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Stable extendibility of vector bundles over RP^n and the stable splitting problem

Yutaka Hemmi [a,*], Teiichi Kobayashi [b]

[a] Department of Mathematics, Faculty of Science, Kochi University, Kochi 780-8520, Japan
[b] Asakura-ki 292-21, Kochi 780-8066, Japan

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ABSTRACT

Let F be the real number field R or the complex number field C, and let RP^n denote the real projective n-space. In this paper, we study the conditions for a given F-vector bundle over RP^n to be stably extendible to RP^m for every $m > n$, and establish the formulas on the power $\zeta^r = \zeta \otimes \cdots \otimes \zeta$ (r-fold) of an F-vector bundle ζ over RP^n. Our results are improvements of the previous papers [T. Kobayashi, H. Yamasaki, T. Yoshida, The power of the tangent bundle of the real projective space, its complexification and extendibility, Proc. Amer. Math. Soc. 134 (2005) 303–310] and [Y. Hemmi, T. Kobayashi, Min Lwin Oo, The power of the normal bundle associated to an immersion of RP^n, its complexification and extendibility, Hiroshima Math. J. 37 (2007) 101–109]. Furthermore, we answer the stable splitting problem for F-vector bundles over RP^n by means of arithmetic conditions.

1. Introduction

Let F be either the real number field R or the complex number field C, and let X be a space and A its subspace. A t-dimensional F-vector bundle ζ over A is said to be stably extendible (respectively extendible) to X if and only if there is a t-dimensional F-vector bundle over X whose restriction to A is stably equivalent (respectively equivalent) to ζ (cf. [4,10]). For simplicity, we use the same letter for a vector bundle and its equivalence class.

In this paper, we study the problem of determining conditions for a given F-vector bundle over RP^n to be stably extendible to RP^m for every $m \geqslant n$. In case $F = R$, the answers for the problem have been obtained when ζ is the power τ^r of the tangent bundle $\tau = \tau(RP^n)$ of RP^n [8] and when ζ is the power ν^r of the normal bundle ν associated to an immersion of RP^n in euclidean space in [2]. These results are as follows.

Let $\otimes$ denote the tensor product and $\phi(n)$ the number of integers q such that $0 < q \leqslant n$ and $q \equiv 0, 1, 2$ or 4 mod 8.

Theorem 1.1. *(Cf. [8, Theorem A].) Let $\tau^r = \tau(RP^n) \otimes \cdots \otimes \tau(RP^n)$ be the r-fold power of the tangent bundle $\tau(RP^n)$. Then τ^r is stably extendible to RP^m for every $m \geqslant n$ if and only if there is an integer x satisfying*

$$(n+2)^r - n^r \leqslant x2^{\phi(n)+1} \leqslant (n+2)^r + n^r.$$

* Corresponding author.
E-mail addresses: hemmi@math.kochi-u.ac.jp (Y. Hemmi), kteiichi@lime.ocn.ne.jp (T. Kobayashi).

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Theorem 1.2. *(Cf. [2, Theorem 3.2].) Let $\nu^r = \nu \otimes \cdots \otimes \nu$ be the r-fold power of the normal bundle ν associated to an immersion of RP^n in euclidean $(n+k)$-space R^{n+k}, where $k > 0$. Then ν^r is stably extendible to RP^m for every $m \geqslant n$ if and only if there is an integer x satisfying*

$$(2n+k+2)^r - k^r \leqslant x2^{\phi(n)+1} \leqslant (2n+k+2)^r + k^r.$$

The first purpose of this paper is to obtain the complete answer for any R-vector bundle over RP^n. Let ξ_n be the canonical line bundle over RP^n. Then, for any R-vector bundle ζ over RP^n, there is an integer s such that ζ is stably equivalent to $s\xi_n$ (cf. [1, Theorem 7.4]). We have

