Decomposable Bilinear Numerical Radii

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ABSTRACT

Let A be an n-square normal matrix over $\mathbb{C}$, and $Q_{m,n}$ be the set of strictly increasing integer sequences of length m chosen from $1,\dots,n$. For $\alpha,\beta\in Q_{m,n}$ denote by $A[\alpha|\beta]$ the submatrix obtained from A by using rows numbered α and columns numbered β. For $k\in\{0,1,\dots,m\}$ write $|\alpha\cap\beta|=k$ if there exists a rearrangement of $1,\dots,m$, say $i_1,\dots,i_k,i_{k+1},\dots,i_m$, such that $\alpha(i_j)=\beta(i_j)$, $j=1,\dots,k$, and $\{\alpha(i_{k+1}),\dots,\alpha(i_m)\}\cap\{\beta(i_{k+1}),\dots,\beta(i_m)\}=\varnothing$. Let $\mathfrak{U}_n$ be the group of n-square unitary matrices. Define the nonnegative number

$$\rho_k(A)=\max_{U\in\mathfrak{U}_n}|\det(U^*AU)[\alpha|\beta]|,$$

where $|\alpha\cap\beta|=k$. Theorem 1 establishes a bound for $\rho_k(A)$, $0\leqslant k<m-1$, in terms of a classical variational inequality due to Fermat. Let A be positive semidefinite Hermitian, $n\geqslant 2m$. Theorem 2 leads to an interlacing inequality which, in the case $n=4$, $m=2$, resolves in the affirmative the conjecture that

$$\rho_m(A)\geqslant\rho_{m-1}(A)\geqslant\cdots\geqslant\rho_0(A).$$

I. INTRODUCTION

Let A be an n-square normal matrix over $\mathbb{C}$ with eigenvalues $\lambda_1,\dots,\lambda_n$. Denote by $Q_{m,n}$ the set of $\binom{n}{m}$ strictly increasing integer sequences of length m chosen from $1,\dots,n$. For $\alpha,\beta,\gamma\in Q_{m,n}$, let $A[\alpha|\beta]$ be the m-square submatrix of A formed by selecting rows numbered α and columns β, and set $\lambda_\gamma=\lambda_{\gamma(1)}\cdots\lambda_{\gamma(m)}$. For $k\in\{0,1,\dots,m\}$, write $|\alpha\cap\beta|=k$ if there exists a rearrangement of $1,\dots,m$, say $i_1,\dots,i_k,i_{k+1},\dots,i_m$, such that $\alpha(i_j)=\beta(i_j)$,

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$j=1,\dots,k$, and $\{\alpha(i_{k+1}),\dots,\alpha(i_m)\}\cap\{\beta(i_{k+1}),\dots,\beta(i_m)\}=\varnothing$. Thus, if $|\alpha\cap\beta|=k$, the submatrix $A[\alpha|\beta]$ intersects the main diagonal of A in k main diagonal places.

Consider the set

$$\Delta^k_{\alpha,\beta}(A)=\{\det(U^*AU)[\alpha|\beta]:U\in\mathfrak{U}_n\},$$

where $\mathfrak{U}_n$ is the group of n-square unitary matrices, and $|\alpha\cap\beta|=k$.

In [6], it is shown that *if X is an n-square matrix, $n\geqslant 2m$ and $|\alpha\cap\beta|=k$, then $\Delta^k_{\alpha,\beta}(X)$ is independent of α and β; that is, $\Delta^k_{\alpha,\beta}(X)=\Delta^k(X)$, where*

$$\Delta^k(X)=\{\det(U^*XU)[12\dots m|12\dots k\,m+1\dots 2m-k]:U\in\mathfrak{U}_n\}.$$

