JOURNAL OF DIFFERENTIAL EQUATIONS 63, 255–263 (1986)

Persistence in Dynamical Systems

GEOFFREY BUTLER*

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

AND

PAUL WALTMAN†

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322

Received November 14, 1985; revised May 24, 1985

1. INTRODUCTION

Let ξ be a locally compact metric space with metric d and let E be a closed subset of ξ with boundary ∂E and interior $\mathring{E}$. Suppose we have a continuous flow $\mathscr{F}$ defined on E such that ∂E is invariant under $\mathscr{F}$. That is, $\mathscr{F} = (E, R, \pi)$ where R is the real numbers, $\pi: E \times R \to E$ is a continuous map such that $\pi(\pi(x, t), s) = \pi(x, t+s)$ for all $x \in E$, s, $t \in R$, and $\pi(\partial E \times R) \subset \partial E$. Denote the restriction of $\mathscr{F}$ to ∂E by $\partial\mathscr{F}$. Such flows are widespread in applications for the modeling of the dynamical behavior of entities that must by their nature always remain nonnegative. For example, we might have $\xi = R^n$; $E = R^n_+$, the nonnegative cone in R^n and $x \in E$ represents n component populations whose interactions are modeled by the flow $\mathscr{F}$.

In this particular context, various definitions of persistence of the system have been given [4, 6, 9], each of which conveys some idea that none of the component populations becomes "extinct". We consider persistence in our more general setting because it is essentially a topological condition—that of the boundary of E in some sense acting as a repeller for the flow—and we shall obtain criteria for persistence. In the more elementary applications, these criteria may be reduced to readily testable hypotheses; in more complicated situations, at least some applications are possible and

* Research partly supported by the Natural Sciences and Engineering Research Council of Canada, Grant A-8130.

† Research supported by NSF Grant MCS-8401746.

0022-0396/86 $3.00

we feel that it may be hard to arrive at a simpler or more amenable set of conditions than the ones that we present.

Broadly speaking, our results show that questions of persistence may be addressed by appealing to suitable conditions on the boundary flow. Of particular interest is the case that $\xi = R^3$ and $E = R^3_+$, for it is the case that the dynamics in the interior of E may a priori be very complicated, whereas the boundary of E is topologically equivalent to R^2 so that one may anticipate a more complete understanding of the boundary flow and therefore a more effective application of the persistence criteria.

This paper is organized as follows: in Section 2, we introduce basic notation and terminology and give a couple of preliminary results to motivate some of the concepts we use. In Section 3, we present our main results, giving the proofs in Section 4. Section 5 provides a brief discussion of persistence for discrete dynamical systems. Section 6 includes some corollaries and applications as well as some concluding remarks.

2. Preliminaries

For the basic definitions and results concerning a flow $\mathscr{F} = (E, R, \pi)$ where E is a closed subset of a locally compact metric space (ξ, d), we refer to [1]. For any subset S of ξ, we shall use S°, ∂S, $\bar{S}$ to denote its interior, boundary and closure, respectively. The orbit, positive semi-orbit and negative semi-orbit of $\mathscr{F}$ through a point x of E will be denoted by $\gamma(x)$, $\gamma^+(x)$, $\gamma^-(x)$, respectively, and the omega and alpha limit sets of the orbit will be denoted by $\Lambda^+(x)$, $\Lambda^-(x)$, respectively.

Definition 2.1. The flow $\mathscr{F}$ is *dissipative* (see, e.g., [7]) if for each $x \in E$, $\Lambda^+(x) \neq \phi$ and the invariant set $\Omega(\mathscr{F}) = \bigcup_{x \in E} \Lambda^+(x)$ has compact closure.

Definition 2.2. A nonempty subset M of E, invariant for $\mathscr{F}$, is called an *isolated invariant set* [1] if it is the maximal invariant set in some neighborhood of itself. The neighborhood is called an isolating neighborhood. An isolated invariant set is necessarily closed, and if it is compact, a compact isolating neighborhood can be found.

Definition 2.3. The *stable set* $W^+(M)$ of an isolated invariant set M is defined to be $\{x \in E: \Lambda^+(x) \neq \phi, \Lambda^+(x) \subset M\}$ and the *unstable set* is defined to be $\{x \in E: \Lambda^-(x) \neq \phi, \Lambda^-(x) \subset M\}$.

Note that if M is compact, $x \in W^+(M)$ is equivalent to $\lim_{t \to \infty} d(\pi(x, t), M) = 0$, with a similar statement holding for $W^-(M)$.

