On the Spectrum of the Transport Operator with Abstract Boundary Conditions in Slab Geometry

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The paper is concerned with the spectral properties of the one-dimensional transport operator with general boundary conditions where an abstract boundary operator relates the incoming and the outgoing fluxes. We first give some existence and nonexistence results of eigenvalues in the half-plane \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda > -\lambda' \} \) where \(-\lambda'\) is the type of the semigroup generated by the streaming operator. Next, we discuss the irreducibility of the transport semigroup. In particular, we show that the transport semigroup is irreducible if the boundary operator is strictly positive. We end the paper by investigating the strict monotonicity of the leading eigenvalue of the transport operator with respect to different parameters of the equation. Finally, an open problem is indicated.

1. INTRODUCTION

This paper is devoted to the investigation of some spectral properties related to the positivity (in the lattice sense) of the one-dimensional transport operator with abstract boundary conditions. More precisely, we consider the unbounded operator

\[
A_H \psi(x, \xi) = -\xi \frac{\partial \psi}{\partial x} - \sigma(\xi) \psi(x, \xi) + \int_{-1}^{1} \kappa(\xi, \xi') \psi(x, \xi') d\xi'
= T_H \psi + K \psi
\]

with the boundary conditions

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 
\end{pmatrix} = \begin{pmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{pmatrix} \begin{pmatrix}
\psi_1^0 \\
\psi_2^0
\end{pmatrix},
\]

where
\begin{align}
\psi_1^j(\xi) &= \psi(-a, \xi), \quad \xi \in (0, 1) \\
\psi_2^j(\xi) &= \psi(a, \xi), \quad \xi \in (-1, 0) \\
\psi_3^j(\xi) &= \psi(-a, \xi), \quad \xi \in (-1, 0) \\
\psi_3^j(\xi) &= \psi(a, \xi), \quad \xi \in (0, 1),
\end{align}

and \( H_{ij} \) with \( 1 \leq i, j \leq 2 \) are bounded linear operators defined on suitable boundary spaces.

Here \( x \in (-a, a) \) and \( \xi \in (-1, 1) \), and \( \psi(x, \xi) \) represents the angular density of particles (for instance gas molecules, photons, or neutrons) in a homogeneous slab of thickness \( 2a \). The functions \( \sigma(\cdot) \) and \( \kappa(\cdot, \cdot) \) are called, respectively, the collision frequency and the scattering kernel.

Our general assumptions are
\[
\sigma(\cdot) \in \mathcal{L}^\infty(-1, 1), \quad K \in \mathcal{L}(L_p([-a, a] \times [-1, 1]; dx \, d\xi)),
\]
\[
(1 \leq p < \infty);
\]

\( K \) is positive in the lattice sense on \( L_p([-a, a] \times [-1, 1]; dx \, d\xi) \);

\( H \) is positive and satisfies \( \|H\| \leq 1 \).

The hypotheses on \( H \) imply that the streaming operator \( T_H \) generates a strongly continuous semigroup on \( \mathcal{D}(L_p([-a, a] \times [-1, 1]; dx \, d\xi)) \) (cf. [2, Theorem 2.2, p. 410, and Theorem 2.3, p. 411]). Following the perturbation theory, \( A_H := T_H + K \) also generates a strongly continuous semigroup on \( \mathcal{D}(L_p([-a, a] \times [-1, 1]; dx \, d\xi)) \) (cf. [7, Theorem 2.1, p. 495]). If we suppose that \( K \) is compact on \( L_p([-1, 1]; d\xi) \) (\( 1 \leq p < \infty \)), it follows from [8, Theorem 2.1] that \( P(A_H) := \sigma(A_H) \cap \{ \Re\lambda < -\lambda^* \} \) \( (-\lambda^* \) denotes the type of the \( C_0 \)-semigroup generated by \( T_H \), \( (e^{tT_H})_{t \geq 0} \) consists of (at most) isolated eigenvalues with finite algebraic multiplicity (see also [9]). It is well known, since the paper by Vidav [15], that if there exists an integer \( n \geq 1 \) such that \( [(\lambda - T_H)^{-1}K]^n \) is compact (\( \Re\lambda > -\lambda^* \)) and if \( \sigma(\cdot) \) is real and \( \kappa(\xi, \cdot) \geq 0 \) a.e. on \([-1, 1] \times [-1, 1] \), then when \( P(A_H) \neq \emptyset \), there exists a real dominant eigenvalue, i.e., less than or equal to the real part of any other eigenvalue of \( A_H \). In neutron transport theory, it is important to know that this real eigenvalue is strictly dominant. Further results about the spectrum of the transport operator can be obtained if one knows that the \( C_0 \)-semigroup generated by the transport operator is irreducible. In fact, if the semigroup generated by \( A_H \), \( (e^{tA_H})_{t \geq 0} \), is irreducible, this eigenvalue is of multiplicity 1 and its associated eigenvector is strictly positive. So, the problems concerning the existence of this eigenvalue and the irreducibility of \( (e^{tA_H})_{t \geq 0} \) are posed.
The main purpose of the present paper is to study these problems as well as that concerning the strict monotonicity of the leading eigenvalue with respect to the parameters of the transport equation.

In the neutron transport setting ($H = 0$) and in all dimensions, questions of irreducibility were treated in [3], [13], and [16]. The problem of existence and nonexistence of eigenvalues was considered in [12]. In addition to the questions about the effective existence of eigenvalues in $\{\text{Re} \lambda : \lambda > -\lambda^*\}$ and the irreducibility of $(e^{tA_H})_{t \geq 0}$, we treat the strict monotonicity problem of the leading eigenvalue with respect to the parameters $H$ and $K$.

Paradoxically, the monotonicity, even in the wide sense, with respect to the size of domain (i.e., if $0 < a_1 < a_2$ then $\lambda(a_1) \leq \lambda(a_2)$, where $\lambda(a_i)$ denotes the leading eigenvalue of $A_H$ on $L_p([-a_i, a_i] \times [-1, 1]; dx \, d\xi)$ with $i = 1, 2$) is not clear and seems to be open.

