Homoclinic orbits and Hopf bifurcations in delay differential systems with T–B singularity

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Received 26 January 2007; revised 7 September 2007
Available online 24 October 2007

Abstract
The paper carries the results on Takens–Bogdanov bifurcation obtained in [T. Faria, L.T. Magalhães, Normal forms for retarded functional differential equations and applications to Bogdanov–Takens singularity, J. Differential Equations 122 (1995) 201–224] for scalar delay differential equations over to the case of delay differential systems with parameters. Firstly, we give feasible algorithms for the determination of Takens–Bogdanov singularity and the generalized eigenspace associated with zero eigenvalue in \( \mathbb{R}^n \). Next, through center manifold reduction and normal form calculation, a concrete reduced form for the parameterized delay differential systems is obtained. Finally, we describe the bifurcation behavior of the parameterized delay differential systems with T–B singularity in detail and present an example to illustrate the results.

MSC: 34C37; 34K17; 34K18; 37G05; 37G10

Keywords: Delay differential system; Takens–Bogdanov singularity; Homoclinic orbit; Hopf bifurcation
1. Introduction

Takens–Bogdanov point represents a class of equilibria with multiple-2 singularity in differential dynamical systems. Takens and Bogdanov firstly revealed the bifurcation phenomena (homoclinic and Hopf bifurcations) caused by T–B singularity in planar systems by the method of versal unfolding [2,3]. This is an important discovery in the bifurcation theory of ordinary differential systems.

Consider the following parameterized dynamical system

\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^p, \]

where \( f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \) is a continuously differentiable function with \( f(\bar{x}, \bar{\alpha}) = 0 \), i.e. \( \bar{x} \) is an equilibrium of system (1.1) for \( \alpha = \bar{\alpha} \). \((\bar{x}, \bar{\alpha})\) is called a T–B point of system (1.1) if it satisfies the following conditions:

1. \( f_x(\bar{x}, \bar{\alpha}) \) has a zero eigenvalue with algebraic multiplicity = 2 and geometric multiplicity = 1,
2. all other eigenvalues of \( f_x(\bar{x}, \bar{\alpha}) \) have nonzero real parts,

meanwhile, we say that system (1.1) has a T–B singularity.

It is obvious that the dimension of an ordinary differential system with T–B singularity is at least 2, but this is not true for delay differential systems. As shown by Faria and Magalhães [1] and Faria [4], the T–B singularity may also happen in scalar delay differential equations, and meanwhile, they also set up a method of normal form reduction for retarded functional differential equations, which can be applied to scalar delay differential equations directly. However, for delay differential systems in \( \mathbb{R}^n \) with \( n \geq 2 \), both to determine the T–B singularity and find an explicit expression of the generalized eigenspace are much more difficult than for the case of scalar delay differential equations. To the best of our knowledge there is no generic result on the problem of T–B bifurcation for delay ordinary differential systems with dimension \( n \geq 2 \) appeared in the existing literature, then we think this problem is still open.

In this paper, firstly, we characterize the T–B singularity for parameterized delay differential systems of the form (2.1) in \( \mathbb{R}^n \) with \( n > 1 \), for the particular case of (1.1) with a single discrete delay, and give feasible algorithms to calculate the explicit expression of the generalized eigenspace associated with the zero eigenvalue of the linearized delay differential system. Next, by following the methods used in [1,5], we reduce the parameterized delay differential systems to the normal form on a center manifold, and a concrete expression of the reduced form is obtained. Lastly, based on a detailed analysis of the normal form, we prove the bifurcation properties caused by T–B singularity for parameterized delay differential systems with dimension \( n \geq 1 \), and we specify the bifurcation curves in the parameter space. An example is also presented to illustrate the results.

2. Description for T–B singularity in delay differential systems

Without loss of generality, we shall study the following delay differential system

\[ \dot{x}(t) = A(\alpha)x(t) + B(\alpha)x(t - 1) + F(x(t), x(t - 1), \alpha), \]

where \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \) is a parameter vector, \( x \in \mathbb{R}^n \).
The basic assumption on system (2.1) is

\((H1)\) \(A(\alpha), B(\alpha)\) are \(C^r\) \((r \geq 2)\) smooth matrix-valued functions from \(\mathbb{R}^2\) to \(\mathbb{R}^{n \times n}\), and 
\(F(x, y, \alpha)\) is a \(C^r\) \((r \geq 2)\) smooth function from \(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^2\) to \(\mathbb{R}^n\) with

\[
F(0, 0, \alpha) = 0, \quad \frac{\partial F}{\partial x}(0, 0, \alpha) = 0, \quad \frac{\partial F}{\partial y}(0, 0, \alpha) = 0, \quad \forall \alpha \in \mathbb{R}^2. \tag{2.2}
\]

From (2.2) we have

\[
\frac{d}{d\alpha} F(0, 0, \alpha) = 0, \quad \frac{d}{d\alpha} \frac{\partial F}{\partial x}(0, 0, \alpha) = 0, \quad \frac{d}{d\alpha} \frac{\partial F}{\partial y}(0, 0, \alpha) = 0, \quad \forall \alpha \in \mathbb{R}^2. \tag{2.3}
\]

Denote \(A = A(0)\) and \(B = B(0)\), then we can write system (2.1) as

\[
\dot{x}(t) = Ax(t) + Bx(t - 1) + \left\{ (A(\alpha) - A)x(t) + (B(\alpha) - B)x(t - 1) + F(x(t), x(t - 1), \alpha) \right\}. \tag{2.4}
\]

Denote the Banach space of continuous mappings from \([-1, 0]\) to \(\mathbb{R}^n\) with norm \(\|\phi\| = \max_{\theta \in [-1, 0]} |\phi(\theta)|\) \((|\cdot|\) is some norm in \(\mathbb{R}^n\)) by \(C = C([-1, 0], \mathbb{R}^n)\), and let

\[
\eta_\alpha(\theta) = \begin{cases} A(\alpha) + B(\alpha), & \theta = 0, \\ B(\alpha), & -1 < \theta < 0, \\ 0, & \theta = -1 \end{cases}
\]

(a bounded variation matrix-valued function on \([-1, 0]\)). Notice that

\[
A(\alpha)x(t) + B(\alpha)x(t - 1) = \int_{-1}^{0} d\eta_\alpha(\theta)x(t + \theta),
\]

then

\[
L(\alpha)x_t = \int_{-1}^{0} d\eta_\alpha(\theta)x_t(\theta)
\]

can be considered as a bounded linear operator from \(C\) to \(\mathbb{R}^n\), where \(x_t(\theta) = x(t + \theta)\). Particularly, when \(\alpha = 0\), we have

\[
L(0)x_t = \int_{-1}^{0} d\eta_0(\theta)x(t + \theta) = Ax(t) + Bx(t - 1) \overset{\text{def}}{=} L_0x_t.
\]

