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On a problem by Beidar concerning the central closure

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Abstract

We give an example of a prime ring with zero center such that its central closure is a simple ring with an identity element. It solves a problem posed by Beidar.

Keywords: Prime ring; Central closure; Brown–McCoy radical

1. Introduction

Köthe's conjecture (whether the sum of two left nil ideals is nil) is one of the most famous problems in Ring Theory. Though the statement looks rather elementary this problem remains open since 1930 when the paper [4] has been published. There are many equivalent formulations of Köthe's problem (see recent survey papers [7,8,11] for details), we will mention just one of them. In 1972 Krempa [5] proved that Köthe's conjecture is equivalent to the problem whether polynomial rings in one indeterminate over nil rings are Jacobson radical.

Recall that a ring R is called *Jacobson radical* if for every $a \in R$ there exists $b \in R$ such that $a - b + ba = 0$. A ring R is called *Brown–McCoy radical* if it cannot be homomorphically mapped onto a ring with an identity element. Clearly, every Jacobson radical ring is Brown–McCoy radical. A natural question [6, Question 13a] whether the polynomial rings in one indeterminate over nil rings are Brown–McCoy radical was open for several years until it was answered positively in 1998 by Puczyłowski and Smoktunowicz [9]. Another natural question [6, Question 13b] whether the same can be obtained for polynomial rings in sets of commuting or non-commuting indeterminates remains open so far, just some partial cases have been considered. In particular,

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Ferrero and Wisbauer showed that the questions for infinitely many commuting and non-commuting indeterminates are equivalent [3]. Smoktunowicz proved that if a matrix ring over a ring R is nil, then the polynomial ring in two commuting indeterminates over R is Brown–McCoy radical [12]. In the recent paper [2] it was shown that the polynomial ring $R[x, y]$ in two commuting indeterminates is Brown–McCoy radical provided that R is nil and $pR = 0$ for some prime p.

The problem of *the existence of a prime ring with zero center whose central closure is a simple ring with an identity element* posed by Beidar has circulated among ring theorists for a while and it was mentioned in several papers on radical theory (see for example [3, p. 223], [8, Question 2.13a] or [10, p. 140]). If such a prime ring would not exist, then every polynomial ring in commuting variables over a nil ring would be Brown–McCoy radical.

The goal of this paper is to prove the following result:

Theorem 1. *There exists a prime ring with zero center such that its central closure is a simple ring with an identity element.*

The construction is a rather interesting matrix algebra which can be of some use for some other examples.

The proof of the theorem will be split into a series of lemmas in the next section.

2. The proof

Let R be a prime ring with extended centroid C and symmetric ring of quotients Q (see [1] for details). Recall that the subring RC of Q is said to be the *central closure of R*.

Let $\mathbb{Q}$ be the field of rational numbers. Throughout this paper $\mathbb{Q}[x, x_0, x_1, \ldots]$ is the ring of polynomials over $\mathbb{Q}$ in indeterminates $x, x_0, x_1, \ldots$, and $\mathbb{Q}\{x, x_0, x_1, \ldots\}$ is the field of fractions of the polynomial ring $\mathbb{Q}[x, x_0, x_1, \ldots]$.

Let M be the ring of $\aleph_0 \times \aleph_0$ row-finite matrices over $\mathbb{Q}\{x, x_0, x_1, \ldots\}$. By e_{ij} we denote an ordinary matrix unit, i.e. a matrix which has 1 in the (i, j)th position and zeros elsewhere.

Let E_{ijk}, $k \geqslant 0$, $1 \leqslant i \leqslant 2^k$, $1 \leqslant j \leqslant 2^k$, be a $2^k \times 2^k$ matrix of the form e_{ij}. Denote by y_{ijk}, $k \geqslant 0$, $1 \leqslant i \leqslant 2^k$, $1 \leqslant j \leqslant 2^k$, an element of the form:

$$y_{ijk} = \begin{pmatrix} xE_{ijk} & 0 & 0 & 0 & \cdots \\ 0 & x_kE_{ijk} & 0 & 0 & \cdots \\ 0 & 0 & xE_{ijk} & 0 & \cdots \\ 0 & 0 & 0 & x_kE_{ijk} & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}.$$

For example,

$$y_{110} = \begin{pmatrix} x & 0 & 0 & 0 & \cdots \\ 0 & x_0 & 0 & 0 & \cdots \\ 0 & 0 & x & 0 & \cdots \\ 0 & 0 & 0 & x_0 & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix} \quad \text{and} \quad y_{121} = \begin{pmatrix} 0 & x & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & x_1 & \cdots \\ 0 & 0 & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let R be a $\mathbb{Q}$-subalgebra of M generated by all y_{ijk}, $k \geqslant 0$, $1 \leqslant i \leqslant 2^k$, $1 \leqslant j \leqslant 2^k$. Note that every element of R is of the form

$$\begin{pmatrix} A_k & 0 & \cdots \\ 0 & A_k & \vdots \\ \vdots & 0 & \ddots \end{pmatrix}$$

where A_k is a $2^k \times 2^k$ matrix with sufficiently large k.