Define the nonnegative number

$$\rho_k(A)=\max_{z\in\Delta^k(A)}|z|.$$

Let $\{e_1,\dots,e_n\}$ be the standard basis in $\mathbf{C}^n$, the space of complex column n-tuples. Denote by $\otimes^m\mathbf{C}^n$ the mth tensor space over $\mathbf{C}^n$, and by $\wedge^m\mathbf{C}^n$ the mth exterior space over $\mathbf{C}^n$. For $\alpha\in Q_{m,n}$, denote by $e_\alpha^\wedge$ the skew-symmetric tensor $e_{\alpha(1)}\wedge\cdots\wedge e_{\alpha(m)}$. If $Q_{m,n}$ is ordered lexicographically, then $\{e_\alpha^\wedge:\alpha\in Q_{m,n}\}$, the standard basis in $\mathbf{C}^{\binom{n}{m}}$, is an ordered orthonormal basis of $\wedge^m\mathbf{C}^n$. Set $G_{m,n}=\{u^\wedge=u_1\wedge\cdots\wedge u_m\in\wedge^m\mathbf{C}^n:\|u^\wedge\|=1\}$, the Grassmannian manifold. By the usual arguments it may be assumed that the vectors $u_1,\dots,u_m$ occurring in a unit length exterior product $u^\wedge$ are orthonormal.

For any n-square matrix X define the induced matrix $\otimes^m X$ by $\otimes^m Xv_1\otimes\cdots\otimes v_m=Xv_1\otimes\dots\otimes Xv_m$ for arbitrary $v_1,\dots,v_m\in\mathbf{C}^n$. Since $\wedge^m\mathbf{C}^n$ is a reducing subspace of $\otimes^m X$ the mth compound of X [1, p. 19] can be defined by $C_m(X)=\otimes^m X|\wedge^m\mathbf{C}^n$. Using the induced inner product in $\wedge^m\mathbf{C}^n$, it can be shown that $(C_m(X)e_\beta^\wedge,e_\alpha^\wedge)=\det X[\alpha|\beta]$. An important property of the mth compound is that $C_m(XY)=C_m(X)C_m(Y)$ for arbitrary X and Y. Clearly for any $u_\alpha^\wedge\in G_{m,n}$ a unitary U can be selected so that $C_m(U)u_\alpha^\wedge=Uu_{\alpha(1)}\wedge\cdots\wedge Uu_{\alpha(m)}=e_\alpha^\wedge$. Hence, for any $\alpha,\beta\in Q_{m,n}$ with $|\alpha\cap\beta|=k$,

$$\Delta^k(A)=\{(C_m(A)u_\beta^\wedge,u_\alpha^\wedge):u_1,\dots,u_n\in\mathbf{C}^n\text{ o.n.}\}.$$

Let $\lambda_{\max}=\max_{\gamma\in Q_{m,n}}|\lambda_\gamma|$, $\lambda_{\min}=\min_{\gamma\in Q_{m,n}}|\lambda_\gamma|$. The normality of A implies that the numerical radius [2, p. 114] of $C_m(A)$ is $\lambda_{\max}$. Since the eigenvectors of $C_m(A)$ are decomposable, $\rho_m(A)=\lambda_{\max}$. Thus $\rho_k(A)$ for $0\leqslant k<m$ are *decomposable bilinear numerical radii.*

The main result of this paper establishes a bound for $\rho_k(A)$, $0\leqslant k<m-1$, in terms of a classical variational inequality due to Fermat [4]. Let A be positive semidefinite, $n\geqslant 2m$. It is conjectured [6] that[1]

$$\rho_m(A)\geqslant\rho_{m-1}(A)\geqslant\cdots\geqslant\rho_0(A). \tag{1}$$

The bound obtained here leads to an interlacing inequality which, in the case $n=4$ and $m=2$, resolves the conjecture (1) in the affirmative.

