On the solution of Falkner–Skan equations

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Abstract

New existence and uniqueness proofs are given for the solutions of the equations governing the self-similar compressible boundary layer (Falkner–Skan equations). The properties of the solutions are studied and some bounds on important quantities are concluded. The paper is restricted to favourable pressure gradients and to wall cooling.

Keywords: Boundary layers; Fixed point; Ordinary differential equations

1. Introduction

Uniqueness and existence proofs for the solutions of the equations of viscous flow are very difficult to obtain. Some proofs were given by Ladyzhenskaya [5] for incompressible flows. More recently Teman [13] studied the subject as well. The boundary layer equations are usually considered more amenable to theoretical treatment. Yet general uniqueness and existence theorems for them are not available either. An easier case is that of the Falkner–Skan equations which govern self-similar solutions of the compressible boundary layer equations for a Prandtl number of unity. Weyl [16] proved existence only for the adiabatic wall case. His work was later extended by Coppel [1] who produced an easier proof for existence and proved uniqueness as well. More work was described by Utz [14] and Veldman and Van der Vooren [15] in which some restrictions were removed and properties of
the solutions were investigated further. McLeod and Serrin [8] and Hastings [3] removed the restriction of adiabatic wall.

Investigating numerically these equations [9], we came upon several interesting properties of the equations that led us to the methods applied in this paper. We present here, a new proof of the existence and uniqueness of solutions for the compressible similar boundary layer (Falkner–Skan) equations subjected to positive wall temperature and positive skin friction at the wall. This is a set of coupled non-linear ordinary differential equations. Generalizing the adiabatic equation and working along similar lines as Coppel [1], we obtained an existence and uniqueness theorem for the generalized equation. Using this theorem we were able to construct a uniformly continuous set of function and applying the Schauder fixed point theorem, we obtained the desired proof, namely, that the set of Falkner–Skan equations with a physical set of boundary conditions have a unique solution.

During the last couple of years, there has been a renewed interest in the mathematical aspects of the Falkner–Skan adiabatic equation [11,12,17]. These papers deal with bifurcation in the equation. Bifurcation in the equation occurs when the boundary conditions are different from those that we consider in this paper, for example when there is a detached flow. Since we view the equations from the physical side and consider boundary conditions, which occur when the flow is laminar and similar, the bifurcation problem is out of the scope of this work. Several authors [6,7,10] considered some physical aspects of the adiabatic equation. These papers are more related to our previous numerical work [9] on the Falkner–Skan equations.

2. Auxiliary uniqueness and existence theorems

A self-similar compressible two-dimensional boundary layer is governed by the following system of non-linear ordinary differential equations (Padé et al. [9]):

\[
\begin{align*}
    y''' + yy'' + \beta(g - y'^2) &= 0, \\
    g'' + yg' &= 0,
\end{align*}
\]  

(1)

with the following boundary conditions:

\[
\begin{align*}
    y(0) &= y'(0) = 0, \\
    g(0) &= g_w > 0, \\
    \lim_{x \to \infty} y'(x) &= \lim_{x \to \infty} g(x) = 1,
\end{align*}
\]

where \(0 \leq \beta < \infty\) is the pressure gradient parameter.

We shall first prove that the first equation in (1) has a unique solution satisfying

\[y'^2 \leq g,\]

where \(g\) is a positive increasing bounded function.

Consider the following equation:

\[
\begin{align*}
    y''' + y'' + \beta(g - y'^2) &= 0, \\
\end{align*}
\]  

(2)

with the initial conditions \(y(0) = 0, y'(0) = 0\).
Let \( g \) be an arbitrary function satisfying
\[
g(0) > 0, \quad \lim_{x \to \infty} g(x) = 1, \quad g'(x) > 0, \quad g''(x) < 0.
\]
(3)

**Proposition 2.1.** Equation (2) has a solution \( y \) satisfying

(a) \( 0 \leq y' \leq \sqrt{g} \).
(b) \( 0 < y'' \).

**Proof.** Write Eq. (2) as a system:
\[
y_1 = y, \\
y_1' = y_2, \\
y_2' = y_3, \\
y_3' = -y_1y_3 - \beta(g - y_2^2).
\]
(4)

The system (4) has a unique solution for any given set of initial conditions (Hartman [2]). In other words, any given set of initial conditions \( \{y_0, y_0', y_0''\} \) define a unique solution of (4). This solution may be viewed as a curve in the \( \langle y_1, y_2, y_3 \rangle \) space, with \( x \) as the curve parameter. Any two of these curves cannot intersect.

