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On the adjoint L-function of the p-adic GSp(4)

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Abstract

We explicitly compute the adjoint L-function of those L-packets of representations of the group GSp(4) over a p-adic field of characteristic zero that contain non-supercuspidal representations. As an application we verify a conjecture of Gross and Prasad and Rallis in this case. The conjecture states that the adjoint L-function is holomorphic at $s = 1$ if and only if the L-packet contains a generic representation.

1. Introduction

Let F be a non-archimedean local field of characteristic zero and let W_F' be the Weil–Deligne group of F. The conjectural local Langlands correspondence for the group $\mathrm{GSp}(4, F)$ assigns to each irreducible admissible representation Π of $\mathrm{GSp}(4, F)$ an L-parameter, i.e., an equivalence class of admissible representations

$$\varphi_\Pi : W_F' \longrightarrow \mathrm{GSp}(4, \mathbb{C}).$$

It was shown in [RS, Section 2.4] that there is a unique way to assign L-parameters to the *non-supercuspidal* irreducible, admissible representations of $\mathrm{GSp}(4, F)$ such that certain desired properties of the local Langlands correspondence hold. In this sense the local Langlands correspondence is known for the non-supercuspidal representations of $\mathrm{GSp}(4, F)$; see Table 1 for a

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complete list of these representations. In a few cases the L-packet of a non-supercuspidal representation is expected to also contain a supercuspidal representation.

The degree 4 and degree 5 L-factors resulting from the non-supercuspidal local Langlands correspondence have been computed and tabulated in [RS, Tables A.8 and A.10]. In this article we treat the next smallest irreducible representation of the dual group, namely the 10-dimensional adjoint representation Ad of $\mathrm{GSp}(4,\mathbb{C})$ on the complex Lie algebra $\mathfrak{sp}(4)$. Thus, given a non-supercuspidal, irreducible, admissible representation Π of $\mathrm{GSp}(4,F)$ with L-parameter φ_Π, we compute

$$L(s,\Pi,\mathrm{Ad}) := L(s,\mathrm{Ad}\circ\varphi_\Pi).$$

This is an easy calculation in most cases, but requires some arguments in a few. The results are tabulated in Table 2 below.

Having explicit formulas for all the adjoint L-functions, we immediately obtain the following case of a general conjecture of Gross and Prasad [GP, Conjecture 2.6] and Rallis [K, Proposition 5.2.2] as a corollary; see Theorem 4 below.[1]

Let Π be a non-supercuspidal irreducible admissible representation of $\mathrm{GSp}(4,F)$. *Then the L-packet of Π contains a generic representation if and only if* $L(s,\Pi,\mathrm{Ad})$ *is holomorphic at* $s=1$.

The analogous statement for $\mathrm{GL}(n,F)$ "has been observed by many people," [K]. For a proof see [JS2, Proposition 7.1].

We note that there is some overlap between Theorem 4 and a result of Jiang and Soudry. In [JS1] and [JS2] they attach to each admissible L-parameter an irreducible, admissible representation of $\mathrm{SO}(2n+1,F)$ and prove that this representation is generic if and only if its associated adjoint L-function is holomorphic at $s=1$ [JS2, Theorem 7.1]. In the special case $n=2$, since $\mathrm{SO}(5,F)\cong\mathrm{PGSp}(4,F)$, the representation of $\mathrm{SO}(5,F)$ corresponds to a representation of $\mathrm{GSp}(4,F)$ with trivial central character. However, it is not immediately clear that this version of the local Langlands correspondence coincides with ours. To mention one difference, the Jiang–Soudry correspondence misses those representations of $\mathrm{GSp}(4,F)$ whose central character is not a square, since such representations are not a twist of a representation with trivial central character. Also, the Jiang–Soudry correspondence does not assign an L-parameter to the non-generic representations of type VIb and XIb (see Table 1), both of which share an L-packet with a generic representation.

2. Notation and definitions

2.1. Group-theoretic definitions

We realize the algebraic $\mathbb{Q}$-group GSp(4) as

$$\mathrm{GSp}(4)=\bigl\{g\in\mathrm{GL}(4)\colon\ {}^tgJg=\lambda(g)J \text{ for some } \lambda(g)\in\mathrm{GL}(1)\bigr\},$$

[1] The same statement is made in a preprint of W. Gan and S. Takeda [GT] which became available after the completion of this work. It is very likely that their L-parameters coincide with ours.

where

$$J=\begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}.$$

The kernel of the *multiplier homomorphism* $g \mapsto \lambda(g)$ is by definition the symplectic group $\mathrm{Sp}(4)$. The Lie algebra of $\mathrm{Sp}(4)$ is 10-dimensional and is given by

$$\mathfrak{sp}(4)=\left\{X\in\mathfrak{gl}(4)\colon\ {}^tXJ+JX=0\right\}.$$

Over the complex numbers, the Lie algebra of $\mathrm{GSp}(4)$ is a direct sum

$$\mathfrak{gsp}(4)=\mathfrak{sp}(4)\oplus\mathfrak{z},\qquad \mathfrak{z}=\mathbb{C}\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix}.$$

The adjoint representation of $\mathrm{GSp}(4,\mathbb{C})$ on $\mathfrak{gsp}(4)$ preserves both summands and, as representations of $\mathrm{GSp}(4,\mathbb{C})$, we have

$$\mathrm{Ad}_{\mathfrak{gsp}}=\mathrm{Ad}_{\mathfrak{sp}}\oplus\mathbf{1}. \tag{1}$$

We use Ad for $\mathrm{Ad}_{\mathfrak{sp}}$ in this article.

The character lattice of $\mathrm{Sp}(4)$ is spanned by

$$e_1:\begin{bmatrix} a&&&\\ &b&&\\ &&b^{-1}&\\ &&&a^{-1} \end{bmatrix}\longmapsto a\quad\text{and}\quad e_2:\begin{bmatrix} a&&&\\ &b&&\\ &&b^{-1}&\\ &&&a^{-1} \end{bmatrix}\longmapsto b. \tag{2}$$

We shall use the following generators for the root spaces in $\mathfrak{sp}(4)$:

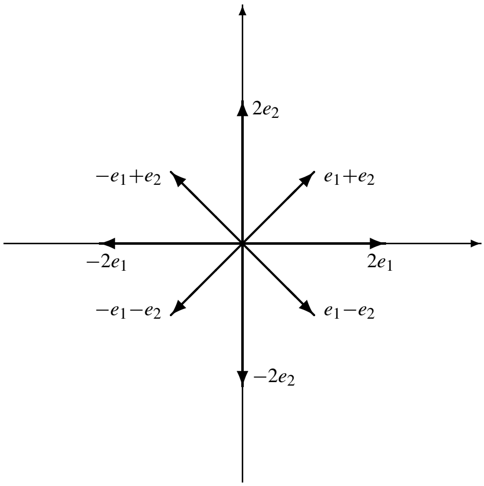
$$L_{e_1-e_2}=\begin{bmatrix} 0&1&&\\ &0&&\\ &&0&-1\\ &&&0 \end{bmatrix},\qquad L_{-e_1+e_2}=\begin{bmatrix} 0&&&\\ 1&0&&\\ &&0&\\ &&-1&0 \end{bmatrix}, \tag{3}$$

$$L_{e_1+e_2}=\begin{bmatrix} 0&&1&\\ &0&&1\\ &&0&\\ &&&0 \end{bmatrix},\qquad L_{-e_1-e_2}=\begin{bmatrix} 0&&&\\ &0&&\\ 1&&0&\\ &1&&0 \end{bmatrix}, \tag{4}$$

$$L_{2e_1}=\begin{bmatrix} 0&&&1\\ &0&&\\ &&0&\\ &&&0 \end{bmatrix},\qquad L_{-2e_1}=\begin{bmatrix} 0&&&\\ &0&&\\ &&0&\\ 1&&&0 \end{bmatrix}, \tag{5}$$

$$L_{2e_2} = \begin{bmatrix} 0 &&& \\ & 0 & 1 & \\ && 0 & \\ &&& 0 \end{bmatrix}, \qquad L_{-2e_2} = \begin{bmatrix} 0 &&& \\ & 0 && \\ & 1 & 0 & \\ &&& 0 \end{bmatrix}. \tag{6}$$