II. STATEMENTS OF RESULTS

THEOREM 1. *Let $m\geqslant 2$, $n\geqslant 2m$; then*

$$\rho_k(A)\leqslant\min_{z\in\mathbf{C}}\begin{cases}\dfrac{1}{4}\displaystyle\sum_{\gamma\in Q_{m,n}}|\lambda_\gamma-z| & \textit{if}\quad k=m-2,\\ \dfrac{1}{2(m-k+1)}\displaystyle\sum_{\gamma\in Q_{m,n}}|\lambda_\gamma-z| & \textit{if}\quad k<m-2.\end{cases} \tag{2}$$

THEOREM 2. *Let A be Hermitian positive semidefinite, $\mu,\nu,\omega\in Q_{m,n}$, $\lambda_\mu=\lambda_{\max}$, and $\lambda_\nu=\lambda_{\min}$. If $m\geqslant 2$, $n\geqslant 2m$, then*

$$\rho_k(A)\leqslant\begin{cases}\dfrac{1}{4}\left\{(\lambda_\mu-\lambda_\nu)+\displaystyle\max_{\gamma,\,\omega\neq\mu,\,\nu}(\lambda_\gamma-\lambda_\omega)\right\} & \textit{if}\quad k=m-2,\\ \dfrac{1}{2(m-k+1)}\left\{(\lambda_\mu-\lambda_\nu)+\displaystyle\max_{\gamma_i,\,\omega_i\neq\mu,\,\nu}\sum_{i=1}^{m-k}(\lambda_{\gamma_i}-\lambda_{\omega_i})\right\} & \\ & \textit{if}\quad k<m-2.\end{cases}$$

III. PROOFS OF RESULTS

The quadratic (Plücker) relations [3, p. 312] yield the following key lemma obtained in [5].

[1] Interesting results concerning (1) may be found in [7].

LEMMA 1. *Let $\alpha, \beta, \gamma \in Q_{m,n}$ $(n \geq 4)$, $|\alpha \cap \beta| = k$. Then for any n-square unitary matrix U,*

$$|\det U[\gamma|\alpha] \det U[\gamma|\beta]| \leq \begin{cases} \frac{1}{4} & \text{if } k = m-2, \\ \dfrac{1}{2(m-k+1)} & \text{if } k < m-2. \end{cases} \tag{3}$$

Now A and U^*AU share a common set of eigenvalues for any $U \in \mathfrak{U}_n$. So to obtain (2) A may be replaced by U^*AU, where A is diagonal. It follows that $C_m(A)$ is diagonal.

Proof of Theorem 1. For any $z \in \mathbb{C}$

$$\begin{aligned} |\det(U^*AU)[\alpha|\beta]| &= \left|\left(C_m(U^*AU)e_\beta^\wedge, e_\alpha^\wedge\right)\right| \\ &= \left|\left(\left\{C_m(U^*AU) - zI_{\binom{n}{m}}\right\}e_\beta^\wedge, e_\alpha^\wedge\right)\right| \qquad (\text{since } \alpha \neq \beta) \\ &= \left|\left(C_m(U^*)\left\{C_m(A) - zI_{\binom{n}{m}}\right\}C_m(U)e_\beta^\wedge, e_\alpha^\wedge\right)\right| \\ &= \left|\sum_{\gamma, \omega \in Q_{m,n}} C_m(U^*)_{\alpha\gamma}\left\{C_m(A) - zI_{\binom{n}{m}}\right\}_{\gamma\omega} C_m(U)_{\omega\beta}\right| \\ &= \left|\sum_\gamma \overline{\det U[\gamma|\alpha]}\,\{\lambda_\gamma - z\}\det U[\gamma|\beta]\right| \\ &\leq \sum_\gamma |\det U[\gamma|\alpha] \det U[\gamma|\beta]||\lambda_\gamma - z| \\ &\leq \begin{cases} \dfrac{1}{4}\sum_\gamma |\lambda_\gamma - z| & \text{if } k = m-2, \\ \dfrac{1}{2(m-k+1)}\sum_\gamma |\lambda_\gamma - z| & \text{if } k < m-2 \end{cases} \end{aligned} \tag{4}$$

from Lemma 1. The theorem follows immediately upon minimizing (4) over $\mathbb{C}$. ■

REMARK. If $U \in \mathfrak{U}_n$, then $C_m(U) \in \mathfrak{U}_{\binom{n}{m}}$. Therefore, the columns of $C_m(U)$ are unit vectors. It follows that $\sum_{\gamma \in Q_{m,n}} |\det U[\gamma|\alpha] \det U[\gamma|\beta]| \leq 1$.

