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Quantum invariants of 3-manifolds: Integrality, splitting, and perturbative expansion ☆

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Abstract

We consider quantum invariants of 3-manifolds associated with arbitrary simple Lie algebras. Using the symmetry principle we show how to decompose the quantum invariant as the product of two invariants, one of them is the invariant corresponding to the projective group. We then show that the projective quantum invariant is always an algebraic integer, if the quantum parameter is a prime root of unity. We also show that the projective quantum invariant of rational homology 3-spheres has a perturbative expansion a la Ohtsuki. The presentation of the theory of quantum 3-manifold invariants is self-contained.

MSC: 57M27; 57N10

Keywords: 3-manifolds; Homology spheres; Quantum invariants; Simple Lie algebras

0. Introduction

0.1. For a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ with Cartan matrix (a_{ij}) let $d = \max_{i \neq j} |a_{ij}|$. Thus $d = 1$ for the *ADE* series, $d = 2$ for *BCF* and $d = 3$ for G_2. The quantum group associated with $\mathfrak{g}$ is a Hopf algebra over $\mathbb{Q}(q^{1/2})$, where $q^{1/2}$ is the quantum parameter. To fix the order let us point out that our q is q^2 in [3,4,20] or v^2 in the book [11]. For example, the quantum integer is given by

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$

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0.2. Modular categories, and hence quantum 3-manifold invariants associated with $\mathfrak{g}$, can be defined only when q a root of unity of order r divisible by d, since this fact guarantees that the so-called S-matrix is invertible. For r not divisible by d, quantum invariants can still be defined, but modular categories might not exist. In this paper we will focus mainly in the more general situation, when r may or may not be divisible by d. The reason is eventually we want r to be a prime number. Note that 3-manifold invariants for the case when r is not divisible by $d \neq 1$ have not been studied earlier.

We will present a self-contained theory of quantum 3-manifold invariants, for arbitrary simple Lie algebra. By making use of an integrality result (Proposition 1.6) we will establish the existence of quantum invariants without using the theory of quantum groups at roots of unity. (We use quantum group and link invariants with general parameter, and only in the last minute, replace q by a root of unity.) The quantum invariant of a manifold M will be denoted by $\tau_M^{\mathfrak{g}}(q)$, considered as a function with domain roots of unity.

0.3. Although the usual construction of modular category might fail, say when r is a prime number and $d \neq 1$, we will show that by using the root lattice instead of the weight lattice in the construction, one can still get a modular category. Actually the construction goes through for a much larger class of numbers r—one needs only that r is coprime with $d\det(a_{ij})$. The corresponding 3-manifolds invariant, denoted by $\tau_M^{P\mathfrak{g}}(q)$, could be considered as the invariant associated to the projective group—the smallest complex Lie group whose Lie algebra is $\mathfrak{g}$. The reason is that the set of all highest weights of modules of the projective group spans the root lattice. As in the case of $\mathfrak{g}$, the invariant $\tau_M^{P\mathfrak{g}}$ can also be defined when r is not coprime with $d\det(a_{ij})$ (although modular categories might not exist).

0.4. We will show that in most cases, $\tau_M^{P\mathfrak{g}}$ is finer than $\tau_M^{\mathfrak{g}}$. More precisely, if r is coprime with $\det(a_{ij})$, then

$$\tau_M^{\mathfrak{g}} = \tau_M^{P\mathfrak{g}} \times \tau_M^{G}, \tag{0.1}$$

where τ_M^G is the 3-manifold invariant associated with the center group G (which is isomorphic to the quotient of the weight lattice by the root lattice) and a naturally defined bilinear form on it. The invariant τ_M^G is a weak invariant, since it is determined by the first homology group and the linking form on its torsion (see [15,2,20]). In some cases $\tau_M^G \equiv 0$, and hence $\tau^{\mathfrak{g}}$ is trivial, although $\tau^{P\mathfrak{g}}$ is not.

The invariant associated to the projective group was first introduced and the splitting (0.1) was obtained by Kirby and Melvin for $\mathfrak{g} = sl_2$, and Kohno and Takata for $\mathfrak{g} = sl_n$. Recently Sawin [18], based on the work of Müger and Bruguierre, established a similar result, but he considered only the case r divisible by d, and the group G cyclic (so he excludes a half of the series D case). When $d = 2$, the result (0.1) complements Sawin's work, i.e., it covers the case that is not considered by Sawin. When $d = 1$, (0.1) overlaps with Sawin's work. But even in this case, our method is quite different, it can be uniformly applied to any simple Lie algebra, and in addition, we get the integrality and perturbative expansion of quantum 3-manifold invariants (see below).

0.5. We will show that unlike $\tau^{\mathfrak{g}}$, the “projective” invariant $\tau_M^{P\mathfrak{g}}$ is always an algebraic integer, provided that the order r is an odd prime. In fact, we will prove that in this case, $\tau_M^{P\mathfrak{g}}(q) \in \mathbb{Z}[q]$. A priori, both $\tau_M^{P\mathfrak{g}}$ and $\tau_M^{\mathfrak{g}}$ are rational functions in a fractional power of q. Integrality of $\tau_M^{P\mathfrak{g}}$ for $\mathfrak{g} = sl_2$ was first established by Murakami [14] by difficult computations, for $\mathfrak{g} = sl_n$ by Takata and Yokota [19] and Masbaum and Wenzl [13], based on an idea of Roberts. We will use a different approach that is good for all simple Lie algebras.

0.6. Finally we will show that $\tau_M^{P\mathfrak{g}}$, with M a rational homology 3-sphere, has a “perturbative expansion a la Ohtsuki”. The function $\tau_M^{P\mathfrak{g}}$ can be defined only at roots of unity, and we want to expand it around $q = 1$. For the case $\mathfrak{g} = sl_2$, Ohtsuki showed that there exists a kind of number-theoretic expansion, which we call perturbative expansion. We proved a similar result for $\mathfrak{g} = sl_n$ in [8] and will extend the result to other Lie algebras here. We borrowed an idea using Gauss integrals from Rozansky’s work [17], although we will not explicitly use Gauss integral.

0.7. The paper is organized as follows. In Section 1 we recall quantum link invariants and their important properties: integrality and symmetry at roots of 1. In Section 2 we present the general theory of 3-manifold quantum invariants (not using the theory of quantum groups at roots of unity). Invariants associated to the projective group are considered in Section 3. Their integrality is proved in Section 4. Section 5 is devoted to the perturbative expansion.

1. Quantum link invariants: Integrality and symmetries

1.1. Lie algebras and quantum groups. We recall here some facts from the theory of Lie algebras and quantum groups, mainly in order to fix notation. For the theory of quantum groups, see [3,11].

1.1.1. Lie algebra. Let $(a_{ij})_{1\leqslant i,j\leqslant \ell}$ be the Cartan matrix of a simple complex Lie algebra $\mathfrak{g}$. There are relatively prime integers $d_1, \ldots, d_\ell$ in $\{1, 2, 3\}$ such that the matrix $(d_i a_{ij})$ is symmetric. Let d be the maximal of (d_i). The values of d, and other data, for various Lie algebras are listed in Table 1.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and basis roots $\alpha_1, \ldots, \alpha_\ell$ in the dual space $\mathfrak{h}^*$. Let $\mathfrak{h}^*_{\mathbb{R}}$ be the $\mathbb{R}$-vector space spanned by $\alpha_1, \ldots, \alpha_\ell$. The root lattice Y is the $\mathbb{Z}$-lattice generated

Table 1

	A_ℓ	B_ℓ ℓ odd	B_ℓ ℓ even	C_ℓ	D_ℓ ℓ odd	D_ℓ ℓ even	E_6	E_7	E_8	F_4	G_2
d	1	2	2	2	1	1	1	1	1	2	3
D	$\ell+1$	2	1	1	4	2	3	2	1	1	1
G	$\mathbb{Z}_{\ell+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	1	1	1
h	$\ell+1$	2ℓ	2ℓ	2ℓ	$2\ell-2$	$2\ell-2$	12	18	30	12	6
$h^\vee$	$\ell+1$	$2\ell-1$	$2\ell-1$	$\ell+1$	$2\ell-2$	$2\ell-2$	12	18	30	9	4

by α_i, $i = 1, \ldots, \ell$. Define the scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ so that $(\alpha_i|\alpha_j) = d_i a_{ij}$. Then $(\alpha|\alpha) = 2$ for every *short* root α.

Let $\mathbb{Z}_+$ be the set of all non-negative integers. The weight lattice X (respectively the set of dominant weights X_+) is the set of all $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ such that $\langle\lambda, \alpha_i\rangle := 2(\lambda|\alpha_i)/(\alpha_i|\alpha_i) \in \mathbb{Z}$ (respectively $\langle\lambda, \alpha_i\rangle \in \mathbb{Z}_+$) for $i = 1, \ldots, \ell$. Let $\lambda_1, \ldots, \lambda_\ell$ be the fundamental weights, i.e., the $\lambda_i \in \mathfrak{h}^*_{\mathbb{R}}$ are defined by $\langle\lambda_i, \alpha_j\rangle = \delta_{ij}$, or $(\lambda_i|\alpha_j) = d_i\delta_{ij}$. Then X is the $\mathbb{Z}$-lattice generated by $\lambda_1, \ldots, \lambda_\ell$. The root lattice Y is a subgroup of the weight lattice X, and the quotient $G = X/Y$ is called the *fundamental group*. If $\mu \in X$ and $\alpha \in Y$, then $(\mu|\alpha)$ is always an integer. On the root lattice Y, the form $(\cdot|\cdot)$ is even.

Let ρ be the half-sum of all positive roots. Then $\rho = \lambda_1 + \cdots + \lambda_\ell \in X_+$, and $2\rho \in Y$.

Let C denote the fundamental chamber:

$$C = \left\{x \in \mathfrak{h}^*_{\mathbb{R}} \mid (x|\alpha_i) \geqslant 0,\ i = 1, \ldots, \ell\right\}.$$

The Weyl group W is the group generated by reflections in the walls of C. In the chamber C there is exactly one root of length $\sqrt{2}$; it is called the short highest root, and denoted by α_0.

For a positive integer r let

$$C_r = \left\{x \in C \mid (x|\alpha_0) < r\right\}.$$

Then the topological closure $\overline{C_r}$ is a simplex. The reflections in the walls of $\overline{C_r}$ generate the affine Weyl group W_r. One also has $W_r = W \ltimes rY$, where rY denotes the translations by vectors ry, $y \in Y$.

Finite-dimensional simple $\mathfrak{g}$-modules are parametrized by X_+: for every $\lambda \in X_+$, there corresponds a unique simple $\mathfrak{g}$-module $\overline{\Lambda}_\lambda$.

The Coxeter number and the dual Coxeter numbers are defined by $h = 1 + (\alpha_0|\rho)$ and $h^\vee = 1 + \max_{\alpha>0} \frac{(\alpha|\rho)}{d}$, see Table 1. Note that $dh^\vee \geqslant h$.