The root system of Sp(4) is of type C_2,



The conjugacy classes of proper parabolic subgroups of GSp(4) are represented by the minimal parabolic subgroup B, the Siegel parabolic subgroup P, and the Klingen parabolic subgroup Q, consisting of matrices in GSp(4) of the following form, respectively:

$$B = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ && * & * \\ &&& * \end{bmatrix}, \qquad P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ && * & * \\ && * & * \end{bmatrix}, \qquad Q = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ &&& * \end{bmatrix}.$$

Setting

$$A' = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} {}^tA^{-1} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \quad \text{for } A \in \mathrm{GL}(2), \tag{7}$$

a typical element of P can be written as $\left[\begin{smallmatrix} A & * \\ & cA' \end{smallmatrix}\right]$ with $c \in \mathrm{GL}(1)$ and $A \in \mathrm{GL}(2)$.

2.2. p-Adic definitions

Let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be its ring of integers and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. We fix a generator ϖ of $\mathfrak{p}$ once and for all. A character χ of $F^\times$ is a continuous homomorphism $F^\times \to \mathbb{C}^\times$. It is unramified if $\chi(\mathfrak{o}^\times) = \{1\}$. A distinguished unramified character is ν, the normalized absolute value of F. It has the property that $\nu(\varpi) = q^{-1}$, where q is the number of elements of the residue class field $\mathfrak{o}/\mathfrak{p}$.

We shall use the notation of [ST] for representations of GSp(4, F) parabolically induced from one of the parabolic subgroups B, P or Q. If χ_1, χ_2 and σ are characters of $F^\times$, then $\chi_1 \times \chi_2 \rtimes \sigma$

denotes the representation of GSp(4, F) obtained via (normalized) parabolic induction from the character

$$\begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} \longmapsto \chi_1(a)\chi_2(b)\sigma(c)$$

of $B(F)$. If σ is a character of $F^\times$ and π is an admissible representation of GL(2, F), we denote by $\pi \rtimes \sigma$ the representation of GSp(4, F) induced from the representation

$$\begin{bmatrix} A & * \\ & cA' \end{bmatrix} \longmapsto \sigma(c)\pi(A)$$

of $P(F)$. If χ is a character of $F^\times$ and π is an admissible representation of GSp(2, F) = GL(2, F), then $\chi \rtimes \pi$ denotes the representation of GSp(4, F) parabolically induced from the representation

$$\begin{bmatrix} x & * & * \\ & A & * \\ & & \det(A)x^{-1} \end{bmatrix} \longmapsto \chi(x)\pi(A)$$

of $Q(F)$.

If Π is an admissible representation of GSp(4, F) and τ is a character of $F^\times$, then the *twist* of Π by τ, denoted $\tau\Pi$, is the representation $g \mapsto \tau(\lambda(g))\Pi(g)$, where λ is the multiplier homomorphism. The effect of twisting on parabolically induced representations is as follows:

$$\tau(\chi_1 \times \chi_2 \rtimes \sigma) = \chi_1 \times \chi_2 \rtimes \tau\sigma, \quad \tau(\pi \rtimes \sigma) = \pi \rtimes \tau\sigma, \quad \tau(\chi \rtimes \pi) = \chi \rtimes \tau\pi.$$

The non-supercuspidal, irreducible, admissible representations of GSp(4, F) have been classified by Sally and Tadić in [ST]. They determined the irreducible subquotients of each representation parabolically induced from an irreducible representation of B, P or Q. In [RS] this information was reorganized in the form of a table, which we reproduce here as Table 1. The representations are organized in cases I–XI. Cases I–VI contain representations supported in B, cases VII–IX contain those supported in Q, and cases X and XI contain representations supported in P. For example, case I contains the irreducible, admissible representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$. We refer to [RS, Section 2.2] for a precise description of the various cases.

2.3. Weil group representations

We recall some basic facts about the Weil group W_F and the Weil–Deligne group W_F' of F, referring to [Roh] and [T] for details. Recall from local Class Field Theory that the abelianized Weil group W_F^{ab} and $F^\times$ are isomorphic, which implies that the characters of W_F and those of $F^\times$ can be identified. We will use the same symbol for a character of $F^\times$ and the corresponding character of W_F. Representations of the Weil–Deligne group W_F' are given by pairs (ρ, N), where ρ is a continuous homomorphism $W_F \to \mathrm{GL}(n, \mathbb{C})$ and N is a nilpotent complex $n \times n$ matrix for which

$$\rho(w)N\rho(w)^{-1} = \nu(w)N \quad \text{for all } w \in W_F.$$

Table 1
Non-supercuspidal representations of GSp(4, F)

		Constituent of	Representation	Centr. char.	Generic
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)		$\chi_1\chi_2\sigma^2$	•
II	a	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2\sigma^2$	•
	b	$(\chi^2 \neq \nu^{\pm 1}, \chi \neq \nu^{\pm 3/2})$	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$		
III	a	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\chi\sigma^2$	•
	b	$(\chi \notin \{1, \nu^{\pm 2}\})$	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$		
IV	a	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	σ^2	•
	b		$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$		
	c		$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		
	d		$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$		
V	a	$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	σ^2	•
	b	$(\xi^2 = 1,\ \xi \neq 1)$	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		
	c		$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$		
	d		$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		
VI	a	$\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$	$\tau(S, \nu^{-1/2}\sigma)$	σ^2	•
	b		$\tau(T, \nu^{-1/2}\sigma)$		
	c		$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		
	d		$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		
VII		$\chi \rtimes \pi$ (irreducible)		$\chi\omega_\pi$	•
VIII	a	$1_{F^\times} \rtimes \pi$	$\tau(S, \pi)$	ω_π	•
	b		$\tau(T, \pi)$		
IX	a	$\nu\xi \rtimes \nu^{-1/2}\pi$	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$\omega_\pi\xi$	•
	b	$(\xi \neq 1,\ \xi\pi = \pi)$	$L(\nu\xi, \nu^{-1/2}\pi)$		
X		$\pi \rtimes \sigma$ (irreducible)		$\omega_\pi\sigma^2$	•
XI	a	$\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	σ^2	•
	b	$(\omega_\pi = 1)$	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		