DEFINITION 2.4. The *weakly stable set* $W_w^+(M)$ of an isolated invariant set M is defined to be $\{x \in E: \Lambda^+(x) \cap M \neq \phi\}$, and the *weakly unstable set* $W_w^-(M)$ is defined to be $\{x \in E: \Lambda^-(x) \cap M \neq \phi\}$.

DEFINITION 2.5. Let M, N be isolated invariant sets (not necessarily distinct). We shall say that *M is chained to N*, written $M \to N$, if there exists $x \notin M \cup N$ such that $x \in W^-(M) \cap W^+(N)$.

DEFINITION 2.6. A finite sequence $M_1, M_2, ..., M_k$ of isolated invariant sets will be called a *chain* if $M_1 \to M_2 \to \cdots \to M_k$ ($M_1 \to M_1$, if $k=1$). The chain will be called a *cycle* if $M_k = M_1$.

Remarks. The simplest example of a cycle is a cycle of saddle-connections of isolated critical points. More generally, cycles may connect critical points, periodic orbits or more complicated examples of isolated invariant sets.

Our Definitions 2.3, 2.4 (but not the terminology) coincide with those given in [1]; for example, our stable and weakly stable sets are called regions of attraction and weak attraction, respectively, in [1]. The definition of stable and unstable sets is, of course, somewhat different in scope from that usually given in the category of smooth flows; see, for example [14]. This reflects the fact that we are not primarily concerned here with smoothness or transversality properties of such sets, nor with hyperbolic structure [8] for the isolated invariant sets. In the simplest cases, however, where E is a smooth manifold and M is a critical point, periodic orbit or periodic surface with hyperbolic structure, our definitions are compatible with the standard ones for stable and unstable manifolds.

The Stable Manifold Theorem [11] ensures (among many other things) that the stable and unstable manifolds of hyperbolic invariant sets consist of more than just the sets themselves. We need to know this for the stable and unstable sets that we have defined, even when hyperbolic structure is not assumed. The following lemma provides a condition for this, namely that the weakly stable set of M is larger than M.

LEMMA 2.1. *Let M be a compact, isolated invariant set. Suppose that $W_w^+(M) \backslash M \neq \phi$. Then $W^+(M) \backslash M \neq \phi$. (A similar result holds with respect to the unstable and weakly unstable sets.)*

Proof. We modify an argument due to Sell and Sibuya [13]. (The authors wish to thank Joseph So for calling this paper to our attention.) Let $x \in W_w^+(M) \backslash M$. If $x \in W^+(M)$, we are done. Otherwise, we may choose a compact neighborhood V of M which the positive semi-orbit $\gamma^+(x)$ enters and leaves infinitely often. We may assume without loss of generality that V is an isolating neighborhood for M. Then we may choose a sequence of points $x_k \in \gamma^+(x)$ and a sequence of negative times t_k such that $\lim_{k \to \infty} d(x_k, M) = 0$, $\pi(x_k, [t_k, 0]) \subset V$, $\pi(x_k, t_k) \in \partial V$. Since M is compact and invariant, continuous dependence of solutions on initial conditions implies that $t_k \to -\infty$ as $k \to \infty$. Let $y_k = \pi(x_k, t_k)$. Since ∂V is compact, we may choose a subsequence y_{n_k} converging to y, say. For each $t > 0$, we have

$$\pi(y, t) = \lim_{k \to \infty} \pi(\pi(x_{n_k}, t_{n_k}), t)$$

Choosing k so large that $t_{n_k} + t < 0$, we see from the definition of the t_{n_k} that $\pi(x_{n_k}, t_{n_k} + t) \in V$, and so $\pi(y, t) \in V$. It follows that the positive semi-orbit $\gamma^+(y) \subset V$. Thus $\Lambda^+(y)$ is a nonempty subset of V. But then the isolating property of V implies that $\Lambda^+(y) \subset M$, i.e., $W^+(M) \backslash M \neq \phi$.

3. Persistence

DEFINITION 3.1. $\mathscr{F}$ will be called *weakly persistent* if for all $x \in E^0$, $\overline{\lim}_{t \to \infty} d(\pi(x, t), \partial E) > 0$.

DEFINITION 3.2. $\mathscr{F}$ will be called *persistent* if for all $x \in E^0$, $\underline{\lim}_{t \to \infty} d(\pi(x, t), \partial E) > 0$.