1.1.2. The quantum group $\mathcal{U}$ and its category of representations. The quantum group $\mathcal{U} = \mathcal{U}_q(\mathfrak{g})$ associated to $\mathfrak{g}$ is a Hopf algebra defined over $\mathbb{Q}[q^{\pm 1/2D}]$, where $q^{1/2}$ is the quantum parameter and D is the least positive integer such that $(\mu|\mu') \in \frac{1}{D}\mathbb{Z}$ for every $\mu, \mu' \in X$ (see [11]). The category $\mathcal{C}$ of finite-dimensional $\mathcal{U}$-modules of type 1 is a ribbon category. In this paper we consider only $\mathcal{U}$-modules of type 1. The introduction of the fractional power $q^{1/2D}$ is necessary for the definition of the braiding. Finite-dimensional simple $\mathcal{U}$-modules of type 1 are also parametrized by X_+: for every $\lambda \in X_+$, there corresponds a unique simple $\mathcal{U}$-module Λ_λ, a deformation of $\overline{\Lambda}_\lambda$. Actually, the Grothendieck ring of finite-dimensional $\mathcal{U}$-modules of type 1 is isomorphic to that of finite-dimensional $\mathfrak{g}$-modules.

The reader should not confuse our q with the quantum parameter used in the definition of quantum groups by several authors. For example, our q is equal to q^2 in [3,4,20], or v^2 in Lusztig book [11].

1.2. Quantum link invariants.

1.2.1. General. Suppose L is a framed oriented link with m ordered components, then the quantum invariant $J_L(V_1, \ldots, V_m)$, for $V_1, \ldots, V_m \in \mathcal{C}$, is defined (since $\mathcal{C}$ is a ribbon category), with values in $\mathbb{Z}[q^{1/2D}]$. The modules $V_1, \ldots, V_m$ are usually called the colors.

The fact that J_L has integer coefficients follows from Lusztig's theory of canonical basis (see a detailed proof in [9]).

We will also use another normalization of the quantum invariant:

$$Q_L(V_1, \dots, V_m) := J_L(V_1, \dots, V_m) \times J_{U^{(m)}}(V_1, \dots, V_m).$$

Here $U^{(m)}$ is the 0 framing trivial link of m components. This normalization is more suitable for the study of quantum 3-manifold invariants, and will help us to get rid of the $\pm$ sign in many formulas.

Since finite-dimensional irreducible $\mathcal{U}$-modules are parametrized by X_+, we define

$$Q_L(\mu_1, \dots, \mu_m) := Q_L(\Lambda_{\mu_1-\rho}, \dots, \Lambda_{\mu_m-\rho}).$$

Note the shift by ρ. This definition is good only for $\mu_j \in \rho + X_+ = X \cap$ (interior of C). We define $Q_L(\mu_1, \dots, \mu_m)$ for arbitrary $\mu_j \in X$ by requiring that $Q_L(\mu_1, \dots, \mu_m) = 0$ if one of the μ_j's is on the boundary of C, and that $Q_L(\mu_1, \dots, \mu_m)$ is component-wise invariant under the action of the Weyl group W, i.e., for every $w_1, \dots, w_m \in W$,

$$Q_L\big(w_1(\mu_1), \dots, w_m(\mu_m)\big) = Q_L(\mu_1, \dots, \mu_m).$$

1.2.2. Example. Suppose $\mathfrak{g} = sl_2$. For a knot K, the invariant $J_K(N)$, with N a positive integer, is known as the colored Jones polynomial. Here N stands for the unique simple sl_2-module of dimension N. Suppose K is the right-hand trefoil, see Fig. 1. Then

$$J_K(N) = [N]\, q^{1-N} \sum_{n=0}^{\infty} q^{-nN}\big(1 - q^{1-N}\big)\big(1 - q^{2-N}\big) \cdots \big(1 - q^{n-N}\big).$$

The sum is actually finite, for any positive integer N. Similar formulas have also been obtained by Gelca and Habiro.

For the figure 8 knot (also obtained by Habiro)

$$\begin{aligned} J_K(N) \;=\; & [N] \sum_{n=0}^{\infty} q^{-nN}\big(1 - q^{N-1}\big)\big(1 - q^{N-2}\big) \cdots \big(1 - q^{N-n}\big) \\ & \times \big(1 - q^{N+1}\big)\big(1 - q^{N+2}\big) \cdots \big(1 - q^{N+n}\big). \end{aligned}$$

1.2.3. The trivial knot. Suppose U is the trivial knot. Then $J_U(V)$ is called the *quantum dimension* of V; its value is well known:

$$J_U(\mu) \;=\; \frac{1}{\psi} \prod_{\text{positive roots } \alpha} \big(q^{(\mu|\alpha)/2} - q^{-(\mu|\alpha)/2}\big) \tag{1.1}$$

$$= \frac{1}{\psi} \sum_{w \in W} \mathrm{sn}(w) q^{(\mu|w(\rho))}, \tag{1.2}$$

where $\mathrm{sn}(w)$ is the sign of w and

$$\psi = \prod_{\alpha>0} \big(q^{(\rho|\alpha)/2} - q^{-(\rho|\alpha)/2}\big) = \sum_{w \in W} \mathrm{sn}(w) q^{(\rho|w(\rho))}. \tag{1.3}$$

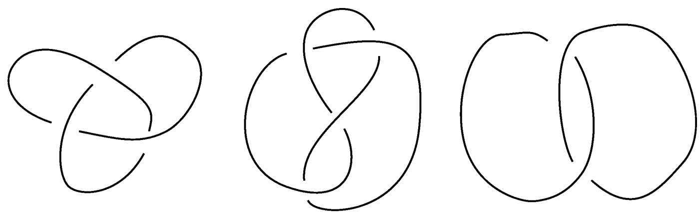


Fig. 1. The trefoil, figure 8 knot and the Hopf link.

Note that $J_U(\rho) = 1$, and $J_U(-\mu) = (-1)^s J_U(\mu)$, where s is the number of positive roots. Also

$$\psi = q^{-|\rho|^2} \prod_{\alpha>0} \big(q^{(\alpha|\rho)} - 1\big). \tag{1.4}$$

Hence if q is a root of unity of order $r \geqslant dh^\vee \geqslant 1 + \max_{\alpha>0}(\alpha|\rho)$, then $\psi \neq 0$.

1.2.4. The Hopf link. Let H be the Hopf link, see Fig. 1, with framing 0 on each component. Then

$$J_H(\mu, \lambda) = \frac{1}{\psi} \sum_{w \in W} \mathrm{sn}(w) q^{(\mu|w(\lambda))}. \tag{1.5}$$

For a proof, see [20]. Note that $J_H(\mu, \rho) = J_U(\mu)$.

1.3. Integrality. In general, J_L and Q_L contain fractional powers of q. The integrality, formulated below, shows that the fractional powers can be factored out.

Theorem 1.1 (Integrality [9]). *Suppose $\mu_1, \ldots, \mu_m \in X_+$. Then $Q_L(\Lambda_{\mu_1}, \ldots, \Lambda_{\mu_m})$ is in $q^{p/2}\mathbb{Z}[q^{\pm 1}]$, where p is a* (*generally fractional*) *number determined by the linking matrix l_{ij} of L*:

$$p = \sum_{1 \leqslant i,j \leqslant m} l_{ij}(\mu_i|\mu_j) + \sum_{1 \leqslant i \leqslant m} l_{ii}(2\rho|\mu_i) \in \frac{1}{D}\mathbb{Z}.$$

If all μ_j's are in the *root lattice*, then the number p is even. Hence we have

Corollary 1.2. *If all the μ_j's are in the* root lattice, *then $Q_L(\Lambda_{\mu_1}, \ldots, \Lambda_{\mu_m})$ is in $\mathbb{Z}[q^{\pm 1}]$.*

1.4. The first symmetry principle. Suppose $f, g \in \mathbb{Z}[q^{\pm 1/2D}]$. We say that f equals g *at rth roots of unity* and write

$$f \overset{(r)}{=} g,$$

if there is a number $a \in \frac{1}{2D}\mathbb{Z}$ such that $f, g \in q^a\mathbb{Z}[q^{\pm 1}]$, and for every rth root of unity ξ, one has

$$q^{-a} f|_{q=\xi} = q^{-a} g|_{q=\xi}.$$

There is no need to fix a $2D$th root of ξ.

An equivalent definition: $f, g \in \mathbb{Z}[q^{\pm 1/2D}]$ are equal rth roots of unity if for every $2Dr$th root ζ of unity, one has $f|_{q^{1/2D}=\zeta} = g|_{q^{1/2D}=\zeta}$.

Recall that the simplex $\overline{C}_r$ is a fundamental domain of the affine Weyl group W_r.

Theorem 1.3 (First symmetry principle, see [9]). *At rth roots of unity, the quantum invariant Q_L is* component-wise *invariant under the action of the affine Weyl group W_r. This means, for every $w_1, \ldots, w_m \in W_r$,*

$$Q_L\big(w_1(\mu_1), \ldots, w_m(\mu_m)\big) \overset{(r)}{=} Q_L(\mu_1, \ldots, \mu_m).$$

If one of the $\mu_1, \ldots, \mu_m$ is on the boundary of $\overline{C}_r$, then $J_L(\mu_1, \ldots, \mu_m) \overset{(r)}{=} 0$.

1.5. The second symmetry principle.

1.5.1. Action of G on $\overline{C}_r$. There is an action of the center group $G = X/Y$ on $\overline{C}_r$, and although Q_L is not really invariant under this action, it almost is. Let us first describe the action. For $\mu \in \overline{C}_r$ and $g \in G = X/Y$ let $\tilde{g} \in X$ be a lift of g, and define

$$g(\mu) = \mu + r\tilde{g} \in (X \bmod W_r) \equiv \overline{C}_r.$$

Another way to look at the action of G is the following. Recall that $W_r = W \ltimes rY$. Note that X is invariant under the action of the Weyl group. Let $\widehat{W}_r$ be the group generated by W and translations by rX. Then $\widehat{W}_r = W \ltimes rX$. If $\lambda \in X$ and $w \in W$, then $w(\lambda) - \lambda$ is in Y. This implies W_r is a *normal subgroup* of $\widehat{W}_r$. We have an exact sequence

$$1 \to W_r \to \widehat{W}_r \to G \to 1.$$

Taking the action of $\widehat{W}_r$ modulo the action of W_r, we get the action of G on $\overline{C}_r$. For more details and examples of actions of G, see [9].

On the group G there is a symmetric bilinear form with values in $\mathbb{Q}/\mathbb{Z}$ defined as follows. Suppose $g_1, g_2 \in G = X/Y$. Let $\tilde{g}_1, \tilde{g}_2 \in X$ be respectively lifts of g_1, g_2. Define $(g_1|g_2) := (\tilde{g}_1|\tilde{g}_2) \in \mathbb{Q}/\mathbb{Z}$. (Actually, $(g_1|g_2) \in \frac{1}{D}\mathbb{Z}/\mathbb{Z}$.) Similarly, for $\mu \in X$ and $g \in G$ one can define $(\mu|g) \in \frac{1}{D}\mathbb{Z}/\mathbb{Z}$.

1.5.2. Second symmetry principle.

Theorem 1.4 [9]. *Suppose $\mu_1, \ldots, \mu_m \in \overline{C}_r$ and $g_1, \ldots, g_m \in G$. Then*

$$Q_L\big(g_1(\mu_1), \ldots, g_m(\mu_m)\big) \overset{(r)}{=} q^{rt/2}\, Q_L(\mu_1, \ldots, \mu_m). \tag{1.6}$$

Here t depends only on the linking matrix (l_{ij}) of L:

$$t = (r-h) \sum_{1 \leqslant i,j \leqslant m} l_{ij}(g_i|g_j) + 2 \sum_{1 \leqslant i,j \leqslant m} l_{ij}(g_i|\mu_j - \rho),$$

with h being the Coxeter number of the Lie algebra $\mathfrak{g}$ (see Table 1).