We will use standard arguments to prove first 2 lemmas.

Lemma 1. *R is a prime ring.*

Proof. Suppose that there exist nonzero elements $a, b \in R$ such that $aRb = 0$. We may assume that there exist k, A_k and B_k such that

$$a = \begin{pmatrix} A_k & 0 & \cdots \\ 0 & A_k & \vdots \\ \vdots & 0 & \ddots \end{pmatrix}, \quad b = \begin{pmatrix} B_k & 0 & \cdots \\ 0 & B_k & \vdots \\ \vdots & 0 & \ddots \end{pmatrix}.$$

Since a and b are nonzero there are nonzero entries α_{ij} of A_k and β_{pq} of B_k. Clearly, $\alpha_{ij}x\beta_{pq}$ is a nonzero element of $\mathbb{Q}[x, x_0, x_1, \ldots]$. Taking $c = y_{jpk}$ we get that $acb \neq 0$, a contradiction. Therefore R is a prime ring. □

Lemma 2. *If $c \in M$ is a row-finite matrix which commutes with all matrices from R, then c is of the form pI, where $p \in \mathbb{Q}\{x, x_0, x_1, \ldots\}$ and I is the identity matrix.*

Proof. Suppose that c has a nonzero entry γ_{ij} with $i \neq j$. Let k be such number that $i, j < 2^k$. Let $a = y_{1ik}$ and $b = y_{j1k}$. We obtain $abc = 0$ and $acb \neq 0$ since the $(1, 1)$th entry of acb is $x\gamma_{ij}x \neq 0$, a contradiction. Hence c is a diagonal matrix.

Suppose that c is not of the form pI. It means that c contains entries $\gamma_{ii} \neq \gamma_{jj}$ for some $i \neq j$. Let k be such number that $i, j < 2^k$. We obtain $cy_{ijk} \neq y_{ijk}c$, a contradiction. The lemma is proved. □

Lemma 3. *R has a zero center.*

Proof. It follows from Lemma 2 that the central elements of R are of the form pI, where $p \in \mathbb{Q}[x, x_0, x_1, \ldots]$.

Let $L = \{y_{ijk}\}$ be the finite set of elements such that pI lies in the $\mathbb{Q}$-algebra generated by L and let $K = \max\{k | y_{ijk} \in L\}$.

We proceed by induction on K. If $K = 0$, then $L = \{y_{110}\}$ and obviously $\mathbb{Q}$-algebra generated by L does not contain a nonzero central element.

Consider now the general case $K > 0$. Write the desired central element in the form $z = z_{K-1} + z_K$, where z_{K-1} is the sum of finite products which do not involve elements of the form y_{ijK}. Observe that z_{K-1} can be presented in the form

$$z_{K-1} = \begin{pmatrix} A_K & 0 & 0 & 0 & \cdots \\ 0 & A_K & 0 & 0 & \cdots \\ 0 & 0 & A_K & 0 & \cdots \\ 0 & 0 & 0 & A_K & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix},$$

where $A_K = A_K(x, x_0, x_1, \ldots, x_{K-1})$ is a $2^K \times 2^K$ matrix. Note that z_K is of the form

$$z_K = \begin{pmatrix} B_K & 0 & 0 & 0 & \ldots \\ 0 & C_K & 0 & 0 & \ldots \\ 0 & 0 & B_K & 0 & \ldots \\ 0 & 0 & 0 & C_K & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix},$$

where $B_K = B_K(x, x_0, x_1, \ldots, x_{K-1})$ and $C_K = C_K(x, x_0, x_1, \ldots, x_{K-1}, x_K)$ are $2^K \times 2^K$ matrices. If C_K depends on the indeterminate x_K, then $z = z_{K-1} + z_K$ cannot be of the form pI. Taking $x_K = 0$ we get by definition of z_K that $C_K = 0$. Now taking $x_K = x$ we obtain

$$0 = C_K = C_K(x, x_0, x_1, \ldots, x_{K-1}, x) = B_K(x, x_0, x_1, \ldots, x_{K-1}),$$

that is $z_K = 0$. Therefore, $z_{K-1}I$ lies in the center, which is impossible by the induction hypothesis. □

Lemma 4. *The extended centroid of R contains the set $\mathbb{Q}\{x, x_0, x_1, \ldots\}I$.*

Proof. We will use some basic properties of the extended centroid that can be found in [1, Section 2.3]. In particular, we will use the well-known facts that it is the center of the symmetric ring of quotients and a field.