If ρ is a semisimple representation, then (ρ, N) is called *admissible*. One attaches an L-factor $L(s, \varphi)$ to the pair $\varphi = (\rho, N)$ as follows. Let $\Phi \in W_F$ be an inverse Frobenius element and let $I = \mathrm{Gal}(\bar{F}/F^{\mathrm{un}}) \subset W_F$ be the inertia subgroup. Let $V_N = \ker(N)$, $V^I = \{v \in V\colon\ \rho(g)v = v$ for all $g \in I\}$ and $V_N^I = V^I \cap V_N$. Then

$$L(s, \varphi) = \det\bigl(1 - q^{-s}\rho(\Phi)\big|V_N^I\bigr)^{-1}. \tag{8}$$

If φ is a one-dimensional representation identified with a character χ of $F^\times$, then

$$L(s, \varphi) = L(s, \chi) = \begin{cases} 1 & \text{if } \chi \text{ is ramified,} \\ (1 - \chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified.} \end{cases}$$

An *L-parameter* for GSp(4, F) is essentially an equivalence class of admissible homomorphisms $W_F' \to \mathrm{GSp}(4, \mathbb{C})$; for the precise definition see [RS, Section 2.4]. The conjectural local Langlands correspondence assigns to each irreducible, admissible representation Π of $\mathrm{GSp}(4,F)$ an L-parameter φ_Π. It was shown in [RS, Section 2.4] that, for the non-supercuspidal representations of $\mathrm{GSp}(4,F)$, there is a unique way to make this assignment in such a way as to satisfy certain desirable properties of the local Langlands correspondence. In what follows we shall always refer to these unique parameters φ_Π when we talk about the local Langlands correspondence for the non-supercuspidal representations of $\mathrm{GSp}(4,F)$. Their explicit forms are given in [RS, Section 2.4] and will be recalled below.

3. Computations of adjoint L-functions

We now go through the list of non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ and compute the adjoint L-functions of the L-parameters of these representations.

3.1. Cases supported in the minimal parabolic subgroup

Case I: These are irreducible representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$, where χ_1, χ_2 and σ are characters of $F^\times$. The condition for irreducibility is that $\chi_1 \neq \nu^{\pm 1}$, $\chi_2 \neq \nu^{\pm 1}$ and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm 1}$. The L-parameter of such a representation is given by the pair (ρ, N), where $N = 0$ and

$$\rho(w) = \begin{bmatrix} (\chi_1\chi_2\sigma)(w) & & & \\ & (\chi_1\sigma)(w) & & \\ & & (\chi_2\sigma)(w) & \\ & & & \sigma(w) \end{bmatrix}.$$

The one-dimensional spaces spanned by the vectors in (3) through (6) are preserved by the action of W_F on the 10-dimensional space $\mathfrak{sp}(4)$ given by $\mathrm{Ad}_{\mathfrak{sp}(4)} \circ \rho$. More precisely, W_F acts on L_α by multiplication with $\alpha(\rho(w))$, for each root α. Furthermore, W_F acts trivially on the diagonal torus of $\mathfrak{sp}(4)$. Thus

$$\begin{aligned} L(s, \chi_1 \times \chi_2 \rtimes \sigma, \mathrm{Ad}) &= L(s, 1_{F^\times})^2 L(s,\chi_1) L\bigl(s,\chi_1^{-1}\bigr) L(s,\chi_2) L\bigl(s,\chi_2^{-1}\bigr) \\ &\quad \cdot L(s,\chi_1\chi_2) L\bigl(s,\chi_1^{-1}\chi_2^{-1}\bigr) L\bigl(s,\chi_1\chi_2^{-1}\bigr) L\bigl(s,\chi_1^{-1}\chi_2\bigr). \end{aligned} \tag{9}$$

Case II: Let χ and σ be characters of $F^\times$ such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$. The induced representation $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ has the two irreducible constituents $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ (type IIa) and $\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$ (type IIb). The L-parameter attached to $\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$ is (ρ, N) with $N = 0$ and

$$\rho(w) = \begin{bmatrix} (\chi^2\sigma)(w) & & & \\ & (\nu^{1/2}\chi\sigma)(w) & & \\ & & (\nu^{-1/2}\chi\sigma)(w) & \\ & & & \sigma(w) \end{bmatrix}.$$

Arguing similarly as in case I above, we obtain

$$\begin{aligned} L(s, \chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma, \mathrm{Ad}) &= L(s, 1_{F^\times})^2 L\bigl(s,\chi^2\bigr) L\bigl(s,\chi^{-2}\bigr) L(s,\nu) L\bigl(s,\nu^{-1}\bigr) \\ &\quad \cdot L\bigl(s,\chi\nu^{-1/2}\bigr) L\bigl(s,\chi^{-1}\nu^{1/2}\bigr) L\bigl(s,\chi\nu^{1/2}\bigr) L\bigl(s,\chi^{-1}\nu^{-1/2}\bigr). \end{aligned} \tag{10}$$

The L-parameter of the IIa type representation $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ has the same semisimple part ρ, but $N = N_1$, where

$$N_1 = \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}. \tag{11}$$

Composing with the adjoint representation, the 10-dimensional representation of W_F' whose L-factor we have to compute is $(\mathrm{Ad}_{\mathfrak{sp}(4)} \circ \rho, \mathrm{ad}(N_1))$. To determine the L-factor we have to consider the restriction of $\mathrm{Ad}_{\mathfrak{sp}(4)} \circ \rho$ to the kernel of $\mathrm{ad}(N_1)$; see (8). It is easy to see that

$$\ker\bigl(\mathrm{ad}(N_1)\bigr) = \left\langle \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}, L_{2e_1}, L_{e_1+e_2}, L_{2e_2}, L_{-e_1+e_2}, L_{-2e_1} \right\rangle. \tag{12}$$

The restriction of $\mathrm{Ad}_{\mathfrak{sp}(4)} \circ \rho$ to this 6-dimensional space decomposes in an obvious way into 1-dimensional invariant subspaces, so that the resulting L-factor is

$$\begin{aligned} L(s, \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma, \mathrm{Ad}) = {} & L(s, 1_{F^\times}) L\bigl(s, \chi^2\bigr) L\bigl(s, \chi^{-2}\bigr) \\ & \cdot L(s, \nu) L\bigl(s, \chi^{-1}\nu^{1/2}\bigr) L\bigl(s, \chi\nu^{1/2}\bigr). \end{aligned} \tag{13}$$

Case III: If χ and σ are characters of $F^\times$ such that $\chi \neq 1$ and $\chi \neq \nu^{\pm 2}$, then the induced representation $\chi \times \nu \rtimes \nu^{-1/2}\sigma$ has two irreducible constituents $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ (type IIIa) and $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$ (type IIIb). The L-parameter of $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$ is (ρ, N) with $N = 0$ and

$$\rho(w) = \begin{bmatrix} (\nu^{1/2}\chi\sigma)(w) & & & \\ & (\nu^{-1/2}\chi\sigma)(w) & & \\ & & (\nu^{1/2}\sigma)(w) & \\ & & & (\nu^{-1/2}\sigma)(w) \end{bmatrix}.$$

Arguing as above, we find that

$$\begin{aligned} L(s, \chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}, \mathrm{Ad}) = {} & L(s, 1_{F^\times})^2 L(s, \chi) L\bigl(s, \chi^{-1}\bigr) L(s, \nu) L\bigl(s, \nu^{-1}\bigr) \\ & \cdot L(s, \chi\nu) L\bigl(s, \chi\nu^{-1}\bigr) L\bigl(s, \chi^{-1}\nu\bigr) L\bigl(s, \chi^{-1}\nu^{-1}\bigr). \end{aligned} \tag{14}$$