From the definition of \(L_0\), it can be easily verified that \(L_0(\zeta) = (A + B)\zeta\), \(L_0(\theta \zeta) = -B\zeta\), \(L_0(\theta^2 \zeta) = B\zeta\), \(\forall \zeta \in \mathbb{R}^n\), and \(L_0(e^{\lambda \beta} \zeta) = (A + Be^{-\lambda})\zeta\), \(\forall \zeta \in \mathbb{R}^n\). These formulae will be used frequently in the rest of this paper.
With the above notation, system (2.1) can be formulated as a retarded functional differential equation
\[ \dot{x}(t) = L(\alpha)x_t + F(x_t, \alpha), \] (2.5)

while system (2.4) has the following form
\[ \dot{x}(t) = L_0 x_t + \left[ L(\alpha)x_t - L_0 x_t + F(x_t, \alpha) \right], \] (2.6)

which can be linearized at \((x_t, \alpha) = (0, 0)\) as
\[ \dot{x}(t) = L_0 x_t. \] (2.7)

We know from [6,7] that the fundamental solution of system (2.7) defines a \(C_0\)-semigroup \(\{T_0(t), t \geq 0\}\) on \(C\) with infinitesimal generator \(A_0 : C \to C\) defined by
\[ A_0 \phi = \dot{\phi}, \]
\[ D(A_0) = \left\{ \phi \in C^1([-1, 0], \mathbb{R}^n) : \dot{\phi}(0) = \int_{-1}^{0} d\eta_0(\theta) \phi(\theta) = L_0 \phi \right\}. \] (2.8)

With the aid of \(A_0\), the linear system (2.7) is equivalent to an abstract ordinary differential equation \(\dot{x} = A_0 x\) in \(C\). Moreover, we know that the spectrum of the operator \(A_0\) consists of its point spectrum, i.e. \(\sigma(A_0) = \sigma_p(A_0)\), and \(\lambda \in \sigma_p(A_0)\) if and only if there exists \(\zeta \in \mathbb{R}^n \setminus \{0\}\) such that
\[ (\lambda I - L_0(e^{\lambda \theta} I))\zeta = 0, \]
or, equivalently, \(\lambda\) satisfies
\[ \Delta_0(\lambda) \overset{\text{def}}{=} \det(\lambda I - L_0(e^{\lambda \theta} I)) = \det(\lambda I - A - Be^{-\lambda}) = 0. \] (2.9)

Usually we call (2.9) the characteristic equation of system (2.7).

The further assumptions on system (2.1) are

(H2) \(\Re \lambda \neq 0\) if \(\lambda \in \sigma_p(A_0) \setminus \{0\}\);

(H3) \(\lambda = 0\) is an eigenvalue of \(A_0\) with algebraic multiplicity = 2 and geometric multiplicity = 1.

We say that system (2.1) has a \(T-B\) singularity if (H1)–(H3) hold, and in this case we call \((x, \alpha) = (0, 0)\) a \(T-B\) point of system (2.1).

The following theorem gives an equivalent description for \(T-B\) singularity in the delay differential system (2.1), which can be used as a feasible algorithm for determining the \(T-B\) singularity.

**Theorem 2.1.** Under assumptions (H1) and (H2), the delay differential system (2.1) has a \(T-B\) singularity if and only if the following conditions hold:

(i) \(\text{rank}(A + B) = n - 1\);

(ii) if \(N(A + B) = \text{span}\{\phi_0^0\}\), then \((B + I)\phi_1^0 \in \mathcal{R}(A + B)\);
(iii) if \((A + B)\phi_2^0 = (B + I)\phi_1^0\), then

\[
(B + I)\phi_2^0 - \frac{1}{2}B\phi_1^0 \notin \mathcal{R}(A + B),
\]

where \(\phi_1^0, \phi_2^0 \in \mathbb{R}^n\).

**Proof.** To prove the theorem, one only needs to show the equivalence of assumption (H3) and conditions (i)–(iii). Here we reformulate assumption (H3) as follows: there exist linearly independent functions \(\phi_1, \phi_2 \in C\) such that

\[
\mathcal{A}_0 \phi_1 = 0, \quad \mathcal{A}_0 \phi_2 = \phi_1
\]

and the following equation

\[
\mathcal{A}_0 \phi = \phi_2
\]

has no solution \(\phi\) in \(C\).

Notice that \(\mathcal{A}_0 \phi_1 = 0\) is equivalent to

\[
\begin{cases}
L_0 \phi_1(\theta) = 0, & \theta = 0, \\
\dot{\phi}_1(\theta) = 0, & -1 \leq \theta \leq 0,
\end{cases}
\]

which is valid if and only if

\[
\phi_1(\theta) \equiv \phi_1^0 \in \mathbb{R}^n \setminus \{0\}, \quad (A + B)\phi_1^0 = 0.
\]

Moreover, from (2.11) we see that the associated eigenspace of \(\mathcal{A}_0\) is a space with dimension 2 spanned by the characteristic function \(\phi_1\) and the generalized characteristic function \(\hat{\phi}_2\), and that any function \(\hat{\phi}_1 \in C\) linearly independent with \(\phi_1\) must obey

\[
\mathcal{A}_0 \hat{\phi}_1 \neq 0,
\]

or

\[
(A + B)\hat{\phi}_1^0 \neq 0
\]

for any \(\hat{\phi}_1^0\) in \(\mathbb{R}^n\) linearly independent with \(\phi_1^0\). As a consequence of (2.14) and (2.16), the condition (i) holds, i.e. \(\text{rank}(A + B) = n - 1\) and \(\mathcal{N}(A + B) = \text{span}\{\phi_1^0\}\).

By the definition of the operator \(\mathcal{A}_0\) and (2.14), \(\mathcal{A}_0 \phi_2 = \phi_1\) is equivalent to

\[
\begin{cases}
L_0 \phi_2(\theta) = \phi_1^0, & \theta = 0, \\
\dot{\phi}_2(\theta) = \phi_1^0, & -1 \leq \theta \leq 0,
\end{cases}
\]

which can be solved by

\[
\phi_2(\theta) = \phi_2^0 + \phi_1^0 \theta,
\]
where $\phi_2^0 \in \mathbb{R}^n$ satisfies

$$L_0(\phi_2^0 + \phi_1^0 \theta) = \phi_1^0,$$  \hspace{1cm} (2.19)

or, the equivalent form

$$(B + I)\phi_1^0 = (A + B)\phi_2^0,$$  \hspace{1cm} (2.20)

i.e. the condition (ii) holds.

Similar to the above discussion, (2.12) is equivalent to

$$\begin{cases}
L_0(\phi_1(\theta)) = \phi_2^0, & \theta = 0, \\
\dot{\phi}(\theta) = \phi_2^0 + \phi_1^0 \theta, & -1 \leq \theta \leq 0.
\end{cases}$$  \hspace{1cm} (2.21)

We see from the second formula that the solution of (2.21) has the form $\phi_3(\theta) = \phi_3^0 + \phi_2^0 \theta + \frac{1}{2} \phi_1^0 \theta^2$, $\phi_3(0) = \phi_3^0 \in \mathbb{R}^n$, and by substituting $\phi_3(\theta)$ into the first formula, we obtain

$$L_0\left(\phi_3^0 + \phi_2^0 \theta + \frac{1}{2} \phi_1^0 \theta^2\right) = \phi_2^0,$$  \hspace{1cm} (2.22)

which namely is

$$(A + B)\phi_3^0 = (B + I)\phi_2^0 - \frac{1}{2} B\phi_1^0.$$  \hspace{1cm} (2.23)

From (2.23) we see that (2.12) has no solution in $C$ is equivalent to $(B + I)\phi_2^0 - \frac{1}{2} B\phi_1^0 \notin \mathcal{R}(A + B)$, i.e. the condition (iii) holds. This completes the proof of the theorem.

