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Inertial Manifolds for Burgers' Original Model System of Turbulence

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Abstract—The existence of inertial manifolds for Burgers' original mathematical model system of turbulence is investigated. The system consists of two equations and enjoys the characteristic quantity: the Reynolds number. Our object in this article is to express the existence in terms of this Reynolds number. The difficulty of first order derivatives is circumvented by the method originally due to M. Kwak.

Keywords—Inertial manifolds, Burgers' system, Reynolds number.

SECTION 1

For the study of the long time dynamics of dissipative nonlinear partial equations, the theory of inertial manifolds has recently called considerable attention. After the pioneering work of C. Foias, G.-R. Sell and R. Temam [1], many authors discussed this topic and much progress has been made. See, for instance, [2–13] and the references therein.

Let H be a Hilbert space and $\{S(t)\}_{t\geq 0}$ the semigroup operators associated to the equation under consideration. Then, we recall that a set $\mathcal{M} \subset H$ is an inertial manifold for the equation if

(1) $\mathcal{M}$ is a finite-dimensional Lipschitz manifold;
(2) $S(t)\mathcal{M} \subset \mathcal{M}$, for all $t \geq 0$;
(3) $\mathcal{M}$ attracts all the orbits with exponential rate.

When $\mathcal{M}$ exists, we see that the long time dynamics of the system can be described by the finite-dimensional dynamics on $\mathcal{M}$, and in particular, $\mathcal{M}$ contains the global attractor.

In this note, we deal with the existence of inertial manifolds for the following physical system derived by J. M. Burgers [14] as a model study of turbulent fluid motion:

$$\left.\begin{aligned} L\,\frac{dU}{dt} &= P - \frac{\nu}{L}U - \frac{1}{L}\int_0^L v^2\,dx, \\ \frac{\partial v}{\partial t} &= \frac{1}{L}Uv + \nu\,\frac{\partial^2 v}{\partial x^2} - 2v\,\frac{\partial v}{\partial x}, \end{aligned}\right\} \qquad \text{in } (x,t) \in [0,L] \times \mathbb{R}^+, \tag{1}$$

with

$$v(0,t) = v(L,t) = 0, \qquad \text{for } t \geq 0,$$

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX where U and v denote velocities corresponding to mean and turbulent motion, respectively. P and ν represent the external force and a kinetic viscosity, respectively. U is independent of the space variable, and note that when $U = 0$, i.e., when the mean motion is 0, the second equation of the system reduces to the famous Burgers equation. The system (1) itself, on the other hand, enjoys the characteristic quantity: Reynolds number Re given by

$$\mathrm{Re} = \frac{UL}{\nu}.$$

Recently Eden [15] discussed the system (1) and established a bound for the dimension of the attractor in terms of the Reynolds number. He employs, instead of Re, the Reynolds number given by

$$\overline{\mathrm{Re}} = \frac{PL^2}{\nu^2},$$

which is justified by the fact that the mean velocity U is asymptotically bounded by PL/ν. The result in [15] then states that the dimension of the global attractor can be estimated from above and below by the square root of $\overline{\mathrm{Re}}$.

Here, we examine the existence of inertial manifolds for the system (1) with a view to clarifying the relation between the Reynolds number $\overline{\mathrm{Re}}$. To see the dependence of $\overline{\mathrm{Re}}$ explicitly, we first make the change of variables:

$$(U, v, x, t) \to \left(\frac{LU}{\nu}, \frac{Lv}{\nu}, \frac{x}{L}, \frac{\nu t}{L}\right),$$

to obtain

$$\begin{aligned} \frac{dU}{dt} &= \overline{\mathrm{Re}} - U - \int_0^1 v^2\,dx, \\ \frac{\partial v}{\partial t} &= Uv + \frac{\partial^2 v}{\partial x^2} - 2v\,\frac{\partial v}{\partial x}, \end{aligned} \qquad \text{in } (x,t) \in [0,1] \times \mathbb{R}^+, \tag{2}$$

with $v(0,\,t) = v(1,\,t) = 0$ for $t \geq 0$. Here, we have used the same letters U, v, x, t for simplicity.

As is well known, however, the presence of the gradient term in the nonlinearity makes the situation worse. Indeed, if we apply directly the well-established procedure developed in [4], we obtain the existence of inertial manifolds for any sufficiently small $\overline{\mathrm{Re}}$. But this leads a trivial results; when $\overline{\mathrm{Re}} < \pi^{-2}$, we easily infer $(U, v) \to \left(\overline{\mathrm{Re}}, 0\right)$ as $t \to \infty$.

