tau,0],\mathbb{R}^n)\to\mathbb{R}^n$ is a continuous, and $u_t(x)$ is an element of $C([-\tau,0],\mathbb{R}^n)$, defined as

$$u_t(x)=u(t+\theta,x),\quad \theta\in[-\tau,0],\ t\geqslant 0,\ x\in\mathbb{R}.$$

Throughout this paper we will assume that

$$f(\hat{0})=f(\hat{K})=0\quad\text{and}\quad f(\hat{u})\neq 0,\quad \text{for } u\in\mathbb{R}^n \text{ with } \mathbf{0}<u<\mathbf{K}, \tag{11}$$

where $\hat{0}$ denotes the function $\varphi \in C([-\tau, 0], \mathbb{R}^n)$ such that $\varphi(\theta) = \mathbf{0} := \{0, 0, \ldots, 0\}^T \in \mathbb{R}^n$ for all $\theta \in [-\tau, 0]$ and $\hat{K}$ denotes the function $\psi \in C([-\tau, 0], \mathbb{R}^n)$ such that $\psi(\theta) = \mathbf{K} := (K_1, K_2, \ldots, K_n)^T \in \mathbb{R}^n$, with given positive K_i, $i = 1, 2, \ldots, n$. In order to use comparison arguments, we use the natural partial order in $\mathbb{R}^n$ to compare two vectors $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, that is $x \geqslant y$ if and only if $x_i \geqslant y_i$, for all $i = 1, \ldots, n$, and if there is at least an i in $\{1, \ldots, n\}$ such that $x_i < y_i$, we write $x < y$. The "interval" $[\mathbf{0}, \mathbf{K}]$ consists of all vectors $v \in \mathbb{R}^n$ such that $0 \leqslant v_i \leqslant K_i$ for all $i = 1, 2, \ldots, n$.

As usual we look for traveling wave solutions of (10) in the form $u(x,t) = \phi(x + ct)$, where $\phi \in C^2(\mathbb{R}, \mathbb{R}^n)$, and $c > 0$ is a constant. Substituting $u(x,t) = \phi(x + ct)$ into (10) leads to the following *wave equation*

$$D\phi''(\xi) - c\phi'(\xi) + f_c(\phi_\xi) = 0, \quad \xi \in \mathbb{R}, \tag{12}$$

where $f_c : X_c := C([-c\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$, defined as

$$f_c(\psi) = f\left(\psi^c\right), \qquad \psi^c(\theta) := \psi(c\theta), \quad \theta \in [-\tau, 0].$$

Throughout this section we assume that there exists a matrix $\beta = \operatorname{diag}(\beta_1, \ldots, \beta_n)$ with $\beta \geqslant 0$ such that

$$f_c(\phi) - f_c(\psi) + \beta\left[\phi(0) - \psi(0)\right] \geqslant \mathbf{0}, \quad \text{for all } \phi, \psi \in X_c,\ \mathbf{0} \leqslant \psi(\theta) \leqslant \phi(\theta) \leqslant \mathbf{K}, \tag{13}$$

where $\mathbf{0} := (0, \ldots, 0)^T \in \mathbb{R}^n$ and $\mathbf{K} := (K, \ldots, K)^T \in \mathbb{R}^n$.

The main purpose of this paper is to look for solutions ϕ of (12) in the following subset of $C(\mathbb{R}, \mathbb{R}^n)$

$$\Gamma := \left\{\varphi \in C\left(\mathbb{R}, \mathbb{R}^n\right) \colon \varphi \text{ is non-decreasing, and } \lim_{\xi \to -\infty} \varphi(\xi) = \mathbf{0},\ \lim_{\xi \to +\infty} \varphi(\xi) = \mathbf{K}\right\}.$$

The solution ϕ is called a *monotone wave front* of (10). Next we define an operator $H : BC(\mathbb{R}, \mathbb{R}^n) \to BC(\mathbb{R}, \mathbb{R}^n)$ by

$$H(\phi)(t) = f_c(\phi_t) + \beta\phi(t), \quad \phi \in C\left(\mathbb{R}, \mathbb{R}^n\right). \tag{14}$$

The following monotonicity lemma was proved in [32].

Lemma 6. *Assume that* (13) *and* (14) *hold. Then, for any $\phi \in \Gamma$, we have that*

(1) *$H(\phi)(t) \geqslant 0$, $t \in \mathbb{R}$,*
(2) *$H(\phi)(t)$ is non-decreasing in $t \in \mathbb{R}$,*
(3) *$H(\psi)(t) \leqslant H(\phi)(t)$ for all $t \in \mathbb{R}$, if $\psi \in C(\mathbb{R}, \mathbb{R}^n)$ is given so that $\mathbf{0} \leqslant \psi(t) \leqslant \phi(t) \leqslant \mathbf{K}$ for all $t \in \mathbb{R}$.*

Our definition of upper solutions, which is found below, requires more smoothness and boundedness than those in [32] and [20].

Definition 7. A function $\rho \in C^2(\mathbb{R}, \mathbb{R}^n)$ with ρ, ρ', ρ'' being bounded on $\mathbb{R}$ is said to be an upper solution of (12) if it satisfies the following

$$D\rho''(t) - c\rho'(t) + f_c(\rho_t) \leqslant \mathbf{0}, \quad \text{for all } t \in \mathbb{R}. \tag{15}$$

Definition 8. A function $\rho \in C^1(\mathbb{R}, \mathbb{R}^n)$ is said to be a quasi-upper solution of (12) if

(1) $$\sup_{t\in\mathbb{R}} \|\rho(t)\| < \infty, \qquad \sup_{t\in\mathbb{R}} \|\rho'(t)\| < \infty,$$