Consider a solution for which \( (y_1(0), y_2(0), y_3(0)) = (0, 0, s) \) with \( s > 0 \). Let \( C(s) \) be the curve describing the solution of (4) satisfying the above initial conditions.

Following Coppel [1] we define a domain \( D \), by
\[
y_1 > 0, \quad 0 < y_2 < \sqrt{g}, \quad y_3 > 0.
\]

For every curve in this domain, \( y_1 \) and \( y_2 \) are increasing functions of \( x \), while \( y_3 \) is a decreasing function. No curve can tend to infinity while \( x \) tends to a finite value. \( y_2 \) and \( y_3 \) are bounded, i.e., \( y_1' \) is bounded so is \( y \) in every finite interval.

No curve can leave \( D \) through the edge \( y_1 > 0, y_2 = \sqrt{g}, y_3 = 0 \). Suppose that it occurs, then there is a point \( x_3 \) such that \( y_1(x_3) = 0 \) and \( y_2(x_3) = \sqrt{g(x_3)} \), clearly from (4) \( y_3'(x_3) = 0 \) and \( y_3 y_3''(x_3) = -\beta g'(x_3) < 0 \), following (3). Hence \( y_3 \) has a maximum at \( x_3 \), contradicting the fact that \( y_3 \) is increasing in \( [0, x_3] \).

We are left with the following three possibilities (modified from [1]):

(a) \( C \) leaves \( D \) through \( y_1 > 0, y_2 = \sqrt{g}, y_3 > 0 \).
(b) \( C \) leaves \( D \) through \( y_1 > 0, 0 < y_2 < \sqrt{g}, y_3 = 0 \).
(c) \( C \) is defined and remains in \( D \) for all \( x > 0 \).

It is clear that if \( s \) is small enough \( y_1 > 0, 0 < y_2 < \sqrt{g} \), if \( x \) is small then \( y_2' \) is small but \( y_1' \) is nearly equal to the negative value of \( -\beta g' \), so that \( C(s) \) is of type (b):
\[
y_2' = -(y_1 y_2)' + y_2^2 - \beta(g - y_2^2)
= -(y_1 y_2)' + (1 + \beta)y_2^2 - \beta g \geq -(y_1 y_2)' - \beta g \geq -(y_1 y_2)' - \beta.
\]
By integration we obtain \( y_3 \geq s - \beta x \). So if \( s \) is large enough, \( C(s) \) must be of type (a). The values of \( s \) for which \( C(s) \) is of type (a) or of type (b), form two open subsets of the half line \((0, \infty)\). Since this half line is a connected set, there must be a value of \( s \) such that \( C(s) \) is of type (c). For a path of type (c) it is clear that since \( y_2' = y_3 \) is a positive decreasing function and \( y_2 \) is bounded, it is clear that we must have \( y_3 \to 0 \) when \( x \to \infty \). Similarly, \( y_2 \) is an increasing function, since \( y_3 \) remains bounded and \( y_2' < -\beta(g - y_2^2) \), we must have \( g - y_2^2 \to 0 \) as \( x \to \infty \). 

\[ \square \]

We have shown that a solution \( y \) of (2) satisfying \( y'' > 0 \) also satisfies \( y'^2 \leq g \). Moreover, since \( y'' > 0 \), \( y' \) is an increasing function. Since \( x \to \infty \Rightarrow g \to 1 \) we obtain also that \( y' \to 1 \).

**Lemma 2.1.** Let \( u, v, \) be two continuously differentiable functions such that \( u/v \) is monotonic non-decreasing in the interval \([0, a]\). If \( v \) is of constant sign in \([0, a]\), then \( \int_0^x u(t) dt / \int_0^x v(t) dt \) is monotonic non-decreasing in \([0, a]\).

**Proof.** Let \( v \) be positive (or negative) in the interval \([0, a]\). Then \( u/v \) is monotonic non-decreasing iff for every \( x_1, x_2, 0 \leq x_1 \leq x_2 \leq a \),

\[
D_{u,v}(x_1, x_2) = \begin{vmatrix}
v(x_1) & v(x_2) \\
u(x_1) & u(x_2)
\end{vmatrix} \geq 0.
\]

This follows directly from the definition of the determinant.