**Remark 2.2.** The conditions (ii) and (iii) in Theorem 2.1 are equivalent to

(ii)’ if $N(A + B) = \text{span}\{\phi_1^0\}$, then $\text{rank}(A + B, (B + I)\phi_1^0) = n - 1$;

(iii)’ if $(A + B)\phi_2^0 = (B + I)\phi_1^0$, then $\text{rank}(A + B, (B + I)\phi_2^0 - \frac{1}{2} B\phi_1^0) = n$.

In certain cases, conditions (ii)’ and (iii)’ might be easier to work with, thereby we can make use of conditions (i), (ii)’ and (iii)’ to verify whether the delay differential system (2.1) has a T–B singularity or not.

**Corollary 2.3.** When $n = 1$ (i.e. the case of scalar delay differential equation) and with assumption (H1), Eq. (2.1) has a T–B singularity if and only if $A = 1$ and $B = -1$.

**Proof.** Here the assumption (H2) can be verified easily, then the proof follows from Theorem 2.1. \hfill $\Box$

A comparison between Theorem 2.1 and Corollary 2.3 indicates that the description for T–B singularity in delay differential systems is much more complicated than that in scalar delay differential equations.
Example 1. Consider the following linear delay differential system with dimension 2

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = A \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + B \begin{bmatrix}
x_1(t - 1) \\
x_2(t - 1)
\end{bmatrix},
\]

(2.24)

where \( A = \begin{bmatrix}
2 & 1 \\
0 & 1
\end{bmatrix} \), \( B = \begin{bmatrix}
1 & 1 \\
0 & -1
\end{bmatrix} \).

Firstly, we verify assumption (H2), i.e. the characteristic equation \( \det(\lambda I - A - Be^{-\lambda}) = 0 \) of (2.24) has no imaginary roots. Notice that

\[
\det(\lambda I - A - Be^{-\lambda}) = 0 \iff \begin{cases}
\lambda - 2 - e^{-\lambda} = 0, \\
\lambda - 1 + e^{-\lambda} = 0.
\end{cases}
\]

If \( i\omega \) is an imaginary root of the above equation, then

\[
\omega + \sin\omega = 0 \quad \text{and} \quad -2 - \cos\omega = 0,
\]

or

\[
-1 + \cos\omega = 0 \quad \text{and} \quad \omega - \sin\omega = 0,
\]

must hold. Evidently this is impossible for \( \omega \neq 0 \), thus we verified (H2).

Now we verify conditions (i)–(iii) in Theorem 2.1.

1) Since \( A + B = \begin{bmatrix}
3 & 2 \\
0 & 0
\end{bmatrix} \), obviously condition (i) is valid.

2) Take \( \phi_0^0 = \begin{bmatrix}
1^{1/3} \\
-1/2
\end{bmatrix} \), then \( \mathcal{N}(A + B) = \text{span}\{\phi_0^0\} \). Let \( \phi_0^2 = \begin{bmatrix}
1/18 \\
0
\end{bmatrix} \), then \( (B + I)\phi_0^0 = \begin{bmatrix}
2 & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
1/3 \\
-1/2
\end{bmatrix} = \begin{bmatrix}
1/6 \\
0
\end{bmatrix} \) and \( (A + B)\phi_0^2 = \begin{bmatrix}
3 & 2 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
1/18 \\
0
\end{bmatrix} = \begin{bmatrix}
1/6 \\
0
\end{bmatrix} \). Therefore \( (B + I)\phi_0^0 \in \mathcal{R}(A + B) \), i.e. condition (ii) holds.

3) Since \( (B + I)\phi_0^0 - \frac{1}{2} B\phi_0^1 = \begin{bmatrix}
7/36 \\
-1/4
\end{bmatrix} \) and \( \text{rank}(A + B, (B + I)\phi_0^0 - \frac{1}{2} B\phi_0^1) = \text{rank}\begin{bmatrix}
3 & 2 \\
0 & 0
\end{bmatrix} = 2 \), we know \( (B + I)\phi_0^0 - \frac{1}{2} B\phi_0^1 \notin \mathcal{R}(A + B) \), therefore condition (iii) holds.

According to Theorem 2.1 we conclude that any delay differential system with linear part as in (2.24) and nonlinear part satisfying condition (H1) has a T–B singularity.

3. Reduction and normal forms for delay differential systems with T–B singularity

In this section, we discuss the reduction and normal forms for the delay differential system (2.1) with T–B singularity, by following the method in [1,5]. We will show that system (2.1) can be reduced to a simple ordinary differential system with dimension 2 on its center manifold.

By virtue of (2.6) we reformulate the parameterized system (2.1) as the following functional differential equation without parameters

\[
\dot{x}(t) = L_0 x_t + \left[L(\alpha) - L_0\right] x_t + F(x_t, \alpha),
\]

\[
\dot{\alpha}(t) = 0,
\]

(3.1)

which takes \( \tilde{C} \) as its phase space. Further, let \( \tilde{x}(t) = (x(t), \alpha(t)) \in \mathbb{R}^n \times \mathbb{R}^2 \) be the solution of (3.1), then (3.1) can be written as

\[
\dot{x}(t) = \tilde{L}_0 \tilde{x}_t + \tilde{F}(\tilde{x}_t),
\]

(3.2)
where \( \tilde{L}_0 \tilde{x}_t = (L_0 x_t, 0) \) is a bounded linear operator from \( \tilde{C} \) to \( \mathbb{R}^n \times \mathbb{R}^2 \), \( \tilde{F}(\tilde{x}_t) = ([L(\alpha(0)) - L_0]x_t + F(x_t, \alpha(0)), 0) \) of (3.3).

By assumption (H3), the dimension of the space \( A \) of (3.3).

Consider the linearization of (3.2) at \( \tilde{x}_t = 0 \)

\[
\dot{\tilde{x}}(t) = \tilde{L}_0 \tilde{x}_t.
\] (3.3)

Denote the infinitesimal generator of the \( C_0 \)-semigroup corresponding to (3.3) by \( \tilde{A}_0 \), then \( \tilde{A}_0 = (A_0, 0) \). The eigenvalues of \( \tilde{A}_0 \) include not only all the eigenvalues of \( A_0 \), but also the two 0-eigenvalues introduced by \( \alpha \). Let \( \tilde{A} \) be the set of all 0-eigenvalues (counting multiplicity) of (3.3).