Recently M. Kwak [5] has found a method to circumvent this difficulty. He regards the derivative and the nonlinear term as independent variables to embed the original equation into a reaction-diffusion system exhibiting the same long time dynamics. Following his idea, we set $z = \frac{\partial v}{\partial x}$, $w := v^2$ and let $\mathrm{u} := J(u) = (U, v, z, w)$, for $u = (U, v)$. It is easy to see that u satisfies

$$\begin{aligned} \frac{dU}{dt} &= -U - \int_0^1 v^2\,dx + \overline{\mathrm{Re}}, \\ \frac{dv}{dt} &= \frac{\partial^2 v}{\partial x^2} + Uv - \frac{\partial w}{\partial x}, \\ \frac{dz}{dt} &= \frac{\partial^2 z}{\partial x^2} + Uz - \frac{\partial^2 w}{\partial x^2}, \\ \frac{dw}{dt} &= \frac{\partial^2 w}{\partial x^2} + 2Uv^2 - 2z^2 - 4v^2 z, \end{aligned} \tag{3}$$

with the boundary condition for (z, w) being

$$\frac{\partial z}{\partial x}(0,t) = \frac{\partial z}{\partial x}(1,t) = 0.$$

Note that the system (3) is no longer dissipative. Under further modification to make (3) dissipative (see (5) below), we can show the existence of inertial manifolds. Although this modified system reflects the long time behaviour for (3) merely partially, its global attractor agrees with the embedded one for (1), and so we content ourselves to prove the existence of inertial manifolds for the modified system. In summary, we state

THEOREM. *For any $\overline{\mathrm{Re}} > 0$, there is an inertial manifold $\mathcal{M}$ for the Burgers' system. The dimension N of $\mathcal{M}$ is estimated by*

$$N < C_1 \overline{\mathrm{Re}}^2 \exp\left(\exp\left(\exp\left(C_2 \overline{\mathrm{Re}}^2\right)\right)\right), \qquad \text{as } \overline{\mathrm{Re}} \to \infty,$$

for some positive real constants C_1, C_2.

Note that the principal part of (3) is no longer self-adjoint. However, it is sectorial and has a compact resolvent; we can appeal to the observation in [12]. Moreover, this time the nonlinearity is without the gradient term, and hence, the required condition for the existence is less stringent than before. Our task is then to compute various constants rather explicitly in terms of $\overline{\mathrm{Re}}$. Unfortunately, the dimension of $\mathcal{M}$ is quite bigger than that of the global attractor, which is the point where the improvement must be needed.

SECTION 2

We give the outline of the proof of the theorem. First, we recall some *a priori* estimates. For any $u_0 = (U_0, v_0) \in \mathbb{R} \times L^2(0,1)$, there exists $t_0 = t(0) > 0$, such that

$$U(t)^2 + \left|v(t)\right|_{L^2}^2 \le 2\overline{\mathrm{Re}}^2, \quad \left|v_x(t)\right|_{L^2}^2 \le \rho_1^2, \quad \left|v_{xx}(t)\right|_{L^2}^2 \le \rho_2^2 \tag{4}$$

holds for all $t \ge t_0$. Here, $u(t) = (U(t), v(t))$ denotes the solution of (2) with the initial value u_0, and we have put

$$\begin{aligned} \rho_1^2 &:= \sqrt{5}\, \overline{\mathrm{Re}} \exp\left(15\, \overline{\mathrm{Re}}^2\right), \\ \rho_2^2 &:= \left(5\overline{\mathrm{Re}}^2 + 9\overline{\mathrm{Re}}^4 + 10\,\rho_1^4 + \rho_1^2\right) \exp\left\{80\left(2\overline{\mathrm{Re}}^2 + 10\,\rho_1^4 + \rho_1^2\right)\right\}. \end{aligned}$$

These estimates follow from the standard argument treated extensively in [13]. See also [15].

In view of (4), we modify the system (3) as follows:

$$\begin{aligned} U_t &= -U - \int_0^1 v^2\,dx + \phi_\rho \overline{\mathrm{Re}}, \\ v_t &= v_{xx} + Uv - w_x - 2v\left(w - v^2\right) - (1 - \phi_\rho)\left(2\overline{\mathrm{Re}} + 4v^2\right) v, \\ z_t &= z_{xx} + \delta_0 z - \delta_0 z + Uz - w_{xx} - 2z\left(w - v^2\right) - \alpha\left(z - v_x\right) \\ &\quad - (1 - \phi_\rho)\left(2\overline{\mathrm{Re}} + 12v^2\right) z, \\ w_t &= w_{xx} + \delta_0 w - \delta_0 w + 2Uv^2 - 2z^2 - 4v^2 z - \left(40\rho^2 + 10v^2\right)\left(w - v^2\right) \\ &\quad - (1 - \phi_\rho)\left(2\overline{\mathrm{Re}} + 4v^2\right) 2v^2, \end{aligned} \tag{5}$$

where $\delta_0 > 0$ and

$$\begin{aligned} \rho^2 &= 2\overline{\mathrm{Re}}^2\left(1 + 2\rho_1^2\right) + 2\rho_1 + 1964, \\ \alpha &= 24\rho^2 + 2\overline{\mathrm{Re}} + 1. \end{aligned}$$