(2) $\rho''(t)$ exists and is continuous on $\mathbb{R} \setminus \{0\}$, and

$$\sup_{t\in\mathbb{R}\setminus\{0\}} \|\rho''(t)\| < \infty,$$

(3) $\lim_{t\to 0^-} \rho''(t)$ and $\lim_{t\to 0^+} \rho''(t)$ exist,

(4) $\rho(t)$ satisfies

$$D\rho''(t) - c\rho'(t) + f_c(\rho_t) \leqslant \mathbf{0}, \quad \text{for all } t \in \mathbb{R} \setminus \{0\}. \tag{16}$$

Similarly, we define the concept of lower and quasi-lower solutions to (12) by switching the inequalities in (15) and (16). Let $\phi \in C(\mathbb{R}, \mathbb{R}^n)$ such that $\mathbf{0} \leqslant \phi(t) \leqslant \mathbf{K}$. By Lemma 6, we have $\mathbf{0} \leqslant H(\phi)(t) \leqslant \mathbf{K}$, so $H(\phi)$ is a bounded function. Note that

$$\lambda_{1i} := \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i} < 0, \qquad \lambda_{2i} := \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i} > 0$$

are the real roots of the equation

$$d_i\lambda^2 - c\lambda - \beta_i = 0, \quad i = 1, 2, \ldots, n.$$

The existence of a unique generalized bounded solution of

$$Dx''(t) - cx'(t) - \beta x(t) + H(\phi)(t) = 0, \quad \text{for all } t \in \mathbb{R}, \tag{17}$$

follows by Perron Theorem, see Lemma 2, is

$$G\big(H(\phi)\big) = \big(G_1\big(H_1(\phi)\big), G_2\big(H_2(\phi)\big), \ldots, G_n\big(H_n(\phi)\big)\big)^T,$$

where

$$G_i\big(H(\phi)\big) := \frac{1}{\lambda_{1i} - \lambda_{2i}} \left(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} H_i(\phi)(s)\, ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)} H_i(\phi)(s)\, ds \right). \tag{18}$$

The monotone iteration scheme is constructed as follows: We start out with a quasi-upper solution ϕ_0, and then use the recurrence formula

$$\phi_n := G\big(H(\phi_{n-1})\big), \quad n = 1, 2, \ldots. \tag{19}$$

Lemma 9. *Assume that $\phi, \phi_0 \in \Gamma$ are, respectively, a quasi-lower and quasi-upper solutions of* (12)*, such that $\phi(t) \leqslant \phi_0(t)$ for all $t \in \mathbb{R}$. Then*

(i) $\phi_1 \in \Gamma$ *for all* $t \in \mathbb{R}$,
(ii) ϕ_1 *is an upper solution of* (12)*, and*

$$\phi(t) \leqslant \phi_1(t) \leqslant \phi_0(t), \quad \textit{for all } t \in \mathbb{R}.$$

Proof. For the proof of (i) we refer to [32, Lemma 3.3]. Next, we prove (ii). Clearly $\phi_1 = G(H(\phi_0)) \in C^2(\mathbb{R}, \mathbb{R}^n)$ and by Lemma 2, ϕ_1 is also the unique bounded (classical) solution of the equation

$$Dx''(t) - cx'(t) - \beta x(t) + H(\phi_0)(t) = \mathbf{0}, \quad \text{for all } t \in \mathbb{R}.$$

By Lemma 6, $H(\phi_0) \geqslant 0$, and so

$$D\phi_1''(t) - c\phi_1'(t) - \beta\phi_1(t) \leqslant \mathbf{0}, \quad \text{for all } t \in \mathbb{R},$$

which means that ϕ_1 is an upper solution of (12) and from its definition $\phi_1 \geqslant \mathbf{0}$. We now show that the sequence $\{\phi_n\}$ is decreasing. To this end, set

$$\begin{aligned} &w(t) := \phi_1(t) - \phi_0(t), \\ &r(t) := -Dw''(t) - cw'(t) + \beta w(t), \quad t \in \mathbb{R}. \end{aligned} \tag{20}$$

By (19) and the assumption that ϕ_0 is a quasi-upper solution of (12) we see that $r_i(t)$ is non-positive and bounded for every $i = 1, \dots, n$ and $t \in \mathbb{R} \setminus \{0\}$. Moreover, $w_i(\cdot)$ and the i-component of (20) satisfy all conditions of Lemma 2. Therefore, it follows that as a bounded generalized solution of (20), w_i satisfies

$$w_i(t) = Gr_i(t) := \frac{1}{\lambda_1 - \lambda_2}\left(\int_{-\infty}^{t} e^{\lambda_1(t-s)} r_i(s)\,ds + \int_{t}^{+\infty} e^{\lambda_2(t-s)} r_i(s)\,ds\right) \leqslant 0,$$

which yields $w(t) = \phi_1 - \phi_0 \leqslant \mathbf{0}$, i.e. $\phi_1 \leqslant \phi_0$. Next we show that $\phi \leqslant \phi_1$. To this end, we again set

$$\begin{aligned} &v(t) := \phi_1(t) - \phi(t), \\ &s(t) := -Dv''(t) - cv'(t) + \beta v(t) \geqslant 0, \quad t \in \mathbb{R}. \end{aligned} \tag{21}$$

By the definition of quasi-lower solution and ϕ_1 and by Lemma 2 we have

$$v_i(t) = G(s_i)(t) := \frac{1}{\lambda_1 - \lambda_2}\left(\int_{-\infty}^{t} e^{\lambda_1(t-\xi)} s_i(\xi)\,d\xi + \int_{t}^{+\infty} e^{\lambda_2(t-\xi)} v_i(\xi)\,d\xi\right) \geqslant 0, \tag{22}$$

which implies $\phi_1(t) \geqslant \phi(t)$. $\square$

Remark 10. As shown in Example 4, without assumption on the continuity of the derivative of ϕ_0, formula (22) may not be true because we know only that w is bounded and it may not be of C^2.