Now we consider the decomposition of the phase space \( C \) of (2.6). Let \( C = P \oplus Q \) with \( P \) the invariant space of \( A_0 \) associated with the eigenvalue zero and \( Q \) the complementary space defined below, and \( C^* = C([0, 1], \mathbb{R}^{n*}) \) be the adjoint space of \( C \), where \( \mathbb{R}^{n*} \) is the \( n \)-dimensional space of row vectors. The adjoint inner product on \( C^* \times C \) is defined by

\[
(\psi, \phi) = \psi(0)\phi(0) - \int_{-1}^{0} \int_{0}^{\theta} \psi(\xi - \theta) d\eta_0(\theta) \phi(\xi) d\xi.
\] (3.4)

By assumption (H3), the dimension of the space \( P \) is 2. Let \( \Phi(\theta) = (\phi_1(\theta), \phi_2(\theta)), -1 \leq \theta \leq 0 \), and \( \Psi(s) = \text{col}(\psi_1(s), \psi_2(s)), 0 \leq s \leq 1 \), be the bases of \( P \) and its dual space \( P^* \), respectively, which satisfy \( (\psi, \phi) = I_2 \), where \( (\psi, \phi) \) is defined by (3.5). Under these preparations, we have the following lemma.

**Lemma 3.1.** The bases of \( P \) and its dual space \( P^* \) have the following representations:

\[
P = \text{span} \Phi, \quad \Phi(\theta) = (\phi_1(\theta), \phi_2(\theta)), \quad -1 \leq \theta \leq 0,
\]

\[
P^* = \text{span} \Psi, \quad \Psi(s) = \text{col}(\psi_1(s), \psi_2(s)), \quad 0 \leq s \leq 1,
\] (3.5)

where \( \phi_1(\theta) = \phi_1^0 \in \mathbb{R}^n \setminus \{0\} \), \( \phi_2(\theta) = \phi_2^0 + \phi_1^0 \theta, \phi_2^0 \in \mathbb{R}^n \), and \( \psi_2(s) = \psi_2^0 \in \mathbb{R}^{n*} \setminus \{0\} \), \( \psi_1(s) = \psi_1^0 - s \psi_2^0, \psi_1^0 \in \mathbb{R}^{n*} \), which satisfy

\[
(1) \ (A + B)\phi_1^0 = 0, \quad (2) \ (A + B)\phi_2^0 = (B + I)\phi_1^0,
\]

\[
(3) \ \psi_2^0(A + B) = 0, \quad (4) \ \psi_1^0(A + B) = \psi_2^0(B + I),
\]

\[
(5) \ \psi_2^0\psi_2^0 - \frac{1}{2}\psi_2^0B\phi_1^0 + \psi_2^0B\phi_2^0 = 1,
\]

\[
(6) \ \psi_1^0\psi_2^0 - \frac{1}{2}\psi_1^0B\phi_1^0 + \psi_1^0B\phi_2^0 + \frac{1}{6}\psi_2^0B\phi_1^0 - \frac{1}{2}\psi_2^0B\phi_2^0 = 0.
\] (3.6)

Here, we can determine the unique vectors \( \phi_1^0, \psi_1^0 \) by (1) and (3), respectively, up to some constant factors; then we can determine \( \phi_2^0, \psi_2^0 \) by (2) and (4), respectively. However (5) and (6) are used to determine the coefficient factors of the vectors \( \phi_1^0 \) and \( \psi_2^0 \).
Proof. Firstly, from the proof of Theorem 2.1 we know that \( \phi_1(\theta) = \phi_1^0 \in \mathbb{R}^n \setminus \{0\}, \phi_2(\theta) = \phi_2^0 + \phi_1^0 \theta, \phi_2^0 \in \mathbb{R}^n \), and (1), (2) in (3.6) hold. Next, we introduce the adjoint operator \( A_0^* : C^* \rightarrow C^* \)

of \( A_0 \) by

\[
A_0^* \psi = -\dot{\psi},
\]

\[
D(A_0^*) = \left\{ \psi \in C^1([0, 1], \mathbb{R}^{n*}) : -\dot{\psi}(0) = \int_{-1}^{0} \psi(-\theta) d\eta_0(\theta) \right\}.
\] (3.7)

From \( A_0^* \psi_2 = 0 \), i.e.

\[
\begin{align*}
-\frac{d\psi_2}{ds}(s) = 0, & \quad 0 \leq s \leq 1, \\
\int_{-1}^{0} \psi_2(-\theta) d\eta_0(\theta) = 0, & \quad s = 0,
\end{align*}
\] (3.8)

we obtain

\[
\psi_2(s) = \psi_2^0 \in \mathbb{R}^{n*} \setminus \{0\} \quad \text{and} \quad \psi_2^0 (A + B) = 0.
\] (3.9)

And then by \( A_0^* \psi_1 = \psi_2 \), i.e.

\[
\begin{align*}
-\frac{d\psi_1}{ds}(s) = \psi_2^0, & \quad 0 \leq s \leq 1, \\
\int_{-1}^{0} \psi_1(-\theta) d\eta_0(\theta) = \psi_2^0, & \quad s = 0,
\end{align*}
\] (3.10)

we obtain

\[
\psi_1(s) = \psi_1^0 - s \psi_2^0, \quad \psi_1^0 \in \mathbb{R}^{n*} \quad \text{and} \quad \psi_1^0 (A + B) = \psi_2^0 (B + I),
\] (3.11)

which means that the conditions (3) and (4) in (3.6) hold. Finally, by the definition of \( \Phi(\theta) \) and \( \Psi(s) \) we have

\[
\begin{align*}
(\psi_1, \phi_1) &= \psi_1^0 \phi_1^0 - \frac{1}{2} \psi_2^0 B \phi_1^0 + \psi_1^0 B \phi_1^0 = 1, \\
(\psi_2, \phi_2) &= \psi_2^0 \phi_2^0 - \frac{1}{2} \psi_2^0 B \phi_2^0 + \psi_2^0 B \phi_2^0 = 1, \\
(\psi_1, \phi_2) &= \psi_1^0 \phi_2^0 - \frac{1}{2} \psi_1^0 B \phi_1^0 + \psi_1^0 B \phi_2^0 + \frac{1}{6} \psi_2^0 B \phi_1^0 - \frac{1}{2} \psi_2^0 B \phi_2^0 = 0, \\
(\psi_2, \phi_1) &= \psi_2^0 \phi_1^0 + \psi_2^0 B \phi_1^0 = 0.
\end{align*}
\] (3.12)

In fact, according to formulae (1) to (4) in (3.6) we know that the fourth formula in (3.12) holds naturally, and the first is equivalent to the second one. Then we can properly choose the
coefficient factors of \( \phi_1^0, \psi_2^0 \) such that all the formulae in (3.12) hold. This completes the proof of the lemma. \( \square \)

It is easy to verify that \( \Phi(\theta) \) satisfies \( \dot{\Phi} = \Phi J \), where \( J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \).