The outline of this paper is as follows. We recall briefly the functional setting of the problem and various notations in Section 2, and Section 3 is devoted to the analysis of the asymptotic spectrum of $A_H$. The effective existence of eigenvalues in $\{\lambda \in \mathbb{C} \text{ such that } \text{Re} \lambda > -\lambda^*\}$ is the main purpose of this section. Existence and nonexistence results of eigenvalue are given. The fourth section deals with the irreducibility of the transport semigroup $(e^{tA_H})_{t \geq 0}$. Sufficient conditions which guarantee the irreducibility of $(e^{tA_H})_{t \geq 0}$ are given. Finally, in Section 5, we treat the problem concerning the strict monotonicity of the leading eigenvalue with respect to the parameters of the equation. We use the comparison results of the spectral radius of positive operators obtained in [11]. These results permit us to show that the leading eigenvalue increases strictly with $H$ and $K$.

2. NOTATIONS AND PRELIMINARY FACTS

In this section, we introduce the different notions and notations which we need in the sequel. Let us first make precise the functional setting of the problem. Let

$$X_p = L_p(D; \, dx \, d\xi),$$

where $D = [-a, a] \times [-1, 1], \,(a > 0),$ and $p \in [1, \infty)$. Define the following sets representing the incoming and the outgoing boundaries of the phase space $D$:

$$D^i = D^i_1 \cup D^i_2 = \{-a\} \times [0, 1] \cup \{a\} \times [-1, 0],$$

$$D^o = D^o_1 \cup D^o_2 = \{-a\} \times [-1, 0] \cup \{a\} \times [0, 1].$$

Moreover, we introduce the boundary spaces

$$X^i_p := L_p(D_i; |\xi| \, d\xi) \sim L_p(D^i_1; |\xi| \, d\xi) \oplus L_p(D^i_2; |\xi| \, d\xi)$$

$$:= X^i_{1,p} \oplus X^i_{2,p},$$
endowed with the norm
\[
\|\psi^i, X^p_p\| = \left(\|\psi^i_1, X^p_{1,p}\|^p + \|\psi^i_2, X^p_{2,p}\|^p\right)^{1/p}
\]
\[
= \left[\int_0^1 |\psi(-a, \xi)|^p |\xi| d\xi + \int_{-1}^0 |\psi(a, \xi)|^p |\xi| d\xi\right]^{1/p}.
\]
\[
X^o_p := L_p(D^0; |\xi| d\xi) \sim L_p(D^0; |\xi| d\xi) \oplus L_p(D^0; |\xi| d\xi)
\]
\[
:= X^o_{1,p} \oplus X^o_{2,p},
\]
endowed with the norm
\[
\|\psi^o, X^p_p\| = \left(\|\psi^o_1, X^o_{1,p}\|^p + \|\psi^o_2, X^o_{2,p}\|^p\right)^{1/p}
\]
\[
= \left[\int_{-1}^0 |\psi(-a, \xi)|^p |\xi| d\xi + \int_0^1 |\psi(a, \xi)|^p |\xi| d\xi\right]^{1/p},
\]
where \(\sim\) means the natural identification of these spaces.

We define the partial Sobolev space \(W_p\) by
\[
W_p = \left\{ \psi \in X_p \text{ such that } \xi \frac{\partial \psi}{\partial x} \in X_p \right\}.
\]
It is well known that any function \(\psi\) in \(W_p\) has traces on \(\{-a\}\) and \(\{a\}\) in \(X^o_p\) and \(X^p_p\) (see, for instance, [1] or [2]). They are denoted, respectively, by \(\psi^o\) and \(\psi^i\) and represent the outgoing and the incoming fluxes (\(o\) for outgoing and \(i\) for incoming).

Let \(H\) be the boundary operator
\[
H : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]
with \(H_{j,k} : X^o_{k,p} \to X^o_{j,p}, H_{j,k} \in \mathcal{L}(X^o_{k,p}; X^i_{j,p}), j, k = 1, 2\), defined such that, on natural identification, the boundary conditions can be written as \(\psi^i = H\psi^o\).

We define, now, the streaming operator \(T_H\) with domain including the boundary conditions,
\[
\begin{pmatrix} T_H : D(T_H) \subseteq X_p \rightarrow X_p \\
\psi \mapsto T_H\psi(x, \xi) = -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi)\psi(x, \xi) \\
D(T_H) = \left\{ \psi \in W_p \text{ such that } \psi^i = H(\psi^o) \right\}
\end{pmatrix}
\]
where \(\sigma(.) \in L^\infty(-1, 1)\), \(\psi^0 = (\psi^0_1, \psi^0_2)^T\), and \(\psi^i = (\psi^i_1, \psi^i_2)^T\) with \(\psi^i_1, \psi^i_2, \psi^i_1\) and \(\psi^i_2\) given by (1.1).
Finally, we denote by $K$ the bounded operator on $X_p$,

$$
K: X_p \rightarrow X_p
$$

$$
\psi \rightarrow \int_{-1}^{1} \kappa(\xi, \xi')\psi(x, \xi')d\xi',
$$

where $\kappa(., .) : [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}$ is assumed to be measurable.

A peculiarity of the collision operator $K$ is that it operates only on the variable $\xi'$, so $x$ may be viewed merely as a parameter in $[-a, a]$. Consider now the operator

$$
\tilde{K}: L_p([-1, 1]; d\xi) \rightarrow L_p([-1, 1]; d\xi)
$$

$$
\zeta \rightarrow \int_{-1}^{1} \kappa(\xi, \xi')\zeta(\xi')d\xi'.
$$

It is easy to see that the boundedness of $K$ on $X_p$ and the fact that $0 < a < \infty$ imply that $\tilde{K} \in \mathcal{L}(L_p([-1, 1]; d\xi))$. On the other hand, the compactness of $\tilde{K}$ on $L_p([-1, 1]; d\xi)$ does not imply that of $K$ on $X_p$.