Take $U\in\mathcal{U}_n$, $\alpha,\beta,\gamma\in Q_{m,n}$, and $|\alpha\cap\beta|=k<m$. The orthogonality of the columns of $C_m(U)$, the Remark, and (3) imply the three following conditions:

$$\sum_{\gamma}\overline{\det U[\gamma|\alpha]}\,\det U[\gamma|\beta]=0,$$

$$\sum_{\gamma}|\det U[\gamma|\alpha]\det U[\gamma|\beta]|\leqslant 1,$$

and

$$|\det U[\gamma|\alpha]\det U[\gamma|\beta]|\leqslant\begin{cases}\frac{1}{4} & \text{if}\quad k=m-2,\\ \dfrac{1}{2(m-k+1)} & \text{if}\quad k<m-2.\end{cases}$$

Lemma 2. *Let $N\geqslant 6$, $a\geqslant 2$, $b=2a<N$ be integers, $l_1\geqslant l_2\geqslant\cdots\geqslant l_N\geqslant 0$ be real numbers, and*

$$\mathcal{D}=\left\{(d_1,\ldots,d_N)\in\mathbb{C}^N:\sum_{i=1}^{N}d_i=0,\ \sum_{i=1}^{N}|d_i|\leqslant 1,\ |d_i|\leqslant\frac{1}{b}\right\}.$$

Then

$$\max_{d\in\mathcal{D}}\left|\sum_{i=1}^{N}l_id_i\right|=\frac{1}{b}\{(l_1-l_N)+\cdots+(l_a-l_{N-a+1})\}.$$

Proof. For any $z=\sum_{i=1}^{N}l_id_i$ there is a ξ, $|\xi|=1$, such that $|z|=\xi z=\sum_{i=1}^{N}l_i\xi d_i\geqslant 0$. Since $\xi(d_1,\ldots,d_N)\in\mathcal{D}$, we may assume $\sum_{i=1}^{N}l_id_i\geqslant 0$. Moreover, $0\leqslant\sum_{i=1}^{N}l_id_i=\mathrm{Re}\sum_{i=1}^{N}l_id_i=\sum_{i=1}^{N}l_i\mathrm{Re}(d_i)$ and $(\mathrm{Re}(d_1),\ldots,\mathrm{Re}(d_N))\in\mathcal{D}$. So we assume d_i is real, $i=1,\ldots,N$. Suppose $l_i=l_j$, some $i\neq j$. Select $\varepsilon>0$, and form $\hat{l}_j=l_j+\varepsilon$, $\hat{l}_i=l_i$, $i\neq j$. Then $|\sum_{i=1}^{N}\hat{l}_id_i-\sum_{i=1}^{N}l_id_i|\leqslant\varepsilon$. Therefore, any maximal sum is arbitrarily close to a sum in which the l_i are distinct. So we may assume $l_1>l_2>\cdots>l_N>0$.

Suppose $\sum_{i=1}^{N}|d_i|<1$. Since $b<N$, we can find $i_1<i_2$ such that $|d_{i_1}|<1/b$ and $|d_{i_2}|<1/b$. Then there exist $\varepsilon>0$, $\hat{d}_{i_1}=d_{i_1}+\varepsilon$, $\hat{d}_{i_2}=d_{i_2}-\varepsilon$, $\hat{d}_i=d_i$ for $i\neq i_1,i_2$ such that $(\hat{d}_1,\ldots,\hat{d}_N)\in\mathcal{D}$, and $\sum_{i=1}^{N}l_i\hat{d}_i=\sum_{i=1}^{N}l_id_i+(l_{i_1}-l_{i_2})\varepsilon>\sum_{i=1}^{N}l_id_i$. So we may assume $\sum_{i=1}^{N}|d_i|=1$.