Theorem A. *Let ζ be a t-dimensional R-vector bundle over RP^n which is stably equivalent to $s\xi_n$, where s is an integer. Then ζ is stably extendible to RP^m for every $m \geqslant n$ if and only if there is an integer a satisfying*

$$-s \leqslant a2^{\phi(n)} \leqslant t - s.$$

As an application to the r-fold power, we have

Theorem B. *Let ζ be a t-dimensional R-vector bundle over RP^n which is stably equivalent to $s\xi_n$, where s is an integer, and let $\zeta^r = \zeta \otimes \cdots \otimes \zeta$ be the r-fold power of ζ. Then ζ^r is stably extendible to RP^m for every $m \geqslant n$ if and only if there is an integer a satisfying*

$$(t-2s)^r - t^r \leqslant a2^{\phi(n)+1} \leqslant (t-2s)^r + t^r.$$

Theorem B is an improvement of Theorem 1.1. In fact, for the tangent bundle $\tau = \tau(RP^n)$, we have $s = n+1$ and $t = n$. Hence we obtain the inequalities of Theorem 1.1 by using x instead of a in the inequalities of Theorem B if r is even, and $-x$ instead of a in the inequalities of Theorem B if r is odd. Furthermore, this is also an improvement of Theorem 1.2. In fact, for the normal bundle associated to an immersion of RP^n in R^{n+k}, we have $s = -n-1$ and $t = k$. Hence we obtain the inequalities of Theorem 1.2 by using x instead of a in the inequalities of Theorem B.

In case $F = C$, the answers for the problem have been obtained when ζ is the complexification $c\tau^r$ of the power τ^r in [8] and when ζ is the complexification $c\nu^r$ of the power ν^r in [2]. These results are as follows.

For a real number z, let $[z]$ denote the largest integer n with $n \leqslant z$.

Theorem 1.3. *(Cf. [8, Theorem B].) Let $c\tau^r = c(\tau(RP^n) \otimes \cdots \otimes \tau(RP^n))$ be the complexification of the r-fold power τ^r of the tangent bundle $\tau(RP^n)$. Then $c\tau^r$ is stably extendible to RP^m for every $m \geqslant n$ if and only if there is an integer y satisfying*

$$(n+2)^r - n^r \leqslant y2^{[n/2]-1} \leqslant (n+2)^r + n^r.$$

Theorem 1.4. *(Cf. [2, Theorem 5.2].) Let $c\nu^r = c(\nu \otimes \cdots \otimes \nu)$ be the complexification of the r-fold power ν^r of the normal bundle ν associated to an immersion of RP^n in R^{n+k}, where $k > 0$. Then $c\nu^r$ is stably extendible to RP^m for every $m \geqslant n$ if and only if there is an integer y satisfying*

$$(2n+k+2)^r - k^r \leqslant y2^{[n/2]+1} \leqslant (2n+k+2)^r + k^r.$$

The second purpose of this paper is to obtain the complete answer for any C-vector bundle over RP^n. Let $c\xi_n$ be the complexification of the canonical line bundle over RP^n. Then, for any C-vector bundle ζ over RP^n, there is an integer s such that ζ is stably equivalent to $sc\xi_n$ (cf. [9, Theorem 3.8]). We have

Theorem C. *Let ζ be a t-dimensional C-vector bundle over RP^n which is stably equivalent to $sc\xi_n$, where s is an integer. Then ζ is stably extendible to RP^m for every $m \geqslant n$ if and only if there is an integer b satisfying*

$$-s \leqslant b2^{[n/2]} \leqslant t - s.$$

As an application to the r-fold power, we have

Theorem D. *Let ζ be a t-dimensional C-vector bundle over RP^n which is stably equivalent to $sc\xi_n$, where s is an integer, and let $\zeta^r = \zeta \otimes \cdots \otimes \zeta$ be the r-fold power of ζ. Then ζ^r is stably extendible to RP^m for every $m \geqslant n$ if and only if there is an integer b satisfying*

$$(t-2s)^r - t^r \leqslant b2^{[n/2]+1} \leqslant (t-2s)^r + t^r.$$

As in the previous case, Theorem D is an improvement of Theorems 1.3 and 1.4.

Finally, we study the problem of determining the conditions for a given t-dimensional F-vector bundle over RP^n to be stably equivalent to a sum of t F-line bundles over RP^n, where $F = R$ or C. This problem is the stable splitting problem for F-vector bundles over RP^n. We answer the problem by arithmetic conditions.