So, we introduce the following definition, which will be constantly used throughout the remainder of this article.

**Definition 2.1.** The collision operator $K$, defined as above, is said to be regular if $\tilde{K}$ is compact on $L_p([-1, 1]; d\xi)$.

For the clarity of our subsequent analysis, we introduce the bounded operators

$$
M_\lambda^i: X_p^i \rightarrow X_p^0, \quad M_\lambda u := (M_\lambda^+ u, M_\lambda^- u) \quad \text{with}
$$

$$
(M_\lambda^+ u)(-a, \xi) := u(-a, \xi) \exp \left(-2a \frac{\lambda + \sigma(\xi)}{\vert \xi \vert}\right), \quad 0 < \xi < 1
$$

$$
(M_\lambda^- u)(a, \xi) := u(a, \xi) \exp \left(-2a \frac{\lambda + \sigma(\xi)}{\vert \xi \vert}\right), \quad -1 < \xi < 0
$$

$$
B_\lambda^i: X_p^i \rightarrow X_p, \quad B_\lambda u := \chi_{(-1,0)}(\xi)B_\lambda^- u + \chi_{(0,1)}(\xi)B_\lambda^+ u \quad \text{with}
$$

$$
(B_\lambda^- u)(-a, \xi) := u(-a, \xi) \exp \left(-\frac{(\lambda + \sigma(\xi))\vert a + x \vert}{\vert \xi \vert}\right), \quad 0 < \xi < 1
$$

$$
(B_\lambda^+ u)(a, \xi) := u(a, \xi) \exp \left(-\frac{(\lambda + \sigma(\xi))\vert a - x \vert}{\vert \xi \vert}\right), \quad -1 < \xi < 0
$$

$$
G_\lambda: X_p \rightarrow X_p^0, \quad G_\lambda \varphi := (G_\lambda^+ \varphi, G_\lambda^- \varphi) \quad \text{with}
$$

$$
G_\lambda^+ \varphi := \frac{1}{\vert \xi \vert} \int_{-a}^{a} \exp \left(-\frac{(\lambda + \sigma(\xi))\vert a - x \vert}{\vert \xi \vert}\right)\varphi(x, \xi)dx, \quad 0 < \xi < 1
$$

$$
G_\lambda^- \varphi := \frac{1}{\vert \xi \vert} \int_{-a}^{a} \exp \left(-\frac{(\lambda + \sigma(\xi))\vert a + x \vert}{\vert \xi \vert}\right)\varphi(x, \xi)dx, \quad -1 < \xi < 0
$$

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and
\[
\begin{cases}
  C_\lambda: X_p \to X_p, & C_\lambda \varphi := \chi_{(-1,0)}(\xi)C_{\lambda}^- \varphi + \chi_{(0,1)}(\xi)C_{\lambda}^+ \varphi \\
  C_{\lambda}^- \varphi := \frac{1}{|\xi|} \int_{-a}^{x} \exp \left( -\frac{(\lambda + \sigma(\xi))|x-x'|}{|\xi|} \right) \varphi(x',\xi)dx', & 0 < \xi < 1 \\
  C_{\lambda}^+ \varphi := \frac{1}{|\xi|} \int_{x}^{a} \exp \left( -\frac{(\lambda + \sigma(\xi))|x-x'|}{|\xi|} \right) \varphi(x',\xi)dx', & -1 < \xi < 0,
\end{cases}
\]
where \( \chi_{(-1,0)}(.) \) and \( \chi_{(0,1)}(.) \) denote, respectively, the characteristic functions of the intervals \((-1, 0)\) and \((0, 1)\).

For \( \lambda \in \mathbb{R} \), these operators are positive with respect to the natural positive cones of the respective spaces. We also have the estimate
\[
\|M_\lambda\| \leq \exp\{-2a(\text{Re}\lambda + \lambda^*)\},
\]
(2.1)
where \( \lambda^* = \lim \inf_{|\xi| \to 0} \sigma(\xi) \).

Let \( \lambda \) be a complex number such that \( \text{Re}\lambda > -\lambda^* \). The use of (2.1) together with the fact that \( \|H\| \leq 1 \) implies that \( \lambda \) lies in the resolvent set of \( T_H \), and \((\lambda - T_H)^{-1}\) is given by
\[
(\lambda - T_H)^{-1} = \sum_{n=0}^{\infty} B_n H(M_\lambda H)^n G_\lambda + C_\lambda,
\]
For more details see [8, Sect. 1].

3. ON THE EFFECTIVE EXISTENCE OF EIGENVALUES

Denote by \( L_p(d\xi) \) the space of functions \( L_p([-1, 0); d\xi] \times L_p([0,1); d\xi] \). Notice that \( L_p(d\xi) \) is a subspace of \( X^1 \) and the imbedding \( L_p(d\xi) \hookrightarrow X^1 \) is continuous. By \( K \) we mean the integral operator on \( X_p \) whose kernel is given by \( \pi(\xi, \xi') = \kappa(\xi, \xi')/|\xi|. \)

THEOREM 3.1. Suppose that the operator \( K \) is a bounded on \( X_{\mu} \) and \( H \) is bounded from \( X^0 \) into \( L_p(d\xi) \) with \( \|H\| < 1 \). Then \( \sigma(A_H) \cap \{\lambda \in \mathbb{C} : \text{Re}\lambda > -\lambda^*\} = \emptyset \) for a small enough.