For the special cases $\mathfrak{g} = sl_2$ and $\mathfrak{g} = sl_n$, the theorem was proved by Kirby and Melvin [5], and Kohno and Takata [6]. In [5,6], the "twisting factor" $q^{rt/2}$ is derived by direct computations. The proof given in [9] for general simple Lie algebra used a tensor product theorem of Lusztig.

Corollary 1.5. *If $\mu_j - \rho$ is in the root lattice and $\mu_j \in \overline{C}_r$, then,*

$$Q_L\bigl(g_1(\mu_1), \dots, g_m(\mu_m)\bigr) \overset{(r)}{=} q^{\frac{r(r-h)}{2}[\sum l_{ij}(g_i|g_j)]} Q_L(\mu_1, \dots, \mu_m).$$

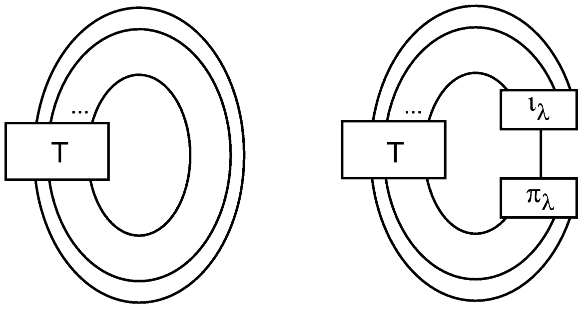
Remark 1.5.1. When $(r, d) \neq 1$, we can strengthen both symmetry principles using the Weyl alcove defined by the *long* highest root, see [9].

1.6. More integrality. The result of this subsection is new and will help us to define quantum invariants of 3-manifold without using the complicated theory of quantum groups at roots of unity.

Every finite-dimensional $\mathcal{U}$-module V decomposes as the direct sum of λ-homogeneous components, $\lambda \in X_+$ (recall that we work with general parameter). Here a λ-homogeneous component E_λ is the maximal submodule isomorphic to the sum of several copies of Λ_λ. Each λ-homogeneous component defines a projection $\pi_\lambda : V \to E_\lambda$ and an inclusion $\iota_\lambda : E_\lambda \to V$.

Suppose a link L is the closure of a (n, n)-tangle T, as shown in Fig. 2(a). Let $V_1, \dots, V_n$ are the colors of the strands shown (some of them may come from the same component). Consider a λ-homogeneous component E_λ of $V_1 \otimes \cdots \otimes V_n$. Cut the n strands and insert 2 "coupons" with operators $\pi_\lambda, \iota_\lambda$ in them; we got a "tangle with coupons" (T, λ), see Fig. 2(b). Ribbon category can be used a define isotopy invariant of objects like (T, λ), denoted by $J_{(T,\lambda)}$, which is generally in the fractional field $\mathbb{Q}(q^{1/2D})$ (see [20]). The following proposition, whose proof is based on the result of [9] and borrows an idea from [13], shows that for the special case (T, λ), the quantum invariant is a Laurent polynomial in $q^{1/2D}$.

Proposition 1.6. *In the above setting, $J_{(T,\lambda)}$ is in the ring $\mathbb{Z}[q^{\pm 1/2D}]$, and in that ring, it is divisible by the quantum dimension $J_U(\Lambda_\lambda)$.*



(a) L (b) (T,λ)

Fig. 2.

Proof. According to the general theory, J_T acts as a $\mathcal{U}$-endomorphism of $V_1 \otimes \cdots \otimes V_n$. Hence J_T commutes with $\iota_\lambda \circ \pi_\lambda$. Note that E_λ must be of the form $\Lambda_\lambda \otimes N$, where N is a vector space over $\mathbb{Q}(q^{1/2D})$. On $E_\lambda = \Lambda_\lambda \otimes N$ the operator J_T acts as $\mathrm{id} \otimes R$, where R is an operator acting on N. It follows that

$$J_{(L,\lambda)} = \mathrm{tr}(R) \times J_U(\Lambda_\lambda). \tag{1.7}$$

Eigenvalues of R are also eigenvalues of J_T which can be represented by a matrix with entries in $\mathbb{Z}[q^{\pm 1/2D}]$ (see [9], the proof used only the theory of quantum groups with general parameter). Hence $\mathrm{tr}(R)$ is in the ring $\mathcal{I}$ of algebraic integers over $\mathbb{Z}[q^{\pm 1/2D}]$. On the other hand, that the decomposition of $V_1 \otimes \cdots \otimes V_n$ into λ-homogeneous components can be done over the fractional field $\mathbb{Q}(q^{1/2D})$ means R can be represented by a matrix with entries in $\mathbb{Q}(q^{1/2D})$. Thus $\mathrm{tr}(R) \in \mathbb{Q}(q^{1/2D})$. Since $\mathcal{I} \cap \mathbb{Q}(q^{1/2D}) = \mathbb{Z}[q^{\pm 1/2D}]$, we have that $\mathrm{tr}(R) \in \mathbb{Z}[q^{\pm 1/2D}]$. The proposition now follows from (1.7). $\square$

Remark 1.6.1. We will use the proposition in the following way. First, since $J_{(T,\lambda)}$ is a Laurent polynomial in $q^{1/2D}$, we can plug any non-zero value of $q^{1/2D}$ in $J_{(T,\lambda)}$. Next, for special values of $q^{1/2D}$ annihilating $J_U(\Lambda_\lambda)$, the value of $J_{(T,\lambda)}$ is 0. Also note that

$$J_L(\mu_1, \ldots, \mu_m) = \sum_\lambda J_{(T,\lambda)}. \tag{1.8}$$

2. Quantum 3-manifold invariants

2.1. Introduction. Quantum invariants of 3-manifolds can be constructed only when q is a root of unity of some order r. In previously known cases, r must be divisible by d, since this will ensure that the so-called S-matrix is invertible. Our construction of 3-manifold invariants is slightly in more general situation: we will get invariants of 3-manifolds even in the case when r is not divisible by d. For this reason we will give a new (but not quite new) proof of the existence of quantum invariants of 3-manifolds. We will use only quantum groups with general parameter to define link invariants, and only on the last step we replace q by a root of unity in *link invariants*. In this paper, a 3-manifold is always closed and oriented.

For the reader to have an idea how the quantum 3-manifold invariant looks like, let us give here the value for the Poincare Homology 3-sphere P (obtained by surgery on a left-hand trefoil with framing -1), with $\mathfrak{g} = sl_2$:

$$\tau_P^{sl_2}(q) = \frac{1}{1-q} \sum_{n=0}^{\infty} q^n \left(1 - q^{n+1}\right)\left(1 - q^{n+2}\right) \cdots \left(1 - q^{2n+1}\right).$$

Here q is a root of unity, and the sum is easily seen to be finite. Similar, but different formula has also been obtained by Lawrence and Zagier, using some calculation involving modular forms, see [7].

When M is the Brieskorn sphere $\Sigma(2,3,7)$ (obtained by surgery on the right-hand trefoil with framing -1):

$$\tau_M^{sl_2}(q) = \frac{1}{1-q}\sum_{n=0}^{\infty} q^{-n(n+2)}\bigl(1-q^{n+1}\bigr)\bigl(1-q^{n+2}\bigr)\cdots\bigl(1-q^{2n+1}\bigr).$$

Again q must be a root of unity for the above expression to have meaning.

2.1.1. Heuristic. The values of $Q_L(\mu_1,\ldots,\mu_m)$ are in $\mathbb{Z}[q^{\pm 1/2D}]$. The infinite sum $\sum_{\mu_j\in X} Q_L(\mu_1,\ldots,\mu_m)$ does not have any meaning. It is believed (and there are reasons for this) that the sum is invariant under the second Kirby move, and hence almost defines a 3-manifold invariant. The problem is to regularize the infinite sum $\sum_{\mu_j\in X} Q_L(\mu_1,\ldots,\mu_m)$. One solution is based on the fact that at rth roots of unity, $Q_L(\mu_1,\ldots,\mu_m)$ is periodic (the first symmetry principle), so we should use the sum with μ_j's run over the set $\overline{C}_r$.

2.2. Sum over $\overline{C}_r$.

2.2.1. General. Let us fix a positive integer $r \geqslant dh^\vee$, called the *shifted level*, and a primitive rth root of unity ξ. At some stage we also need a primitive $2Dr$th root ζ of 1 such that $\zeta^{2D} = \xi$. Let

$$F_L^{\mathfrak{g}}(\xi;\zeta) = \sum_{\mu_j\in\overline{C}_r\cap X} Q_L(\mu_1,\ldots,\mu_m)\big|_{q^{1/2D}=\zeta}.$$

Certainly ζ determines ξ, but we prefer to keep ξ in the notation $F_L^{\mathfrak{g}}(\xi;\zeta)$ since in later cases, the whole thing depends only on ξ but not ζ, and in those cases we will drop ζ in the notation.

Recall that $h = (\rho|\alpha_0)+1$ is the Coxeter number. Let $k = r-h$. Then we have (recalling the shift by ρ and the fact that $Q_L \overset{(r)}{=} 0$ on the boundary of $\overline{C}_r$)

$$F_L^{\mathfrak{g}}(\xi;\zeta) = \sum_{\mu_j\in\overline{C}_k} Q_L(\Lambda_{\mu_1},\ldots,\Lambda_{\mu_m})\big|_{q^{1/2D}=\zeta}. \tag{2.1}$$

The following half-open parallelepiped P_r is a fundamental domain of the group rY:

$$P_r = \bigl\{x = c_1\alpha_1+\cdots+c_\ell\alpha_\ell \in \mathfrak{h}_{\mathbb{R}}^* \mid 0\leqslant c_1,\ldots,c_\ell < r\bigr\}.$$

Since $\overline{C}_r$ is a fundamental domain of $W_r = W \ltimes rY$, we have, due to the first symmetry principle,

$$F_L^{\mathfrak{g}}(\xi;\zeta) = \frac{1}{|W|}\sum_{\mu_j\in\overline{P}_r\cap X} Q_L(\mu_1,\ldots,\mu_m)\big|_{q^{1/2D}=\zeta}. \tag{2.2}$$

2.2.2. The Gauss sum. The following is a quadratic Gauss sum on the abelian group $X/rY \equiv P_r\cap X$, with ζ used to define fractional powers of ξ:

$$\gamma^{\mathfrak{g}}(\xi,\zeta) := \sum_{\mu\in P_r\cap X} \xi^{\frac{1}{2}(|\mu|^2-|\rho|^2)}.$$

Criteria for vanishing of a Gauss sum are known, see, e.g., [2]. Using the explicit structure of simple Lie algebras and the criterion one can prove the following.

Proposition 2.1. *The Gauss sum $\gamma^{\mathfrak{g}}(\xi,\zeta)=0$ if and only if r is odd and $\mathfrak{g}$ is either C_ℓ with arbitrary ℓ, or B_ℓ with even ℓ.*

The following lemma uses the well known trick of completing the square.