For $F = R$, combining Theorem 1 of [5] with Theorem A, we have

Theorem E. *Let ζ be a t-dimensional R-vector bundle over RP^n which is stably equivalent to $s\xi_n$, where s is an integer. Then ζ is stably equivalent to a sum of t R-line bundles over RP^n if and only if there is an integer a satisfying*

$$-s \leqslant a2^{\phi(n)} \leqslant t - s.$$

For $F = C$, combining Theorem 2 of [5] with Theorem C, we have

Theorem F. *Let ζ be a t-dimensional C-vector bundle over RP^n which is stably equivalent to $sc\xi_n$, where s is an integer. Then ζ is stably equivalent to a sum of t C-line bundles over RP^n if and only if there is an integer b satisfying*

$$-s \leqslant b2^{[n/2]} \leqslant t - s.$$

This paper is arranged as follows. In Section 2 we prove Theorem A, establish the formula in $KO(RP^n)$ on the r-fold power ζ^r of the R-vector bundle ζ over RP^n, and prove Theorem B. In Section 3 we prove Theorem C, establish the formula in $K(RP^n)$ on the r-fold power ζ^r of the C-vector bundle ζ over RP^n, and prove Theorem D. In Section 4 we study the stable splitting problem for F-vector bundles over RP^n.

2. Proofs of Theorems A and B

We recall the following result on stable non-extendibility of an R-vector bundle over RP^n.

Theorem 2.1. *(Cf. [6, Theorem 4.1].) Let α be a k-dimensional R-vector bundle over RP^n. Assume that there is a positive integer ℓ such that α is stably equivalent to $(k+\ell)\xi_n$ and $k+\ell < 2^{\phi(n)}$. Then $n < k+\ell$ and α is not stably extendible to RP^m for every $m \geqslant k+\ell$.*

Proof of Theorem A. The proof of the "if" part: By the assumption we have $\zeta = s\xi_n + t - s$ in $KO(RP^n)$. By Theorem 7.4 of [1] the equality $a2^{\phi(n)}(\xi_n - 1) = 0$ holds in $\widetilde{KO}(RP^n)$ for any integer a. Hence we obtain the equality

$$\zeta = \left(a2^{\phi(n)} + s\right)\xi_n + t - s - a2^{\phi(n)}$$

in $KO(RP^n)$. Set $X = a2^{\phi(n)} + s$ and $Y = t - s - a2^{\phi(n)}$. Then we may take a so that $X \geqslant 0$ and $Y \geqslant 0$ by the assumption, and $\zeta = X\xi_n + Y$ in $KO(RP^n)$. Since the Whitney sum $X\xi_n \oplus Y$ is extendible to RP^m for every $m \geqslant n$, ζ is stably extendible to RP^m for every $m \geqslant n$.

The proof of the "only if" part: We prove the contraposition. Assume that every integer a satisfies

$$a2^{\phi(n)} < -s \quad \text{or} \quad t - s < a2^{\phi(n)}.$$

Let A be the maximum integer such that $A2^{\phi(n)} < -s$. Then, since $(A+1)2^{\phi(n)} \geqslant -s$, we have $t - s < (A+1)2^{\phi(n)}$ by the assumption. Put $\alpha = \zeta$, $k = t$ and $\ell = (A+1)2^{\phi(n)} - t + s$ in Theorem 2.1. Then $\ell > 0$, $k + \ell = (A+1)2^{\phi(n)} + s < 2^{\phi(n)}$ and $(k+\ell)\xi_n = \{(A+1)2^{\phi(n)} + s\}\xi_n = s\xi_n + (A+1)2^{\phi(n)}$ in $KO(RP^n)$ by Theorem 7.4 of [1]. Hence we see that $n < (A+1)2^{\phi(n)} + s$ and that ζ is not stably extendible to RP^m for every $m \geqslant (A+1)2^{\phi(n)} + s$. □

In the next theorem we establish the formula in $KO(RP^n)$ on the power ζ^r of ζ.

Theorem 2.2. *Let ζ be a t-dimensional R-vector bundle over RP^n which is stably equivalent to $s\xi_n$, where s is an integer, and let $\zeta^r = \zeta \otimes \cdots \otimes \zeta$ be the r-fold power of ζ. Then the following holds in $KO(RP^n)$.*

$$\zeta^r = -2^{-1}\left\{(t-2s)^r - t^r\right\}\xi_n + 2^{-1}\left\{(t-2s)^r + t^r\right\}.$$

Proof. Since $\zeta = s\xi_n + t - s$ in $KO(RP^n)$, the equality clearly holds for $r = 1$.