Proof. Let \( \psi \in X_p \) and put \( \varphi = K\psi. \) Then we have
\[
|C_\lambda \varphi|^p \leq (2a)^{p/q} \int_{-a}^{a} \frac{|\varphi(x,\xi)|^p}{|\xi|^p} \, dx,
\]
and so,
\[
\int_{-a}^{a} \int_{-1}^{1} |C_\lambda \varphi(x,\xi)|^p \, dx \, d\xi \leq (2a)^{(p+1)/q} \int_{-a}^{a} \int_{-1}^{1} \frac{|\varphi(x,\xi)|^p}{|\xi|^p} \, dx \, d\xi
\]
\[
= (2a)^p \int_{-a}^{a} \int_{-1}^{1} |K\varphi(x,\xi)|^p \, dx \, d\xi,
\]
where $q$ is the conjugate of $p$. Thus, we can write
\[
\left[ \int_{-a}^{a} \int_{-1}^{1} |C_{\lambda} \varphi|^p dx \, d\xi \right]^{1/p} \leq 2a \|K\| \|\psi\|
\]
which gives the estimate
\[
\|C_{\lambda} K\| \leq 2a \|K\|. \tag{3.1}
\]

The expressions of the operators $G_{\lambda}^+$ and $G_{\lambda}^-$ may be written, respectively,
\[
G_{\lambda}^+ \varphi = \int_{0}^{2a/|\xi|} e^{-(\lambda + \sigma(\xi))s} \varphi(a - s\xi, \xi) \, ds
\]
and
\[
G_{\lambda}^- \varphi = \int_{0}^{2a/|\xi|} e^{-(\lambda + \sigma(\xi))s} \varphi(-a - s\xi, \xi) \, ds.
\]
Consequently,
\[
|G_{\lambda}^+ \varphi| \leq \int_{0}^{2a/|\xi|} e^{-(\text{Re} \lambda + \sigma(\xi))s} |\varphi(a - s\xi, \xi)| \, ds
\]
\[
\leq \int_{0}^{\infty} \chi_{[0, 2a/|\xi|]}(s) |\tilde{\varphi}(a - s\xi, \xi)| \, ds,
\]
with
\[
\tilde{\varphi}(x, \xi) = \begin{cases} 
\varphi(x, \xi) & \text{if } x \in (-a, a) \\
0 & \text{if } x \notin (-a, a).
\end{cases}
\]
Using Hölder’s inequality one obtains
\[
|G_{\lambda}^+ \varphi| \leq \left[ \int_{0}^{\infty} \chi_{[0, 2a/|\xi|]}(s) \right]^{1/q'} \left[ \int_{0}^{\infty} \chi_{[0, 2a/|\xi|]}(s) |\tilde{\varphi}(a - s\xi, \xi)|^p \, ds \right]^{1/p}
\]
\[
= \left( \frac{2a}{|\xi|} \right)^{1/q'} \left[ \int_{0}^{\infty} \chi_{[0, 2a/|\xi|]}(s) |\tilde{\varphi}(a - s\xi, \xi)|^p \, ds \right]^{1/p}
\]
\[
\leq \left( \frac{2a}{|\xi|} \right)^{1/q'} \left[ \int_{-a}^{a} |\varphi(x, \xi)|^p \, dx \right]^{1/p}.
\]
Finally, we have the estimate
\[
\|G_{\lambda} K\| \leq (2a)^{1/q} \|K\|. \tag{3.2}
\]
On the other hand, the hypothesis on $H$ together with the estimate (2.1) gives
\[
\|M_{\lambda} H\| < 1 \text{ uniformly on } \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq -\lambda^* \}\]
which implies

\[
\|(I - M_\lambda H)^{-1}\| \leq \frac{1}{1 - \|H\|}, \quad (\Re \lambda \geq -\lambda^*). \quad (3.3)
\]

Next, a simple calculation leads to

\[
\|B_\lambda\|_{\mathcal{L}(L_p(\xi), X_p)} \leq (2a)^{1/p}.
\]

Using the estimates (3.1), (3.2), (3.3), and (3.4) and the hypothesis on \(H\) (i.e., \(\|Hu\|_{L_p(d\xi)} \leq \rho \|u\|_{X_p'}, \rho > 0\)), we may write

\[
\|\lambda - T_H\|^{-1}_K \leq \frac{(2a)^{1/p} \rho (2a)^{1/q} \|K\|}{1 - \|H\|} + 2a\|K\|
\]

\[
= 2\left[\rho\|K\| + (1 - \|H\|)\|K\|\right]a
\]

\[
= f(a).
\]

But \(f\) is a continuously increasing function on \([0, \infty]\) which verifies \(f(0) = 0\) and \(\lim_{a \to \infty} f(a) = +\infty\). So, we infer that there exists \(a_0 > 0\) such that \(f(a_0) < 1\). This completes the proof. Q.E.D.

In what follows, we turn our attention to the bounded part of the transport operator \(A_H\), which we denote by \(B\). It is defined by

\[
\begin{aligned}
B : L_p([-1, 1]; d\xi) &\longrightarrow L_p([-1, 1]; d\xi) \\
\varphi &\longrightarrow B\varphi(\xi) = -\sigma(\xi) \varphi(\xi) + \int_{-1}^{1} \kappa(\xi, \xi') \varphi(\xi')d\xi'.
\end{aligned}
\]

We shall discuss the relationship between the real eigenvalues of \(A_H\) and those of \(B\). Let us first denote by \(P(A_H)\) (resp. \(P(B)\)) the set

\[
\sigma P(A_H) \cap \{\lambda \in \mathbb{C} : \Re \lambda > -\lambda^*\} \quad (\text{resp. } \sigma P(B) \cap \{\lambda \in \mathbb{C} : \Re \lambda > -\lambda^*\})
\]

where \(\sigma P(A_H)\) (resp. \(\sigma P(B)\)) stands for the point spectrum of the operator \(A_H\) (resp. \(B\)).

**Theorem 3.2.** Suppose that \(K\) is regular, \(H_{11} = H_{22} = 0\) and \(H_{i,j} \leq \text{Id, } i \neq j; \ i, j = 1, 2\). Then, if \(P(B) = \emptyset\), then \(P(A_H) = \emptyset \forall a > 0\), and the leading eigenvalue of \(A_H\) is less than or equal to that of \(B\). Moreover, the latter is less than or equal to \(-\lambda^* + r_\sigma(K)\) (\(r_\sigma(K)\) denotes the spectral radius of \(K\)).