Further, we consider the decomposition \( \widetilde{C} = \widetilde{P} \oplus \widetilde{Q} \), where \( \widetilde{P} = P \times \mathbb{R}^2 \) is the invariant space of \( \mathcal{A}_0 \) associated with \( \mathcal{A} \), \( \widetilde{Q} = Q \times W \), \( W = \{ v \in C_2; v(0) = 0 \} \). Here, the bases of \( \widetilde{P} \) and its dual space \( \widetilde{P}^* \) are formed by the columns of \( \Phi = \begin{bmatrix} \phi & 0 \\ 0 & I_2 \end{bmatrix} \) and the rows of \( \Psi = \begin{bmatrix} \psi & 0 \\ 0 & I_2 \end{bmatrix} \), respectively, which satisfy \( (\Phi, \Psi) = I_4 \), and \( \Phi = \Phi J \) with \( J = \text{diag}(J, 0_2) \). Moreover, the center manifold of the functional differential equation (3.1) at the origin can be represented as \( \mathcal{Y}(y(\alpha), v(\alpha)) \): \( \mathbb{R}^2 \times \mathbb{R}^2 \to \widetilde{Q} = Q \times W \), which satisfies \( y(0, 0) = v(0, 0) = 0 \), \( D_y(0, 0) = D_v(0, 0) = 0 \).

As shown in [1,5], for considering the normal forms for (2.5) with fixed \( \alpha \), a suitable phase space is the following enlarged space of \( C \):

\[
BC = \{ \phi : [-1, 0] \to \mathbb{R}^n, \phi \text{ uniformly continuous on } [-1, 0] \text{ and with a possible jump discontinuity at 0} \},
\]

which is identified with the space \( C \times \mathbb{R}^n \). Similarly, for considering the normal forms for (3.2), we extend \( \mathcal{C} \) to \( B \mathcal{C} = BC \times BC_2 \), which is identified with the space \( \mathcal{C} \times \mathbb{R}^{n+2} \).

Let \( X_0 \) and \( Y_0 \) be the matrix-valued jump functions on \([-1, 0]\) defined as follows:

\[
X_0(\theta) = \begin{cases} I_n, & \theta = 0, \\ 0, & -1 \leq \theta < 0, \end{cases} \quad Y_0(\theta) = \begin{cases} I_2, & \theta = 0, \\ 0, & -1 \leq \theta < 0, \end{cases}
\]

and

\[
\pi : BC \to P, \quad \pi(\phi + X_0\mu) = \Phi \left[ (\Psi, \phi) + \Psi(0)\mu \right],
\]

where \( \phi \in C, \mu \in \mathbb{R}^n \). Define

\[
\widetilde{\pi} : B\mathcal{C} \to \mathcal{P},
\]

\[
\pi(\phi + X_0\mu, \psi + Y_0v) = \Phi \left[ \left( \begin{bmatrix} \psi \\ \phi \end{bmatrix} \right) + \Psi(0) \left[ \begin{bmatrix} \mu \\ v \end{bmatrix} \right] \right]
\]

\[
= (\pi(\phi + X_0\mu), \psi(0) + v),
\]

where \( \phi \in C, \psi \in C_2, \mu \in \mathbb{R}^n, v \in \mathbb{R}^2 \). By decomposing the solution \( \tilde{x} \) of (3.2) with respect to \( B\mathcal{C} = \mathcal{P} \oplus \mathcal{N}(\widetilde{\pi}) \), we have

\[
\begin{bmatrix} \frac{\dot{z}}{\alpha} \\ \frac{\dot{\alpha}}{\alpha} \end{bmatrix} = \tilde{J} \begin{bmatrix} z \\ \alpha \end{bmatrix} + \tilde{\Psi}(0)\tilde{F} \left( \Phi \begin{bmatrix} z \\ \alpha \end{bmatrix} + \begin{bmatrix} y \\ v \end{bmatrix} \right),
\]

\[
\frac{dy}{dt} = \tilde{A}_0 \begin{bmatrix} y \\ v \end{bmatrix} + (I - \tilde{\pi})[X_0 Y_0] \tilde{F} \left( \Phi \begin{bmatrix} z \\ \alpha \end{bmatrix} + \begin{bmatrix} y \\ v \end{bmatrix} \right),
\]

(3.13)
where \( z \in \mathbb{R}^2, \alpha \in \mathbb{R}^2, \ y \in Q^1 \equiv Q \cap C^1, \ v \in W^1 \equiv W \cap C^1_2, \) and \( \tilde{A}_{\tilde{Q}^1} \) is an operator from \( \tilde{Q}^1 \equiv \tilde{Q} \cap \tilde{C}^1 = Q^1 \times W^1 \) to \( \mathcal{N}(\tilde{\pi}) \) defined as follows

\[
\tilde{A}_{\tilde{Q}^1} \left[ \begin{array}{c} \phi \\ \psi \end{array} \right] = \left[ \begin{array}{c} \phi \\ \psi \end{array} \right] + [X_0 \ Y_0] \left\{ \tilde{L}_0 \left[ \begin{array}{c} \phi \\ \psi \end{array} \right] - \left[ \begin{array}{c} \phi(0) \\ \psi(0) \end{array} \right] \right\}.
\]

The Taylor expansion of \( \hat{F}(x_t, \alpha) \) with respect to \( x_t \) and \( \alpha \) reads

\[
\hat{F}(x_t, \alpha) = \sum_{j \geq 2} \frac{1}{j!} \hat{F}_j (x_t, \alpha), \quad (3.14)
\]

where the first term \((j = 2)\) can be expressed in the form

\[
\frac{1}{2} \hat{F}_2 (x_t, \alpha) = A_1 \alpha_1 x(t) + A_2 \alpha_2 x(t) + B_1 \alpha_1 x(t - 1) + B_2 \alpha_2 x(t - 1)
\]

\[
+ \sum_{i=1}^{n} E_i x_i(t) x(t - 1) + \sum_{i=1}^{n} F_i x_i(t) x(t)
\]

\[
+ \sum_{i=1}^{n} G_i x_i(t - 1) x(t - 1) \quad (3.15)
\]

with \( A_i, B_i \ (i = 1, 2), \ E_i, F_i, G_i \ (i = 1, 2, \ldots, n) \) coefficient matrices, and there is no terms of \( O(\alpha^2) \) in \( \hat{F}_2 (x_t, \alpha) \) since \( \hat{F}(0, \alpha) = 0, \ \forall \alpha \in \mathbb{R}^2. \)

Defining \( f_j^1 (z, y, \alpha) = \Psi(0) \hat{F}_j (\Phi z + y, \alpha), \ f_j^2 (z, y, \alpha) = (I - \pi) X_0 \hat{F}_j (\Phi z + y, \alpha), \) and noticing that \( v(0) = 0, \) (3.13) can be reduced to the following equation on \( BC = P \oplus \mathcal{N}(\pi) \)

\[
\dot{z} = Jz + \sum_{j \geq 2} \frac{1}{j!} f_j^1 (z, y, \alpha),
\]

\[
d \frac{dy}{dt} = A_{Q^1} y + \sum_{j \geq 2} \frac{1}{j!} f_j^2 (z, y, \alpha), \quad (3.16)
\]

where \( A_{Q^1} : Q^1 \subset \mathcal{N}(\pi) \to \mathcal{N}(\pi) \) is defined by \( A_{Q^1} \phi = \dot{\phi} + X_0 [L_0 \phi - \dot{\phi}(0)]. \)