Lemma 2.2. *Suppose $\beta\in X$. Then*

$$\sum_{\mu\in P_r\cap X}\xi^{\frac{1}{2}(|\mu|^2-|\rho|^2)}\xi^{(\beta|\mu)}=\gamma^{\mathfrak{g}}(\xi,\zeta)\times\xi^{-\frac{1}{2}|\beta|^2}.$$

Proof. Completing the square, we see that

$$\frac{1}{2}\big(|\mu|^2-|\rho|^2\big)+(\beta|\mu)=\big(|\mu+\beta|^2-|\rho|^2\big)-\frac{1}{2}|\beta|^2.$$

It remains to notice that everything is invariant under the translation rY, and both P_r and $P_r+\beta$ are fundamental domains of rY. □

2.3. Invariance under the second Kirby move.

Proposition 2.3. *Suppose that the order r of ξ is greater than or equal to $dh^\vee$. Then $F_L^{\mathfrak{g}}(\xi,\zeta)$ does not depend on the orientation of L and is invariant under the second Kirby move.*

Proof. Using linearity we extend the invariant J_L to the case when the colors are elements of the $\mathbb{Z}[q^{\pm 1/2D}]$-module freely generated by Λ_λ, $\lambda\in X_+$. Then

$$F_L^{\mathfrak{g}}(\xi,\zeta)=J_L(\omega,\dots,\omega)|_{q^{1/2D}=\zeta},$$

where

$$\omega=\sum_{\mu\in\mathrm{int}(C_r)\cap X}J_U(\mu)\,\Lambda_{\mu-\rho}=\sum_{\mu\in\overline{C}_k\cap X}J_U(\Lambda_\mu)\,\Lambda_\mu.$$

The independence of orientation is simple: If we reverse the orientation of one component, and at the same time change the color from V to the dual V^*, then the quantum link invariant remains the same. It is known that the alcove $\overline{C}_k$, (here $k=r-h$), is invariant under taking dual, i.e., the dual of Λ_μ, $\mu\in\overline{C}_k$, is another Λ_{μ^*}, with μ^* again in $\overline{C}_k$. Moreover, $J_U(\Lambda_\mu)=J_U(\Lambda_\mu^*)$. Hence ω is invariant under $\mu\to\mu^*$, and $J_L(\omega,\dots,\omega)$ is unchanged if we reverse the orientation of one component.

Let us consider the 2nd Kirby move $L\to L'$, as described in Fig. 3, with blackboard framing. In both L, L' let K be the singled out unknot component with framing 1.

Then we have to show that

$$J_L(\omega,\dots,\omega)\overset{(r)}{=}J_{L'}(\omega,\dots,\omega).$$

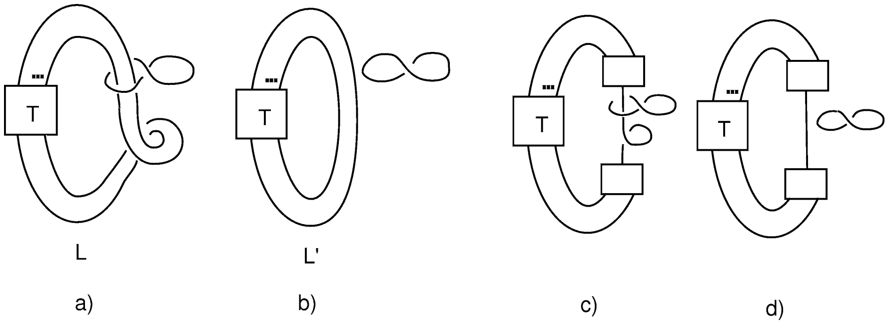


Fig. 3.

Fig. 4.

It is enough to show that for every $\mu_1, \ldots, \mu_m \in X_+$,

$$J_L(\mu_1, \ldots, \mu_m, \omega) \overset{(r)}{=} J_{L'}(\mu_1, \ldots, \mu_m, \omega).$$

Here we suppose L and L' have $m+1$ components with K being the $(m+1)$st.

Suppose the colors of the n strands coming out from the box T are $V_1, \ldots, V_n$. (Each V_i is one of $\Lambda_{\mu_j-\rho}$ or their duals.) The module $V_1 \otimes \cdots \otimes V_n$ is completely reducible over $\mathbb{Q}(q^{1/2D})$, so we decompose it into homogeneous components. Using (1.8) to decompose J_L and $J_{L'}$ into sums of quantum invariants of "tangles with coupons", see Fig. 3(c), (d). In each tangle with coupons there is only one strand, with color a homogeneous component, piercing through K. Now put $q^{1/2D} = \zeta$ (see Remark 1.6.1). We see that it's remain to prove the following lemma, which is essentially the statement of the proposition for the case when $n = 1$. $\square$

Lemma 2.4. *Suppose* $J_U(\lambda)|_{q^{1/2D}=\zeta} \neq 0$. *Then*

$$J_Z(\lambda, \omega)|_{q^{1/2D}=\zeta} = J_{Z'}(\lambda, \omega)|_{q^{1/2D}=\zeta},$$

where Z, Z' *are the* $(1,1)$*-tangles in Fig.* 4.

Proof. Note that both sides of $J_Z(\lambda, \omega)$ and $J_{Z'}(\lambda, \omega)$ are scalar operator acting on $\Lambda_{\lambda-\rho}$. Closing Z, we get the Hopf link H_+ with framing 1 on both components. Similarly, closing Z' we get the trivial link U_2 with framing 1 on the second component. Since $J_U(\lambda)|_{q^{1/2D}=\zeta} \neq 0$, the identity to prove is equivalent to

$$J_{H_+}(\lambda, \omega) \overset{(r)}{=} J_{U_2}(\lambda, \omega). \tag{2.3}$$

Let us first calculate

$$u = \sum_{\mu \in P_r \cap X} q^{\frac{|\mu|^2 - |\rho|^2}{2}} J_U(\mu)\, J_H(\lambda, \mu)|_{q^{1/2D} = \zeta}.$$

We have, with $q^{1/2D} = \zeta$,

$$\begin{aligned}
u &= \frac{1}{\psi^2} \sum_{\mu \in P_r \cap X} \sum_{w, w' \in W} \operatorname{sn}(ww')\, q^{\frac{|\mu|^2 - |\rho|^2}{2}} q^{(\mu | w(\lambda) + w'(\rho))} \quad \text{by formula (1.5)} \\
&= \frac{1}{\psi^2} \gamma^{\mathfrak{g}}(\xi, \zeta) \sum_{w, w' \in W} \operatorname{sn}(ww')\, q^{-\frac{1}{2}(|\lambda|^2 + |\rho|^2 + 2(\lambda | w^{-1} w'(\rho))} \quad \text{by Lemma 2.2} \\
&= \frac{1}{\psi^2} \gamma^{\mathfrak{g}}(\xi, \zeta)\, q^{-\frac{|\rho|^2 + |\lambda|^2}{2}} |W| \sum_{w \in W} \operatorname{sn}(w) q^{(-\lambda | w(\rho))} \\
&= \frac{1}{\psi} q^{-\frac{|\rho|^2 + |\lambda|^2}{2}} |W|\, \gamma^{\mathfrak{g}}(\xi, \zeta)\, J_U(-\lambda) \quad \text{by (1.2).}
\end{aligned}$$

Thus

$$\sum_{\mu \in P_r \cap X} q^{\frac{|\mu|^2 - |\rho|^2}{2}} J_U(\mu)\, J_H(\lambda, \mu) \overset{(r)}{=} \frac{(-1)^s |W|}{\psi} q^{-\frac{|\rho|^2 + |\lambda|^2}{2}} |W| \gamma^{\mathfrak{g}}(\xi, \zeta)\, J_U(\lambda). \tag{2.4}$$

Recall that for Q_L, increasing by 1 the framing of a component colored by $\Lambda_{\mu - \rho}$ results in a factor $q^{(|\mu|^2 - |\rho|^2)/2}$. The left-hand side of (2.3) is

$$LHS = \frac{1}{|W|} q^{\frac{|\mu|^2 - |\rho|^2}{2}} \sum_{\mu \in P_r \cap X} q^{\frac{|\mu|^2 - |\rho|^2}{2}} J_U(\mu)\, J_H(\lambda, \mu)\, |_{q^{1/2D} = \zeta}.$$

The right-hand side is

$$\begin{aligned}
RHS &= J_U(\lambda) \frac{1}{|W|} \sum_{\mu \in P_r \cap X} q^{\frac{|\mu|^2 - |\rho|^2}{2}} J_U(\mu)^2\, |_{q^{1/2D} = \zeta} \\
&= J_U(\lambda) \frac{1}{|W|} \sum_{\mu \in P_r \cap X} q^{\frac{|\mu|^2 - |\rho|^2}{2}} J_H(\mu, \rho)\, J_U(\mu)\, |_{q^{1/2D} = \zeta}.
\end{aligned}$$

Hence it follows from (2.4) that $LHS = RHS$. □

Let us record here the formula for $F_{U_+}(\xi; \zeta)$, where U_+ is the unknot with framing 1.

$$F^{\mathfrak{g}}_{U_+}(\xi; \zeta) = \frac{1}{|W|} \sum_{\mu \in P_r \cap X} q^{\frac{|\mu|^2 - |\rho|^2}{2}} J_H(\rho, \mu) J_U(\mu)\, |_{q^{1/2D} = \zeta},$$

and hence (2.4) gives

$$F^{\mathfrak{g}}_{U_+}(\xi; \zeta) = \frac{\gamma^{\mathfrak{g}}(\xi; \zeta)}{\prod_{\alpha > 0} (1 - q^{(\alpha | \rho)})}. \tag{2.5}$$

Remark 2.4.1. In the proof we used the first symmetry principle, whose proof required the theory of quantum groups at roots of unity. However, if we defined F_L using the sum over $P_r \cap X$ at the beginning, then we would not have to use the first symmetry principle.

2.4. Quantum invariants.

2.4.1. Definition. Suppose $U_\pm$ are the unknot with framing ± 1. Note that $F^{\mathfrak{g}}_{U_\pm}(\xi;\zeta)$ are complex conjugate to each other. If $F^{\mathfrak{g}}_{U_\pm}(\xi;\zeta) \neq 0$, then one can define invariant of the 3-manifold M obtained by surgery along L by the formula:

$$\tau^{\mathfrak{g}}_M(\xi;\zeta) := \frac{F_L(\xi;\zeta)}{F_{U_+}(\xi;\zeta)^{\sigma_+} F_{U_-}(\xi;\zeta)^{\sigma_-}}.$$

Here σ_+, σ_- are the number of positive and negative eigenvalues of the linking matrix of L. If $F^{\mathfrak{g}}_{U_\pm}(\xi;\zeta) = 0$, then let $\tau^{\mathfrak{g}}_M(\xi;\zeta) = 0$ for every 3-manifold M.

Here are the cases when $F^{\mathfrak{g}}_{U_\pm}(\xi;\zeta) = 0$.

Proposition 2.5. *Suppose the order r of the root ξ satisfies $r \geqslant dh^\vee$. Then $F^{\mathfrak{g}}_{U_\pm}(\xi;\zeta) = 0$ if and only r is odd and $\mathfrak{g}$ is either B_ℓ with even ℓ or C_ℓ with arbitrary ℓ. In particular, if r is divisible by d, then $F^{\mathfrak{g}}_{U_\pm}(\xi;\zeta)$ is not equal to 0.*

Proof. The proposition follows from formula (2.5) and Proposition 2.1. □

Remark 2.5.1. Only in the two cases listed in the proposition are the invariants trivial. But $F_{U_\pm} \neq 0$ does mean that the so-called S-matrix is invertible.