Let C be the extended centroid of R. To prove that $c \in \mathbb{Q}\{x, x_0, x_1, \ldots\}I$ is an element of C it is enough to find two nonzero elements $r_1, r_2 \in R$ such that $r_2 = cr_1$.

Note that $y_{110}y_{111} = xy_{111}$ and so $xI \in C$. Next, $y_{110}y_{221} = x_0y_{221}$ implies $x_0I \in C$.

Finally, $y_{11k}(y_{110}y_{11k} - y_{11k}^2) = x_k(y_{110}y_{11k} - y_{11k}^2)$, so $x_kI \in C$ for all $k > 0$. Since C is a field we get that C contains the set $\mathbb{Q}\{x, x_0, x_1, \ldots\}I$. □

Lemma 5. *Let $Z = \mathbb{Q}\{x, x_0, x_1, \ldots\}I$. The ring RZ is a simple ring with an identity.*

Proof. Note that every nonzero element a of RZ can be written in the form

$$a = \begin{pmatrix} A_k & 0 & \ldots \\ 0 & A_k & \vdots \\ \vdots & 0 & \ddots \end{pmatrix} \tag{2.1}$$

where A_k is a $2^k \times 2^k$ matrix with sufficiently large k. Setting

$$A_{k+1} = \begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix}$$

if necessary we may assume that $k > 1$.

Our goal is to show that the ideal J of the ring RZ generated by any nonzero element $a \in RZ$ contains the identity I.

Step 1. J contains an element ϵ_{11l} of the form

$$\epsilon_{11l} = \begin{pmatrix} E_{11l} & 0 & \ldots \\ 0 & E_{11l} & \vdots \\ \vdots & 0 & \ddots \end{pmatrix}$$

with sufficiently large $l > 1$.

Let a be a nonzero element of J of the form (2.1). We may assume that A_k contains a nonzero entry αe_{ij} with $\alpha \in \mathbb{Q}\{x, x_0, x_1, \ldots\}$. Note that $\alpha^{-1} y_{1ik} a y_{j1k} \in J$ is of the form

$$b = \begin{pmatrix} x^2 E_{11k} & 0 & 0 & 0 & \cdots \\ 0 & x_k^2 E_{11k} & 0 & 0 & \cdots \\ 0 & 0 & x^2 E_{11k} & 0 & \cdots \\ 0 & 0 & 0 & x_k^2 E_{11k} & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}.$$

Computing $[(x^2 - x_k x)x^2]^{-1}(y_{11k}^2 - x_k y_{11k})b \in J$ we get the desired matrix $\epsilon_{11,k+1}$.

Step 2. For every $l > 1$ and i with $1 \leqslant i \leqslant 2^l$, the ring RZ contains elements ϵ_{1il} and ϵ_{i1l}.

We will prove only that $\epsilon_{1il} \in RZ$, the second statement can be proved analogously.

First, let i be in the interval $1 \leqslant i \leqslant 2^{l-1}$. Note that

$$\epsilon_{1il} = (x^2 - x_{l-1}x)^{-1}[y_{11,l-1} y_{1i,l-1} - x_{l-1} y_{1i,l-1}] \in RZ.$$

Now let i be in the interval $2^{l-1} + 1 \leqslant i \leqslant 2^l$. We obtain

$$\epsilon_{1il} = (x_{l-1}^2 - x_{l-1}x)^{-1}[y_{11,l-1} y_{1i,l-1} - x y_{1i,l-1}] \in RZ$$

which completes the step.

Step 3. J contains the identity I.

By Step 1 $\epsilon_{11l} \in J$ and by Step 2 $\epsilon_{1il}, \epsilon_{i1l} \in RZ$. For every i with $1 \leqslant i \leqslant 2^l$ we get $\epsilon_{iil} = \epsilon_{i1l}\epsilon_{11l}\epsilon_{1il} \in J$. To complete the proof just observe that $I = \sum_{i=1}^{2^l} \epsilon_{iil}$. □

Let C' be the extended centroid of RZ. Clearly C is a subset of C', so by Lemma 4 we have the inclusion $Z \subseteq C \subseteq C'$ and consequently $RZ \subseteq RC \subseteq RC'$. Since RZ is a simple ring with an identity we get $RZ = RC'$ and so $RZ = RC$. Therefore RC is a simple ring with an identity as desired.

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