DEFINITION 3.3. $\mathscr{F}$ will be called *uniformly persistent* if there exists $\varepsilon_0 > 0$ such that for all $x \in E^0$, $\underline{\lim}_{t \to \infty} d(\pi(x, t), \partial E) \geqslant \varepsilon_0$.

Of these concepts, that of uniform persistence is evidently the most desirable from the point of view of applications, since together with dissipativeness it provides a global attractor for the flow in the interior of E, which is at a positive distance from the boundary of E. On the other hand, it is generally easier to give testable criteria for weak persistence or persistence. We are now able to link these varying degrees of persistence because of a result given in [2].

We impose three basic conditions on the flow $\mathscr{F}$. The first is quite natural from the point of view of applications; it is that $\mathscr{F}$ is dissipative.

The remaining two conditions are primarily concerned with the boundary flow $\partial\mathscr{F}$. They are both robust in a way we shall classify later; in the category of smooth flows, they make some statement about the hyperbolic structure of invariant sets in ∂E. Before stating these conditions, we require some further definitions. Given that F is dissipative, the sets $\Omega(\mathscr{F})$ and $\Omega(\partial\mathscr{F})$ (see Definition 2.1) have compact closure. $\overline{\Omega(\partial\mathscr{F})}$ is a compact, isolated invariant set for $\partial\mathscr{F}$.

DEFINITION 3.4. $\partial\mathscr{F}$ is *isolated* if there exists a covering $\mathscr{M}$ of $\Omega(\partial\mathscr{F})$ by pairwise disjoint, compact, isolated invariant sets $M_1, M_2, ..., M_k$ for $\partial\mathscr{F}$ such that each M_i is also isolated for $\mathscr{F}$. $\mathscr{M}$ is then called an *isolated* covering.

DEFINITION 3.5. $\partial\mathscr{F}$ will be called *acyclic* if there exists some isolated covering $\mathscr{M} = \bigcup_{i=1}^{k} M_i$ of $\Omega(\partial\mathscr{F})$ such that no subset of the $\{M_i\}$ forms a cycle (see Definition 2.6). (Otherwise, $\partial\mathscr{F}$ will be called *cyclic*.) An isolated covering satisfying this condition will also be called acyclic.

We are now in a position to state our main result:

THEOREM 3.1. *Let $\mathscr{F}$ be a continuous flow on a locally compact metric space E with invariant boundary. Assume that the flow $\mathscr{F}$ is dissipative and the boundary flow $\partial\mathscr{F}$ is isolated and is acyclic with acyclic covering $\mathscr{M}$. Then $\mathscr{F}$ is uniformly persistent if and only if*

(H) *for each $M_i \in \mathscr{M}$, $W^+(M_i) \cap \mathring{E} = \phi$.*

4. PROOFS

To prove Theorem 3.1, we first need to prove

THEOREM 4.1. *Let M be a compact isolated invariant set for any continuous flow $\mathscr{F}$ on a locally compact metric space. Then for any $x \in W_w^+(M)\backslash W^+(M)$, it follows that $\Lambda^+(x) \cap W^+(M)\backslash M \neq \phi$, $\Lambda^+(x) \cap W^-(M)\backslash M \neq \phi$.* (A similar statement holds for Λ^-.)

Proof. Let $x \in W_w^+(M)\backslash W^+(M)$. Then there exists a compact isolating neighborhood V of M such that $\gamma(x)$ enters and leaves V infinitely often as $t \to \infty$. Without loss of generality, we may assume that $x \in V$. Choose $t_k \to \infty$ such that $d(x_k, M) \to 0$ as $k \to \infty$, where $x_k = \pi(x, t_k)$. Choose $\tau_k < 0$ so that $\pi(x_k, [\tau_k, 0]) \subset V$, $\pi(x_k, \tau_k) \in \partial V$. Since M is invariant and compact, it follows from the continuity of π that $\tau_k \to -\infty$ as $k \to \infty$. Since $W_w^+(M)\backslash M \neq \phi$, Lemma 2.1 shows that $W^+(M)\backslash M \neq \phi$. Clearly $W^+(M) \cap V \neq \phi$ and $W^+(M) \not\subset V$, otherwise the isolating property of V is violated by the invariant set $W^+(M) \cup M$.