The L-parameter of $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ is (ρ, N_4) with the same ρ and

$$N_4 = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}. \tag{15}$$

Composing with the adjoint representation, we obtain the representation of W_F' given by $(\mathrm{Ad}_{\mathfrak{sp}(4)} \circ \rho, \mathrm{ad}(N_4))$. It is easily computed that

$$\ker\bigl(\mathrm{ad}(N_4)\bigr) = \left\langle \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, L_{-2e_2}, L_{e_1-e_2}, L_{2e_1} \right\rangle. \tag{16}$$

Using the definition (8) it follows that

$$L(s, \chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma, \mathrm{Ad}) = L(s, 1_{F^\times}) L(s, \nu) L(s, \nu\chi) L\bigl(s, \nu\chi^{-1}\bigr). \tag{17}$$

Case IV: Representations of type IV are the subquotients of $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$, where σ is a character of $F^\times$. The Langlands quotient is $\sigma \mathbf{1}_{\mathrm{GSp}(4)}$, a twist of the trivial representation (type IVd). Its L-parameter is given by (ρ, N) with $N = 0$ and

$$\rho(w) = \begin{bmatrix} (\nu^{3/2}\sigma)(w) & & & \\ & (\nu^{1/2}\sigma)(w) & & \\ & & (\nu^{-1/2}\sigma)(w) & \\ & & & (\nu^{-3/2}\sigma)(w) \end{bmatrix}.$$

Arguing as before, we obtain

$$\begin{aligned} L(s, \sigma \mathbf{1}_{\mathrm{GSp}(4)}, \mathrm{Ad}) = {} & L(s, 1_{F^\times})^2 L(s, \nu)^2 L\bigl(s, \nu^{-1}\bigr)^2 L\bigl(s, \nu^2\bigr) L\bigl(s, \nu^{-2}\bigr) \\ & \cdot L\bigl(s, \nu^3\bigr) L\bigl(s, \nu^{-3}\bigr). \end{aligned} \tag{18}$$

The L-parameter of the IVc type representation $L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$ is (ρ, N_1) with N_1 as in (11). It follows from (12) that

$$\begin{aligned} L\bigl(s, L\bigl(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma\bigr), \mathrm{Ad}\bigr) = {} & L(s, 1_{F^\times}) L(s, \nu) L\bigl(s, \nu^{-1}\bigr) L\bigl(s, \nu^2\bigr) \\ & \cdot L\bigl(s, \nu^3\bigr) L\bigl(s, \nu^{-3}\bigr). \end{aligned} \tag{19}$$

The L-parameter of the IVb type representation $L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$ is (ρ, N_4) with N_4 as in (15). It follows from (16) that

$$L\bigl(s, L\bigl(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)}\bigr), \mathrm{Ad}\bigr) = L(s, 1_{F^\times}) L(s, \nu) L\bigl(s, \nu^{-1}\bigr) L\bigl(s, \nu^3\bigr). \tag{20}$$

The L-parameter of the IVa type representation $\sigma \mathrm{St}_{\mathrm{GSp}(4)}$ is (ρ, N_5) with

$$N_5 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}. \tag{21}$$

Easy computations show that

$$\ker\bigl(\mathrm{ad}(N_5)\bigr) = \langle L_{2e_1}, L_{2e_2} + L_{e_1-e_2} \rangle. \tag{22}$$

Thus

$$L(s,\sigma \mathrm{St}_{\mathrm{GSp}(4)},\mathrm{Ad}) = L(s,\nu)L\big(s,\nu^3\big). \tag{23}$$

Case V: These are the irreducible subquotients of an induced representation of the form $\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$, where ξ is a non-trivial quadratic character of $F^\times$ and σ is an arbitrary character of $F^\times$. One of these subquotients is $L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$ (type IVd), and its L-parameter is (ρ, N) with $N = 0$ and ρ given by

$$\rho(w) = \begin{bmatrix} (\nu^{1/2}\sigma)(w) & & & \\ & (\nu^{1/2}\xi\sigma)(w) & & \\ & & (\nu^{-1/2}\xi\sigma)(w) & \\ & & & (\nu^{-1/2}\sigma)(w) \end{bmatrix}.$$

As in the other cases with $N = 0$ one computes

$$\begin{aligned} L\big(s, L\big(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma\big), \mathrm{Ad}\big) &= L(s, 1_{F^\times})^2 L(s,\nu)^2 L\big(s,\nu^{-1}\big)^2 \\ &\quad \cdot L(s,\xi)^2 L(s,\nu\xi) L\big(s,\nu^{-1}\xi\big). \end{aligned} \tag{24}$$

The L-parameter attached to the Vc type representation $L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$ is (ρ, N_2) with the same ρ and

$$N_2 = \begin{bmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}. \tag{25}$$

Computations show that

$$\ker\big(\mathrm{ad}(N_2)\big) = \left\langle \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{bmatrix}, L_{2e_1}, L_{e_1+e_2}, L_{2e_2}, L_{-2e_2}, L_{e_1-e_2} \right\rangle. \tag{26}$$

Hence

$$\begin{aligned} L\big(s, L\big(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma\big), \mathrm{Ad}\big) &= L(s, 1_{F^\times}) L(s,\nu)^2 L\big(s,\nu^{-1}\big) \\ &\quad \cdot L(s,\xi) L(s,\nu\xi). \end{aligned} \tag{27}$$

The representation $L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ of type Vb is a ξ-twist of Vc. Since adjoint L-functions are invariant under twists, its adjoint L-function is the same as in (27). The essentially square-integrable Va type representation $\delta([\xi,\nu\xi], \nu^{-1/2}\sigma)$ has L-parameter (ρ, N_3) with ρ as before and

$$N_3 = \begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}. \tag{28}$$

It is easy to compute that

$$\ker\bigl(\mathrm{ad}(N_3)\bigr) = \langle L_{2e_1}, L_{e_1+e_2}, L_{2e_2}, L_{e_1-e_2} - L_{-e_1+e_2}\rangle. \tag{29}$$

It follows that

$$L\bigl(s, \delta\bigl([\xi, \nu\xi], \nu^{-1/2}\sigma\bigr), \mathrm{Ad}\bigr) = L(s,\nu)^2 L(s,\xi) L(s,\nu\xi). \tag{30}$$

Case VI: These are the irreducible subquotients of an induced representation of the form $\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$, where σ is a character of $F^\times$. One of these irreducible subquotients is the VId type representation $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$. Its L-parameter is (ρ, N) with $N = 0$ and

$$\rho(w) = \begin{bmatrix} (\nu^{1/2}\sigma)(w) &&& \\ & (\nu^{1/2}\sigma)(w) && \\ && (\nu^{-1/2}\sigma)(w) & \\ &&& (\nu^{-1/2}\sigma)(w) \end{bmatrix}.$$

The resulting adjoint L-function is

$$L\bigl(s, L\bigl(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma\bigr), \mathrm{Ad}\bigr) = L(s, 1_{F^\times})^4 L(s,\nu)^3 L\bigl(s,\nu^{-1}\bigr)^3. \tag{31}$$

The L-parameter of the VIc type representation $L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ is (ρ, N_1) with N_1 as in (11). By (12),

$$L\bigl(s, L\bigl(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma\bigr), \mathrm{Ad}\bigr) = L(s, 1_{F^\times})^2 L(s,\nu)^3 L\bigl(s,\nu^{-1}\bigr). \tag{32}$$