Assume that the equality holds for $r \geqslant 1$. Then

$$\begin{aligned}
\zeta^{r+1} &= \zeta \otimes \zeta^r \\
&= (s\xi_n + t - s)\left[-2^{-1}\left\{(t-2s)^r - t^r\right\}\xi_n + 2^{-1}\left\{(t-2s)^r + t^r\right\}\right] \\
&= -2^{-1}\left\{(t-2s)^{r+1} - t^{r+1}\right\}\xi_n + 2^{-1}\left\{(t-2s)^{r+1} + t^{r+1}\right\}
\end{aligned}$$

since $\xi_n \otimes \xi_n = 1$. Hence the desired equality holds for any positive integer r by induction on r. □

Theorem 2.2 is an improvement of Lemma 2.1 of [8] and Theorem 2.1 of [2].

Proof of Theorem B. The dimension of ζ^r is t^r and, by Theorem 2.2, ζ^r is stably equivalent to $2^{-1}\{t^r - (t-2s)^r\}\xi_n$. Hence the result follows from Theorem A. $\square$

Using Theorem 2.2, we have the next theorem that is an improvement of Theorem 2.4 of [8] and Theorem 2.2 of [2].

Theorem 2.3. *Under the assumption of Theorem 2.2, the following holds in $KO(RP^n)$ for any integer a.*

$$\zeta^r = 2^{-1}\{a2^{\phi(n)+1} - (t-2s)^r + t^r\}\xi_n + 2^{-1}\{(t-2s)^r + t^r - a2^{\phi(n)+1}\}.$$

Proof. Adding $a2^{\phi(n)}(\xi_n - 1) = 0$ (cf. [1, Theorem 7.4]) to the equality in Theorem 2.2, we have the desired equality. $\square$

Using Theorem 2.3, we have the next theorem that is an improvement of Theorem 2.3 of [2].

Theorem 2.4. *Assume that there is an integer a satisfying the inequalities of Theorem* B. *Then, under the assumption of Theorem* 2.2, *the Whitney sum decomposition*

$$\zeta^r = 2^{-1}\{a2^{\phi(n)+1} - (t-2s)^r + t^r\}\xi_n \oplus 2^{-1}\{(t-2s)^r + t^r - a2^{\phi(n)+1}\}$$

holds as R-vector bundles if $n < t^r$.

Proof. Set $X = 2^{-1}\{a2^{\phi(n)+1} - (t-2s)^r + t^r\}$ and $Y = 2^{-1}\{(t-2s)^r + t^r - a2^{\phi(n)+1}\}$. Then, by the assumption, $X \geqslant 0$ and $Y \geqslant 0$, and, by Theorem 2.3, $\zeta^r = X\xi_n + Y$ in $KO(RP^n)$. If $n(= \dim RP^n) < t^r(= \dim \zeta^r = \dim(X\xi_n \oplus Y))$, then we have $\zeta^r = X\xi_n \oplus Y$ as R-vector bundles (cf. [3, Theorem 1.5, p. 100]). $\square$

As for extendibility, we have

Theorem 2.5. *In addition to the assumption of Theorem* B, *assume that $n < t^r$. Then ζ^r is extendible to RP^m for every $m \geqslant n$ if and only if there is an integer a satisfying the inequalities of Theorem* B.

Proof. By Theorem 2.2 of [7], for $m \geqslant n$, ζ^r is extendible to RP^m if and only if ζ^r is stably extendible to RP^m, provided $n < t^r$. Hence the result follows from Theorem B. $\square$

This result is an improvement of the results on extendibility obtained from [8, Theorem A] and [2, Theorem A].

3. Proofs of Theorems C and D

We recall the following result on stable non-extendibility of a C-vector bundle over RP^n.

Theorem 3.1. *(Cf. [6, Theorem 2.1].) Let α be a k-dimensional C-vector bundle over RP^n. Assume that there is a positive integer ℓ such that α is stably equivalent to $(k+\ell)c\xi_n$ and $k + \ell < 2^{[n/2]}$. Then $n < 2k + 2\ell$ and α is not stably extendible to RP^m for every $m \geqslant 2k + 2\ell$.*

Proof of Theorem C. The proof of the "if" part: By the assumption we have $\zeta = sc\xi_n + t - s$ in $K(RP^n)$. By Theorem 3.8 of [9] the equality $b2^{[n/2]}(c\xi_n - 1) = 0$ holds in $\widetilde{K}(RP^n)$ for any integer b. Hence we obtain the equality

$$\zeta = \big(b2^{[n/2]} + s\big)c\xi_n + t - s - b2^{[n/2]}$$

in $K(RP^n)$. Set $V = b2^{[n/2]} + s$ and $W = t - s - b2^{[n/2]}$. Then we may take b so that $V \geqslant 0$ and $W \geqslant 0$ by the assumption, and $\zeta = Vc\xi_n + W$ in $K(RP^n)$. Since the Whitney sum $Vc\xi_n \oplus W$ is extendible to RP^m for every $m \geqslant n$, ζ is stably extendible to RP^m for every $m \geqslant n$.