Below we will use the notation $BC_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n) := \{g \in BC(\mathbb{R},\mathbb{R}^n) \mid \mathbf{0} \leqslant g(t) \leqslant \mathbf{K}, \forall t \in \mathbb{R}\}$ as a closed convex subspace of the Banach space $BC(\mathbb{R},\mathbb{R}^n)$ equipped with the sup-norm. We are now ready to state the monotone iteration method.

Theorem 11. *Assume that*

(1) *there exist a quasi-upper solution of* (12) $\overline{\phi}_0 \in \Gamma$ *and a non-zero non-decreasing quasi-lower solution* $\underline{\phi_0}$ *such that* $0 < \underline{\phi_0}(t) \leqslant \overline{\phi_0}(t)$ *for all* $t \in \mathbb{R}$,
(2) H *is continuous on* $BC_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$, *and*

$$\sup_{\varphi \in \Gamma} \left\| H(\varphi) \right\| < \infty. \tag{23}$$

Then, the following assertions hold:

(i) *the sequence* $\{\phi_n\}_{n=1}^{\infty}$, *defined as above, is a decreasing sequence in* Γ,
(ii) $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ *is a monotone wave front of* (12). *Moreover, this limit is uniform on each compact interval of the real line.*

Proof. By Lemma 9 claim (i) is clear. To prove (ii) we use Arzela–Ascoli Theorem. Note that by Lemma 9, all $\{\phi_n\}_{n=1}^{\infty}$ are C^2 functions. From

$$\begin{aligned}(\phi_{n+1})_i(t) &= G_i\big(H(\phi_n)\big)(t) \\ &= \frac{1}{\lambda_{1i} - \lambda_{2i}} \left(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} H_i(\phi_n)(s)\,ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)} H_i(\phi_n)(s)\,ds \right),\end{aligned}$$

we deduce

$$\left|(\phi_{n+1})_i'(t)\right| \leqslant \frac{2\sup_{\varphi\in\Gamma} \|H(\varphi)\|}{|\lambda_{1i} - \lambda_{2i}|}. \tag{24}$$

Therefore, on each interval $[-N, N]$, the set $\{\phi_n\}_{n=1}^{\infty}$ is equicontinuous and by Arzela–Ascoli Theorem, we have a uniformly convergent subsequence $\{\phi_{n_k}\}_k$ on $[-N,\ N]$. Since the sequence is monotone we can conclude that the sequence $\{\phi_n\}_{n=1}^{\infty}$ converges uniformly to a continuous and non-decreasing function ϕ such that $\mathbf{0} \leqslant \phi(t) \leqslant \mathbf{K}$ on every compact interval of the real line. Since $\phi \in BC_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$ and H is continuous on $BC_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$, for every fixed (t,s), we have

$$\lim_{n\to\infty} e^{\lambda_{1i}(t-s)} H_i(\phi_n)(s) = e^{\lambda_{1i}(t-s)} H_i(\phi)(s).$$

Lebesgue's Dominated Convergence Theorem then yields

$$\begin{aligned}\big(\phi(t)\big)_i &= \lim_{n\to\infty}\big(\phi_{n+1}(t)\big)_i = \frac{1}{\lambda_{1i}-\lambda_{2i}}\left(\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}H_i(\phi)(s)\,ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)}H_i(\phi)(s)\,ds\right)\\ &= \big(G\big(H(\phi)\big)(t)\big)_i.\end{aligned}$$

This shows in particular that ϕ is a classical solution of the equation

$$D\phi''(t) - c\phi'(t) - \beta\phi(t) + H(\phi)(t) = \mathbf{0}.$$

It remains to see that $\phi \in \Gamma$. First since $\phi(t) \leqslant \phi_n(t)$ for all n and $t \in \mathbb{R}$

$$\mathbf{0} \leqslant \limsup_{t\to-\infty}\phi(t) \leqslant \limsup_{t\to-\infty}\phi_n(t) = \mathbf{0}.$$

Next to show that $\lim_{t\to\infty}\phi(t) = \mathbf{K}$, we use $\mathbf{0} \neq \underline{\phi_0}(t) \leqslant \phi(t)$ for all t, and the fact that ϕ is non-decreasing means that $\lim_{t\to\infty}\phi(t)$ exists as a vector $\mathbf{L} \in [\mathbf{0}, \mathbf{K}]$. Obviously, $\mathbf{L} \neq \mathbf{0}$, and so $\mathbf{0} < \mathbf{L}$. We now use the delay and as in [32, Proposition 2.1], we must have $f_c(\mathbf{L}) = 0$. But the assumption on the function f makes it impossible if $\mathbf{L} \neq \mathbf{K}$. So, $\lim_{t\to\infty}\phi(t) = \mathbf{K}$, and $\phi \in \Gamma$. □

3. Traveling waves for the predator–prey model

Consider the predator–prey model with diffusion and delay in time

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1\dfrac{\partial^2 u(x,t)}{\partial x^2} + ru(x,t)\left[\left(1 - \dfrac{1}{P}u(x,t)\right) - av(x,t)\right],\\[2ex] \dfrac{\partial v(x,t)}{\partial t} = d_2\dfrac{\partial^2 v(x,t)}{\partial x^2} + v(x,t)\big[-\nu + bu(x,t-\tau)\big], \end{cases} \tag{25}$$

where $u(x,t)$, $v(x,t)$ are scalar functions, $(x,t) \in \mathbb{R}\times[\tau,\infty)$, and $d_1, d_2, r, P, a, b, \nu, \tau$ are all positive constants. Define $\mathbb{R}$-valued functions $f_1(\phi,\psi)$ and $f_2(\phi,\psi)$ as follows: For each $\phi, \psi \in C([-\tau,0],\mathbb{R})$, the functionals

$$\begin{aligned} f_1(\phi,\psi) &= r\phi(0)\left[\left(1 - \frac{\phi(0)}{P} - a\psi(0)\right)\right],\\ f_2(\phi,\psi) &= \psi(0)\big(-\nu + b\phi(-\tau)\big). \end{aligned}$$