The remaining irreducible subquotients are the generic $\tau(S, \nu^{-1/2}\sigma)$ and the non-generic $\tau(T, \nu^{-1/2}\sigma)$. Both of these are tempered representations and they constitute an L-packet. Their common L-parameter is (ρ, N_3) with ρ as above and N_3 as in (28). By (29),

$$L\bigl(s, \tau\bigl(S/T, \nu^{-1/2}\sigma\bigr), \mathrm{Ad}\bigr) = L(s, 1_{F^\times}) L(s,\nu)^3. \tag{33}$$

3.2. Cases supported in the Klingen parabolic subgroup

Case VII: These representations are the irreducible representations of the form $\chi \rtimes \pi$, where χ is a character of $F^\times$ and π is a supercuspidal irreducible admissible representation of $\mathrm{GL}(2,F)$. If $\mu : W_F \to \mathrm{GL}(2,\mathbb{C})$ is the L-parameter of π, then $\chi \rtimes \pi$ has L-parameter (ρ, N) with $N = 0$ and

$$\rho(w) = \begin{bmatrix} \chi(w)\det(\mu(w))\mu(w)' & \\ & \mu(w) \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C}). \tag{34}$$

To compute the adjoint L-function of this parameter, we identify the Siegel Levi M_P in $\mathrm{GSp}(4,\mathbb{C}) = \widehat{\mathrm{GSp}}(4,F)$ with $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ via

$$(A, x) \longmapsto \begin{bmatrix} xA' & \\ & A \end{bmatrix} \quad \bigl(A \in \mathrm{GL}(2,\mathbb{C}),\ x \in \mathrm{GL}(1,\mathbb{C})\bigr). \tag{35}$$

We have to decompose the Lie algebra $\mathfrak{sp}(4)$ into irreducible representations of M_P. It is easy to see that

$$\mathfrak{sp}(4)=\underbrace{\mathbb{C}\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix}}_{\text{invariant}}\oplus\underbrace{\mathbb{C}L_{-e_1+e_2}\oplus\mathbb{C}\begin{bmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{bmatrix}\oplus\mathbb{C}L_{e_1-e_2}}_{\text{invariant}}$$
$$\oplus\underbrace{\mathbb{C}L_{2e_2}\oplus\mathbb{C}L_{e_1+e_2}\oplus\mathbb{C}L_{2e_1}}_{\text{invariant}}\oplus\underbrace{\mathbb{C}L_{-2e_1}\oplus\mathbb{C}L_{-e_1-e_2}\oplus\mathbb{C}L_{-2e_2}}_{\text{invariant}}. \tag{36}$$

The representation on the 1-dimensional invariant subspace is the trivial representation. The representation on $\mathbb{C}L_{2e_2}\oplus\mathbb{C}L_{e_1+e_2}\oplus\mathbb{C}L_{2e_1}$ is

$$\underbrace{\big(\det{}^{-2}\otimes\mathrm{Sym}^2\big)}_{\text{representation of }\mathrm{GL}(2,\mathbb{C})}\otimes\,\mathrm{std}_{\mathrm{GL}(1)}.$$

The representation on $\mathbb{C}L_{-e_1+e_2}\oplus\mathbb{C}\begin{bmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{bmatrix}\oplus\mathbb{C}L_{e_1-e_2}$ is

$$\big(\det{}^{-1}\otimes\mathrm{Sym}^2\big)\otimes\mathrm{triv}_{\mathrm{GL}(1)}.$$

The representation on $\mathbb{C}L_{-2e_1}\oplus\mathbb{C}L_{-e_1-e_2}\oplus\mathbb{C}L_{-2e_2}$ is

$$\mathrm{Sym}^2\otimes\mathrm{std}_{\mathrm{GL}(1)}^{-1}.$$

Using $\mathrm{Sym}^2=\det\otimes\mathrm{Ad}_{\mathrm{GL}(2)}$ as representations of $\mathrm{GL}(2,\mathbb{C})$, we can rewrite these three-dimensional representations as

$$\begin{aligned}\big(\det{}^{-1}\otimes\mathrm{Ad}_{\mathrm{GL}(2)}\big)&\otimes\mathrm{std}_{\mathrm{GL}(1)},\\ \mathrm{Ad}_{\mathrm{GL}(2)}&\otimes\mathrm{triv}_{\mathrm{GL}(1)},\\ (\det\otimes\mathrm{Ad}_{\mathrm{GL}(2)})&\otimes\mathrm{std}_{\mathrm{GL}(1)}^{-1}.\end{aligned}$$

Via the identification (35), we consider ρ as a homomorphism $W_F\to\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$. As such we have $\rho=\mu\times\chi\omega_\pi$; note that $\det\circ\mu=\omega_\pi$. For the resulting L-functions we have the following lemma.

Lemma 1. *For a character χ of $F^\times$ and an irreducible admissible representation π of* $\mathrm{GL}(2,F)$*, let*

$$L_2(s,\pi,\chi)=\frac{L(s,(\chi\pi)\times\tilde{\pi})}{L(s,\chi)},$$

as in [GJ]. *Then*

$$L\big(s,\big(\big(\det{}^{-1}\otimes\mathrm{Ad}_{\mathrm{GL}(2)}\big)\otimes\mathrm{std}_{\mathrm{GL}(1)}\big)\circ\big(\mu\times(\chi\omega_\pi)\big)\big)=L_2(s,\pi,\chi)$$

and

$$L\big(s,\big(\big(\det\otimes\mathrm{Ad}_{\mathrm{GL}(2)}\big)\otimes\mathrm{std}_{\mathrm{GL}(1)}^{-1}\big)\circ\big(\mu\times(\chi\omega_\pi)\big)\big)=L_2\big(s,\pi,\chi^{-1}\big).$$

Here, $\mu: W_F'\to\mathrm{GL}(2,\mathbb{C})$ *is the* L*-parameter of* π.

Proof. We have

$$\mathrm{std}_{\mathrm{GL}(2)}\otimes\mathrm{std}_{\mathrm{GL}(2)}=\det\otimes(\mathrm{Ad}_{\mathrm{GL}(2)}\oplus\mathbf{1}_{\mathrm{GL}(2)}),$$

and hence

$$\begin{aligned}
L\big(s,(\chi\pi)\times\tilde\pi\big)&=L\big(s,\big(\chi\omega_\pi^{-1}\big)\pi\times\pi\big)\\
&=L\big(s,\chi\cdot\big(\det{}^{-1}\circ\mu\big)(\mu\otimes\mu)\big)\\
&=L\big(s,\chi\cdot(\mathrm{Ad}_{\mathrm{GL}(2)}\oplus\mathbf{1}_{\mathrm{GL}(2)})\circ\mu\big)\\
&=L(s,\chi)L\big(s,\chi\cdot(\mathrm{Ad}_{\mathrm{GL}(2)}\circ\mu)\big)\\
&=L(s,\chi)L\big(s,(\chi\omega_\pi)\cdot\big(\big(\det{}^{-1}\mathrm{Ad}_{\mathrm{GL}(2)}\big)\circ\mu\big)\big)\\
&=L(s,\chi)L\big(s,\big(\big(\det{}^{-1}\mathrm{Ad}_{\mathrm{GL}(2)}\big)\otimes\mathrm{std}_{\mathrm{GL}(1)}\big)\circ\big(\mu\times(\chi\omega_\pi)\big)\big).\qquad\square
\end{aligned}$$

Remark 2. One can write the L-function L_2 in the standard notation of Langlands L-functions as

$$L_2(s,\pi,\chi)=L\big(s,\pi,\mathrm{Sym}^2\otimes\big(\omega_\pi^{-1}\chi\big)\big)=L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\chi),$$

where ω_π denotes the central character of π and we use the same symbol for both characters of $F^\times$ and the corresponding characters of W_F as in 2.3. This means that L_2 is a twisted symmetric square, or equivalently, a twisted adjoint L-function of GL(2). (The adjoint L-function is sometimes also referred to as the adjoint square L-function.) We will use the latter notation in our final formulas below.