Denote the linear space of homogeneous polynomials of \( (z, \alpha) = (z_1, z_2, \alpha_1, \alpha_2) \) with degree 2 and coefficients in \( \mathbb{R}^2 \) by

\[
V_2^4 (\mathbb{R}^2) = \left\{ \sum_{(q, l) \in \mathbb{N}_0^4, c(q, l) \in \mathbb{R}^2} c(q, l) z^q \alpha^l : (q, l) \in \mathbb{N}_0^4 \right\},
\]

and let \( M_2^1 \) be the operator defined on \( V_2^4 (\mathbb{R}^2) \) by

\[
(M_2^1 p)(z, \alpha) = D_z p(z, \alpha) Jz - Jp(z, \alpha), \quad \forall p \in V_2^4 (\mathbb{R}^2),
\]

then we decompose \( V_2^4 (\mathbb{R}^2) \) as \( \mathcal{R}(M_2^1) \oplus \mathcal{R}(M_2^2)^c \) and denote the map from \( V_2^4 (\mathbb{R}^2) \) to \( \mathcal{R}(M_2^1) \) by \( P_{1,2}. \)
According to assumption (H2), it can be verified that \((q, \bar{\lambda}) \neq \mu\) for any \(\mu \in \sigma(\tilde{A}_0) \setminus \tilde{A}\) and \(q \in \mathbb{N}_0^4\), where \(\bar{\lambda} = (0, 0, 0, 0)\) is the vector constructed by the elements (counting multiplicity) in \(\tilde{A}\). In other words, (3.2) satisfies the nonresonance condition relative to \(\tilde{A}\) (see [5]). Therefore, from (3.16) we know that the normal form for (3.2) on the center manifold corresponding to the space \(P\) (see [1,5]) can be written as

\[
\dot{z} = Jz + \frac{1}{2} g_2^1(z, 0, \alpha) + h.o.t.,
\]  

(3.17)

where \(g_2^1 = (I - P_{1,2}) f_2^1\).

A concrete normal form (3.17) for system (2.1) on the center manifold can be obtained through the following computation.

Evidently, the canonical basis of \(V_2^4(\mathbb{R}^2)\) is composed by the elements

\[
\begin{pmatrix}
  z_1^2 \\
  0 \\
  z_2^2 \\
  0 \\
  \alpha_1^2 \\
  0 \\
  \alpha_2^2 \\
  0 \\
  z_1z_2 \\
  0 \\
  z_1 \alpha_1 \\
  0 \\
  z_1 \alpha_2 \\
  0 \\
  z_2 \alpha_1 \\
  0 \\
  z_2 \alpha_2 \\
  0 \\
  \end{pmatrix}
\]

and the images of these elements under \(M_2^1\) are respectively

\[
\begin{pmatrix}
  2z_1 z_2 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  \end{pmatrix}
\]

Therefore, a basis of \(\mathcal{R}(M_2^1)^c\) can be taken as the set composed by the elements

\[
\begin{pmatrix}
  0 \\
  z_1^2 \\
  0 \\
  \alpha_1^2 \\
  \alpha_2^2 \\
  0 \\
  z_1z_2 \\
  0 \\
  z_1 \alpha_1 \\
  0 \\
  z_1 \alpha_2 \\
  0 \\
  z_2 \alpha_1 \\
  0 \\
  z_2 \alpha_2 \\
  0 \\
  \end{pmatrix}
\]

Denoting by \(\phi_{ji}\) the \(i\)th element of \(\phi_j\), we have from (3.15)

\[
\frac{1}{2} \hat{F}_2(\phi z, \alpha) = A_1 \alpha_1 \begin{pmatrix} \phi(0) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + A_2 \alpha_2 \begin{pmatrix} \phi(0) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

\[
+ B_1 \alpha_1 \begin{pmatrix} \phi(-1) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + B_2 \alpha_2 \begin{pmatrix} \phi(-1) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

\[
+ \sum_{i=1}^{n} E_i \begin{pmatrix} \phi_{1i}(0), \phi_{2i}(0) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \begin{pmatrix} \phi(-1) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]
\[
+ \sum_{i=1}^{n} F_i[\phi_{1i}(0), \phi_{2i}(0)] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{bmatrix} \Phi(0) \\ \Phi(-1) \end{bmatrix} \\
+ \sum_{i=1}^{n} G_i[\phi_{1i}(-1), \phi_{2i}(-1)] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{bmatrix} \Phi(0) \\ \Phi(-1) \end{bmatrix} \\
= \begin{bmatrix} A_1 \phi_1(0) + B_1 \phi_1(-1) \end{bmatrix} \alpha_1 z_1 + \begin{bmatrix} A_2 \phi_2(0) + B_2 \phi_1(-1) \end{bmatrix} \alpha_2 z_1 \\
+ \begin{bmatrix} A_1 \phi_2(0) + B_1 \phi_2(-1) \end{bmatrix} \alpha_1 z_2 + \begin{bmatrix} A_2 \phi_2(0) + B_2 \phi_2(-1) \end{bmatrix} \alpha_2 z_2 \\
+ \sum_{i=1}^{n} \{ E_i \phi_1(-1) \phi_{1i}(0) + F_i \phi_1(0) \phi_{1i}(0) + G_i \phi_1(-1) \phi_{1i}(-1) \} z_1^2 \\
+ \sum_{i=1}^{n} \{ E_i \phi_2(-1) \phi_{1i}(0) + \phi_1(-1) \phi_{2i}(0) \} + F_i \phi_2(0) \phi_{1i}(0) + \phi_1(0) \phi_{2i}(0) \\
+ G_i \phi_2(-1) \phi_{1i}(-1) + \phi_1(-1) \phi_{2i}(-1) \} z_1 z_2 \\
+ \sum_{i=1}^{n} \{ E_i \phi_2(-1) \phi_{2i}(0) + F_i \phi_2(0) \phi_{2i}(0) + G_i \phi_2(-1) \phi_{2i}(0) \} z_2^2.
\]

Since \( \phi_1(0) = \phi_1(-1) = \phi^0_1, \), \( \phi_2(0) = \phi^0_2, \), \( \phi_2(-1) = \phi^0_2 - \phi^0_1, \), \( \psi_1(0) = \psi^0_1, \), \( \psi_2(0) = \psi^0_2, \), and \( f^1_2(z, 0, \alpha) = \Psi(0) \hat{F}_2(\Phi z, \alpha), \) then \( g^1_2 = (I - P^1_{1,2}) f^1_2 \) can be computed by the above formulae and the basis of \( \mathcal{R}(M^1_2)^c \). In this way, we obtain the following theorem in terms of (3.17):