Suppose there were a subsequence $n_k \to \infty$ for which $t_{n_k} + \tau_{n_k} < 0$. On letting $k \to \infty$ and using the definitions of the t_{n_k}, τ_{n_k}, it would follow that $\gamma^+(x) \subset V$, implying that $\Lambda^+(x)$ is a nonempty subset of V. Since $\Lambda^+(x) \not\subset M$, $M \cup \Lambda^+(x)$ would be an invariant set which violates the isolating property of V. Thus, for sufficiently large k, we may suppose that $t_k + \tau_k > 0$. Let $y_k = \pi(x, t_k + \tau_k)$. Then $y_k \in \partial V$. By compactness, we may choose a subsequence which we relabel by k, so that $\lim_{k\to\infty} y_k = y \in \partial V$. Since $y_k = \pi(x_k, \tau_k)$, $\tau_k \to -\infty$, the arguments used in the proof of Lemma 2.1 show that $\gamma^+(y) \subset V$, and therefore $\Lambda^+(y) \subset V$. This implies that $y \in W^+(M)$, by the isolating property of V. Since $t_k + \tau_k > 0$ for sufficiently large k, one possibility is the sequence $t_k + \tau_k$ has a bounded subsequence converging to τ, say, in which case $y = \pi(x, \tau)$ which implies that $x \in W^+(M)$, since $W^+(M)$ is invariant, contradicting $x \not\subset W^+(M)$. The other possibility is that $t_k + \tau_k \to \infty$ as $k \to \infty$, which implies that $y \in \Lambda^+(x)$, and so $\Lambda^+(x) \cap (W^+(M)\backslash M) \neq \phi$.

Now choose σ_k so that $\pi(x_k, [0, \sigma_k]) \subset V$, $\zeta_k = \pi(x_k, \sigma_k) \in \partial V$. Then $\lim_{k\to\infty} \sigma_k = \infty$. Choose a subsequence, which we relabel k again, so that $\lim_{k\to\infty} \zeta_k = \zeta \in \partial V$. Again arguing as in the proof of Lemma 2.1, we see that $\gamma^-(\zeta) \subset V$, and so $\zeta \in \Lambda^-(M)\backslash M$ (otherwise the isolating property of V is violated). Since $\zeta = \pi(x, t_k + \sigma_k)$ and $t_k + \sigma_k \to \infty$ as $k \to \infty$, it follows that $\zeta \in \Lambda^+(x)$, and so $\Lambda^+(x) \cap W^-(M)\backslash M \neq \phi$. This completes the proof.

Remark. Theorem 4.1 considerably extends Lemma A1 of [4] where M was a hyperbolic critical point and the method of isolating blocks was used.

Proof of Theorem 3.1. If (H) fails to hold, then there exists $M_i \in \mathcal{M}$ with $W^+(M_i) \cap \mathring{E} \neq \phi$, i.e., there exists $x \in \mathring{E}$ such that $\Lambda^+(x) \subset M_i \subset \partial E$, and $\mathcal{F}$ is not weakly persistent (therefore not uniformly persistent). Thus (H) is necessary.

Now suppose that (H) holds. If $\mathcal{F}$ were not persistent, there would exist $x \in \mathring{E}$ such that $\Lambda^+(x) \cap \partial E \neq \phi$. By compactness and invariance, $\Lambda^+(x) \cap \Omega(\partial\mathcal{F}) \neq \phi$. Therefore we can select i_1 so that $\Lambda^+(x) \cap M_{i_1} \neq \phi$. By (H), $W^+(M_{i_1}) \subset \partial E$ and so $x \in W^+_w(M_{i_1})\backslash W^+(M_{i_1})$. By Theorem 4.1, $\Lambda^+(x) \cap W^+(M_{i_1})\backslash M_{i_1} \neq \phi$. Let $p_{i_1} \in \Lambda^+(x) \cap W^+(M_{i_1})\backslash M_{i_1}$. Since the M_i are pairwise disjoint, we can ensure the p_{i_1} is disjoint from all the M_i. By compactness and invariance, $\Lambda^-(p_{i_1})$ is a nonempty, compact, connected subset of, $\Lambda^+(x) \cap \partial E$. Thus $\Lambda^+(\Lambda^-(p_{i_1}))$ is a nonempty subset of $\Omega(\partial\mathcal{F})$. It follows that $\Lambda^-(p_{i_1}) \cap \bigcup M_i \neq \phi$. There are two cases to consider.