2.4.2. Comparison with known cases. In the literature, only the case r divisible by d was considered. In that case $r = dr'$, and the number $r' - h^\vee$ is called the level of the theory (see [4]). Also in this case one can construct a modular category, and a topological quantum field theory.

Here we consider both cases when r is or is not divisible by d. In the latter case, the level should be $r - h$.

In the book [20] modular category, and hence quantum invariants, was constructed for simple Lie algebras with $d = 1$. Later work of [1] established the existence of modular category for every simple Lie algebra, at shifted level r divisible by d, see a rigorous proof in [4]. We will explain here why the invariant of [4] is coincident with ours, when r is divisible by d.

If $d = 1$, then the set of modules Λ_μ, with $\mu + \rho \in C_r$ forms a modular category (see [1,4]), hence the 3-manifold invariant derived from the modular category is exactly our $\tau^{\mathfrak{g}}_M(\xi;\zeta)$.

Suppose $d > 1$, and r is divisible by d. In this case the above set of modules does not form a modular category. There is a smaller simplex $C'_r \subset C_r$ with the corresponding affine Weyl group W'_r such that C_r consists of several copies of C'_r under the action of W'_r; and the modular category consists of Λ_μ with $\mu + \rho \in C'_r$. The corresponding 3-manifold invariant is thus obtained by taking the sum over the smaller simplex C'_r. The first symmetry principle is valid if C_r, W_r are replaced with C'_r, W'_r (for details see [9]). Due to this symmetry, the sum of Q_L over the bigger simplex C_r is simply a constant times the sum over C'_r. This is the reason why we can use C_r to define the same 3-manifold invariant. This smaller simplex C'_r is constructed using the *long* highest root.

3. Quantum invariant of the projective group

3.1. Preliminaries. There is a simply-connected complex Lie group $\mathcal{G}$ corresponding to $\mathfrak{g}$. The invariant $\tau_M^{\mathfrak{g}}$ is associated to $\mathcal{G}$. Let G be the center group of $\mathcal{G}$. It is known that G is isomorphic to X/Y, and $|G| = \det(a_{ij})$. For every subgroup $G' \subset G$, there corresponds a Lie group $\mathcal{G}/G'$, and there is a quantum invariant associated with this quotient group. We will describe here a method to construct them, focusing on the extreme case when $G' = G$. We will see that there are many shifted levels r for which the invariant $\tau_M^{\mathfrak{g}}$ is trivial, but at the same time the invariant of the projective group, denoted by $\tau_M^{P\mathfrak{g}}$, is non-trivial, and even defined by a modular category. We will see that if r and $\det(a_{ij})$ are coprime, i.e., $(r, \det(a_{ij})) = 1$, then the invariant associated to the projective group is not trivial.

3.1.1. The lattice $\rho + Y$.

Lemma 3.1. *For every positive integer r, the lattice $\rho + Y$ is invariant under the action of W_r.*

Proof. Recall that $W_r = W \ltimes rY$. The fact that $\rho + Y$ is invariant under rY is obvious. That $\rho + Y$ is invariant under the action of W follows from the fact that $w(\rho) - \rho$ belongs to Y. (Actually, $w(\mu) - \mu \in Y$ for every $\mu \in X$.) $\square$

3.1.2. Sums over the root lattice. Let $k = r - h$, and ξ is a root of unity of order $r \geqslant dh^\vee \geqslant h$. Let

$$F_L^{P\mathfrak{g}}(\xi) = \sum_{\mu_j \in (\overline{C}_k \cap Y)} Q_L(\Lambda_{\mu_1}, \dots, \Lambda_{\mu_m})|_{q=\xi}.$$

The definition is the same as in (2.1), except that we sum over μ_j's which are in the *root lattice*. Note that there is no need to fix a $2D$th root of ξ, since by Corollary 1.2, there is no fractional power of q.

Recalling the shift by ρ, we have

$$F_L^{P\mathfrak{g}}(\xi) = \sum_{\mu_j \in \overline{C}_r \cap (\rho + Y)} Q_L(\mu_1, \dots, \mu_m)|_{q=\xi}.$$

Lemma 3.1 and the first symmetry principle show that

$$F_L^{P\mathfrak{g}}(\xi) = \frac{1}{|W|} \sum_{\mu_j \in \rho + (P_r \cap Y)} Q_L(\mu_1, \dots, \mu_m)|_{q=\xi}.$$

3.1.3. Gauss sum. We will encounter a Gauss sum on the group Y/rY. From now on let $\sum_r$ stands for $\sum_{\mu \in \rho + (P_r \cap Y)}$. Put

$$\gamma^{P\mathfrak{g}}(\xi) := \sum_r \xi^{\frac{|\mu|^2 - |\rho|^2}{2}}.$$

Then, the same proof of Lemma 2.2 gives us:

Lemma 3.2. *Suppose $\beta \in Y$. Then*

$$\sum_r \xi^{\frac{|\mu|^2-|\rho|^2}{2}} \xi^{(\mu|\beta)} = \gamma^{P\mathfrak{g}}(\xi) \times \xi^{-\frac{|\beta|^2}{2}}.$$

3.2. Definition of invariants associated to the projective group. Recall that H is the Hopf link. Let $S_{\lambda,\mu} = J_H(\lambda,\mu)|_{q=\xi}$. Let the matrix S have entries $S_{\lambda,\mu}$ with $\lambda,\mu \in \mathrm{Interior}(C_r) \cap (\rho + Y)$.

Theorem 3.3.

(a) *Suppose the order r of ξ is greater than or equal to $dh^\vee$. Then $F_L^{P\mathfrak{g}}(\xi)$ is invariant under the* 2*nd Kirby move and does not depend on the orientation of L.*
(b) *If r is coprime with $d\,\det(a_{ij})$, then the matrix S is invertible.*
(c) *If r is coprime with $\det(a_{ij})$, then $F_{U_\pm}^{P\mathfrak{g}}(\xi) \neq 0$.*

Proof. Notice that if $\lambda,\mu \in Y$, then in the decomposition of $\Lambda_\lambda \otimes \Lambda_\mu$ into irreducible modules one encounters only Λ_ν with $\nu \in Y$. This is a well known fact: The irreducible modules of the group $\mathcal{G}/G$ have highest weights in $X_+ \cap Y$, and finite-dimensional $\mathcal{G}/G$-modules are completely reducible.

Using this fact one can repeat the proof of Proposition 2.3 to get a proof of part (a).

(b) We will show that $S\overline{S}$ is a non-zero constant times the identity matrix. Here $\overline{S}$ is the complex conjugate. We know that $\psi \neq 0$ when $q = \xi$, since $r \geqslant dh^\vee$ (see 1.2.3). Using (1.5) and $\sum_{\mu \in \overline{C}_r \cap (\rho+Y)} = \frac{1}{|W|}\sum_r$, we have

$$\begin{aligned} |W|\psi^2\big(S\overline{S}\big)_{\lambda,\nu} &= \sum_r \sum_{w,w' \in W} \mathrm{sn}(ww')\xi^{(\mu|w(\lambda)-w'(\nu))} \\ &= \sum_{w,w' \in W} \mathrm{sn}(ww')\left[\sum_r \xi^{(\mu|w(\lambda)-w'(\nu))}\right]. \end{aligned}$$

Let Y^* be the lattice dual to Y, over $\mathbb{Z}$, with respect to the scalar product. If $w(\lambda) - w'(\nu) \notin rY^*$, then there is a fundamental root α_i such that $(\alpha_i | w(\lambda) - w'(\nu)) \notin r\mathbb{Z}$. It follows that the sum in the square bracket is 0, since $\xi^n + \xi^{2n} + \cdots + \xi^{(r-1)n} = 0$ if n is not divisible by r, and $\sum_r$ is the sum over a fundamental domain of rY.

We will find out when $w(\lambda) - w'(\nu) \in rY^*$. Note first that $w(\lambda) - w'(\nu) \in Y$. We'll find the intersection $rY^* \cap Y$.

The lattice Y^* is spanned by $\lambda_1/d_1, \ldots, \lambda_\ell/d_\ell$, where $\lambda_1, \ldots, \lambda_\ell$ are the fundamental weights. Thus the order of Y^*/X is $d_1 d_2, \ldots, d_\ell$, a factor of d^ℓ. The order of X/Y is $\det(a_{ij})$. Thus the group $rY^*/rY \equiv Y^*/Y$ has order a factor of $d^\ell \times \det(a_{ij})$.

The group Y/rY has order r^ℓ. By assumption, the orders of two groups rY^*/rY and Y/rY are co-prime. Their intersection must be trivial. Hence $w(\lambda) - w'(\nu)$, belonging to both rY^* and Y, must belong to rY. But this means λ and ν are in the same W_r-orbit. This could happen for $\lambda, \nu \in \mathrm{Int}(C_r)$ if and only if $w = w'$ and $\lambda = \nu$.

When $w = w'$ and $\lambda = \nu$, the sum in the square bracket is r^ℓ. Thus $(S\overline{S})$ is a non-zero constant times the identity.

(c) One can prove (c) directly using the criterion of vanishing of Gauss sums. Or one can use the following arguments. If, in addition, r is coprime with d, then by (b), the S matrix is non-degenerate. In this case it is known that $F_{U_\pm}^{P\mathfrak{g}}(\xi) \neq 0$ (see [20]). Suppose now $(r,d) \neq 1$. Then r is divisible by d. Formula (3.2) below shows that $F_{U_+}^{P\mathfrak{g}}(\xi)$ is a factor of $F_{U_+}^{\mathfrak{g}}(\xi,\zeta)$, which is not 0 by Proposition 2.5 (for some $2D$th root ζ of ξ). Hence $F_{U_\pm}^{P\mathfrak{g}}(\xi) \neq 0$. $\square$

If $F_{U_\pm}^{P\mathfrak{g}}(\xi) = 0$, we define $\tau_M^{P\mathfrak{g}}(\xi) = 0$, otherwise, let

$$\tau_M^{P\mathfrak{g}}(\xi) := \frac{F_L^{P\mathfrak{g}}(\xi)}{(F_{U_+}^{P\mathfrak{g}}(\xi))^{\sigma_+}(F_{U_-}^{P\mathfrak{g}}(\xi))^{\sigma_-}}, \tag{3.1}$$

where M is obtained from S^3 by surgery along the framed link L.

Remark 3.3.1. (a) Theorem 3.3, part (b) shows that when r is co-prime with $d\det(a_{ij})$, the set of all modules Λ_μ, with $\mu \in \overline{C}_k \cap Y$ (note the root lattice Y here), generates a modular category. Here one has to use the reduced quotient structure as in [1,4]. At the same time, if Y is replaced by X, then the resulting category, usually considered by algebraists (say, in earlier papers of H. Andersen) might not be a modular category. The reason is the S-matrix might not be invertible. There are values of r when the S-matrix is invertible for the Y case, but not for the X case.

(b) Whenever $F_{U_\pm}^{P\mathfrak{g}}(\xi) \neq 0$, one has non-trivial invariants. In addition to the cases described in the theorem, there are other cases when $F_{U_\pm}^{P\mathfrak{g}}(\xi) \neq 0$. For example, using the criterion for the vanishing of Gauss sum, one can also prove that whenever r is odd (for all $\mathfrak{g}$), $F_{U_\pm}^{P\mathfrak{g}}(\xi) \neq 0$. On the other hand, there are cases when $F_{U_\pm}^{P\mathfrak{g}}(\xi) = 0$: Examples include the case $\mathfrak{g} = sl_2$, r is divisible by 4.