Let \( 0 \leq x_1 \leq x_2 \leq a \). For every \( t, s \) such that \( 0 \leq t \leq x_1 \leq s \leq x_2 \leq a \), \( D_{u,v}(t, s) \geq 0 \),

\[
E(x_1, s) = \int_0^{x_1} D_{u,v}(t, s) dt = \begin{vmatrix}
\int_t^{x_1} v(t) dt & v(s) \\
\int_t^{x_1} u(t) dt & u(s)
\end{vmatrix} \geq 0, \quad \forall s, \ 0 \leq x_1 \leq x_2 \leq a.
\]

Adding the first column of the above matrix to the second and we obtain

\[
H(x_1, x_2) = \begin{vmatrix}
\int_0^{x_1} v(t) dt & \int_0^{x_2} v(s) ds \\
\int_0^{x_1} u(t) dt & \int_0^{x_2} u(s) ds
\end{vmatrix} \geq 0.
\]

By the first part of the proof, \( \int_0^x u(t) dt / \int_0^x v(t) dt \) is monotonic non-decreasing in \([0, a]\). \( \square \)

**Proposition 2.2.** Equation (2) has a unique solution \( y \) satisfying \( y'' > 0 \).

**Proof.** Equation (2) is equivalent to the following equation:

\[
\frac{1}{y'} \frac{d^2}{dx^2} (\log y') = -\frac{1}{y'} \left[ \frac{d}{dx} (\log y') \right]^2 - \frac{y}{y'} \frac{d}{dx} (\log y') + \beta \left( 1 - \frac{g}{y'^2} \right).
\]

Suppose that there are two solutions to (2), \( y \) and \( z \). Without loss of generality, \( y''(0) > z''(0) \). It follows directly from (2) that

\[ \text{Dr. E. Lapidot of Rafael suggested the proof given here. I am grateful to him for his help.} \]
\[ y'''(0) = z'''(0) = -\beta g(0), \]
\[ y^{(4)}(0) = z^{(4)}(0) = -\beta g'(0). \]

First we will show that \( y/z \) and \( y'/z' \) are increasing functions in some neighborhood of 0. We will do it by showing that the derivative of \( y'/z' \) is positive near 0 and then using Lemma 2.1:
\[
\frac{d}{dx} \frac{y'}{z'} = \frac{y'' z' - y' z''}{z'^2}.
\]
Both numerator and denominator tend to 0 as \( x \to 0 \). By l'Hospital rule
\[
\lim_{x \to 0} \frac{y'' z' - y' z''}{z'^2} = \lim_{x \to 0} \frac{y'''(0)z''(0) - y''(0)z'''(0)}{2z''(0)^2} \\
= \frac{\beta g(0)[y''(0) - z''(0)]}{2z''(0)^2} > 0.
\]

It is clear now that \( y'/z' \) is increasing from 0.

Clearly
\[
\lim_{x \to 0} \frac{y'}{z'} = \frac{y''(0)}{z''(0)} > 1 \quad \text{and} \quad \lim_{x \to \infty} \frac{y'}{z'} = 1.
\]
So it is clear that \( y'/z' \) and its logarithm has a maximum point. By Lemma 2.1 and the boundary conditions \( y(0) = z(0) = 0 \), \( y/z \) is increasing in the same interval that \( y'/z' \) is increasing.

Let \( x_1 \) be a maximum point for \( y'/z' \) so it is also a maximum point for \( \log(y'/z') \). Then, for \( x = x_1 \),
\[
\frac{d}{dx} \log \left( \frac{y}{z} \right) > 0 \quad \Rightarrow \quad \frac{y'}{y} > \frac{z'}{z} \quad \Rightarrow \quad -\frac{y}{y'} > -\frac{z}{z'}
\]
and
\[
\frac{d}{dx} \log y' = \frac{d}{dx} \log z'.
\]

In the neighborhood of \( x_1, y'>z' \Rightarrow -1/y' > -1/z'.

From the above we obtain the following inequality:
\[
-\frac{1}{y'} \left[ \frac{d}{dx} \log y' \right]^2 - \frac{y}{y'} \frac{d}{dx} \log y' + \beta \left( 1 - \frac{g}{y'^2} \right) \\
> -\frac{1}{z'} \left[ \frac{d}{dx} \log z' \right]^2 - \frac{z}{z'} \frac{d}{dx} \log z' + \beta \left( 1 - \frac{g}{z'^2} \right).
\]
We have shown that \( y'/z' \) increases from 0, and it is bigger then 1 at zero, so clearly, since \( x_1 \) is a maximum, we see that
\[
y'(x_1) > z'(x_1) \quad \Rightarrow \quad -\frac{1}{y'(x_1)} > -\frac{1}{z'(x_1)} \quad \text{and} \quad \frac{d}{dx} (\log y')(x_1) = \frac{d}{dx} (\log z')(x_1).
\]
and since \( y/z \) is increasing at \( x_1 \), \( y'(x_1)z(x_1) - y(x_1)z'(x_1) > 0 \). Hence,
\[
\frac{y}{y'} < \frac{z}{z'} \quad \Rightarrow \quad -\frac{y}{y'} > -\frac{z}{z'}.
\]