The proof of the "only if" part: We prove the contraposition. Assume that every integer b satisfies

$$b2^{[n/2]} < -s \quad \text{or} \quad t - s < b2^{[n/2]}.$$

Let B be the maximum integer such that $B2^{[n/2]} < -s$. Then, since $(B+1)2^{[n/2]} \geqslant -s$, we have $t - s < (B+1)2^{[n/2]}$ by the assumption. Put $\alpha = \zeta$, $k = t$ and $\ell = (B+1)2^{[n/2]} - t + s$ in Theorem 3.1. Then $\ell > 0$, $k + \ell = (B+1)2^{[n/2]} + s < 2^{[n/2]}$ and $(k+\ell)c\xi_n = \{(B+1)2^{[n/2]} + s\}c\xi_n = sc\xi_n + (B+1)2^{[n/2]}$ in $K(RP^n)$ by Theorem 3.8 of [9]. Hence we see that $n < (B+1)2^{[n/2]+1} + 2s$ and that ζ is not stably extendible to RP^m for every $m \geqslant (B+1)2^{[n/2]+1} + 2s$. $\square$

In the next theorem we establish the formula in $K(RP^n)$ on the power ζ^r of ζ.

Theorem 3.2. *Let ζ be a t-dimensional C-vector bundle over RP^n which is stably equivalent to $sc\xi_n$, where s is an integer, and let $\zeta^r = \zeta \otimes \cdots \otimes \zeta$ be the r-fold power of ζ. Then the following holds in $K(RP^n)$.*

$$\zeta^r = -2^{-1}\{(t-2s)^r - t^r\}c\xi_n + 2^{-1}\{(t-2s)^r + t^r\}.$$

Proof. Since $\zeta = sc\xi_n + t - s$ in $K(RP^n)$ and since $c\xi_n \otimes c\xi_n = c(\xi_n \otimes \xi_n) = 1$, the proof is parallel to that of Theorem 2.2. □

Theorem 3.2 is an improvement of Lemma 4.1 of [8] and Theorem 4.1 of [2].

Proof of Theorem D. The dimension of ζ^r is t^r and, by Theorem 3.2, ζ^r is stably equivalent to $2^{-1}\{t^r - (t-2s)^r\}c\xi_n$. Hence the result follows from Theorem C. □

Using Theorem 3.2, we have the next theorem that is an improvement of Theorem 4.3 of [8] and Theorem 4.2 of [2].

Theorem 3.3. *Under the assumption of Theorem 3.2, the following holds in $K(RP^n)$ for any integer b.*

$$\zeta^r = 2^{-1}\{b2^{[n/2]+1} - (t-2s)^r + t^r\}c\xi_n + 2^{-1}\{(t-2s)^r + t^r - b2^{[n/2]+1}\}.$$

Proof. Adding $b2^{[n/2]}(c\xi_n - 1) = 0$ (cf. [9, Theorem 3.8]) to the equality in Theorem 3.2, we have the desired equality. □

Using Theorem 3.3, we have the next theorem that is an improvement of Theorem 4.3 of [2].

Theorem 3.4. *Assume that there is an integer b satisfying the inequalities of Theorem D. Then, under the assumption of Theorem 3.2, the Whitney sum decomposition*

$$\zeta^r = 2^{-1}\{b2^{[n/2]+1} - (t-2s)^r + t^r\}c\xi_n \oplus 2^{-1}\{(t-2s)^r + t^r - b2^{[n/2]+1}\}$$

holds as C-vector bundles if $n/2 \leqslant t^r$.