**Theorem 3.2.** Assume that conditions (H1)–(H3) hold. Then the delay differential system (2.1) can be reduced to the following two-dimensional system of ODE on the center manifold at \((x_t, \alpha) = (0, 0)\):

\[
\begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= \kappa_1 z_1 + \kappa_2 z_2 + a z_1^2 + b z_1 z_2 + h.o.t.,
\end{align*}
\]

where

\[
\kappa_1 = \psi^0_2 (A_1 + B_1) \phi^0_1 \alpha_1 + \psi^0_2 (A_2 \phi^0_2 + B_2 \phi^0_1) \alpha_2, \\
\kappa_2 = \left[ \psi^0_1 (A_1 + B_1) \phi^0_1 + \psi^0_2 ((A_1 + B_1) \phi^0_2 - B_1 \phi^0_1) \right] \alpha_1 \\
+ \left[ \psi^0_1 (A_2 \phi^0_2 + B_2 \phi^0_1) + \psi^0_2 ((A_2 + B_2) \phi^0_2 - B_2 \phi^0_1) \right] \alpha_2, \\
a = \psi^0_2 \sum_{i=1}^{n} (E_i + F_i + G_i) \phi^0_1 \phi^0_{1i}, \\
b = 2 \psi^0_1 \sum_{i=1}^{n} (E_i + F_i + G_i) \phi^0_1 \phi^0_{1i} \\
+ \psi^0_2 \left[ \sum_{i=1}^{n} (E_i + F_i + G_i) (\phi^0_2 \phi^0_{1i} + \phi^0_1 \phi^0_{2i}) - \sum_{i=1}^{n} (E_i + 2G_i) \phi^0_1 \phi^0_{1i} \right].
\]
Denoting by
\[ \Pi = \begin{bmatrix} \psi_1^0(A_1 + B_1)\phi_1^0 + \psi_2^0(A_1 + B_1)\phi_2^0 - B_1\phi_1^0 \\ \psi_2^0(A_2\phi_2^0 + B_2\phi_1^0) \end{bmatrix} \begin{bmatrix} \psi_1^0(A_2\phi_2^0 + B_2\phi_1^0) \\ \psi_2^0((A_2 + B_2)\phi_2^0 - B_2\phi_1^0) \end{bmatrix}, \]
we have
\[ \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \Pi \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \quad (3.19) \]

The coefficients \( \kappa_1 \) and \( \kappa_2 \) in the reduced system (3.18) will be regarded as new parameters. To guarantee that they are independent, it is required that (3.19) is a non-singular linear transformation. Therefore, in addition we assume

(H4) \( \det \Pi \neq 0. \)

4. Homoclinic orbits and Hopf bifurcations near T–B point

In this section, we give the bifurcation structure for the delay differential system (2.1) by investigating the reduced system (3.18). It is known that for the reduced system (3.18), if \( a \cdot b \neq 0 \) its local bifurcation behavior about \( (\kappa_1, \kappa_2, z_1, z_2) = (0, 0, 0, 0) \) is determined by the linear and quadratic terms, and not by the higher order terms in the equation (see [2,8]). After neglecting the terms of order higher than two in (3.18), we obtain

\[ \begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= \kappa_1 z_1 + \kappa_2 z_2 + az_1^2 + bz_1 z_2.
\end{align*} \quad (4.1) \]

Here, we mainly investigate the homoclinic orbits and Hopf bifurcations in system (4.1).

**Lemma 4.1.** Let \( \lambda_j(\kappa_1, \kappa_2), j = 1, 2, \) be the eigenvalues of the Jacobian of system (4.1) at \( (-\kappa_1/a, 0) \). Then, when \( a \cdot b \neq 0 \) and \( \kappa_1 > 0, \)

(i) \( \lambda_1(\kappa_1, \kappa_2) = \lambda_2(\kappa_1, \kappa_2) = i\sqrt{\kappa_1}, \) for \( \kappa_2 = b/a\kappa_1, \)

(ii) \( \text{Re} \lambda_j(\kappa_1, \kappa_2) = \frac{1}{2}(\kappa_2 - \frac{b}{a}\kappa_1), j = 1, 2. \)

Hence we conclude that each point on the half line \( l_h = \{(\kappa_1, \kappa_2): \kappa_2 = \frac{b}{a}\kappa_1, \kappa_1 > 0\} \) in the parameter plane \( (\kappa_1, \kappa_2) \) is a Hopf bifurcation point of system (4.1).

**Proof.** Notice that system (4.1) has two equilibria: \( (0, 0) \) and \( (-\kappa_1/a, 0) \), and it is easy to verify that \( (0, 0) \) is a hyperbolic equilibrium as \( \kappa_1 > 0. \) Let \( v_1 = \kappa_1, v_2 = \kappa_2 - \frac{b}{a}\kappa_1, \) then the Jacobian of (4.1) at \( (-\kappa_1/a, 0) \) is \( J(\kappa_1, \kappa_2) = \begin{bmatrix} 0 & 1 \\ -v_1 & v_2 \end{bmatrix}. \) Consequently, \( \lambda_j(\kappa_1, \kappa_2), j = 1, 2, \) solve the equation \( \lambda^2 - v_2\lambda + v_1 = 0, \) and satisfy

\[ \begin{align*}
\lambda_1(\kappa_1, \kappa_2) &= \lambda_2(\kappa_1, \kappa_2), \\
\text{Re} \lambda_j(\kappa_1, \kappa_2) &= \frac{1}{2}v_2 = \frac{1}{2}\left(\kappa_2 - \frac{b}{a}\kappa_1\right), \quad j = 1, 2.
\end{align*} \]
Moreover, \( \lambda_1(\kappa_1, \kappa_2) = \tilde{x}_2(\kappa_1, \kappa_2) = i\sqrt{\kappa_1} \) as \( \kappa_1 = v_1 > 0 \) and \( \kappa_2 = \frac{b}{a}\kappa_1 \) (i.e. \( v_2 = 0 \)). Thus (i) and (ii) hold. From (ii) we can see that when \((\kappa_1, \kappa_2)\) changes from one side of \( l_h \) to the other side, in the parameter plane the eigenvalues \( \lambda_j(\kappa_1, \kappa_2) \) will cross the imaginary axis from one side to the other side. Therefore, we conclude from the Hopf bifurcation theorem that (i) and (ii) hold. From (ii) we can see that when \( \kappa_2 = \frac{b}{a}\kappa_1 > 0 \) and is close to \( l_h \), \( (4.1) \) has a periodic orbit with center \((-\frac{b}{a}, 0)\). In other words, the half line \( l_h \) in the parameter plane \((\kappa_1, \kappa_2)\) is a Hopf point branch of system \((4.1)\). \( \Box \)

**Lemma 4.2.** *(See [9].)* Assume \( a \cdot b \neq 0 \), then there exist a constant \( \kappa_1^0 > 0 \) and a continuously differentiable function \( \mu = \mu(\varsigma) \) with \( \mu(0) = \frac{b}{a}\mu^{-1} \), such that when \( 0 < \kappa_1 < \kappa_1^0 \) and \((\kappa_1, \kappa_2)\) is located on the curve

\[
l_\infty = \{(\kappa_1, \kappa_2) : \kappa_2 = \mu(\sqrt{\kappa_1})\kappa_1, \ k_1 > 0\},
\]

the planar system \((4.1)\) has a unique homoclinic orbit connecting the origin \((z_1, z_2) = (0, 0)\).