It follows that
\[
\frac{1}{y/\,dx^2}(\log y') > \frac{1}{z/\,dx^2}(\log z') \quad \Rightarrow \quad \frac{d^2}{dx^2}(\log y') > \frac{y'}{z}(\frac{d^2}{dx^2}(\log z')).
\]

which means that \( x_1 \) is a minimum and we obtained the desired contradiction. \( \square \)

We will use the following lemma [1].

Lemma 2.2. Let \( z(x), w(x) \) be continuous functions in the interval \( [a, b] \). Assume that \( z''(x) \leq F(x, z, z') \) and \( w''(x) \geq F(x, w, w') \) where \( F \) is an increasing function of its second argument. If \( w(a) \leq z(a) \) and \( w(b) \leq z(b) \) then \( w(x) \leq z(x) \) for \( a \leq x \leq b \).

Lemma 2.3. Let \( y \) be the solution of (2) and let \( y_0 \) be the solution of (2) for \( \beta = 0 \). Then \( y \geq y_0 \).

Proof. It is clear from (2) that for every \( \beta > 0 \), \( y''' \geq -yy'' \). Equality exists for \( y = y_0 \).

We will use the following transformation:
\[
y' = z(y).
\]
Then
\[
y'' = z\,\frac{dz}{dy}, \quad y''' = z\left(\frac{dz}{dy}\right)^2 + z^2\left(\frac{d^2z}{dy^2}\right).
\]

From (2) we obtain
\[
y''' + y''y = -\beta(g - y'^2) \quad \Rightarrow \quad y''' + y''y \leq 0 \quad \Rightarrow \quad y''' \leq -y''y.
\]

Substitution of (5) and (6) yields
\[
z\left(\frac{dz}{dy}\right)^2 + z^2\left(\frac{d^2z}{dy^2}\right) \leq -yz\,\frac{dz}{dy} \quad \Rightarrow \quad z\left(\frac{d^2z}{dy^2}\right) \leq -z\left(\frac{dz}{dy}\right)^2 - yz\,\frac{dz}{dy}.
\]

Now, \( z > 0 \), so dividing by \( z \) we obtain
\[
\frac{d^2z}{dy^2} \leq -\frac{1}{z}\left(\frac{dz}{dy}\right)^2 - \frac{y}{z}\,\frac{dz}{dy}.
\]

Let \( v(y) \) satisfy (7) and let \( w(y) \) be the solution of
\[
\frac{d^2z}{dy^2} = -\frac{1}{z}\left(\frac{dz}{dy}\right)^2 - \frac{y}{z}\,\frac{dz}{dy}.
\]

Define
\[
F\left(y, z, \frac{dy}{dz}\right) = -\frac{1}{z}\left(\frac{dz}{dy}\right)^2 - \frac{y}{z}\,\frac{dz}{dy}.
\]
Then
\[ \frac{d^2v}{dy^2} \leq F(y, v, dv/dy), \]
\[ \frac{d^2w}{dy^2} \geq F(y, w, dw/dy), \]
\[ w(0) = v(0), \]
\[ \lim_{y \to \infty} w(y) = \lim_{y \to \infty} v(y) = 1, \]
so that by Lemma 2.2 \( w(y) \leq v(y) \).

From (6), it follows that for every solution of (2)
\[ \frac{y''}{y'} \leq \frac{y''}{y'} \Rightarrow \log y' \leq \log y' \Rightarrow y' \leq y' \Rightarrow y_0 \leq y. \]

\[ \therefore \]

3. Main uniqueness and existence theorem

**Proposition 3.1.** The system (1) has a solution \( (y, g) \) satisfying \( 0 \leq y' \leq 1, 0 \leq y'', g_w \leq g \leq 1, 0 < g' \).