If we assume that

$$1 > \frac{\nu}{Pb} \quad \Leftrightarrow \quad P > \frac{\nu}{b},$$

then the model has a non-trivial positive steady state

$$\left(\frac{\nu}{b}, \frac{1}{a}\left(1 - \frac{\nu}{Pb}\right)\right)^T.$$

The above model with $\tau = 0$ has been treated in [4,5,27] (for related results see [22–24]). The case with delay $\tau > 0$ was first considered in [20]. As noted in Remark 5, the smoothness was overlooked in the proof of [20, Lemma 3.5]. This gap is due to the failure of Perron Theorem for this type of "supersolutions." Hence, the question on the existence of traveling waves for this model with delay remains open. A quick remedy is provided by the results of the previous section, which require us to construct quasi-upper and quasi-lower solutions which have more smoothness than "supersolutions" in [20].

3.1. Wave equations and upper, lower solutions

Let us agree to denote by $\varphi_1(x+ct) := u(x,t)$, $\varphi_2(x+ct) := v(x,t)$, and $\xi = x+ct$. We recast (25) into

$$c\varphi_1'(\xi) = d_1\varphi_1''(\xi) + r\varphi_1(\xi)\left[\left(1-\frac{\varphi_1(\xi)}{P}\right) - a\varphi_2(\xi)\right] = 0,$$
$$c\varphi_2'(\xi) = d_2\varphi_2''(\xi) + \varphi_2(\xi)\left[-\nu + b\varphi_1(\xi - c\tau)\right] = 0,$$

which can be re-written as

$$\begin{aligned} d_1\varphi_1''(\xi) - c\varphi_1'(\xi) + r\varphi_1(\xi)\left[\left(1-\frac{\varphi_1(\xi)}{P}\right) - a\varphi_2(\xi)\right] = 0, \\ d_2\varphi_2''(\xi) - c\varphi_2'(\xi) + \varphi_2(\xi)\left[-\nu + b\varphi_1(\xi - c\tau)\right] = 0. \end{aligned} \tag{26}$$

We will find monotone solutions of the above wave equation such that

$$\lim_{t\to-\infty}\varphi_1(t) = 0, \qquad \lim_{t\to+\infty}\varphi_1(t) = p := \frac{\nu}{b}, \tag{27}$$

$$\lim_{t\to-\infty}\varphi_2(t) = 0, \qquad \lim_{t\to+\infty}\varphi_2(t) = q := \frac{1}{a}\left(1-\frac{\nu}{Pb}\right). \tag{28}$$

We now check the quasi-monotonicity of $(f_1, f_2)^T$, that is, there are positive numbers β_1, β_2 such that

$$f_1(\phi_1,\psi_2) - f_1(\psi_1,\psi_2) + \beta_1\big(\phi(0) - \psi(0)\big) \geqslant 0,$$
$$f_2(\phi_1,\phi_2) - f_2(\psi_1,\psi_2) + \beta\big(\phi_2(0) - \psi_2(0)\big) \geqslant 0,$$

for all

$$\mathbf{0} \leqslant \psi(s) \leqslant \phi(s) \leqslant \mathbf{K} := \left(\frac{\nu}{b}, \frac{1}{a}\left(1-\frac{\nu}{Pb}\right)\right)^T.$$

Using the definition of the functions $(f_1, f_2)^T$ we can easily come up with a pair of positive numbers β_1, β_2 that satisfy

$$\begin{cases} \beta_1 \geqslant \max\left(-r + \dfrac{r\nu}{Pb} + \dfrac{\nu}{b}, 0\right), \\ \beta_2 \geqslant \nu. \end{cases}$$

3.2. Upper solutions

Let us define $\varphi(t) := (\varphi_1(t), \varphi_2(t))^T$, where

$$\varphi_1(t) = \begin{cases} \frac{\nu}{2b} e^{\lambda_1 t}, & t \leqslant 0, \\ \frac{\nu}{b} - \frac{\nu}{2b} e^{-\lambda_1 t}, & t > 0, \end{cases} \quad \text{and} \quad \varphi_2(t) = \begin{cases} \frac{q}{2} e^{\lambda_2 t}, & t \leqslant 0, \\ q - \frac{q}{2} e^{-\lambda_2 t}, & t > 0, \end{cases} \tag{29}$$

where the parameters are chosen so that

$$\lambda_1 = \frac{c + \sqrt{c^2 - 4d_1 r}}{2d_1}, \qquad \lambda_2 = \frac{c}{2d_2}, \qquad p = \frac{\nu}{b}, \qquad q = \frac{1}{a}\left(1 - \frac{\nu}{Pb}\right).$$

Therefore, λ_1 and λ_2 are simply the roots of the quadratics

$$d_1\lambda^2 - c\lambda + r = 0, \qquad d_2\lambda^2 - c\lambda = 0. \tag{30}$$

Lemma 12. *Assume that $\nu < b$ and $d_2 > 2d_1$. Then, there exists a constant $c^* = c^*(a, b, r, \nu, P) > 0$ such that if $c > c^*$, $\overline{\phi_0}(t) := (\varphi_1(t), \varphi_2(t))^T$ is a monotone quasi-upper solution of the wave equation.*

Proof. It is easily seen that both φ_1 and φ_2 are continuously differentiable, monotone and obviously

$$\lim_{t\to-\infty} \varphi_1(t) = 0, \qquad \lim_{t\to+\infty} \varphi_1(t) = p := \frac{\nu}{b},$$
$$\lim_{t\to-\infty} \varphi_2(t) = 0, \qquad \lim_{t\to+\infty} \varphi_2(t) = q := \frac{1}{a}\left(1 - \frac{\nu}{Pb}\right).$$