It follows that

$$\begin{aligned}
L(s,\chi\rtimes\pi,\mathrm{Ad})=L(s,1_{F^\times})&L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)})\\
&\cdot L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\chi)L\big(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\chi^{-1}\big).
\end{aligned}\tag{37}$$

Case VIII: If π is a supercuspidal irreducible admissible representation of $\mathrm{GL}(2,F)$, then the induced representation $1_{F^\times}\rtimes\pi$ is a direct sum of two irreducible constituents $\tau(S,\pi)$ (type VIIIa) and $\tau(T,\pi)$ (type VIIIb). Both irreducible constituents are tempered, but only VIIIa is generic. These two representations constitute an L-packet. Their common L-parameter is (ρ,N) with $N=0$ and

$$\rho(w)=\begin{bmatrix}\det(\mu(w))\mu(w)' & \\ & \mu(w)\end{bmatrix}\in\mathrm{GSp}(4,\mathbb{C}).$$

Here, $\mu : W_F \to \mathrm{GL}(2,\mathbb{C})$ is the parameter of π. The calculation of the adjoint L-function of this parameter is exactly as in case VII. The result is

$$L(s, 1_{F^\times} \rtimes \pi, \mathrm{Ad}) = L(s, 1_{F^\times}) L(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)})^3. \tag{38}$$

Case IX: These are the irreducible constituents of induced representations of the form $\nu\xi \rtimes \nu^{-1/2}\pi$, where ξ is a non-trivial quadratic character of $F^\times$, and where π is a supercuspidal representation of $\mathrm{GL}(2,F)$ for which $\xi\pi = \pi$. The generic constituent is denoted by $\delta(\nu\xi, \nu^{-1/2}\pi)$ (type IXa), and the non-generic constituent is denoted by $L(\nu\xi, \nu^{-1/2}\pi)$ (type IXb). The L-parameter of $L(\nu\xi, \nu^{-1/2}\pi)$ is (ρ, N), where $N = 0$ and

$$\rho(w) = \begin{bmatrix} \xi(w)\nu^{1/2}(w)\det(\mu(w))\mu'(w) & \\ & \nu^{-1/2}(w)\mu(w) \end{bmatrix}. \tag{39}$$

Here, $\mu : W_F \to \mathrm{GL}(2,\mathbb{C})$ is the L-parameter of π. The computation of the adjoint L-function of this representation is very similar to type VII above. The result is

$$\begin{aligned} L\bigl(s, L\bigl(\nu\xi, \nu^{-1/2}\pi\bigr), \mathrm{Ad}\bigr) &= L(s, 1_{F^\times}) L\bigl(s, \nu^{-1/2}\pi, \mathrm{Ad}_{\mathrm{GL}(2)}\bigr) \\ &\quad \cdot L_2\bigl(s, \nu^{-1/2}\pi, \xi\nu\bigr) L_2\bigl(s, \nu^{-1/2}\pi, \xi\nu^{-1}\bigr) \\ &= L(s, 1_{F^\times}) L(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)}) \\ &\quad \cdot L(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)} \otimes \xi\nu) L\bigl(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)} \otimes \xi\nu^{-1}\bigr). \end{aligned} \tag{40}$$

The L-parameter of $\delta(\nu\xi, \nu^{-1/2}\pi)$ is (ρ, N), where ρ is as above and N is defined as follows. By [RS, Lemma 2.4.1] there exists a symmetric invertible matrix $S \in \mathrm{GL}(2,\mathbb{C})$ such that

$${}^t\mu(w) S \mu(w) = \xi(w)\det\bigl(\mu(w)\bigr) S \quad \text{for all } w \in W_F. \tag{41}$$

Then $N = \left[\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right]$ with $B = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] S$. We have to consider the action of W_F on $\ker(\mathrm{ad}(N))$ via $\mathrm{Ad} \circ \rho$. It is clear that $\ker(\mathrm{ad}(N))$ contains the subspace $\mathbb{C}L_{2e_2} \oplus \mathbb{C}L_{e_1+e_2} \oplus \mathbb{C}L_{2e_1}$ appearing in (36). The operator $\mathrm{ad}(N)$ induces a linear map

$$\mathfrak{sp}(4) \supset \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \subset \mathfrak{sp}(4). \tag{42}$$

The domain of this linear map is 4-dimensional, and the target space is 3-dimensional. It is easy to see that, since S is invertible, this linear map is surjective. It follows that there exists a non-zero matrix $A_0 \in M(2 \times 2, \mathbb{C})$, unique up to scalars, for which

$$A_0 B = -B \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} {}^t A_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{43}$$

In fact, a calculation verifies that

$$A_0 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} S \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \tag{44}$$

is such a matrix. Furthermore, we get $\dim(\ker(\mathrm{ad}(N))) \geqslant 4$ and $\dim(\mathrm{im}(\mathrm{ad}(N))) \geqslant 3$. In fact, we claim that

$$\dim\bigl(\ker\bigl(\mathrm{ad}(N)\bigr)\bigr) = 4 \quad \text{and} \quad \dim\bigl(\mathrm{im}\bigl(\mathrm{ad}(N)\bigr)\bigr) = 6.$$

By what we already proved, it is enough to show that $\dim(\mathrm{im}(\mathrm{ad}(N))) \geqslant 6$. It is easy to see that $\mathrm{ad}(N)$ induces an injective linear map

$$\mathfrak{sp}(4) \supset \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \subset \mathfrak{sp}(4). \tag{45}$$

It follows that the intersection of $\mathrm{im}(\mathrm{ad}(N))$ with the Siegel Levi is at least 3-dimensional. Since $\mathrm{im}(\mathrm{ad}(N))$ also contains the image of the map (42), it follows that we have indeed $\dim(\mathrm{im}(\mathrm{ad}(N))) \geqslant 6$. This proves our claim. We showed that

$$\ker\bigl(\mathrm{ad}(N)\bigr) = \left\langle L_{2e_1}, L_{e_1+e_2}, L_{2e_2}, \begin{bmatrix} A_0 & \\ & A_0' \end{bmatrix} \right\rangle, \quad A_0' = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} {}^tA_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The action of W_F preserves the Siegel Levi of $\mathfrak{sp}(4)$, and therefore the one-dimensional space spanned by $\left[\begin{smallmatrix} A_0 & \\ & A_0' \end{smallmatrix}\right]$. Hence,

$$\rho(w) \begin{bmatrix} A_0 & \\ & A_0' \end{bmatrix} \rho(w)^{-1} = \eta(w) \begin{bmatrix} A_0 & \\ & A_0' \end{bmatrix}$$

for a character η of W_F. In fact, using (41), it is easy to see that $\eta = \xi$. This one-dimensional subspace therefore contributes a factor $L(s, \xi)$ to the L-function. The L-factor resulting from the action of W_F on $\mathbb{C}L_{2e_2} \oplus \mathbb{C}L_{e_1+e_2} \oplus \mathbb{C}L_{2e_1}$ has been computed before; see Lemma 1. We finally get

$$\begin{aligned} L\bigl(s, \delta\bigl(\nu\xi, \nu^{-1/2}\pi\bigr), \mathrm{Ad}\bigr) &= L(s, \xi) L_2\bigl(s, \nu^{-1/2}\pi, \xi\nu\bigr) \\ &= L(s, \xi) L(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)} \otimes \xi\nu). \end{aligned} \tag{46}$$