**Remark 3.1.** 1. Theorem 3.2 is a generalization to abstract boundary conditions of a result proved for vacuum boundary ones (cf. [12]).

2. For sufficient conditions, in terms of collision operators, ensuring \(P(B) = \emptyset\), we refer the reader to [12, p. 942].
Proof of Theorem 3.2. In fact, assume that \( P(A_H) \neq \emptyset \). Let \( \bar{\lambda} \) be the leading eigenvalue of \( A_H \) and denote by \( \bar{\psi} \) an associated positive eigenvector to \( \bar{\lambda} \). Hence

\[
A_H \bar{\psi} = \bar{\lambda} \bar{\psi}.
\]

This equation may be written in the form

\[
-\xi \frac{d\bar{\psi}}{dx}(x, \xi) - (\bar{\lambda} + \sigma(\xi))\bar{\psi}(x, \xi) + \int_{-1}^{1} \kappa(\xi, \xi')\bar{\psi}(x, \xi')d\xi' = 0. \tag{3.5}
\]

Set \( \varphi(\xi) = \int_{-1}^{a} \bar{\psi}(x, \xi)dx \). It is clear that \( \varphi \geq 0 \) and \( \varphi \neq 0 \). By integrating (3.5) with respect to \( x \), we get

\[
-\xi [\bar{\psi}(a, \xi) - \bar{\psi}(-a, \xi)] - \sigma(\xi)\varphi(\xi) + \int_{-1}^{1} \kappa(\xi, \xi')\varphi(\xi')d\xi' = \bar{\lambda} \varphi(\xi). \]

Taking into account the hypotheses and the sign of \( \bar{\psi} \) we get

\[
-\xi [\bar{\psi}(a, \xi) - \bar{\psi}(-a, \xi)] \leq 0 \quad \forall \xi \in [-1, 1]. \tag{3.6}
\]

If \( \xi < 0 \), then

\[
\bar{\psi}(a, \xi) - \bar{\psi}(-a, \xi) = \bar{\psi}_2^0 - \bar{\psi}_1^0 = H_{21}\bar{\psi}_1^0 - \bar{\psi}_1^0 = (H_{21} - I)\bar{\psi}_1^0 \leq 0,
\]

and therefore

\[
-\xi[\bar{\psi}(a, \xi) - \bar{\psi}(-a, \xi)] \leq 0.
\]

If \( \xi > 0 \), then

\[
\bar{\psi}(a, \xi) - \bar{\psi}(-a, \xi) = \bar{\psi}_2^0 - \bar{\psi}_1^0 = H_{12}\bar{\psi}_2^0 = (I - H_{12})\bar{\psi}_2^0,
\]

which implies

\[
-\xi [\bar{\psi}(a, \xi) - \bar{\psi}(-a, \xi)] \leq 0.
\]

Now, (3.5) and (3.6) lead to

\[
-\sigma(\xi) \varphi + K \varphi \geq \bar{\lambda} \varphi.
\]

Therefore, we get

\[
\int_{-1}^{1} \frac{\kappa(\xi, \xi')}{\bar{\lambda} + \sigma(\xi)} \varphi \geq \varphi. \tag{3.7}
\]

Let \( \lambda \in \mathbb{R} \) and define the operator \( K_\lambda \) on \( L_p([-1, 1]; d\xi) \) by

\[
\begin{cases}
K_\lambda \colon L_p([-1, 1]; d\xi) \rightarrow L_p([-1, 1]; d\xi) \\
\varphi \mapsto K_\lambda \varphi = \int_{-1}^{1} \frac{\kappa(\xi, \xi')}{\bar{\lambda} + \sigma(\xi)} \varphi(\xi')d\xi'.
\end{cases}
\]
Notice that $K_λ$ is a positive compact operator on $L_ρ([-1, 1]; dξ)$. It follows from the Krein–Rutman theorem that $r_σ(K_λ)$ is an eigenvalue of $K_λ$ depending continuously on $λ$. On the other hand, (3.7) implies that $r_σ(K_λ) ≥ 1$. Since $\lim_{λ→+∞} r_σ(K_λ) = 0$, there exists $λ_0 ≥ 1$ such that $r_σ(K_λ) = 1$. Consequently, there exists $ϕ_0 ≠ 0$ in $L_ρ([-1, 1]; dξ)$ satisfying

$$K_λϕ_0 = ϕ_0.$$ (3.8)

This leads to $Bϕ_0 = λ_0ϕ_0$ and proves the first part of the theorem.

On the other hand, Eq. (3.8) allows us to write

$$\int_{-1}^{1} κ(ξ, ξ')|ϕ_0(ξ')|dξ' = (λ_0 + λ)|ϕ_0(ξ)| ≥ (λ_0 + λ^*)|ϕ_0(ξ)|,$$

and therefore $r_σ(K) ≥ λ^* + λ_0$. So,

$$λ_0 ≤ −λ^* + r_σ(K).$$

Q.E.D.

Remark 3.2 If $K$ is quasinilpotent (i.e., $r_σ(K) = 0$), then $P(A_H) = ∅ ∀a > 0$, regardless of $∥K∥$. The operator $K$ is quasinilpotent if, for example, $κ(ξ, ξ')$ satisfies,

$$κ(ξ, ξ') = 0 \quad \text{for} \quad |ξ| > |ξ'|.$$ 

Thus, we get an extension to abstract boundary conditions of a result of Jörgens established for vacuum boundary conditions [6, Theorem 6.3].

Corollary 3.1. Suppose that the hypotheses of Theorem 3.2 hold. If the operator $B$ is subcritical (i.e., $P(B) ⊆ \{λ ∈ ℝ : λ < 0\}$), then the transport operator $A_H$ is subcritical $∀a > 0$.