Next, we have

$$\varphi_1'(t) = \begin{cases} \frac{p\lambda_1}{2} e^{\lambda_1 t}, & t \leqslant 0, \\ \frac{p\lambda_1}{2} e^{-\lambda_1 t}, & t > 0, \end{cases} \qquad \varphi_1''(t) = \begin{cases} \frac{p\lambda_1^2}{2} e^{\lambda_1 t}, & t \leqslant 0, \\ \frac{-p\lambda_1^2}{2} e^{-\lambda_1 t}, & t > 0, \end{cases}$$

$$\varphi_2'(t) = \begin{cases} \frac{q\lambda_2}{2} e^{\lambda_2 t}, & t \leqslant 0, \\ \frac{q\lambda_2}{2} e^{-\lambda_2 t}, & t > 0, \end{cases} \qquad \varphi_2''(t) = \begin{cases} \frac{q\lambda_2^2}{2} e^{\lambda_2 t}, & t \leqslant 0, \\ \frac{-q\lambda_2^2}{2} e^{-\lambda_2 t}, & t > 0. \end{cases}$$

A crucial property of the functions φ_1, φ_2 is their smoothness. We now look for conditions on the parameters so that they form upper solutions. In doing so, we examine the cases when $t \leqslant 0$ and $t > 0$ separately. Substituting the above expressions into the first equation of (26) and using (30) yield for all $t \leqslant 0$

$$\begin{aligned} &d_1 \frac{p\lambda_1^2}{2} e^{\lambda_1 t} - c\frac{p\lambda_1}{2} e^{\lambda_1 t} + r\frac{p}{2} e^{\lambda_1 t} - \frac{rp}{2P} e^{\lambda_1 t} - \frac{arpq}{2} e^{\lambda_1 t} e^{\lambda_2 t} \\ &= -\frac{rp}{2P} e^{\lambda_1 t} - \frac{arpq}{2} e^{\lambda_1 t} e^{\lambda_2 t} \\ &\leqslant 0. \end{aligned}$$

Similarly for $t > 0$, we have

$$
\begin{aligned}
&-d_1\frac{p\lambda_1^2}{2}e^{-\lambda_1 t} - c\frac{p\lambda_1}{2}e^{-\lambda_1 t} + r\left(p - \frac{p}{2}e^{-\lambda_1 t}\right) - \frac{r}{P}\left(p - \frac{p}{2}e^{-\lambda_1 t}\right)\\
&\quad - ar\left(p - \frac{p}{2}e^{-\lambda_1 t}\right)\left(q - \frac{q}{2}e^{-\lambda_2(t-c\tau)}\right)\\
&\quad = \left(rp - \frac{rp}{P} - arpq\right) + \left(\frac{rp}{2P} + \frac{arpq}{2} - cp\lambda_1\right)e^{-\lambda_1 t} + \frac{arpq}{2}e^{-\lambda_2 t} - \frac{arpq}{4}e^{-(\lambda_1+\lambda_2)t}.
\end{aligned}
$$

To check the sign of the above expression, we first factor out rp, which is positive,

$$
\left(1 - \frac{1}{P} - aq\right) + \left(\frac{1}{2P} + \frac{aq}{2} - c\frac{\lambda_1}{r}\right)e^{-\lambda_1 t} + \frac{aq}{2}e^{-\lambda_2 t} - \frac{aq}{4}e^{-(\lambda_1+\lambda_2)t}. \tag{31}
$$

A sufficient condition on P can be obtained from (31) and use if $\lambda_2 > \lambda_1$, which holds if $d_2 > 2d_1$

$$
\begin{aligned}
&2\left(1 - \frac{1}{P} - aq\right) + \left(\frac{1}{P} + aq - c\frac{2\lambda_1}{r}\right)e^{-\lambda_1 t} + aqe^{-\lambda_2 t} - \frac{1}{2}aqe^{-(\lambda_1+\lambda_2)t}\\
&\quad = \frac{2}{P}\left(\frac{\nu}{b} - 1\right) + \left(\frac{1}{P} + 2aq - 2c\frac{\lambda_1}{r}\right)e^{-\lambda_1 t} \leqslant 0
\end{aligned}
$$

to be non-positive for all $t > 0$, is

$$
\frac{\nu}{b} < 1, \tag{32}
$$

$$
\frac{1}{P} + 2aq < 2c\frac{\lambda_1}{r}. \tag{33}
$$

For (33) to hold it is sufficient to take c large enough. More precisely, if $c > \sqrt{4d_1 r}$, then $\lambda_1 > 2c$ which, in turn, leads to $c > \frac{1}{2}\sqrt{(\frac{1}{P} + 2aq)r}$. Thus the value of c^* can be estimated by

$$
c^* = \max\left(\frac{1}{2}\sqrt{\left(\frac{1}{P} + 2aq\right)r},\ \sqrt{4d_1 r}\right).
$$

Corollary 13. *In case $\frac{\nu}{b} < P < \frac{1}{2}$, then there $c^* = 0$.*

Proof. It is enough to see that (31) can also be written as

$$
\begin{aligned}
&= \left(1 - \frac{1}{P}\right) + \left(\frac{1}{2P} - c\frac{\lambda_1}{r}\right)e^{-\lambda_1 t} + aq\left(\frac{e^{-\lambda_2 t} + e^{-\lambda_1 t}}{2} - 1 - \frac{e^{-(\lambda_1+\lambda_2)t}}{4}\right)\\
&= 1 - \frac{1}{2P} - c\frac{\lambda_1}{r}e^{-\lambda_1 t} + \left(\frac{e^{-\lambda_1 t} - 1}{2P}\right) + aq\left(\frac{e^{-\lambda_2 t} + e^{-\lambda_1 t}}{2} - 1 - \frac{e^{-(\lambda_1+\lambda_2)t}}{4}\right) \leqslant 0.
\end{aligned} \tag{34}
$$