**Proof.** Define
\[ G_0 = \left\{ g \mid g \text{ continuous, } g(0) = g_w, g \leq 1, g' \geq 0, \lim_{x \to \infty} g(x) = 1 \right\}. \]
\[ G_0 \subset C_0(R_+). \]

If \( g \in G_0 \), then by Proposition 2.1 the equation
\[ y'''' + yy'' + \beta(g - y'^2) = 0 \]
has a unique solution \( y_g \) such that \( y_g'^2 \leq g \) and \( y_g'' > 0 \). Define an operator \( L \) on \( G_0 \) by
\[ Lg = g_w + g'(0) \int_0^x \exp \left[ - \int_0^t y_g(s) \, ds \right] \, dt, \]
where
\[ g'(0) = \frac{1 - g_w}{\int_0^\infty \exp \left( - \int_0^x y_g(t) \, dt \right) \, dx}. \]

For \( g \in G_0, y_g' \leq 1 \), hence \( y_g \leq x \). Thus we have the following inequality:
\[ \int_0^\infty \exp \left[ - \frac{x}{2} \right] \, dx \geq \int_0^\infty \exp(-x^2/2) \, dx = \sqrt{\frac{\pi}{2}} \]
(9)
From (9) we obtain the inequality
\[ g'(0) \leq (1 - g_w) \sqrt{\frac{2}{\pi}} < 1. \]
By (8) and (9) we obtain

$$\left| Lg(s) - Lg(t) \right| = g'(0) \left| \int_s^t \exp\left[ - \int_0^x y_g(r) \, dr \right] \, dx \right| \leq g'(0) |s - t|. \quad (10)$$

We wish to show that $LG_0$ is compact in $C_0(R_+)$. From (10) it follows that the functions in $LG_0$ are equicontinuous in every finite subinterval of $R_+$. In order to show compactness we shall have to prove that the functions in $LG_0$ are equicontinuous at infinity:

$$\left| \int_s^t \exp\left[ - \int_0^x y_g(r) \, dr \right] \, dx \right| \leq \left| \int_s^\infty \exp\left[ - \int_0^x f(r) \, dr \right] \, dx \right|. \quad (11)$$

Let $f$ be the solution of (2) for $\beta = 0$, such that $f'' > 0$. From Lemma 2.3 we know that $y_g \geq f$, which proves (11). The right hand side of (11) is independent of $g$ and can be made as small as we wish for $s$ and $t$ large enough. Using (11) we now see that the functions in $LG_0$ are equicontinuous at infinity. We found that $LG_0$ is a set a closed bounded set of equicontinuous functions on the one point compactification of $R_+$, and by Ascoli–Arzela theorem [4], $LG_0$ is a compact set.

Clearly $LG_0 \subseteq G_0$, and since $L$ is a compact operator in $G_0$, then by the Schauder fixed point theorem [4] $L$ has a fixed point $y_0$. The meaning of $Ly_0 = y_0$ is that the corresponding $y_0$ is the desired solution.

Lemma 3.1. The function $g - y'^2$ has one maximum for every solution of (1).

**Proof.** $g'(0) > 0$ and $y'(0) = 0$, hence $g' - 2y'y'' > 0$ in some neighborhood of 0, i.e., $g - y'^2$ is increasing in that neighborhood. If $x \to \infty$ then $g - y'^2 \to 0$, so that $g - y'^2$ has at least one maximum. We will show now that $g' - 2y'y'' = 0$ only once.

From (1)

$$g'(x) = g'(0) \exp\left( - \int_0^x y \, dt \right),$$

$$y''(x) = \exp\left( - \int_0^x y \, dt \right) \left[ y''(0) - \beta \int_0^x \exp\left( - \int_0^y f \, ds \right) (g - y'^2) \, ds \right]$$

$$= \frac{g'(x)}{g'(0)} \left[ y''(0) - \beta \int_0^x \frac{g'(0)}{g'(s)} (g - y'^2) \, ds \right].$$

Hence

$$g' - 2y'y'' = g' \left[ 1 - 2y' \frac{y''(0)}{g'(0)} + 2\beta y' \int_0^x \frac{g - y'^2}{g} \, ds \right].$$

$g'$ is positive so that in order to prove the lemma we have to show that the expression in the bracket can be 0 only once. The function $1 - 2y' [y''(0)/g'(0)]$ is a decreasing function,
equals 1 for \( x = 0 \). The second function \( 2\beta y' \int [(g - y^2)/g'] \, ds \) is an increasing function, equals 0 for \( x = 0 \). It is clear now that they cannot have more then one intersection point, hence, they have exactly one. \( \Box \)