Summing up Theorem 3.2, Lemmas 4.1 and 4.2, and noticing that system \((3.18)\) is locally topologically equivalent near the origin to system \((4.1)\), we obtain the following result.

**Theorem 4.3.** Assume that \((H1)-(H4)\) hold, and \( a \cdot b \neq 0 \) \((a, b \in \mathbb{R} \) defined in Theorem 3.2). Then there exists a constant \( \kappa_1^0 > 0 \), such that when \( 0 < \kappa_1(\alpha_1, \alpha_2) < \kappa_1^0 \), in the parameter plane \((\alpha_1, \alpha_2)\) near the origin there exist two curves \( \tilde{l}_h \) and \( \tilde{l}_\infty \):

1. The curve \( \tilde{l}_h \), which has the following local representation:

\[
\tilde{l}_h = \{(\alpha_1, \alpha_2) : \kappa_2(\alpha_1, \alpha_2) = \frac{b}{a}\kappa_1(\alpha_1, \alpha_2) + h.o.t. = 0, \ k_1(\alpha_1, \alpha_2) > 0\},
\]

is a Hopf point branch of the delay differential system \((2.1)\), where \( h.o.t. = o(||(\alpha_1, \alpha_2)||) \), i.e. \( \tilde{l}_h \) consists of Hopf bifurcation points of \((2.1)\).

2. The curve \( \tilde{l}_\infty \), which has the following local representation:

\[
\tilde{l}_\infty = \{(\alpha_1, \alpha_2) : h(\alpha_1, \alpha_2) + h.o.t. = 0, \ k_1(\alpha_1, \alpha_2) > 0\},
\]

is a homoclinic branch of the delay differential system \((2.1)\), where \( h(\alpha_1, \alpha_2) = \kappa_2(\alpha_1, \alpha_2) - \mu(\sqrt{\kappa_1(\alpha_1, \alpha_2)})\kappa_1(\alpha_1, \alpha_2), \ \mu(\cdot) \) is defined in Lemma 4.2 and \( h.o.t. = o(||(\alpha_1, \alpha_2)||) \). In other words, system \((2.1)\) has a unique homoclinic orbit connecting the origin for each \((\alpha_1, \alpha_2) \in \tilde{l}_\infty\).

**Example 2.** Consider the following 2-dimensional delay differential system

\[
\dot{x}(t) = A(\alpha)x(t) + B(\alpha)x(t-1) + F(x(t), x(t-1), \alpha),
\]

where \( A(\alpha) = \left[ \begin{array}{cc} 2 & 1 \\ a & 1+a \end{array} \right], \ B(\alpha) = \left[ \begin{array}{cc} 1+a & 1 \\ 0 & -1+a \end{array} \right], \ F(x(t), x(t-1), \alpha) = -x_1^2(t) + x_2^2(t+1)\right]^T. \) The discussion in Example 1 shows that \((4.4)\) has a T–B singularity at \((x, \alpha) = (0, 0)\).
Let \( \hat{F}(x(t), x(t-1), \alpha) = (A(\alpha) - A(0))x(t) + (B(\alpha) - B(0))x(t-1) + F(x(t), x(t-1), \alpha) \), which can be expanded as

\[
\hat{F}(x(t), x(t-1), \alpha) = \frac{1}{2} \hat{F}_2(x(t), x(t-1), \alpha)
\]

\[
= A_1 \alpha_1 x(t) + A_2 \alpha_2 x(t) + B_1 \alpha_1 x(t-1) + B_2 \alpha_2 x(t-1) + F_1 x_1(t) x(t) + G_2 x_2(t-1) x(t-1),
\]

(4.5)

where
\[
A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},
A_2 = B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
F_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},
G_2 = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}.
\]

We choose the basis functions \( \Phi(\theta) \) and \( \Psi(s) \) according to Lemma 3.1, where \( \phi_1^0 = (1/3, -1/2)^T, \phi_2^0 = (1/18, 0)^T, \psi_1^0 = (0, -4), \psi_2^0 = (0, -4/3) \). By virtue of Theorem 3.2, it follows that system (4.4) can be reduced to

\[
\begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= \left( -\frac{4}{3} \alpha_1 + 2\alpha_2 \right) z_1 + \left( -\frac{2}{3} \alpha_1 - \frac{4}{3} \alpha_2 \right) z_2 + 4z_1^2 - \frac{16}{3} z_1 z_2.
\end{align*}
\]

(4.6)

Because \( \det \Pi = \det \left( \begin{array}{cc} -4/3 & 2/3 \\ 2/3 & -4/3 \end{array} \right) = \frac{28}{9} \neq 0 \), condition (H4) holds. Moreover \( a = 4, b = -\frac{16}{3} \), then \( a \cdot b = -\frac{64}{3} \neq 0 \). Thus, according to Theorem 4.3 we have the following results:

1. There exists a Hopf point branch \( \tilde{l}_h \) of the delay differential system (4.4) emanating from origin in the 1st quadrant of the parameter plane \((\alpha_1, \alpha_2)\), which has the following local representation:

\[
\tilde{l}_h = \left\{ (\alpha_1, \alpha_2): \alpha_2 = \frac{11}{6} \alpha_1 + \text{h.o.t.}, \alpha_1 > 0, \alpha_2 > 0 \right\}.
\]

2. There exists a homoclinic branch \( \tilde{l}_\infty \) of the delay differential system (4.4) emanating from origin in the 1st quadrant of the parameter plane \((\alpha_1, \alpha_2)\), which has the following local representation:

\[
\tilde{l}_\infty \approx \frac{-8}{7} \kappa_1, \quad \kappa_2 \approx \frac{-4}{3} \kappa_1.
\]

Fig. 1. The bifurcation diagram of system (4.4) in the parameter plane \((\kappa_1, \kappa_2)\).
\[ l_\infty = \{ (\alpha_1, \alpha_2): \kappa_2(\alpha_1, \alpha_2) - \mu(\sqrt{\kappa_1(\alpha_1, \alpha_2)}) \kappa_1(\alpha_1, \alpha_2) + h.o.t. = 0, \kappa_1(\alpha_1, \alpha_2) > 0 \}, \]

where \( \kappa_1(\alpha_1, \alpha_2) = -\frac{4}{3} \alpha_1 + 2 \alpha_2, \kappa_2(\alpha_1, \alpha_2) = -\frac{2}{3} \alpha_1 - \frac{4}{3} \alpha_2, \mu(\cdot) \) is defined in Lemma 4.2, \( \mu(0) = -\frac{8}{7} \).

In Figs. 1 and 2, we present the bifurcation diagram of the 2-dimensional delay differential system (4.4) in the parameter plane (\( \kappa_1, \kappa_2 \)) and (\( \alpha_1, \alpha_2 \)), respectively, where \( l_\infty, \tilde{l}_\infty \) and \( l_h, \tilde{l}_h \) represent the homoclinic branch and the Hopf point branch, respectively.

Acknowledgment

The authors are grateful to the referee for his/her valuable comments and suggestions which have led to an improvement of the presentation.

References