Case 1. Suppose that $\Lambda^-(p_{i_1})$ is not contained in any one of the M_i. Choose i_2 so that $\Lambda^-(p_{i_1}) \cap M_{i_2} \neq \phi$. Then $p_{i_1} \in W^-_w(M_{i_2})\backslash W^-(M_{i_2})$. By Theorem 4.1, there exists $q_{i_2} \in \Lambda^-(p_{i_1}) \cap W^-(M_{i_2})\backslash M_{i_2}$. Now $q_{i_2} \in \partial E$ and so $\Lambda^+(q_{i_2}) \subset \Omega(\partial\mathcal{F}) \subset \bigcup M_i$. Since $\Lambda^+(q_{i_2})$ is connected, there exists i_3 so that $\Lambda^+(q_{i_2}) \subset M_{i_3}$. If we had $q_{i_2} \in M_{i_3}$, then $\Lambda^-(q_{i_2}) \subset M_{i_3}$, by invariance, which implies that $M_{i_3} = M_{i_2}$ and $q_i \in M_{i_2}$, a contradiction. Therefore we have $q_{i_2} \in W^-(M_{i_2}) \cap W^+(M_{i_3})$, $q_{i_2} \notin M_{i_2} \cup M_{i_3}$, i.e., $M_{i_2} \to M_{i_3}$. Now $q_{i_2} \in \Lambda(p_{i_1})$, so by compactness and invariance, we have $\Lambda^+(q_{i_2}) \subset \Lambda^-(p_{i_1})$, i.e., $p_{i_1} \in W^-_w(M_{i_3})\backslash W^-(M_{i_3})$.

Repeating the above argument, we find q_{i_3} and M_{i_4} such that

$q_{i_3} \in W^-(M_{i_3}) \cap W^+(M_{i_4})$, $q_{i_3} \notin M_{i_3} \cup M_{i_4}$, i.e., we have $M_{i_2} \to M_{i_3} \to M_{i_4}$. Continuing with this argument, we must eventually arrive at a cycle, since there are only finitely many M_i.

Case 2. Suppose that $\Lambda^-(p_{i_1}) \subset M_{j_1}$ for some j_1. Since p_i is disjoint from all of the M_i, we have $M_{j_1} \to M_{i_1}$. By compactness and invariance, $\Lambda^+(x) \cap M_{j_1} \neq \phi$ and so $x \in W_w^+(M_{j_1}) \backslash W^+(M_{j_1})$, and appealing to Theorem 4.1, we find $p_{j_1} \in \Lambda^+(x) \cap W^+(M_{j_1}) \backslash M_{j_1}$, with p_{j_1} disjoint from all of the M_j.

Arguing as before, with p_{i_1} replaced by p_{j_1}, we either find ourselves back in Case 1 or we remain in Case 2 and find k_1 such that $M_{k_1} \to M_{j_1} \to M_{i_1}$.

Repeating the preceding arguments, we must eventually achieve a cycle either by getting into Case 1 or by remaining in Case 2. The existence of a cycle contradicts the fact that M is an acyclic covering. Thus we have shown that (H) implies that $\mathscr{F}$ is persistent. It was shown in [2] that if $\mathscr{F}$ is dissipative and weakly persistent with isolated, acyclic boundary flow, then $\mathscr{F}$ is in fact uniformly persistent. Thus (H) implies uniform persistence. This completes the proof of the theorem. (The authors wish to thank J. Reineck of Northwestern University for his comments on a previous version of this proof.)

5. Discrete Dynamical Systems

If we have a discrete dynamical system $\mathscr{F} = (E, Z, \pi)$ and furnish the corresponding definitions for dissipative systems, stable set, etc., we may carry over to such systems all the results obtained above for continuous flows by a suspension argument. Thus, we may conclude

Theorem 5.1. *Let $\mathscr{F}$ be a discrete dynamical system on a locally compact metric space E with invariant boundary. Assume that $\mathscr{F}$ is dissipative and that the discrete dynamical system on the boundary, $\partial\mathscr{F}$, is isolated and acyclic with acyclic covering $\mathscr{M}$. Then $\mathscr{F}$ is uniformly persistent if and only if* (H) *holds.*

6. Applications

The class of applied problems which motivated the abstract question considered in this paper are best realized in the context of R^3_+ and the system of population equations of the form

$$\begin{aligned} u' &= uf(u, v, w), \\ v' &= vg(u, v, w), \\ w' &= wh(u, v, w), \end{aligned} \tag{6.1}$$

where f, g, h are C^1 functions. The form of the equations guarantee that the ∂R^3_+ is invariant and that each axis is invariant. This type of ecological model is associated with the name of Kolmogoroff [3]. While the two dimensional subsystems on ∂R^3_+ are well studied under various hypotheses — competitive, cooperative, predator–prey, etc., determining the asymptotic behavior in the interior would appear to be hopeless, yet the principal ecological question (as with any ecosystem) is whether all components of the system survive. This is the motivation of the definition of persistence. An analysis of several problems of ecological interest for (6.1) were given in [4] and [5].