ϕ_ρ is the cut-off function defined by

$$\phi_\rho := \phi \frac{1}{\rho^2}\left(U^2 + \left|v\right|_{L^2}^2 + \left|v^2\right|_{L^2}^2 + \left|v_x\right|_{L^2}^2 + \left|z\right|_{L^2}^2 + \left|w\right|_{L^2}^2\right),$$

where $\phi : [0,\infty) \to [0,1]$ is a given smooth nonincreasing function such that

$$\phi(s) = \begin{cases} 1, & \text{for } 0 \le s \le 1, \\ 0, & \text{for } s \ge 2, \end{cases}$$
$$|\phi'(s)| \le 2, \qquad \text{for } 0 \le s < \infty.$$

The system (5) can be rewritten in the following abstract form:

$$\frac{d\mathfrak{u}}{dt} = -\mathfrak{A}\mathfrak{u} + \mathfrak{F}(\mathfrak{u}),$$

where $\mathfrak{u} = (U, v, z, w)$ and

$$\mathfrak{A} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\dfrac{\partial^2}{\partial x^2} & 0 & -\dfrac{\partial}{\partial x} \\ 0 & -k\dfrac{\partial}{\partial x} & -\dfrac{\partial^2}{\partial x^2} + \delta_0 & -\dfrac{\partial^2}{\partial x^2} \\ 0 & 0 & 0 & -\dfrac{\partial^2}{\partial x^2} + \delta_0 \end{pmatrix}, \tag{6}$$
$$\mathfrak{F}(\mathfrak{u}) := (F_1, F_2, F_3, F_4),$$

where

$$F_1 := -\int_0^1 v^2 dx + \phi_\rho \overline{\mathrm{Re}},$$
$$F_2 := Uv - 2v\left(w - v^2\right) - (1 - \phi_\rho)\left(2\overline{\mathrm{Re}} + 4v^2\right) v,$$
$$F_3 := \delta_0 z - kz + Uz - 2z\left(w - v^2\right) - \alpha\left(z - v_x\right) - (1 - \phi_\rho)\left(2\overline{\mathrm{Re}} + 12v^2\right) z,$$
$$F_4 := \delta_0 w + 2Uv^2 - 2z^2 - 4v^2 z - \left(40\rho^2 + 10v^2\right)\left(w - v^2\right) - (1 - \phi_\rho)\left(2\overline{\mathrm{Re}} + 4v^2\right) 2v^2.$$

The domain of $\mathfrak{A}$, $D(\mathfrak{A})$, is defined by

$$D(\mathfrak{A}) := \mathbb{R} \times \left(H_0^1(0,1) \cap H^2(0,1)\right) \times \left\{y \in H^2(0,1);\ y_x(0) = y_x(1) = 0\right\}^2.$$

We notice that the operator $\mathfrak{A}$ is not self-adjoint but it is sectorial, i.e., $-\mathfrak{A}$ generates an analytic semigroup on $\mathfrak{H} := \mathbb{R} \times L^2(0,1)^3$. (See the corresponding argument of [5, Lemma 2.6].) We can, therefore, define the fractional powers of $\mathfrak{A}$. Moreover, we easily see that the nonlinear term, $\mathfrak{F}$: $D\left(\mathfrak{A}^{1/2}\right) \to \mathfrak{H}$ is locally Lipschitz continuous. Hence, if $\mathfrak{u}(0) = \mathfrak{u}_0 \in D\left(\mathfrak{A}^{1/2}\right)$, then there exists a unique (strong) solution $\mathfrak{u} \in C\left([0,T);\ D\left(\mathfrak{A}^{1/2}\right)\right)$ of (5).

The next lemma shows the relation between the global attractor $\mathcal{A}_{or}$ for (2) and the one $\mathcal{A}_{rd}$ for (5). Its meaning is that the long time dynamics for (5) partly reflects the one for (2). See also [5, Theorem 3.5].