It turns out that we have an upper solution, provided $c > 0$ and

$$1 - \frac{1}{2P} < 0, \quad \text{i.e.} \quad \frac{\nu}{b} < P < \frac{1}{2} \tag{35}$$

no restrictions on d_1, d_2. $\square$

Now we check the second equation of the wave equation. Starting again with $t \leqslant 0$ and substituting the expressions for φ_1, φ_2 into the second equation, together with (30) yield

$$d_2 \frac{q\lambda_2^2}{2} e^{\lambda_2 t} - c\frac{q\lambda_2}{2} e^{\lambda_2 t} - \frac{q}{2} e^{\lambda_2 t}\left(\nu - b\frac{\nu}{2b} e^{\lambda_1 (t-c\tau)}\right) = -\frac{q\nu}{4} e^{\lambda_2 t}\left(2 - e^{\lambda_1 (t-c\tau)}\right) \leqslant 0,$$

for all $t \leqslant 0$ since $t - c\tau < 0$.

Next, for $t > 0$, substituting into the second equation of the wave equation yields

$$\begin{aligned}
&-d_2 \frac{q\lambda_2^2}{2} e^{-\lambda_2 t} - c\frac{q\lambda_2}{2} e^{-\lambda_2 t} - \nu\left(q - \frac{q}{2} e^{-\lambda_2 t}\right) + b\left(q - \frac{q}{2} e^{-\lambda_2 t}\right)\left(p - \frac{p}{2} e^{-\lambda_1 (t-c\tau)}\right) \\
&\quad = -(\nu - pb)\frac{q}{2} + (\nu - pb)\frac{q}{2}\left(e^{-\lambda_2 t} - 1\right) - 2c\lambda_2 q e^{-\lambda_2 t} - \frac{pqb}{2} e^{-\lambda_1 (t-c\tau)} \\
&\qquad + \frac{pqb}{4} e^{-\lambda_1 (t-c\tau)} e^{-\lambda_2 t}.
\end{aligned}$$

Since

$$pb = \nu$$

the above expression reduces to

$$\begin{aligned}
&-2c\lambda_2 q e^{-\lambda_2 t} - \frac{pqb}{2} e^{-\lambda_1 (t-c\tau)} + \frac{pqb}{4} e^{-\lambda_1 (t-c\tau)} e^{-\lambda_2 t} \\
&\quad \leqslant -2c\lambda_2 q e^{-\lambda_2 t} - \frac{q\nu}{2} e^{-\lambda_1 (t-c\tau)} + \frac{q\nu}{4} e^{-\lambda_1 (t-c\tau)} e^{-\lambda_2 t} \\
&\quad = -2cq\lambda_2 e^{-\lambda_2 t} - \frac{q\nu}{2} e^{-\lambda_1 (t-c\tau)}\left(1 - \frac{1}{2} e^{-\lambda_2 t}\right) \\
&\quad < 0,
\end{aligned}$$

and therefore the lemma is proved. $\square$

3.3. Lower solutions

We will construct a quasi-lower solutions to (26) as follows: Set

$$\nu_1 := \frac{c}{d_1},$$

which is a root of

$$d_1\nu_1^2 - c\nu_1 = 0.$$

Let us define functions $\psi_2(t) = 0$ for all $t \in \mathbb{R}$, and

$$\psi_1(t) = \begin{cases} \alpha k e^{\nu_1 t}, & t < 0, \\ k - \alpha k e^{-\nu_1 t}, & t \geqslant 0, \end{cases} \tag{36}$$

where the constants k and α are positive, sufficiently small and will be determined at the end. A simple computation yields

$$\psi_1'(t) := \begin{cases} \nu_1 \alpha k e^{\nu_1 t}, & t < 0, \\ \nu_1 \alpha k e^{-\nu_1 t}, & t \geqslant 0, \end{cases} \qquad \psi_1''(t) := \begin{cases} \nu_1^2 \alpha k e^{\nu_1 t}, & t < 0, \\ -\nu_1^2 \alpha k e^{-\nu_1 t}, & t \geqslant 0, \end{cases}$$

and the first equation of (26) with $t \leqslant 0$ gives

$$\begin{aligned} &\big(d_1 \alpha k \nu_1^2 - c\alpha k \nu_1\big) e^{\nu_1 t} + r\alpha k e^{\nu_1 t}\left[1 - \frac{1}{P}\alpha k e^{\nu_1 t}\right] \\ &= r\alpha k e^{\nu_1 t}\left[1 - \frac{1}{P}\alpha k e^{\nu_1 t}\right] \\ &\geqslant 0, \end{aligned}$$

provided $0 < \alpha k < P$.

For $t > 0$, substituting the derivatives of ψ_1 into the first equation of (26) leads to

$$\begin{aligned} &-d_1 \nu_1^2 \alpha k e^{-\nu_1 t} - c\nu_1 \alpha k e^{-\nu_1 t} + r\big(k - \alpha k e^{-\nu_1 t}\big)\left[1 - \frac{1}{P}\big(k - \alpha k e^{-\nu_1 t}\big)\right] \\ &= rk\left(1 - \frac{k}{P}\right) + \alpha k\left(\frac{2rk}{P} - 2c\nu_1 - r\right)e^{-\nu_1 t} - \frac{r\alpha^2 k^2}{P} e^{-2\nu_1 t} \\ &\geqslant 0, \end{aligned}$$

provided that c and k are fixed, $0 < k < P$, and α is sufficiently small.