The possible interior critical points in the faces are of the form $(u^*, 0, w^*)$, $(0, \hat{v}, \hat{w})$, and $(\tilde{u}, \tilde{v}, 0)$. One needs that these critical points repel orthogonal to the plane that contains them. In view of the form of the system (6.1), this condition can be expressed as

$$f(u^*, 0, w^*) > 0,$$
$$g(0, \hat{v}, \hat{w}) > 0,$$
$$h(\tilde{u}, \tilde{v}, 0) > 0,$$

for each critical point of appropriate type. This is a readily testable condition. If in addition, there were no limit cycles in the faces (which can often be eliminated by the Dulac Criterion), this is all that is required to verify (H). If limit cycles occur, the condition becomes one on the Floquet exponent of the linearization about that limit cycle.

The boundary flow on R^3_+ is isolated if critical points and limit cycles are generic (no eigenvalue or Floquet exponent with zero real part). The work in applying the theorem then is to check the acyclic condition, and the approach to this question depends on the type of populations being modeled (competitive, predator–prey, etc.). One interpretation of the work in [4, 5] is that the principal effort was equivalent to verifying the acyclic condition.

If F is dissipative with isolated boundary flow ∂F, but ∂F is not acyclic, then the question of persistence may be quite a delicate matter. A simple, but nontrivial case arises with a three-dimensional Lotka–Volterra competitive system in the so-called nontransitive case (no one of the three populations out-competes each of the other two in the two-dimensional subsystems). Persistence may or may not occur, depending upon the precise parameter values [10, 12].

Note added in proof. Since this paper was accepted for publication the authors have learned of an independently obtained result of Dunbar, Rybakowski, and Schmitt on semiflows which replaces the local compactness assumption in Theorem 4.1 with relative compactness of the forward orbit of x. Their result will appear in this journal.

REFERENCES

1. N. P. BHATIA AND G. P. SZEGŐ, "Dynamical Systems: Stability Theory and Applications," Lecture Notes in Math., Vol. 35, Springer-Verlag, New York, 1967.
2. G. BUTLER, H. FREEDMAN, AND P. WALTMAN, Uniformly persistent systems, *Proc. Amer. Math. Soc.* **96** (1986), 425–430.
3. H. I. FREEDMAN, "Deterministic Mathematical Models in Population Ecology," Marcel Dekker, New York, 1980.
4. H. I. FREEDMAN AND P. WALTMAN, Persistence in models of three interacting predator-prey populations, *Math. Biosci.* **68** (1984), 213–231.
5. H. I. FREEDMAN AND P. WALTMAN, Persistence in a model of three competitive populations, *Math. Biosci.* **73** (1985), 89–101.
6. T. C. GARD AND T. G. HALLAM, Persistence in food webs. I. Lotka–Volterra food chains, *Bull. Math. Biol.* **41** (1979), 877–891.
7. J. HALE, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.
8. M. W. HIRSCH AND C. PUGH, Stable manifolds and hyperbolic sets, *in* "Global Analysis: Proc. Symp. Pure Math.," Vol. 14, pp. 133–165, Amer. Math. Soc., Providence, R.I., 1970.
9. V. HUTSON AND G. T. VICKERS, A criterion for permanent coexistence of species, with an application to a two-prey one-predator system, *Math. Biosci.* **63** (1983), 253–269.
10. R. M. MAY AND W. J. LEONARD, Nonlinear aspects of competition between three species, *SIAM J. Appl. Math.* **29** (1975), 243–253.
11. Z. NITECKI, "Differentiable Dynamics," M.I.T. Press, Cambridge, 1971.
12. P. SCHUSTER, K. SIGMUND AND R. WOLFF, On ω-limits for competition between three species, *SIAM J. Appl. Math.* **37** (1979), 49–54.
13. G. SELL AND Y. SIBUYA, Behaviour of solutions near a critical point, *in* "Proc. United States-Japan Seminar on Differential and Functional Equations" (W. A. Harris, Jr. and Y. Sibuya, Eds.), pp. 501–506, Benjamin, New York, 1967.
14. S. SMALE, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747–817.