3.3. Cases supported in the Siegel parabolic subgroup

Case X: This case consists of the irreducible admissible representations of $\mathrm{GSp}(4, F)$ of the form $\pi \rtimes \sigma$, where π is a supercuspidal, irreducible representation of $\mathrm{GL}(2, F)$ and σ is a character of $F^\times$. The condition for irreducibility is that the central character ω_π of π is not equal to $\nu^{\pm 1}$. If $\mu : W_F \to \mathrm{GL}(2, \mathbb{C})$ is the L-parameter of π, then the L-parameter of $\pi \rtimes \sigma$ is (ρ, N) with $N = 0$ and

$$\rho(w) = \begin{bmatrix} \sigma(w)\det(\mu(w)) & & \\ & \sigma(w)\mu(w) & \\ & & \sigma(w) \end{bmatrix}. \tag{47}$$

In particular, the image of ρ is contained in M_Q, the standard Levi subgroup of the Klingen parabolic. It is easy to see that the restriction of the adjoint representation of $\mathrm{GSp}(4, \mathbb{C})$ to M_Q decomposes into the following invariant subspaces:

$$\begin{aligned}\mathfrak{sp}(4) = {}& \underbrace{\mathbb{C}\begin{bmatrix}1&&&\\&0&&\\&&0&\\&&&-1\end{bmatrix}}_{\text{invariant}} \oplus \underbrace{\mathbb{C}L_{2e_2} \oplus \mathbb{C}\begin{bmatrix}0&&&\\&1&&\\&&-1&\\&&&0\end{bmatrix} \oplus \mathbb{C}L_{-2e_2}}_{\text{invariant}} \\ & \oplus \underbrace{\mathbb{C}L_{e_1+e_2} \oplus \mathbb{C}L_{e_1-e_2}}_{\text{invariant}} \oplus \underbrace{\mathbb{C}L_{-e_1+e_2} \oplus \mathbb{C}L_{-e_1-e_2}}_{\text{invariant}} \\ & \oplus \underbrace{\mathbb{C}L_{2e_1}}_{\text{invariant}} \oplus \underbrace{\mathbb{C}L_{-2e_1}}_{\text{invariant}}.\end{aligned} \tag{48}$$

The action of W_F via $\mathrm{Ad} \circ \rho$ on the first invariant subspace is trivial. The action on the second invariant subspace is $\mathrm{Ad}_{\mathfrak{sl}(2)} \circ \mu$. The action on the third invariant subspace is $\mathrm{std}_{\mathrm{GL}(2)} \circ \mu$. The action on the fourth invariant subspace is the twist of the previous one by $\det \circ \mu^{-1}$. And the action on the last two invariant subspaces is via $\det \circ \mu$ and its inverse, respectively. Hence we get

$$L(s, \pi \rtimes \sigma, \mathrm{Ad}) = L(s, 1_{F^\times}) L(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)}) L(s, \pi) L\big(s, \omega_\pi^{-1}\pi\big) L(s, \omega_\pi) L\big(s, \omega_\pi^{-1}\big).$$

Since π is supercuspidal, $L(s, \pi) = L(s, \omega_\pi^{-1}\pi) = 1$, so that

$$L(s, \pi \rtimes \sigma, \mathrm{Ad}) = L(s, 1_{F^\times}) L(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)}) L(s, \omega_\pi) L\big(s, \omega_\pi^{-1}\big). \tag{49}$$

Case XI: Let π be a supercuspidal representation of $\mathrm{GL}(2, F)$ with $\omega_\pi = 1$ and σ a character of $F^\times$. Then $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ decomposes into the XIa type representation $\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ and the XIb type representation $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$. The Langlands quotient $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ has L-parameter (ρ, N) with $N = 0$ and

$$\rho(w) = \begin{bmatrix}\sigma(w)\nu^{1/2}(w)&&\\&\sigma(w)\mu(w)&\\&&\sigma(w)\nu^{-1/2}(w)\end{bmatrix}. \tag{50}$$

The computation of the adjoint L-function is the same as in case X. The result is

$$L\big(s, L\big(\nu^{1/2}\pi, \nu^{-1/2}\sigma\big), \mathrm{Ad}\big) = L(s, 1_{F^\times}) L(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)}) L(s, \nu) L\big(s, \nu^{-1}\big). \tag{51}$$

The L-parameter of the XIa type representation $\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ is (ρ, N_2) with the same ρ as above and N_2 as defined in (25). By (26), we have to consider the restriction of $\mathrm{Ad} \circ \rho$ to the second, third and fourth invariant subspace in (48). It follows that

$$L\big(s, \delta\big(\nu^{1/2}\pi, \nu^{-1/2}\sigma\big), \mathrm{Ad}\big) = L(s, \nu) L(s, \pi, \mathrm{Ad}_{\mathrm{GL}(2)}). \tag{52}$$

4. Generic criterion

As a corollary of our computations we prove Theorem 4 below, which is a special case of a conjecture of Gross and Prasad and Rallis for non-supercuspidal representations of $\mathrm{GSp}(4, F)$.

Lemma 3. *Let π be a supercuspidal representation of* $\mathrm{GL}(2,F)$*. Then the L-function $L_2(s,\pi,\chi)=L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\chi)$ in Lemma* 1 *has a pole at $s=1$ if and only if $\chi=\nu^{-1}\xi$ with ξ a non-trivial quadratic character for which $\xi\pi\cong\pi$. In case of a pole, that pole must be simple.*

Proof. Assume that

$$L_2(s,\pi,\chi)=\frac{L(s,(\chi\pi)\times\tilde{\pi})}{L(s,\chi)}$$

has a pole at $s=1$. Then $L(s,(\chi\pi)\times\tilde{\pi})$ has a pole at $s=1$. By [GJ, Proposition (1.2)] this implies that $\nu\chi\pi\cong\pi$. Taking central characters shows that $\chi=\nu^{-1}\xi$ with a quadratic character ξ. Since the pole of $L(s,(\chi\pi)\times\tilde{\pi})$ is simple by [GJ, Proposition (1.2)], our hypothesis implies that the function $L(s,\chi)$ cannot have a pole at $s=1$. Hence ξ is non-trivial.

Conversely, if $\chi=\nu^{-1}\xi$ with ξ a non-trivial quadratic character for which $\xi\pi\cong\pi$, then $L_2(s,\pi,\chi)$ has a simple pole at $s=1$ by [GJ, Proposition (1.2)]. □

Theorem 4. *Let φ be the L-parameter of a non-supercuspidal, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *as above. Then the L-function $L(s,\varphi,\mathrm{Ad})$ is holomorphic at $s=1$ if and only if one of the L-indistinguishable representations with L-parameter φ listed in Table* 1 *is generic.*

Proof. Among the representations listed in Table 1 in each group the top one (type “a”) is generic. We verify that their adjoint L-functions are holomorphic at $s=1$ while the adjoint L-function of all the other representations do indeed have poles at $s=1$. Recall that the local factor $L(s,\chi)$ is always non-zero and it has a pole at $s=1$ if and only if $\chi=\nu^{-1}$. We now go through the list and determine the order of the possible pole at $s=1$ using the irreducibility conditions for each case. The results are summarized in Table 2.