Proof. Set $V = 2^{-1}\{b2^{[n/2]+1} - (t-2s)^r + t^r\}$ and $W = 2^{-1}\{(t-2s)^r + t^r - b2^{[n/2]+1}\}$. Then, by the assumption, $V \geqslant 0$ and $W \geqslant 0$, and, by Theorem 3.3, $\zeta^r = Vc\xi_n + W$ in $K(RP^n)$. If $\langle n/2\rangle (= \langle(\dim RP^n)/2\rangle) \leqslant t^r (= \dim \zeta^r = \dim(Vc\xi_n \oplus W))$, then we have $\zeta^r = Vc\xi_n \oplus W$ as C-vector bundles (cf. [3, Theorem 1.5, p. 100]), where $\langle x\rangle$ denotes the smallest integer q with $x \leqslant q$. Since t^r is an integer, the condition $\langle n/2\rangle \leqslant t^r$ is equivalent to $n/2 \leqslant t^r$. Thus, we have the desired result. □

As for extendibility, we have

Theorem 3.5. *In addition to the assumption of Theorem D, assume that $n/2 \leqslant t^r$. Then ζ^r is extendible to RP^m for every $m \geqslant n$ if and only if there is an integer b satisfying the inequalities of Theorem D.*

Proof. By Theorem 2.3 of [7], for $m \geqslant n$, ζ^r is extendible to RP^m if and only if ζ^r is stably extendible to RP^m, provided $\langle n/2\rangle \leqslant t^r$, which is equivalent to $n/2 \leqslant t^r$ since t^r is an integer. Hence the result follows from Theorem D. □

This result is an improvement of the results on extendibility obtained from [8, Theorem B] and [2, Theorem B].

4. The stable splitting problem for vector bundles over RP^n

For a positive integer i write $i = q2^{\nu(i)}$, where q is some odd integer, and define, for a positive integer k,

$$\beta(k) = \min\{i - \nu(i) - 1 \mid k < i\}.$$

In [5], we call $\beta(k)$ the Schwarzenberger number.

For $F = R$, the following theorem is proved by Kobayashi and Yoshida.

Theorem 4.1. *(See [5, Theorem 1].) Let ζ be a t-dimensional R-vector bundle over RP^n, where $t > 0$, and consider the following four conditions.*

(1) *ζ is stably extendible to RP^m for every $m \geqslant n$.*
(2) *ζ is stably extendible to RP^m, where $m \geqslant n$, $m \geqslant 2t-1$ and $\phi(m) \geqslant \phi(n) + \beta(t)$.*
(3) *ζ is stably extendible to RP^m, where $m = 2^{\phi(n)} - 1$.*
(4) *ζ is stably equivalent to a sum of t R-line bundles over RP^n.*

Then all the four conditions are equivalent. Moreover, when $t = 1$ or $n = 1, 3$ or 7, the conditions always hold.

Combining the above theorem with Theorem A, we have

Theorem 4.2. *Let ζ be a t-dimensional R-vector bundle over RP^n, where $t > 0$. Then each condition in Theorem 4.1 is equivalent to that there is an integer a satisfying $-s \leqslant a2^{\phi(n)} \leqslant t-s$, where $\zeta = s\xi_n + t - s$ in $KO(RP^n)$. Moreover, when $t = 1$ or $n = 1, 3$ or 7, this condition always holds.*

For $F = C$, the following theorem is also proved by Kobayashi and Yoshida.

Theorem 4.3. *(See [5, Theorem 2].) Let ζ be a t-dimensional C-vector bundle over RP^n, where $t > 0$, and consider the following four conditions.*

(1) *ζ is stably extendible to RP^m for every $m \geqslant n$.*
(2) *ζ is stably extendible to RP^m, where $m \geqslant n$, $m \geqslant 4t-1$ and $\phi(m) \geqslant [n/2] + \beta(2t) + 1$.*
(3) *ζ is stably extendible to RP^m, where $m = 2^{[n/2]+1} - 1$.*
(4) *ζ is stably equivalent to a sum of t C-line bundles over RP^n.*

Then all the four conditions are equivalent. Moreover, when $t = 1$ or $n = 1, 2$ or 3, the conditions always hold.

Combining the above theorem with Theorem C, we have

Theorem 4.4. *Let ζ be a t-dimensional C-vector bundle over RP^n, where $t > 0$. Then each condition in Theorem 4.3 is equivalent to that there is an integer b satisfying $-s \leqslant b2^{[n/2]} \leqslant t-s$, where $\zeta = sc\xi_n + t - s$ in $K(RP^n)$. Moreover, when $t = 1$ or $n = 1, 2$ or 3, this condition always holds.*

Theorems E and F are contained in Theorems 4.2 and 4.4 respectively.

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