(c) The invariant $\tau_M^{P\mathfrak{g}}$ is the invariant associated with the projective group, since the root lattice is spanned by highest weights of finite-dimensional irreducible modules of $\mathcal{G}/G$. If G' is a subgroup of G, then one can construct invariant associated with $\mathcal{G}/G'$ by using the lattice Y' generated by the set of all highest weights of $\mathcal{G}/G'$. The construction is similar.

3.3. Invariant associated to a finite abelian group with a bilinear form. On the group $G = X/Y$ there is defined the symmetric bilinear form $(\cdot|\cdot)$ with values in $\frac{1}{D}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. For any such group there is a way to define invariants of 3-manifolds which carry only the information about the homology groups and the linking form on the torsion of the first homology group, see [15,2,20]. We will present here the theory in the form most convenient for us.

Again ζ is root of unity of order $2Dr$, and $\xi = \zeta^{2D}$. Define

$$F_L^G(\xi;\zeta) := \sum_{g_i,g_j \in G} \xi^{r(r-h)\times\frac{1}{2}\sum l_{ij}(g_i|g_j)},$$

where l_{ij} is the linking matrix of L. Here we use ζ to define fractional powers of ξ.

Then

$$F^G_{U_\pm}(\xi;\zeta)=\sum_{g\in G}\xi^{r(r-h)(g|g)/2}$$

is a Gauss sum. If $F^G_{U_\pm}(\xi;\zeta)=0$, we define $\tau^G_M(\xi;\zeta)=0$, otherwise we define, for

$$\tau^G_M(\xi;\zeta):=\frac{F^G_L(\xi;\zeta)}{(F^G_{U_+})^{\sigma_+}(F^G_{U_-})^{\sigma_-}},$$

for M obtained by surgery on a framed link L. It is a 3-manifold invariants. In general, $\zeta^{r(r-h)}$ is a root of unity of order $2D$. If $\zeta^{r(r-h)}=\exp(2\pi \mathrm{i}/2D)$, then our invariant is coincident with those in [2].

3.4. Splitting.

Lemma 3.4. *Suppose* $(r,\det(a_{ij}))=1$. *Then*

(a) *G acts freely on the set $\overline{C}_r\cap X$.*
(b) *In each G-orbit of $\overline{C}_r\cap X$ there is exactly one element in $\rho+Y$.*

Proof. (a) Note that $\overline{C}_r\cap X$ is a finite set. Suppose $g(\mu)=\mu$ for some $\mu\in\overline{C}_r\cap X$, we will show that g is the identity of G. There is a lift $\tilde g\in X$ of G such that

$$r\tilde g+\mu=\mu \ (\mathrm{mod}\ W_r),$$

which, due to $W_r=W\ltimes rY$, means there is $w\in W$ such that

$$r\tilde g+\mu\in w(\mu)+rY.$$

Since $w(\mu)-\mu\in Y$, it follows that $r\tilde g\in Y$, or $rg=0$ in G. Because $(r,|G|)=1$, this implies $g=0$ in G.

(b) Using $X/Y=rX/Y$ (since $(r,|G|)=1$), we have

$$X=Y+rX.$$

Hence $(\rho+Y)+rX=X$. This shows that in each G-orbit there is at least one element in $\rho+Y$. The proof of part (a) shows that each G-orbit contains at most one element in $\rho+Y$. □

Suppose $(r,\det(a_{ij}))=1$. By the above lemma and the second symmetry principle (see Corollary 1.5), one has

$$F^{\mathfrak g}_L(\xi;\zeta)=F^G_L(\xi;\zeta)F^{P\mathfrak g}_L(\xi). \tag{3.2}$$

Hence we have the following splitting theorem

Theorem 3.5. *Suppose* $(r,\det(a_{ij}))=1$ *and ζ is a $2Dr$th root of unity, $\xi=\zeta^{2D}$. Then*

$$\tau^{\mathfrak g}_M(\xi;\zeta)=\tau^{P\mathfrak g}_M(\xi)\,\tau^G_M(\xi;\zeta).$$

Remark 3.5.1. (a) The invariant $\tau_M^G(\xi;\zeta)$ carries only the information about the first homology group and the linking form on its torsion; it is a weak invariant, and sometimes it is equal to 0, in which case $\tau_M^{\mathfrak{g}}(\xi;\zeta)=0$. Hence $\tau_M^{P\mathfrak{g}}(\xi)$ is in general a finer invariant. For example, if $\mathfrak{g}=B_\ell$, and r is odd, then $\tau_M^{\mathfrak{g}}(\xi;\zeta)=0$, but $\tau_M^{P\mathfrak{g}}(\xi)$ is in general not 0.

(b) When r is not coprime with $\det(a_{ij})$, there are cases when both $\tau^{\mathfrak{g}}$ and $\tau^{P\mathfrak{g}}$ are non-trivial, but there is no simple relation between the two invariants. Examples of such case are: $\mathfrak{g}=sl_n$ and $(r,n)\neq 1$, $\mathfrak{g}=D_\ell$ and r even, and $\mathfrak{g}=C_\ell$ with ℓ odd and r even.

(c) The splitting of Theorem 3.5 fits very well with the Gussarov–Habiro theory of finite type 3-manifold invariants: In that theory one has first to partition the set of 3-manifolds into subset of ones with the same homology and linking form, then defines finite type invariants in each subset using a suitable filtration. The invariant τ^G corresponds to homology and the linking form, and $\tau^{P\mathfrak{g}}$ can be expanded into power series, at least for rational homology 3-spheres (see below), that gives rise to finite type invariants.

(d) The projective quantum invariants were defined and the splitting theorem was proved in Kirby and Melvin [5] for $\mathfrak{g}=sl_2$ and Kohno and Takata [6] for $\mathfrak{g}=sl_n$. For the case when r is divisible by d, a similar splitting has also been obtained by Sawin [18], but his proof does not go through for all simple Lie algebras, he has to exclude a half of D series. For $d=2$, Sawin's result and Theorem 3.5 address *different* cases, and hence they complement each other.

4. Integrality

Theorem 4.1. *Suppose that $r\geqslant dh^\vee$ is a prime and not a factor of $|W|\det(a_{ij})$, and ξ a primitive r-th root of unity. Then $\tau_M^{P\mathfrak{g}}(\xi)$ is in $\mathbb{Z}[\xi]=\mathbb{Z}[\exp(2\pi\mathrm{i}/r)]$.*

The theorem was proved in the $\mathfrak{g}=sl_2$ case by Murakami [14] (see also [12]) and $\mathfrak{g}=sl_n$ by Takata and Yokota [19] and Masbaum and Wenzl [13]. It is conjectured that even when r is a not prime, one also has $\tau_M^{P\mathfrak{g}}(\xi)\in\mathbb{Z}[\xi]$. The remaining part of this section is devoted to a proof of this theorem.

4.1. General facts. For $a,b\in\mathbb{Z}[\xi]$, we write $a\sim b$ if there is a unit u in $\mathbb{Z}[\xi]$ such that $a=ub$. Suppose r is an odd prime. It is known that $(\xi-1)$ is prime in $\mathbb{Z}[\xi]$, and $r\sim(\xi-1)^{r-1}$. It follows that $(r-1)!$ is coprime with $(\xi-1)$. If $(n,r)=1$ then $(\xi^n-1)\sim(\xi-1)$.

Formula (1.1) shows that for every $\lambda\in Y$,

$$J_U(\Lambda_\lambda)|_{q=\xi}\sim 1.$$

To prove the theorem, we have to show that the numerator of the right-hand side of (3.1) is divisible by the denominator. First we will show that the denominator is just a power of $(\xi-1)$, then we show that the numerator can be decomposed as a sum of simple terms, each divisible by that power of $(\xi-1)$.

4.2. Gauss sum again. Suppose b is an integer. Recall that $\sum_r$ stands for $\sum_{\mu\in\rho+(P_r\cap Y)}$. Let

$$\gamma_b^{P\mathfrak{g}}(\xi)=\sum_r \xi^{b\frac{|\mu|^2-|\rho^2|}{2}}.$$

Note that for $\mu\in\rho+Y$, $|\mu|^2-|\rho|^2$ is always an even number. Hence $\gamma_b^{P\mathfrak{g}}(\xi)\in\mathbb{Z}[\xi]$.

Lemma 4.2. *Suppose r is an odd prime not a factor of $d\det(a_{ij})$. Then $\gamma_b^{P\mathfrak{g}}(\xi)$ is divisible by $(\xi-1)^{\frac{r-1}{2}\ell}$. Moreover, if b is not divisible by r, then*

$$\gamma_b^{P\mathfrak{g}}(\xi)\sim(\xi-1)^{\frac{r-1}{2}\ell}.$$

Proof. If b is divisible by r, then $\gamma_b^{P\mathfrak{g}}(\xi)=r^\ell\sim(\xi-1)^{\ell(r-1)}$, and we are done.

Suppose b is not divisible by r. Then ξ^b is a root of 1 of order r. Hence there is a Galois automorphism σ of the field $\mathbb{Z}(\xi)$ over $\mathbb{Q}$ such that $\sigma(\xi^b)=\exp(2\pi\mathrm{i}/r)$. Since $\sigma(\xi-1)\sim\xi-1$, it's enough to prove the lemma in the case $\xi^b=\exp(2\pi\mathrm{i}/r)$. In this case

$$\gamma_b^{P\mathfrak{g}}(\xi)=\sum_{\mu\in P_r\cap Y}\exp\left[\frac{\pi\mathrm{i}}{r}\left(|\mu+\rho|^2-|\rho|^2\right)\right].$$

Since r is odd, and $(\mu|\rho)\in\mathbb{Z}$, one has

$$|\mu+\rho|^2-|\rho|^2\equiv\left|\mu+(r+1)\rho\right|^2-(r+1)^2|\rho|^2\ (\mathrm{mod}\ 2r).$$

It follows that

$$\gamma_b^{P\mathfrak{g}}(\xi)=\exp\left[\frac{-\pi\mathrm{i}}{r}(r+1)^2|\rho|^2\right]\sum_{\mu\in P_r\cap Y}\exp\left[\frac{\pi\mathrm{i}}{r}\left(|\mu+(r+1)\rho|^2\right)\right].$$

Notice that $(r+1)\rho\in Y$ since $r+1$ is even, and use the rY-invariance, we have

$$\gamma_b^{P\mathfrak{g}}(\xi)=\exp\left[-\frac{\pi\mathrm{i}}{r}(r+1)^2|\rho|^2\right]\sum_{\mu\in P_r\cap Y}\exp\frac{\pi\mathrm{i}|\mu|^2}{r}.$$

The first factor is a unit in $\mathbb{Z}[\xi]$. If P is the matrix (d_ia_{ij}) (so that $(\alpha_i|\alpha_j)=P_{ij}$), then the second factor is

$$\sum_{\mu\in P_r\cap Y}\exp\frac{\pi\mathrm{i}|\mu|^2}{r}=\sum_{\vec{k}\in(\mathbb{Z}/r\mathbb{Z})^\ell}\exp\left[\frac{\pi\mathrm{i}}{r}\vec{k}^tP\vec{k}\right].$$

It is known that this Gauss sum is $\sim(\xi-1)^{\frac{r-1}{2}\ell}$. (This fact can be proved by diagonalizing the matrix P and use the value of the 1-variable Gauss sum. The matrix P is non-degenerate over $\mathbb{Z}/r\mathbb{Z}$ since $\det P$ and r are coprime.) □

Lemma 4.3.