Suppose there is an i_0 with $|d_{i_0}| \notin \{0, 1/b\}$. Then there are at least two indices i_1, i_2 with $i_0 \in \{i_1, i_2\}$, $i_1 < i_2$, and

$$|d_{i_1}|, |d_{i_2}| \notin \left\{0, \frac{1}{b}\right\}. \tag{5}$$

Otherwise, $0 = \sum_{i=1}^N d_i = d_{i_0}$. Moreover, d_{i_1} and d_{i_2} may be chosen so that $d_{i_1} \cdot d_{i_2} > 0$. For if not, then

$$d_{i_1} \cdot d_{i_2} < 0, \tag{6}$$

and we cannot find $|d_{i_3}| < 1/b$ such that $d_{i_1} \cdot d_{i_3} > 0$ or $d_{i_2} \cdot d_{i_3} > 0$. Thus $|d_i| \in \{0, 1/b\}$ for $i \neq i_1, i_2$, and $1 = \sum_{i \neq i_1, i_2}^N |d_i| + |d_{i_1}| + |d_{i_2}| = (b-1)/b + |d_{i_1}| + |d_{i_2}|$. Since b is even,

$$0 = \sum_{i \neq i_1, i_2}^N d_i + d_{i_1} + d_{i_2} = \pm \frac{1}{b} + d_{i_1} + d_{i_2}. \tag{7}$$

But (5) and (6) imply

$$\begin{aligned} |d_{i_1} + d_{i_2}| &= \big| |d_{i_1}| - |d_{i_2}| \big| \\ &< \begin{cases} \dfrac{1}{b} - |d_{i_2}| & \text{if} \quad |d_{i_1}| \geq |d_{i_2}|, \\ \dfrac{1}{b} - |d_{i_1}| & \text{if} \quad |d_{i_2}| \geq |d_{i_1}| \end{cases} \\ &< \frac{1}{b}, \end{aligned}$$

which contradicts (7). Therefore $d_{i_1} \cdot d_{i_2} > 0$, and for any $0 < \delta < \min\{|d_{i_1}|, |d_{i_2}|$, $|d_{i_1} + \delta| + |d_{i_2} - \delta| = |d_{i_1}| + |d_{i_2}|$. As above, there exist $\varepsilon > 0$, $\hat{d}_{i_1} = d_{i_1} + \varepsilon$, $\hat{d}_{i_2} = d_{i_2} - \varepsilon$, $\hat{d}_i = d_i$, $i \neq i_1, i_2$, such that $(\hat{d}_1, \ldots, \hat{d}_N) \in \mathfrak{D}$, and $\sum_{i=1}^N l_i \hat{d}_i > \sum_{i=1}^N l_i d_i$. So we assume $|d_i| \in \{0, 1/b\}$, $i = 1, \ldots, N$. Since $\sum_{i=1}^N d_i = 0$, the d_i's must pair off with opposite signs. In other words, there exist $i_1, \ldots, i_a, i_1', \ldots, i_a'$ such that

$$\begin{aligned} \sum_{i=1}^N l_i d_i &= \sum_{j=1}^a d_{i_j}(l_{i_j} - l_{i_j'}) \\ &\leq \frac{1}{b}\{(l_1 - l_N) + \cdots + (l_a - l_{N-a+1})\}. \end{aligned}$$