In case I the irreducibility conditions $\chi_1\neq\nu^{\pm1}$, $\chi_2\neq\nu^{\pm1}$ and $\chi_1\neq\nu^{\pm1}\chi_2^{\pm1}$ imply that the L-function (9) has no pole at $s=1$.

In case IIb, the factor $L(s,\nu^{-1})$ in (10) contributes a simple pole at $s=1$ and the conditions $\chi^2\neq\nu^{\pm1}$ and $\chi\neq\nu^{\pm3/2}$ imply that none of the other factors contributes a pole at $s=1$. Also, it follows from (13) that the adjoint L-function of a generic representation of type IIa has no pole at $s=1$.

The L-function in (14) for case IIIb has a double pole at $s=1$ if $\chi=\nu^{\pm1}$, and a simple pole otherwise. Since $\chi\neq\nu^{\pm2}$, it follows from (17) that the adjoint L-function of a generic representation of type IIIa has no pole at $s=1$.

The adjoint L-function for cases IVa–IVd are, respectively, given in (23), (20), (19), and (18). Clearly, the first has no pole, the second and third have simple poles, and the fourth has a double pole at $s=1$.

Similarly the adjoint L-function for case Va is given in (30), for cases Vb and Vc in (27), and for case Vd in (24). Again, the first has no pole, the second and third have a simple pole, and the last a double pole at $s=1$.

The representations in VIa and VIb are in the same L-packet. Their adjoint L-function, given in (33), is holomorphic at $s=1$. On the other hand, the adjoint L-function of VIc is given in (32) and has a simple pole at $s=1$. The adjoint L-function of VId is given in (31) with a triple pole at $s=1$.

Table 2
The adjoint L-function $L(s,\Pi,\mathrm{Ad})$

		$L(s,\Pi,\mathrm{Ad})$	$\mathrm{ord}_{s=1}$
I		$L(s,1_{F^\times})^2L(s,\chi_1)L(s,\chi_1^{-1})L(s,\chi_2)L(s,\chi_2^{-1})$ $\cdot L(s,\chi_1\chi_2)L(s,\chi_1^{-1}\chi_2^{-1})L(s,\chi_1\chi_2^{-1})L(s,\chi_1^{-1}\chi_2)$	0
II	a	$L(s,1_{F^\times})L(s,\chi^2)L(s,\chi^{-2})L(s,\nu)L(s,\chi^{-1}\nu^{1/2})L(s,\chi\nu^{1/2})$	0
	b	$L(s,1_{F^\times})^2L(s,\chi^2)L(s,\chi^{-2})L(s,\nu)L(s,\nu^{-1})$ $\cdot L(s,\chi\nu^{-1/2})L(s,\chi^{-1}\nu^{1/2})L(s,\chi\nu^{1/2})L(s,\chi^{-1}\nu^{-1/2})$	1
III	a	$L(s,1_{F^\times})L(s,\nu)L(s,\nu\chi)L(s,\nu\chi^{-1})$	0
	b	$L(s,1_{F^\times})^2L(s,\chi)L(s,\chi^{-1})L(s,\nu)L(s,\nu^{-1})$ $\cdot L(s,\chi\nu)L(s,\chi\nu^{-1})L(s,\chi^{-1}\nu)L(s,\chi^{-1}\nu^{-1})$	1 or 2
IV	a	$L(s,\nu)L(s,\nu^3)$	0
	b	$L(s,1_{F^\times})L(s,\nu)L(s,\nu^{-1})L(s,\nu^3)$	1
	c	$L(s,1_{F^\times})L(s,\nu)L(s,\nu^{-1})L(s,\nu^2)L(s,\nu^3)L(s,\nu^{-3})$	1
	d	$L(s,1_{F^\times})^2L(s,\nu)^2L(s,\nu^{-1})^2L(s,\nu^2)L(s,\nu^{-2})L(s,\nu^3)L(s,\nu^{-3})$	2
V	a	$L(s,\nu)^2L(s,\xi)L(s,\nu\xi)$	0
	b	$L(s,1_{F^\times})L(s,\nu)^2L(s,\nu^{-1})L(s,\xi)L(s,\nu\xi)$	1
	c	$L(s,1_{F^\times})L(s,\nu)^2L(s,\nu^{-1})L(s,\xi)L(s,\nu\xi)$	1
	d	$L(s,1_{F^\times})^2L(s,\nu)^2L(s,\nu^{-1})^2L(s,\xi)^2L(s,\nu\xi)L(s,\nu^{-1}\xi)$	2
VI	a, b	$L(s,1_{F^\times})L(s,\nu)^3$	0
	c	$L(s,1_{F^\times})^2L(s,\nu)^3L(s,\nu^{-1})$	1
	d	$L(s,1_{F^\times})^4L(s,\nu)^3L(s,\nu^{-1})^3$	3
VII		$L(s,1_{F^\times})L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)})L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\chi)L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\chi^{-1})$	0
VIII	a, b	$L(s,1_{F^\times})L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)})^3$	0
IX	a	$L(s,\xi)L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\xi\nu)$	0
	b	$L(s,1_{F^\times})L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)})L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\xi\nu)L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\xi\nu^{-1})$	1
X		$L(s,1_{F^\times})L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)})L(s,\omega_\pi)L(s,\omega_\pi^{-1})$	0
XI	a	$L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)})L(s,\nu)$	0
	b	$L(s,1_{F^\times})L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)})L(s,\nu)L(s,\nu^{-1})$	1

For case VII note that if we had $\chi=\nu^{-1}\xi$ with a non-trivial quadratic character ξ for which $\xi\pi\cong\pi$, then $\chi\rtimes\pi$ would reduce and would therefore not be of type VII, but of type IX. Therefore, Lemma 3 implies that $L(s,\chi\rtimes\pi,\mathrm{Ad})$, given by (37), has no pole at $s=1$ (note that $L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)})$ is holomorphic at $s=1$ since π is generic).

Cases VIIIa and VIIIb constitute an L-packet with VIIIa generic. Their adjoint L-function, given by (38), is holomorphic at $s=1$ by Lemma 3.

Case IXb has the adjoint L-function given in (40). By Lemma 3 this L-function has a simple pole at $s=1$, coming from the factor $L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\xi\nu^{-1})$. The adjoint L-function of case IXa is given in (46). By Lemma 3 the factor $L(s,\pi,\mathrm{Ad}_{\mathrm{GL}(2)}\otimes\xi\nu)$, and therefore $L(s,\delta(\nu\xi,\nu^{-1/2}\pi),\mathrm{Ad})$, has no pole at $s=1$.

The adjoint L-function for case X is given in (49). Since $\omega_\pi \neq \nu^{\pm 1}$, this function is holomorphic at $s = 1$.

Finally, the adjoint L-functions for cases XIa and XIb are given in (52) and (51), respectively. The former is holomorphic at $s = 1$ while the latter has a simple pole there. □

Remark 5. Cases Va and XIa are expected to have non-generic supercuspidal representations in their L-packets. Also, cases VIa and VIb as well as VIIIa and VIIIb constitute L-packets. L-packets of all the other representations in Table 1 are singletons.

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