(a) *Suppose b is an integer coprime with r. Let b^* be an integer such that $bb^* \equiv 1 \pmod r$. Then*

$$\sum_r \xi^{b\frac{|\mu|^2-|\rho^2|}{2}}\,\xi^{(\mu|\beta)} = \xi^{-b^*\beta^2/2}\,\gamma_b^{P\mathfrak{g}}(\xi). \tag{4.1}$$

(b) *Suppose r is an odd prime, then the left-hand side of* (4.1) *is divisible by* $(\xi-1)^{\frac{r-1}{2}\ell}$.

Proof. (a) The proof is similar to that of Lemma 2.2, using the trick of completing the square.

(b) If b is not divisible by r, then the statement follows from part (a) and Lemma 4.2. Suppose b is divisible by r. Then the *LHS* is either 0 or r^ℓ, which is $\sim(\xi-1)^{\ell(r-1)}$. □

4.3. Unknots and simple lens spaces. Let U_b be the unknot with framing b. We will first find the prime factors of $F_{U_b}^{P\mathfrak{g}}$.

Proposition 4.4.

(a) *Suppose $r \geqslant dh^\vee$ and is coprime with b. Let b^* be an integer such that $bb^* \equiv 1 \pmod r$. Then*

$$F_{U_b}^{P\mathfrak{g}}(\xi) = \frac{\xi^{(1-b^*)|\rho|^2}\,\gamma_b^{P\mathfrak{g}}(\xi)\,J_U(b^*\rho)}{\prod_{\alpha>0}(1-\xi^{(\alpha|\rho)})}. \tag{4.2}$$

(b) *If, in addition, r is an odd prime, then* $F_{U_b}^{P\mathfrak{g}}(\xi) \sim (\xi-1)^{(r\ell-\dim\mathfrak{g})/2}$.

Proof. (a) The proof is similar to that of (2.5): with $q=\xi$ in ψ,

$$\begin{aligned} F_{U_b}^{P\mathfrak{g}}(\xi) &= \frac{1}{|W|\,\psi^2}\sum_r \xi^{b\,\frac{|\mu|^2-|\rho|^2}{2}}\Bigl(\sum_{w\in W}\operatorname{sn}(w)\xi^{(\mu|w(\rho))}\Bigr)^2 \\ &= \frac{1}{|W|\,\psi^2}\sum_r \xi^{b\,\frac{|\mu|^2-|\rho|^2}{2}}\sum_{w,w'\in W}\operatorname{sn}(ww')\xi^{(\mu|w(\rho)+w'(\rho))}. \end{aligned}$$

Since $w(\rho)\in\rho+Y$ we have $w(\rho)+w'(\rho)\in 2\rho+Y=Y$. Using (4.1),

$$\begin{aligned} F_{U_b}^{P\mathfrak{g}}(\xi) &= \frac{1}{|W|\psi^2}\gamma_b^{P\mathfrak{g}}(\xi)\,\xi^{-b^*|\rho|^2}\sum_{w,w'\in W}\operatorname{sn}(ww')\xi^{-b^*(w(\rho)|w'(\rho))} \\ &= \frac{\gamma_b^{P\mathfrak{g}}\,\xi^{-b^*|\rho|^2}}{|W|\psi^2}|W|\sum_{w\in W}\operatorname{sn}(w)\xi^{(-b^*\rho|w(\rho))} \\ &= \frac{\gamma_b^{P\mathfrak{g}}(\xi)\,\xi^{-b^*|\rho|^2}}{\psi}J_U\bigl(-b^*\rho\bigr) \end{aligned}$$

$$= \frac{\gamma_b^{P\mathfrak{g}}(\xi)\, \xi^{(1-b^*)|\rho|^2} J_U(b^*\rho)}{\prod_{\alpha>0}(1-\xi^{(\alpha|\rho)})} \quad \text{by (1.4).}$$

(b) follows from part (a), Lemma 4.2, and the fact that $s = (\dim \mathfrak{g} - \ell)/2$. □

Corollary 4.5. *Suppose $r \geqslant dh^\vee$ is an odd prime, and b is not divisible by r. Then $\tau_M^{P\mathfrak{g}}(\xi) \sim 1$, i.e., $\tau_M^{P\mathfrak{g}}(\xi)$ is a unit in $\mathbb{Z}[\xi]$, for the lens space M obtained by surgery along U_b.*

Remark 4.5.1. The actual value of $\tau_M^{P\mathfrak{g}}(\xi)$, where M is obtained by surgery on U_b is (again here b is an integer not divisible by the odd prime r)

$$\tau_M^{P\mathfrak{g}}(\xi) = \left(\frac{|b|}{r}\right)^{\ell} \xi^{(\frac{\mathrm{sn}(b)-b}{2}|\rho|^2)^\sim} \prod_{\alpha>0} \frac{1-\xi^{-(b^*\rho|\alpha)}}{1-\xi^{-(\mathrm{sn}(b)\rho|\alpha)}}. \tag{4.3}$$

Here $(\frac{|b|}{r})$ is the Legendre symbol, $(\frac{x}{y})^\sim$ is the reduction modulo r, i.e., $(\frac{x}{y})^\sim = xy^*$.

4.4. Expansion of quantum link invariants.

Lemma 4.6. *For each positive integer N one has*

$$Q_L(\mu_1,\dots,\mu_m) = \sum_{n=0}^{N-1} p_n(\mu_1,\dots,\mu_m)(q-1)^n + R,$$

where R is in $\mathbb{Z}[q^{\pm 1}]$ and divisible by $(q-1)^N$, $p_n(\mu_1,\dots,\mu_m)$ is a polynomial function on $\mathfrak{h}_\mathbb{R}^*$ which takes integer values when $\mu_j \in Y$. Moreover the degree of p_n satisfies

$$\deg(p_n) \leqslant 2n + m(\dim \mathfrak{g} - \ell). \tag{4.4}$$

Proof. This follows easily from a counting argument in the theory of the Kontsevich integral, using the fact that J_L is obtained from the Kontsevich integral by substituting the Lie algebra into the chord diagrams (see [10,3]). The fact that p_k takes integer values when $\mu_1,\dots,\mu_m \in Y$ follows from the integrality of the coefficients of J_L. Let us briefly sketch the idea.

Expanding J_L using $q = e^\hbar$, with $\hbar$ a new variable, we get

$$J_L(\mu_1,\dots,\mu_m)|_{q=\exp\hbar} = \sum_{n=0}^{\infty} p'_n(\mu_1,\dots,\mu_m)\hbar^n,$$

where p'_n is a function on $(\mathfrak{h}_\mathbb{R}^*)^m$. The Kontsevich integral theory will show that p'_n is a polynomial function with degree at most $2n + \deg(\dim(\mu_1)) + \dots + \deg(\dim(\mu_m))$, where $\dim(\mu)$ is the function which gives the dimension of the module $\Lambda_{(\mu-\rho)}$. By Weyl formula, $\dim(\mu)$ is a polynomial function of degree s—the number of positive roots. Hence

$$\deg(p'_n) \leqslant 2n + ms.$$

Thus for $Q_L = J_L J_{U^m}$ we have

$$Q_L(\mu_1,\dots,\mu_m)|_{q=\exp\hbar} = \sum_{n=0}^{\infty} p''_n(\mu_1,\dots,\mu_m)\hbar^n,$$

where the degree of p_n'' is less than or equal to $2n + 2ms$.

Change the variable from $\hbar$ to $q - 1 = e^{\hbar} - 1$ (or $\hbar = \ln[(q-1)+1]$, we get

$$Q_L(\mu_1, \dots, \mu_m) = \sum_{k=0}^{\infty} p_n(\mu_1, \dots, \mu_m)(q-1)^n,$$

with $\deg(p_n) \leqslant 2n + 2ms$. It remains to notice that $s = (\dim \mathfrak{g} - \ell)/2$. □

4.5. A technical lemma.

Lemma 4.7. *Suppose r is an odd prime, p a polynomial function taking values in $\mathbb{Z}$ when $\mu_1, \dots, \mu_m \in Y$. Let*

$$x = \sum_{\mu_j \in \rho + (P_r \cap Y)} p(\mu_1, \dots, \mu_m)$$

and

$$y = (\xi - 1)^{\ell m \frac{r-1}{2} - \lfloor \frac{\deg p}{2} \rfloor},$$

where $\lfloor z \rfloor$ is the greatest integer less than or equal to z. Then $x/y \in \mathbb{Z}[\xi]$. (Note that y may not be in $\mathbb{Z}[\xi]$.)

Proof. In [8, Corollary 4.14] we proved that the quotient x/y is in $\mathbb{Z}[\xi, \frac{1}{(r-1)!}]$. But y is coprime with $(r-1)!$, hence the quotient must be in $\mathbb{Z}[\xi]$. □

4.6. Proof of the theorem. Let $N = m\frac{r\ell - \dim \mathfrak{g}}{2}$. Then the denominator of (3.1) is a factor of $(\xi - 1)^N$ by Proposition 4.2. We will prove that the numerator is divisible by $(\xi - 1)^N$.

Applying Lemma 4.6

$$Q_L(\mu_1, \dots, \mu_m) = \sum_{n=0}^{N-1} p_n(\mu_1, \dots, \mu_m)(q-1)^n + R.$$

We sum over $\mu_j \in \rho + (P_r \cap Y)$ to get $F_L^{P\mathfrak{g}}(\xi)$. The term involving R is certainly divisible by $(\xi - 1)^N$. For each n the term involving p_n, by Lemma 4.7 is divisible by

$$(\xi - 1)^{\ell m \frac{r-1}{2} - \lfloor \frac{\deg p_n}{2} \rfloor} \times (\xi - 1)^n,$$

which, by (4.4), is divisible by $(\xi - 1)^N$. This completes the proof of the theorem.

5. Perturbative expansion

5.1. General. Unlike the link case, quantum 3-manifold invariants can be defined only at roots of unity, i.e., the domain of the function $\tau_M^{\mathfrak{g}}(q)$ is the set of rational points on the unit circle in the complex plane $\mathbb{C}$. For many manifolds, eg the Poincare sphere or the Brieskorn sphere $\Sigma(2,3,7)$, there is no analytic extension of the function $\tau_M^{\mathfrak{g}}$ around $q = 1$. In perturbative theory, we want to expand the function $\tau_M^{\mathfrak{g}}$ around $q = 1$ into power series.

For rational homology 3-spheres, i.e., manifolds M with 0 rational homology, and for $\mathfrak{g} = sl_2$, Ohtsuki showed that there is a number-theoretic expansion of $\tau_M^{P\mathfrak{g}}$ around $q = 1$, see [16]. We established the same result for the case $\mathfrak{g} = sl_n$, see [8]. The proof in [8] is readily applied to any simple Lie algebra: In [8] we had to use some integrality properties of quantum link invariants and quantum 3-manifold invariants, and there we established these properties for the special case $\mathfrak{g} = sl_n$. For the general simple Lie algebras, these integrality properties are the results of [9] and Theorem 4.1.