Remark 3.3 Let $λ$ be in $ρ(A_H)∩ρ(A_0)$ such that $r_σ((λ - T_H)^{-1}K) < 1$ (spectral radius). Consequently,

$$(λ - T_H - K)^{-1} = \sum_{κ≥0}[(λ - T_H)^{-1}K]^n(λ - T_H)^{-1}.$$

The positivity of $K$ and the fact that $(λ - T_H)^{-1} ≥ (λ - T_0)^{-1} ≥ 0$ imply

$$[(λ - T_H)^{-1}K]^n(λ - T_H)^{-1} ≥ [ (λ - T_0)^{-1}K ]^n(λ - T_0)^{-1} ≥ 0,$$

and therefore,

$$R(λ, A_H) ≥ R(λ, A_0) ≥ 0.$$ (3.9)

Next, using (3.9) and [14, Proposition 2.5, p. 67], it follows that if $P(A_0) ≠ ∅$, then $P(A_H) ≠ ∅$. For sufficient conditions ensuring the existence of eigenvalues of $A_0$, we refer to [12, Theorem 6].
4. THE IRREDUCIBILITY OF THE SEMIGROUP \((e^{tA_H})_{t \geq 0}\)

In this section we are interested in the irreducibility of the \(C^0\)-semigroup \((e^{tA_H})_{t \geq 0}\) generated by the transport operator \(A_H\). In fact, if \((e^{tA_H})_{t \geq 0}\) is irreducible, then the leading eigenvalue (if it exists) is strictly dominant with multiplicity 1 and the associated eigenprojection is strictly positive (= positivity improving). Thus, if this eigenvalue is strictly dominant, we obtain a simple description of the time asymptotic behavior \((t \to \infty)\) of the solution of the Cauchy problem

\[
\frac{d\psi}{dt} = A_H \psi, \quad \psi(0) = \psi_0 \geq 0.
\]

Let \((V(t))_{t \geq 0}\) be a positive \(C^0\)-semigroup. In the literature, there are various equivalent definitions of irreducibility of \((V(t))_{t \geq 0}\) in \(L_p(\Omega)\) (or more generally in arbitrary Banach lattices) (see, for instance, [3], [4], [14, Definition 3.1, p.306], or [16]).

We now recall the following definitions, which we use throughout this section.

**Definition 4.1.** Let \(Q\) be a positive operator on \(L_p(\Omega)\). \(Q\) is called strictly positive if \(Qf > 0\) a.e. on \(\Omega\) for all \(f \geq 0, f \neq 0\).

**Definition 4.2.** Let \((V(t))_{t \geq 0}\) be a positive \(C^0\)-semigroup on \(L_p(\Omega)\) and let \(A\) be its infinitesimal generator. \((V(t))_{t \geq 0}\) is irreducible on \(L_p(\Omega)\) if for all \(\lambda > \sigma(A)\) (where \(\sigma(A)\) denotes the spectral bound of \(A\)) and all \(f \in L_p(\Omega)\), \(f \geq 0, f \neq 0\), we have \((\lambda - A)^{-1}f\) is strictly positive a.e. on \(\Omega\).

In the neutron transport setting, sufficient conditions of irreducibility of \(e^{tA}\) were proposed by many authors. In particular, we dispose of two practical criteria. The first is due to Voigt [16] and the second is due to Mokhtar–Kharroubi [13]. Voigt’s result is the following.

**Criterion 1** [16, Theorem 3.2]. Suppose that \(D\) is convex and \(V\) contains an annulus \(V_0 = \{\xi : 0 \leq v_1 < |\xi| < v_2 \leq +\infty\}\) with

\[
\kappa(x, \xi, \xi') > 0 \quad \text{on} \quad (D \times V_0 \times V) \cup (D \times V \times V_0).
\]

Then \(V(t)f > 0\) a.e. on \(D \times V\) for all \(f \geq 0, f \neq 0\) and \(t > d/v_2\) (where \(d\) is the diameter of \(D\)).

The second criterion is another approach of the irreducibility of the semigroup \((V(t))_{t \geq 0}\) generated by \(A = T + K\) where \(K \in \mathcal{L}(L_p(\Omega))\). It is based on Definition 4.2.

**Criterion 2** [13, Theorem 6]. We suppose that \(\sigma(A) \cap \{\Re \lambda > \eta\} \neq \emptyset\) (\(\eta\) denotes the type of \((e^{tT})_{t \geq 0}\)). If there exists an integer \(r\) such that \([\lambda T]^{-r}K\) is strictly positive, then \((e^{tA})_{t \geq 0}\) is irreducible.
In what follows we shall discuss the influence of the boundary operators on the irreducibility of the \( C^0 \)-semigroup on \( X_p, 1 \leq p < \infty \).

We denote by \((V_{H,K}(t))_{t \geq 0}\) the semigroup generated by \( A_H = T_H + K \). Following [8, Lemma 3.2], we have the estimate

\[
V_{H,K}(t) \geq V_{0,K}(t), \quad (\forall t \geq 0).
\]  

(4.1)

On the other hand, the positivity of the operators \( K, V_{0,K}(t), \) and \( V_{H,K}(t) \) shows that

\[
V_{H,K}(t) \geq V_{H,0}(t), \quad (\forall t \geq 0).
\]  

(4.2)

As an immediate consequence of (4.1) we have

**Theorem 4.1.** If \((V_{0,K}(t))_{t \geq 0}\) is irreducible, then \((V_{H,K}(t))_{t \geq 0}\) is irreducible.

**Remark 4.1.** Sufficient conditions of the irreducibility of the semigroup \((V_{0,K}(t))_{t \geq 0}\) (neutron transport semigroup) were given by many authors (see, for example, [3], [5, Chap. 12], [13], or [16]).

In the following, we shall derive some sufficient conditions of the irreducibility of \((V_{H,K}(t))_{t \geq 0}\) in terms of the boundary operators.

**Lemma 4.1.** Suppose that the operator \( H(I - M_\lambda H)^{-1} \) is strictly positive. Then the semigroup \((V_{H,0}(t))_{t \geq 0}\) generated by \( T_H \) is irreducible.