LEMMA. *The system (5) admits the global attractor $\mathcal{A}_{rd}$, which agrees with the embedding of the global attractor $\mathcal{A}_{or}$ for (2), i.e.,*

$$J\left(\mathcal{A}_{or}\right) = \mathcal{A}_{rd}.$$

Moreover, the injection J gives a flow homeomorphism between them.

The crucial step for the proof of the lemma is to compute the time derivative of

$$\left|z - v_x\right|_{L^2}^2 + \left|w - v^2\right|_{L^2}^2 \tag{7}$$

so that one sees (7) tending to 0 as $t \to \infty$. The rest of the proof then consists of standard *a priori* estimates.

Finally, we examine the existence of inertial manifolds for the modified system (5). Let $\mathcal{B}_N$ be the linear subspace of $\mathbb{R} \times L^2(0,1)^3$ spanned by the first N eigenfunctions of $\mathfrak{A}$ and let $\mathcal{Q}_N$ be the orthogonal complement. It is easy to see that $\mathcal{B}_N$ and $\mathcal{Q}_N$ are invariant under $\mathfrak{A}$ and $\dim \mathcal{B}_N = 3(N+1)$. The latter can be seen from the fact that $\mathcal{B}_N$ has a canonical basis

$$(1,0,0,0)\,,\ (0,\phi_n,0,0)\,,\ (0,0,\psi_n,0)\,,\ (0,0,0,\psi_n)\,,$$

where

$$\phi_n = \sin \pi n x, \qquad \psi_n = \cos \pi n x,$$

for $n = 1,2,3,\dots, \psi_0 = 1$, taking into account (6). Now, we can apply Theorem 3.4 in [12] with $\beta = 1$, and we obtain an inertial manifold $\mathcal{M}_{rd}$ for (5) as a graph of a Lipschitz continuous mapping from $\mathcal{B}_N$ into $\mathcal{Q}_N \cap D\left(\mathfrak{A}^{1/2}\right)$. Although the required *a priori* estimate is stronger than that needed in the previous lemma, the verification may be safely omitted. The constant C_2 in (3.12) in [12] is related to the nonlinearity $\mathfrak{F}(\mathrm{u})$ of the modified system (5); it involves the third derivative of u, from which $\exp^3$ comes. This completes the proof of the theorem.

REFERENCES

1. C. Foias, G.R. Sell and R. Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Diff. Eqs.* **73**, 309–353 (1988).
2. P. Constantin, C. Foias, B. Nicolaenko and R. Temam, Integral manifolds and inertial manifolds for dissipative partial differential equations, *Appl. Math. Sci.*, Springer-Verlag, New York, (1989).
3. A. Debussche, Inertial manifolds and Sacker's equation, *Diff. Int. Eqs.* **3**, 467–486 (1990).
4. C. Foias, B. Nicolaenko, G.R. Sell and R. Temam, Inertial manifolds for the Kuramoto-Sivashinsky equation and an estimate of their lowest dimension, *J. Math. Pures Appl.* **67**, 197–226 (1988).
5. M. Kwak, Finite-dimensional description of convective reaction-diffusion equations, *J. Dyn. Diff. Eqs.* **4**, 515–543 (1992).
6. M. Kwak, Finite dimensional inertial forms for the 2-D Navier-Stokes equations, *Indiana Univ. Math. J.* **41**, 927–981 (1992).
7. J. Mallet-Paret and G.R. Sell, Inertial manifolds for reaction diffusion equations in higher space dimensions, *J. Amer. Math. Soc.* **1**, 805–866 (1988).
8. M. Marion, Approximate inertial manifolds for reaction-diffusion equations high space dimensions, *J. Dyn. Diff. Eqs.* **1**, 245–267 (1989).
9. M. Miklavčič, A sharp condition for existence of an inertial manifold, *J. Dyn. Diff. Eqs.* **3**, 437–456 (1991).
10. Y. Morita, H. Ninomiya and E. Yanagida, Nonlinear perturbation of boundary values for reaction diffusion systems: Inertial manifolds and their applications, (Preprint).
11. H. Ninomiya, Some remarks on inertial manifolds, *J. Math. Kyoto Univ.* **32**, 667–688 (1992).
12. G.R. Sell and Y. You, Inertial manifolds: The non-self-adjoint case, *J. Diff. Eqs.* **96**, 203–255 (1992).
13. R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, *Appl. Math. Sciences*, Volume 68, Springer-Verlag, (1988).
14. J.M. Burgers, A mathematical model illustrating the theory of turbulence, *Advances in Appl. Mech.* **1**, 171–199 (1948).
15. A. Eden, On Burgers' original mathematical model of turbulence, *Nonlinearity* **3**, 557–566 (1990).