5.2. The number-theoretic expansion. Suppose r is a big enough prime, and $\xi = \exp(2\pi \mathrm{i}/r)$. By the integrality (Theorem 4.1),

$$\tau_M^{P\mathfrak{g}}(\xi) \in \mathbb{Z}[\xi] = \mathbb{Z}[q]/\big(1 + q + q^2 + \cdots + q^{r-1}\big).$$

Choose a representative $f(q) \in \mathbb{Z}[q]$ of $\tau_M^{P\mathfrak{g}}(\xi)$. Formally substitute $q = e^{\hbar}$ in $f(q)$:

$$f(q) = c_{r,0} + c_{r,1}\hbar + \cdots + c_{r,n}\hbar^n + \cdots$$

The rational numbers $c_{r,n}$ depend on r and the representative $f(q)$. Their denominators must be a factors of $n!$, by Theorem 4.1. Hence if $n < r - 1$, we can reduce $c_{r,n}$ modulo r and get an element of $\mathbb{Z}/r\mathbb{Z}$. It is easy to see that these reductions $c_{r,n} \pmod r$ do not depend on the representative $f(q)$ and hence are invariants of the 3-manifolds. The dependence on r is a big drawback. The theorem below says that there is a number c_n, not depending on r, such that $c_{r,n} \pmod r$ is the reduction of c_n, or $-c_n$, modulo r, for sufficiently large prime r. It is easy to see that if such c_n exists, it must be unique.

Theorem 5.1. *For every rational homology* 3*-sphere* M, *there are a sequence of numbers* $c_n \in \mathbb{Z}[\frac{1}{(2n+2s)!|H_1(M,\mathbb{Z})|}]$, *such that for sufficiently large prime* r (*actually any prime* $r > \max(|H_1(M,\mathbb{Z}), \dim\mathfrak{g} - \ell)$ *is enough*),

$$c_{r,n} \equiv \left(\frac{|H_1(M,\mathbb{Z})|}{r}\right)^{\ell} c_n \pmod r,$$

where $(\frac{|H_1(M,\mathbb{Z})|}{r}) = \pm 1$ *is the Legendre symbol.*

The series $\mathfrak{t}_M^{P\mathfrak{g}}(\hbar) = \sum_{n=0}^{\infty} c_n \hbar^n$ can be considered as the perturbative expansion of the function $\tau_M^{P\mathfrak{g}}$ at $q = 1$. As mentioned above, the proof is just similar to the one for the case $\mathfrak{g} = sl_n$ in [8].

5.3. Some calculation. Let us describe here how to calculate the power series $\mathfrak{t}_M$, and sketch the ideas behind the calculation.

5.3.1. The $\mathfrak{g} = sl_2$ *and surgery on a knot case.* In this case let the positive integer N stand for the unique $\mathfrak{g}$-module of dimension N. The invariant $J_L(N_1, \ldots, N_m)$ is known as the colored Jones polynomial. Suppose M is obtained by surgery along a knot K with framing 1. Let K^0 be the same knot with framing 0. Then

$$Q_{K^0}(N)|_{q=e^{\hbar}} = \sum_{2 \leqslant j \leqslant n+2} c_{j,n} N^j \hbar^n. \tag{5.1}$$

The restriction $2j \leqslant n+2$ follows from the fact that K^0 has framing 0. It is known that there is no odd order of N: j must be even.

To obtain $\mathfrak{t}_M(\hbar)$, all one needs is to replace N^{2j} in (5.1) by $(-2)^j(2j-1)!!\,\hbar^{-j}$, then multiply by a universal constant:

$$\mathfrak{t}_M(\hbar) = z\sum c_{2j,n}(-2)^j(2j-1)!!\,\hbar^{n-j},$$

where $z = z^{sl_2} = (1-q)/2 = (1-e^{\hbar})/2$.

Presumably this was first obtained by Rozansky [17].

5.3.2. The case of general simple Lie algebra and surgery along a knot. Again assume that M is obtained by surgery along the knot K with framing 1, and K^0 is the same knot with framing 0. It is known that every polynomial function $p(\mu)$ on $\mathfrak{h}^*_{\mathbb{R}}$ are linear combinations of functions of the form β^j, $\beta \in Y$. Here $\beta^j(\mu) := (\beta|\mu)^j$. Thus one has

$$Q_{K^0}(\mu)|_{q=e^{\hbar}} = \sum_{2s\leqslant j\leqslant n+2s,\ \beta\in Y} c_{\beta;\,j;n}\beta^j(\mu)\,\hbar^n. \tag{5.2}$$

Here for each degree n the sum is finite. Again the restriction $j \leqslant 2s+n$ comes from the fact that K^0 has framing 0.

To obtain $\mathfrak{t}_M(h)$, all one needs is to replace $\beta^j(\mu)$ in (5.2) by 0 if j is odd, $\beta^{2j}(\mu)$ by

$$(2j-1)!!\,\hbar^{-j}\big(-|\beta|^2\big)^j, \tag{5.3}$$

then multiply by a universal constant:

$$\mathfrak{t}_M^{P\mathfrak{g}}(\hbar) = \sum c_{\beta;2j;n}(2j-1)!!\,\big(-|\beta|^2\big)^j\,\hbar^{n-j} \times \frac{1}{|W|}\prod_{\alpha>0}\big(1-q^{(\alpha|\rho)}\big). \tag{5.4}$$

5.3.3. A sketch of the main idea. The main idea is to separate the framing part, and consider the sum $\sum_r$ as a discreet Gauss integral. This was first used by Rozansky (for sl_2) in his series of important work on quantum invariants.

Recall how we define $\tau_M^{P\mathfrak{g}}(\xi)$. One gets Q_K by multiplying Q_{K^0} by $q^{(|\mu|^2-|\rho|^2)/2}$. Summing Q_K over $\mu \in \rho + (P_r \cap Y)$, we get F_K. Then we have to divide F_K by F_{U_+}. The result is $\tau_M^{P\mathfrak{g}}(\xi)$. A look at formula (5.2) shows that if we understand the perturbative expansion of

$$\frac{\sum_r q^{\frac{|\mu|^2-|\rho|^2}{2}}\beta^j(\mu)}{F_{U_+}}, \tag{5.5}$$

then we will know the perturbative expansion of $\tau_M^{P\mathfrak{g}}$.

If we replace $\beta^j(\mu) = (\beta|\mu)^j$ in (5.5) by $q^{\beta}(\mu) := q^{(\beta|\mu)}$, then the perturbative expansion is easy to calculate:

$$\begin{aligned}\frac{\sum_r \xi^{\frac{|\mu|^2-|\rho|^2}{2}}\xi^{(\beta|\mu)}}{F_{U_+}} &= \frac{1}{F_{U_+}}\gamma^{P\mathfrak{g}}(\xi)\,\xi^{-|\beta|^2/2} \qquad \text{by (4.1)}\\ &= \xi^{-|\beta|^2/2}\prod_{\alpha>0}\big(1-\xi^{(\alpha|\rho)}\big) \quad \text{by (4.2)}.\end{aligned} \tag{5.6}$$

Thus the perturbative expansion of the left-hand side of (5.6) is $q^{-|\beta|^2/2}z$, with $z = \prod_{\alpha>0}(1-q^{(\alpha|\rho)})$ and $q = e^{\hbar}$.

Now if we expand $q^{(\beta|\mu)} = \exp[\hbar(\beta|\mu)]$, we can see the term $(\beta|\mu)^j$ there:

$$\exp\bigl[\hbar(\beta|\mu)\bigr] = \sum_{j\geqslant 0}\frac{\hbar^j(\beta|\mu)^j}{j!}.$$

To obtain the perturbative expansion of (5.5), we expand $q^{-|\beta|^2/2}z$ into power series of $\hbar$, and keep only the part of degree j in μ. It is easy to see that if j is odd, there is no part of degree j, and if j is even, then the part of degree j is given by the formula (5.3). (In this argument we consider μ as a variable. To be more precise, one replace μ by $t\mu$, with $t\in\mathbb{R}$ a variable, then compare the terms of same degree of t.)

5.3.4. Special lens spaces. Suppose M is obtained by surgery on the unknot U_b, with $b\neq 0$. Then from (4.3) it follows that

$$\mathfrak{t}_M^{P\mathfrak{g}}(\hbar) = q^{\frac{\mathrm{sn}(b)-b}{2}|\rho|^2}\prod_{\alpha>0}\frac{1-q^{-(\rho|\alpha)/b}}{1-q^{-\mathrm{sn}(b)(\rho|\alpha)}}\,\Big|_{q=\exp\hbar}.$$

5.3.5. Link with diagonal linking matrix. Suppose L is a framing link whose linking matrix is diagonal, with non-zero integers $b_1,\dots,b_m$ on the diagonal. Let L^0 be the same link with 0 framing, and M the 3-manifold obtained by surgery along L, which is a rational homology 3-sphere. Expanding $q=e^{\hbar}$ in Q_{L^0} we get

$$\begin{aligned}&Q_{L^0}(\mu_1,\dots,\mu_m)|_{q=e^{\hbar}}\\ &\quad=\sum_{\beta_1,\dots,\beta_m\in Y;\ j_1,\dots,j_m\in Z_+;\ n\in\mathbb{Z}_+} c_{\beta_1,\dots,\beta_m;j_1,\dots,j_m;n}\,\beta_1^{j_1}(\mu_1)\cdots\beta_m^{j_m}(\mu_m)\,\hbar^n.\end{aligned}$$

There are some restrictions on β_i, j_i, for a fixed n. Then to obtain $\mathfrak{t}_M^{P\mathfrak{g}}(\hbar)$ one needs to replace $\beta_i^j(\mu_i)$ by 0 if j is odd, $\beta_i^{2j}(\mu_i)$ by

$$z_{b_i}\,b_i^{-j}(2j-1)!!\,\bigl(-|\beta_i|^2\bigr)^j\,\hbar^{-j},$$

where

$$z_{b_i} = \frac{1}{|W|}\,q^{\frac{|\rho|^2}{2}(\mathrm{sn}(b)-b)}\prod_{\alpha>0}\bigl(1-q^{\mathrm{sn}(b)(\alpha|\rho)}\bigr).$$

Thus,

$$\begin{aligned}\mathfrak{t}_M^{P\mathfrak{g}}(\hbar) \;=\; & z_{b_1}\cdots z_{b_m}\sum c_{\beta_1,\dots,\beta_m;\,2j_1,\dots,2j_m;n}\\ &\times\prod_{i=1}^m(2j_i-1)!!\left(\frac{-|\beta_i|^2}{b_i}\right)^{j_i}\hbar^{n-j_1-\cdots-j_m}.\end{aligned}$$

The restriction on $j_1,\dots,j_m$ will guarantee that the right-hand side is a formal power series in h.

5.3.6. General case. Suppose now M is an arbitrary rational homology 3-sphere. Ohtsuki showed that there are lens spaces $M_1, \ldots, M_l$, each obtained by surgery on an unknot with non-zero framing, such that $M' = M \# M_1 \# \cdots \# M_l$ can be obtained surgery along a link with diagonal linking matrix, see [16]. Then one has

$$\mathfrak{t}_M^{P\mathfrak{g}}(\hbar) = \mathfrak{t}_{M'}^{P\mathfrak{g}}(\hbar)\big(\mathfrak{t}_{M_1}^{P\mathfrak{g}}(\hbar)\big)^{-1} \cdots \big(\mathfrak{t}_{M_1}^{P\mathfrak{g}}(\hbar)\big)^{-1}.$$

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