**Proof.** In view of the positivity of the operator \( C_\lambda \) we have the estimate

\[
(\lambda - T_H)^{-1} \geq B_\lambda H(I - M_\lambda H)^{-1} G_\lambda.
\]

(4.3)

On the other hand, \( G_\lambda \) is a strictly positive operator from \( X_p \) into \( X_p^0 \) and \( B_\lambda \) is a multiplication operator by a strictly positive function on \((-a, a) \times (-1, 1)\). Then, we infer that \( \forall f \geq 0 \) and \( f \neq 0 \), \( B_\lambda H(I - M_\lambda H)^{-1} G_\lambda f > 0 \) a.e. on \((-a, a) \times (-1, 1)\). The result follows from Eq. (4.3) and Definition 4.2. Q.E.D.

**Corollary 4.1.** The semigroup \((V_{H,0}(t))_{t \geq 0}\) is irreducible if one of the following conditions is satisfied:

1. \( H \) is strictly positive.
2. There exists an integer \( n_0 \in \mathbb{N} \) such that \((M_\lambda H)^{n_0}\) is strictly positive.

**Remark 4.2.** Note that \( M_\lambda \) is a multiplication operator by a strictly positive function.
Proof of Corollary 4.1. In view of Remark 4.2, it is clear that $H(I - M_\lambda H)^{-1}$ is strictly positive if and only if $M_\lambda H(I - M_\lambda H)^{-1}$ is strictly positive. On the other hand, $M_\lambda H(I - M_\lambda H)^{-1} = \sum_{n \geq 1} (M_\lambda H)^n$. So,

$$M_\lambda H(I - M_\lambda H)^{-1} \geq (M_\lambda H)^n. \quad (4.4)$$

Thus, if one of the two conditions 1 and 2 is satisfied, then the result follows immediately from (4.3) and (4.4).

Q.E.D.

Theorem 4.2. Assume that one of the two conditions of Corollary 4.1 is satisfied. Then $(V_{H,0}(t))_{t \geq 0}$ is irreducible.

Proof. As is seen above, the conditions of Corollary 3.1 imply that the semigroup $(V_{H,0}(t))_{t \geq 0}$ is irreducible. Now the use of (4.2) gives the result.

Q.E.D.

5. STRICT MONOTONICITY OF THE LEADING EIGENVALUE

The objective of this section is to study the strict growth properties of the leading eigenvalue with respect to the parameters of the equation. The analysis is based on a comparison result of the spectral radius of positive operators owing to Marek [11]. It is clear that if there are two bounded positive operators $\chi_1$ and $\chi_2$ satisfying $\chi_1 \leq \chi_2$ (i.e., $(\chi_2 - \chi_1) f \geq 0$ if $f \geq 0$), then $r_\sigma(\chi_1) \leq r_\sigma(\chi_2)$ ($r_\sigma(\chi_i)$ denotes the spectral radius of the operator $\chi_i$, $i = 1, 2$). Marek’s result gives sufficient conditions under which the latter inequality is strict. In the following, we need only a particular version of this result.

Let $\Omega$ be an open subset of $\mathbb{R}^m$ and let $E = L_p(\Omega)$ with $1 \leq p < \infty$. Consider two positive operators $\chi_1$ and $\chi_2$ in $\mathbb{D}(E)$, and assume that they are power compact (i.e., $(\chi_i)^n$ compact for some integer $n \geq 1$, $i = 1, 2$).

Theorem 5.0 [11, Theorem 4.3]. We suppose that $\chi_1 \leq \chi_2$. If $r_\sigma(\chi_1) > 0$ and if there exists an integer $N$ such that $(\chi_2)^N$ is strictly positive, then

$$r_\sigma(\chi_1) < r_\sigma(\chi_2) \quad \text{if} \quad \chi_1 \neq \chi_2.$$

As already mentioned above, the keys of the subsequent analysis are Theorem 5.0 and Gohberg–Shmul’yan’s theorem (cf. [5, p. 258]).

We start our study by discussing the influence of the boundary operators on the monotonicity of the leading eigenvalue. To this end, we consider two boundary operators $H_1$ and $H_2$ such that $H_1 \leq H_2$ and $H_1 \neq H_2$. We denote by $\lambda(H)$ the leading eigenvalue of $A_H$ (when it exists).
Now we are in the position to state

**Theorem 5.1.** Suppose that $K$ is regular and $\lambda(H_1)$ exists. Then $\lambda(H_2)$ exists and $\lambda(H_1) \leq \lambda(H_2)$. Furthermore, if one of the following conditions is satisfied, then $\lambda(H_1) < \lambda(H_2)$.

1. There exists an integer $n \geq 1$ such that $[C_{\lambda(H_1)}K]^n$ is strictly positive.
2. There exists an integer $n \geq 1$ such that $[B_{\lambda(H_1)}H_2(I - M_{\lambda(H_1)}H_2)^{-1}]G_{\lambda(H_1)}K]^n$ is strictly positive.

**Remark 5.1.** More practical criteria are given in Corollary 5.1.

**Proof of Theorem 5.1.** Let us first remark that, for any $\lambda > -\lambda^*$, the positivity of the operators $H_1, H_2,$ and $K$ and the fact that $H_1 \leq H_2$ imply the following inequality:

\[(\lambda - T_{H_2})^{-1}K \leq (\lambda - T_{H_1})^{-1}K.\] (5.1)

Applying the Krein–Rutman theorem, we check easily that $\lambda(H_1)$, the leading eigenvalue of $T_{H_1} + K$, is characterized by

\[r_{\sigma}[(\lambda(H_1) - T_{H_1})^{-1}K] = 1.\] (5.2)

Set $\chi_1 = (\lambda(H_1) - T_{H_1})^{-1}K$ and $\chi_2 = (\lambda(H_1) - T_{H_1})^{-1}K$. Since $K$ is regular, it follows from [8, Theorem 2.1] that $\chi_1$ and $\chi_2$ are compact on $X_p$. If one of the conditions above is satisfied, then the operator $\chi_2$ has a strictly positive power. Next, (5.1), (5.2) and Theorem 5.0 give

\[r_{\sigma}(\chi_2) = r_{\sigma}[(\lambda(H_1) - T_{H_1})^{-1}K] > 1.\]

But the function $\lambda^* + \infty [\exists \lambda \rightarrow r_{\sigma}\left[(\lambda - T_{H_1})^{-1}K\right]$ is strictly decreasing. Hence, there exists an unique $\lambda' > \lambda(H_1)$ such that

\[r_{\sigma}\left[(\lambda' - T_{H_1})^{-1}K\right] = 1.\]

This immediately implies that $\lambda' = \lambda(H_2)$, which completes the proof. Q.E.D.