■

Proof of Theorem 2. Take $N=\binom{n}{m}$. If $k=m-2$, set $a=2$ and $b=2a=4<6\leqslant N$. If $0\leqslant k<m-2$, set $a=m-k+1\geqslant 4$ and $b=2a=2(m-k+1)$; since $n\geqslant 2m$, it follows that $b<N$. Select $\gamma_i\in Q_{m,n}$ so that $\lambda_{\gamma_i}=l_i$, where $l_1\geqslant l_2\geqslant\cdots\geqslant l_N\geqslant 0$, and let $d_i(U)=\det U[\gamma_i|\alpha]\ \det U[\gamma_i|\beta]$, $i=1,\dots,N$, where $U\in\mathcal{U}_n$, $|\alpha\cap\beta|=k$. Hence from Lemma 1 and Lemma 2

$$\rho_k(A)=\max_{U\in\mathcal{U}_n}\left|\sum_{i=1}^{N}\lambda_{\gamma_i}\overline{\det U[\gamma_i|\alpha]}\det U[\gamma_i|\beta]\right|$$

$$=\max_{U\in\mathcal{U}_n}\left|\sum_{i=1}^{N}l_i d_i(U)\right|$$

$$\leqslant\begin{cases}\frac{1}{4}\{(l_1-l_N)+(l_2-l_{N-1})\}, & k=m-2,\\ \dfrac{1}{2(m-k+1)}\{(l_1-l_N)+\cdots+(l_{m-k+1}-l_{N-m+k})\}, & k<m-2.\end{cases}$$

The result follows immediately upon replacing the l_i's with the λ_γ's. ■

IV. APPLICATIONS

It is shown in [5] that *if A is an n-square normal matrix*, $m\geqslant 2$, $n\geqslant 2m$, *then*

$$\rho_k(A)\leqslant\begin{cases}\dfrac{E_m(|\lambda_1|,\dots,|\lambda_n|)}{4} & \textit{if}\quad k=m-2,\\ \dfrac{E_m(|\lambda_1|,\dots,|\lambda_n|)}{2(m-k+1)} & \textit{if}\quad k<m-2,\end{cases}\tag{8}$$

where $E_m(t_1,\dots,t_m)=\sum_{\gamma\in Q_{m,n}}\prod_{i=1}^m t_{\gamma(i)}$ *is the mth elementary symmetric polynomial.* Since $\min_{z\in\mathbf{C}}\sum_\gamma|\lambda_\gamma-z|\leqslant E_m(|\lambda_1|,\dots,|\lambda_n|)$, (2) refines (8).

Let A be Hermitian, $k\in\{0,1,\dots,m-1\}$. From Mirsky [8] it is immediate that $\rho_k(A)\leqslant\frac{1}{2}(\lambda_{\max}-\lambda_{\min})$. In [6], (1) is conjectured for positive semidefinite A. This conjecture is resolved here in the affirmative for the case $n=4$, $m=2$.

Assume $A=\operatorname{diag}(\lambda_1,\dots,\lambda_4)$, $\lambda_1\geqslant\cdots\geqslant\lambda_4\geqslant 0$, $\lambda_{ij}=\lambda_i\lambda_j$ for $1\leqslant i,j\leqslant 4$. Since the eigenvectors of $C_2(A)$ may be chosen from $G_{2,4}$, we have $\rho_2(A)=$

$\lambda_{\max}=\lambda_{12}$. Clearly $\lambda_{12}\geqslant\frac{1}{2}(\lambda_{12}-\lambda_{34})$, so $\rho_2(A)\geqslant\rho_1(A)$. If

$$U_0=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\end{bmatrix},$$

then $\det(U_0^*AU_0)[12|13]=\frac{1}{4}\{(\lambda_{12}-\lambda_{34})+(\lambda_{13}-\lambda_{24})\}\in\Delta^1(A)$. Therefore

$$\rho_2(A)\geqslant\rho_1(A)\geqslant\tfrac{1}{4}\{(\lambda_{12}-\lambda_{34})+(\lambda_{13}-\lambda_{24})\}$$
$$\geqslant\rho_0(A)\qquad\text{from Theorem 2.}$$

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Received 20 October 1980; revised 22 June 1981