Next, we need to choose α, k so that

$$\psi_1(t) \leqslant \phi_1(t).$$

In fact, a simple computation shows that we may choose $k \leqslant \nu/(4b)$ and $\alpha < (4Pb)/\nu$.

Obviously, this function $(\psi_1, \psi_2)^T$ is a quasi-lower solution to (26). Using the same procedure as for upper solutions we will obtain a smooth lower solution as desired.

Therefore, we have proved

Lemma 14. *Let all assumptions of Lemma* 12 *hold. Then, for every fixed $c > c^*$, where c^* is as in Lemma* 12*, there are* (*sufficiently small*) *constants α and k such that the function $\underline{\phi_0}(t) := (\psi_1(t), 0)^T$, where $\psi_1(t)$ is defined by* (36) *is a quasi-lower solution of* (26) *that satisfies*

$$0 < \underline{\phi_0}(t) \leqslant \overline{\phi_0}(t), \quad t \in \mathbb{R}, \tag{37}$$

where $\overline{\phi_0}$ is the quasi-upper solution that is defined by (29) *and mentioned in Lemma* 12.

Finally, by Theorem 11 we have

Theorem 15. *Assume that $\nu < b$ and $d_2 > 2d_1$. Then, there exists a constant $c^* = c^*(a,b,r,\nu,P) > 0$ such that if $c > c^*$, Eq.* (25) *has a wave front solution $u(x,t) = \varphi_1(x+ct)$, $v(x,t) = \varphi_2(x+ct)$. Moreover, if $\frac{\nu}{b} < P < \frac{1}{2}$, c^* may be chosen to be* 0.

4. Belousov–Zhabotinskii equations

In this section we will apply the results in Section 2 to prove the existence of traveling waves to Belousov–Zhabotinskii equations

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = \dfrac{\partial^2}{\partial x^2}u(x,t) + u(x,t)\big[1 - u(x,t) - rv(x,t-\tau)\big], \\ \dfrac{\partial}{\partial t}v(x,t) = \dfrac{\partial^2}{\partial x^2}v(x,t) - bu(x,t)v(x,t), \end{cases} \tag{38}$$

where $r > 0$, $b > 0$ are constants, u and v are scalar functions. We refer the reader to [27] for more details on the history as well as applications of this kind of equations. Traveling waves in these models without delay were considered in various papers (see e.g. [17,18,29,35]). In the recent papers [20,32], the models with delay were first studied as an application of the monotone iteration method, which was extended by Wu and Zou. As we have noted earlier, in the previous model, the monotone iteration method reduces to finding a pair of upper and lower solutions to the corresponding wave equation. Again due to the failure of Perron Theorem for the concepts of "upper solutions" and "supersolutions" in [20,32], the question of existence of traveling waves in these models remained open. Below, Perron Theorem and Theorem 11 will be used to answer this question by constructing suitable quasi-upper and quasi-lower solutions to the wave equation.

The wave equation associated with Belousov–Zhabotinskii equations can be modified (see also [20,32]) so that it takes the form

$$\begin{cases} \varphi_1''(t) - c\varphi_1'(t) + \varphi_1(t)\big((1-r) - \varphi_1(t) + r\varphi_2(t-c\tau)\big) = 0, \\ \varphi_2''(t) - c\varphi_2'(t) + b\varphi_1(t)\big(1 - \varphi_2(t)\big) = 0. \end{cases} \tag{39}$$

We seek monotone solutions $(\varphi_1, \varphi_2)^T$ such that

$$\lim_{t\to-\infty}\varphi_1(t) = 0, \qquad \lim_{t\to+\infty}\varphi_1(t) = 1,$$
$$\lim_{t\to-\infty}\varphi_2(t) = 0, \qquad \lim_{t\to+\infty}\varphi_2(t) = 1.$$

To this end we recast the wave equations in the following form

$$\begin{aligned} \big[\varphi_1''(t) - c\varphi_1'(t) + \varphi_1(t)\big] - r\varphi_1(t)\big(1 - \varphi_2(t-c\tau)\big) - \varphi_1^2(t) = 0, \\ \big[\varphi_2''(t) - c\varphi_2'(t)\big] + b\varphi_1(t) - b\varphi_1(t)\varphi_2(t) = 0. \end{aligned} \tag{40}$$

Define the numbers λ_1 and μ_1 as

$$\lambda_1 = \frac{c+\sqrt{c^2-4}}{2}, \qquad \mu_1 = \frac{c+\sqrt{c^2-4b}}{2},$$

which are the roots of the characteristic equations

$$\lambda^2 - c\lambda + 1 = 0, \qquad \mu^2 - c\mu + b = 0,$$

respectively. Observe that

$$b < 1 \quad \Rightarrow \quad \lambda_1 < \mu_1. \tag{41}$$

Let us define functions φ_1 and φ_2 as follows:

$$\varphi_1(t) := \begin{cases} \frac{1}{2}e^{\lambda_1 t}, & t \leqslant 0, \\ 1 - \frac{1}{2}e^{-\lambda_1 t}, & t > 0, \end{cases} \qquad \varphi_2(t) := \begin{cases} \frac{1}{2}e^{\mu_1 t}, & t \leqslant 0, \\ 1 - \frac{1}{2}e^{-\mu_1 t}, & t > 0, \end{cases}$$

observe that $0 < \varphi_1(t) < 1$ and similarly for $0 < \varphi_2(t) < 1$.