**Lemma 3.2.** \( y'g' - 2gy'' \leq 0 \) for every solution of the system (1).

**Proof.** \( y'g' - 2gy'' \leq y'g' - 2y^2y'' = y'(g' - 2y'y'') \leq 0 \), where \( g - y^2 \) decreases. If \( y'g' - 2gy'' = 0 \), it must be in an interval where \( g - y^2 \) increases. By Lemma 3.1 \( g - y^2 \) has one maximum at \( y^* \) so that \( g' - 2y'y'' \leq 0 \) for \( x \leq y^* \). Clearly, \( y'g' - 2gy'' = 0 \) in some neighborhood of 0, therefore, there must be an interval \([a, b]\) such that \( y'g' - 2gy'' > 0 \) for \( 0 < a < x < b \leq y^* \) and \( y'g' - 2gy'' = 0 \) for \( x = a \) and for \( x = b \), and since \( g' > 0 \), the same is true for \( y' - 2gy''/g' \). It follows that \( y' - 2gy''/g' \) has a maximum at some \( x^* \in [a, b] \).

We have seen that
\[
y'' = \exp\left( -\int_0^x y \, dt \right) \left[ y''(0) - \beta \int_0^x \exp\left( \int_0^s y \, dt \right) (g - y^2) \, ds \right],
\]
\[
g' = g'(0) \exp\left( -\int_0^x y \, dt \right).
\]

From these relations we obtain
\[
y'' = \frac{g'}{g'(0)} \left[ y''(0) - \beta \int_0^x \exp\left( \int_0^s y \, dt \right) (g - y^2) \, ds \right].
\]

It is easy to see that
\[
\left[ y' - 2g\frac{y''}{g'} \right]' = -y'' + \frac{2\beta}{g'(0)} g(g - y^2) \exp\left( \int_0^x y \, dt \right), \tag{12}
\]
so that for \( x = y^* \)
\[
-y'' + \frac{2\beta}{g'(0)} g(g - y^2) \exp\left( \int_0^x y \, dt \right) \begin{cases} > 0 & \text{for } x < y^*, \\ = 0 & \text{for } x = y^*, \\ < 0 & \text{for } x > y^*. \end{cases} \tag{13}
\]

But in \([a, b]\), \( g - y^2 \) is an increasing function and so is \(-y''\), so if the derivative (12) is positive in \([a, x^*]\), it must be positive in \([a, b]\). This is a contradiction to (13) which proves the lemma. \( \Box \)

**Theorem 3.1.** The system (1) has a unique solution.

**Proof.** Using the transformations \( y' = z(y) \) and \( g = h(y) \) we obtain from (1) the following system:
\[
\frac{d^2z}{dy^2} = -\frac{1}{z} \left( \frac{dz}{dy} \right)^2 - \frac{y}{z} \frac{dz}{dy} + \beta \left( 1 - \frac{h}{z^2} \right),
\]
\[
\frac{d^2h}{dy^2} = -\left( \frac{1}{z} \frac{dz}{dy} + \frac{y}{z} \right) \frac{dh}{dy}.
\]
(14)

Define the following function:
\[
F\left(y, z, \frac{dz}{dy}\right) = -\left( \frac{dz}{dy} \right)^2 - \frac{y}{z} \frac{dz}{dy} + \beta \left( 1 - \frac{h}{z^2} \right).
\]
(15)

It is easy to see that \( h \) is a function of \( y, z \) and \( dz/dy \). In order to use Lemma 2.2, we have to show that \( -h/z^2 \) is an increasing function of \( z \), i.e., to show that \( h/z^2 \) is a decreasing function if \( z \), i.e., that
\[
\frac{d}{dz}\left( \frac{h}{z^2} \right) = \frac{1}{z^3} \left( \frac{dh}{dz} - 2h \right) \leq 0.
\]

Meaning that we have to show that \( z(dh/dz) - 2h \leq 0 \):
\[
z \frac{dh}{dz} - 2h = \frac{dh}{dy} \frac{dz}{dy} - 2h = \frac{g'/y'}{y''/y'} - 2g = \frac{y'g' - 2gy''}{y''}.
\]

The denominator of the right hand side is positive and the numerator is negative by Lemma 3.2. If the system (14) have two solutions \( z \) and \( w \), satisfying the same boundary conditions, then applying Lemma 2.2 to (14) and (15) would show that \( z = w \), which proves uniqueness. \( \square \)

References