We deduce the following corollary, which provides a more practical criterion of monotonicity of $\lambda(H)$.

**Corollary 5.1.** Assume that $K$ is regular and $\lambda(H_1)$ exists, then $\lambda(H_2)$ exists and $\lambda(H_1) \leq \lambda(H_2)$. Furthermore, if one of the following conditions is satisfied, then $\lambda(H_1) < \lambda(H_2)$.

1. $H_2$ is strictly positive and $\ker(K) \cap \{\varphi \in X_p, \varphi \geq 0\} = \{0\}$. 

2. There exists \( n_0 \in \mathbb{N} \) such that \((M_{\lambda(H_1)}H_2)^{n_0}\) is strictly positive and \(\ker(K) \cap \{\varphi \in X_p, \varphi \geq 0\} = \{0\}\).

Proof. Observe that Remark 4.2, (5.1), and the fact that \(B_{\lambda(H_1)}\) is a multiplication operator by a strictly positive function imply that \(\chi_2 = (\lambda(H_1) - T_{H_1})^{-1}K\) is strictly positive if one of the conditions above is satisfied. Now, a reasoning similar to that of Theorem 5.1 achieves the proof. Q.E.D.

Remark 5.2. We first observe that an immediate consequence of the relationship between the monotonicity of the leading eigenvalue of \(A_{H_2}\) and the irreducibility of \((e^{tA_{H_2}})_{t \geq 0}\).

In the following, we shall study the strict monotonicity of the leading eigenvalue of \(A_{H_1}\) with respect to the collision operators. In fact, consider \(K_1\) and \(K_2\), two regular collisions operators satisfying \(K_1 \leq K_2\) and \(K_1 \neq K_2\). We denote by \(\lambda(K)\) the leading eigenvalue of \(A_{H_1} = T_{H_1} + K\) (when it exists).

**Theorem 5.2.** Suppose that \(\lambda(K_1)\) exists; then \(\lambda(K_1) \leq \lambda(K_2)\). Furthermore, if one of the following conditions is satisfied, then \(\lambda(K_1) < \lambda(K_2)\).

1. There exists an integer \(n \geq 1\) such that \([C_{\lambda(K_1)}K_2]^n\) is strictly positive.
2. There exists an integer \(n \geq 1\) such that \([B_{\lambda(K_1)}H(I - M_{\lambda(K_1)}H)^{-1}G_{\lambda(K_1)}K_2]^n\) is strictly positive.

Proof. As in the proof of Theorem 5.1, from the positivity of the operators \(H, K_1, K_2\) and the fact that \(K_1 \leq K_2\) we deduce, for any \(\lambda > -\lambda^*\), the inequality

\[
(\lambda - T_{H_1})^{-1}K_2 \geq (\lambda - T_{H_1})^{-1}K_1.
\]

(5.3)

We recall that the leading eigenvalue \(T_{H_1} + K_1, \lambda(K_1)\), is characterized by

\[
r_{\sigma}[\left(\lambda(K_1) - T_{H_1}\right)^{-1}K_1] = 1.
\]

(5.4)

Next, the use of [8, Theorem 2.1] implies that \(\chi_1\) and \(\chi_2\) are compact. On the other hand, if one of the two conditions above is satisfied, the operator \(\chi_2\) has a strictly positive power. Therefore, the use of (5.3), (5.4), and Theorem 5.0 implies

\[
r_{\sigma}(\chi_2) = r_{\sigma}[\left(\lambda(K_1) - T_{H_1}\right)^{-1}K_2] > 1.
\]

Since the function \(] - \lambda^*, +\infty[ : \lambda \rightarrow r_{\sigma}[\left(\lambda - T_{H_1}\right)^{-1}K_2]\) is strictly decreasing, there exists a unique \(\lambda' > \lambda(K_1)\) such that \(r_{\sigma}(\lambda' - T_{H_1})K_2 = 1\). But this equation characterizes the leading eigenvalue of \(T_{H_1} + K_2\), so we have \(\lambda' = \lambda(K_2)\). This completes the proof of the theorem. Q.E.D.
As an immediate consequence of Theorem 5.2, we have

**Corollary 5.2.** Assume that $\lambda(K_1)$ exists; then $\lambda(K_2)$ exists and $\lambda(K_1) \leq \lambda(K_2)$. Furthermore, if one of the following conditions is satisfied, then $\lambda(K_1) < \lambda(K_2)$.

1. $H$ is strictly positive and $\ker(K_2) \cap \{\varphi \in X_p, \varphi \geq 0\} = \{0\}$.

2. There exists $n_0 \in \mathbb{N}$ such that $(M_{\lambda(K_1)}H)^{n_0}$ is strictly positive and $\ker(K_2) \cap \{\varphi \in X_p, \varphi \geq 0\} = \{0\}$.

We conclude this section with these remarks.

**Remark 5.3.**

1. Results similar to those of Theorems 5.1 and 5.2 may be derived by varying simultaneously the operators $H$ and $K$.

2. Recently, Mokhtar-Karroubi showed that the leading eigenvalue of the transport operator with vacuum boundary conditions $(H = 0)$ increases strictly with the size of the domain [13, Theorem 4]. Unfortunately, in our framework, the monotonicity with respect to the size of the domain (even in the wide sense) remains an open problem.

**REFERENCES**