Lemma 16. *There exists a positive number $c^* = c^*(b, r)$ such that if $c > c^*$, then $(\varphi_1, \varphi_2)^T$ defined as above is a smooth monotone upper solution of the wave equation.*

Proof. First, it is easily seen that

$$\varphi_1'(t) = \begin{cases} \frac{\lambda_1}{2}e^{\lambda_1 t}, & t \leqslant 0, \\ \frac{\lambda_1}{2}e^{-\lambda_1 t}, & t > 0, \end{cases} \qquad \varphi_1''(t) = \begin{cases} \frac{\lambda_1^2}{2}e^{\lambda_1 t}, & t \leqslant 0, \\ \frac{-\lambda_1^2}{2}e^{-\lambda_1 t}, & t > 0, \end{cases}$$

$$\varphi_2'(t) = \begin{cases} \frac{\mu_1}{2}e^{\mu_1 t}, & t \leqslant 0, \\ \frac{\mu_1}{2}e^{-\mu_1 t}, & t > 0, \end{cases} \qquad \varphi_2''(t) = \begin{cases} \frac{\mu_1^2}{2}e^{\mu_1 t}, & t \leqslant 0, \\ \frac{-\mu_1^2}{2}e^{-\mu_1 t}, & t > 0. \end{cases}$$

The second derivative exists almost everywhere. Substituting the above expressions into the first equation of (40) we have for $t \leqslant 0$

$$\begin{aligned} &\big[\varphi_1''(t) - c\varphi_1'(t) + \varphi_1(t)\big] - r\varphi_1(t)\big(1 - \varphi_2(t - c\tau)\big) - \varphi_1^2(t) \\ &\quad = -re^{\lambda_1 t}\big(1 - \varphi_2(t - c\tau)\big) - \frac{1}{4}e^{2\lambda_1 t} \leqslant 0 \end{aligned}$$

since $(1 - \varphi_2(t - c\tau)) > 0$.

For the second component we have for $t \leqslant 0$

$$\begin{aligned} \varphi_2''(t) - c\varphi_2'(t) + b\varphi_1(t)\big(1 - \varphi_2(t)\big) &= \frac{\mu_1^2}{2}e^{\mu_1 t} - c\frac{\mu_1}{2}e^{\mu_1 t} + \frac{b}{2}e^{\lambda_1 t}\left(1 - \frac{1}{2}e^{\mu_1 t}\right) \\ &= \frac{b}{2}\big(e^{\lambda_1 t} - e^{\mu_1 t}\big) - \frac{b}{4}e^{(\lambda_1 + \mu_1)t}. \end{aligned}$$

Thus by (41)

$$\lambda_1 \leqslant \mu_1 \quad \Rightarrow \quad \varphi_2''(t) - c\varphi_2'(t) + b\varphi_1(t)\big(1 - \varphi_2(t)\big) \leqslant -\frac{b}{4} < 0.$$

On the other hand if $t \geqslant 0$, then we have

$$\begin{aligned}
&\big[\varphi_1''(t) - c\varphi_1'(t) + \varphi_1(t)\big] - r\varphi_1(t)\big(1 - \varphi_2(t - c\tau)\big) - \varphi_1^2(t) \\
&\quad = \left[\frac{-\lambda_1^2}{2}e^{-\lambda_1 t} - c\frac{\lambda_1}{2}e^{-\lambda_1 t} + 1 - \frac{1}{2}e^{-\lambda_1 t}\right] - r\left(1 - \frac{1}{2}e^{-\lambda_1 t}\right)\big(1 - \varphi_2(t - c\tau)\big) - \varphi_1^2(t) \\
&\quad = -r\left(1 - \frac{1}{2}e^{-\lambda_1 t}\right)\big(1 - \varphi_2(t - c\tau)\big) - \lambda_1^2 e^{-\lambda_1 t} - \frac{1}{4}e^{-2\lambda_1 t} \\
&\quad < 0.
\end{aligned}$$

Similarly the second equation yields

$$\begin{aligned}
&\varphi_2''(t) - c\varphi_2'(t) + b\varphi_1(t)\big(1 - \varphi_2(t)\big) \\
&\quad = \frac{-\mu_1^2}{2}e^{-\mu_1 t} - c\frac{\mu_1}{2}e^{-\mu_1 t} + b\left(1 - \frac{1}{2}e^{-\lambda_1 t}\right) - b\varphi_1(t)\varphi_2(t) \\
&\quad = -ce^{-\mu_1 t} + \frac{b}{2}\big(e^{-\mu_1 t} - e^{-\lambda_1 t}\big) + \frac{b}{2}e^{-\mu_1 t} + \frac{b}{2}e^{-\lambda_1 t} - \frac{b}{4}e^{-(\lambda_1 + \mu_1)t} \\
&\quad = \left[(b - c) - \frac{b}{4}e^{-\lambda_1 t}\right]e^{-\mu_1 t},
\end{aligned}$$

so, when $t \geqslant 0$,

$$b - c \leqslant 0 \quad \Rightarrow \quad \varphi_2''(t) - c\varphi_2'(t) + b\varphi_1(t)\big(1 - \varphi_2(t)\big) < 0.$$

To summarize the situation we have

$$\left.\begin{array}{l} b \leqslant c, \\ \lambda_1 \leqslant \mu_1 \end{array}\right\} \quad \Rightarrow \quad \varphi_2''(t) - c\varphi_2'(t) + b\varphi_1(t)\big(1 - \varphi_2(t)\big) < 0 \quad \forall t \in \mathbb{R},$$

and by (41) the condition reduces to

$$\begin{cases} b \leqslant c, \\ b < 1 \end{cases} \quad \Rightarrow \quad \begin{cases} b \leqslant c, \\ \lambda_1 \leqslant \mu_1, \end{cases}$$

which is satisfied if

$$b < \min\{1, c\}. \qquad \square$$

Finally, since the procedure of constructing quasi-lower solutions is similar for the predator–prey models we leave it to the reader. To conclude this section we have

Theorem 17. *There exists a positive number $c^* = c^*(b, r)$ such that if $c > c^*$, then Eq.* (38) *has a wave front solution $u(x, t) = \varphi_1(x + ct)$, $v(x, t) = \varphi_2(x + ct)$.*

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