The canonical genus for Whitehead doubles of a family of alternating knots

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A B S T R A C T

For any given integer $r \geqslant 1$ and a quasitoric braid $\beta_r = (\sigma_r^{-\epsilon} \sigma_{r-1}^{\epsilon} \cdots \sigma_1^{(-1)^r \epsilon})^3$ with $\epsilon = \pm 1$, we prove that the maximum degree in z of the HOMFLYPT polynomial $P_{W_2(\hat{\beta}_r)}(v, z)$ of the doubled link $W_2(\hat{\beta}_r)$ of the closure $\hat{\beta}_r$ is equal to $6r - 1$. As an application, we give a family $\mathcal{K}^3$ of alternating knots, including $(2, n)$-torus knots, 2-bridge knots and alternating pretzel knots as its subfamilies, such that the minimal crossing number of any alternating knot in $\mathcal{K}^3$ coincides with the canonical genus of its Whitehead double. Consequently, we give a new family $\mathcal{K}^3$ of alternating knots for which Tripp's conjecture holds.

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1. Introduction

A knot is an ambient isotopy class of an oriented 1-sphere S^1 smoothly embedded in the 3-sphere S^3 with a fixed standard orientation. Satellite construction is one of frequently used machineries to obtain a new knot from an arbitrary given knot. One of famous families of satellite knots is that of m-twisted positive Whitehead doubles $W_+(K, m)$ and negative Whitehead doubles $W_-(K, m)$ ($m \in \mathbb{Z}$), which are the satellites of knots K with positive Whitehead-clasp W_+ and negative Whitehead-clasp W_- as patterns, respectively (see Section 2).

A remarkable feature of Whitehead doubles is well known facts that the Alexander polynomial and the signature invariant of the 0-twisted Whitehead double of an arbitrary given knot are identical to those of the trivial knot. Also, they have genus one and have unknotting number one. In fact, Whitehead doubles are characterized as follows: A non-trivial knot is a Whitehead double of a knot if and only if its minimal genus and unknotting number are both 1 [17].

In 2002, Tripp [18] showed that the canonical genus of a Whitehead double of a torus knot $T(2, n)$ of type $(2, n)$ is equal to n, the minimal crossing number of $T(2, n)$, and conjectured that the minimal crossing number of any knot coincides with the canonical genus of its Whitehead double. In [15], Nakamura has extended Tripp's argument to show that for 2-bridge

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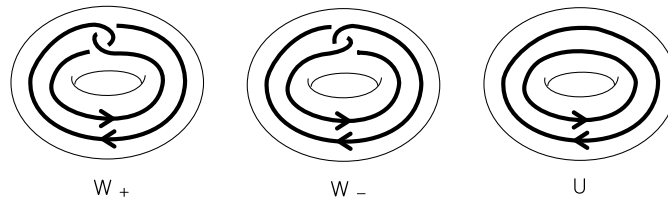
Fig. 1. Whitehead-clasp.

knots, Tripp's conjecture holds. He also found a non-alternating knot of which the minimal crossing number is not equal to the canonical genus of its Whitehead double and so he modified Tripp's conjecture to the following:

Conjecture 1.1. *The minimal crossing number of any alternating knot coincides with the canonical genus of its Whitehead double.*

In [1], Brittenham and Jensen showed that Conjecture 1.1 holds for alternating pretzel knots $P(k_1, \ldots, k_n)$, $k_1, \ldots, k_n \geqslant 1$ [1, Theorem 1]. To prove this, they used Morton's inequality [13] and provided a method for building new knots K satisfying $\max \deg_z P_{W_{\pm}(K,m)}(v, z) = 2c(K)$ from old ones K' (for more details, see Section 3 or [1]). Actually, Brittenham and Jensen gave a larger class of alternating knots than the class including $(2, n)$-torus knots, 2-bridge knots, and alternating pretzel knots. In addition, Gruber [5] extended Nakamura's result to algebraic alternating knots in Conway's sense in a different way.

The main purpose of this paper is to give a new infinite family of alternating knots for which Conjecture 1.1 holds, which is an extension of the previous results of Tripp [18], Nakamura [15] and Brittenham and Jensen [1].

This paper is organized as follows. In Section 2, we review Whitehead double of a knot and some known preliminary results for the canonical genus of Whitehead double of a knot. In Section 3, we review the Morton's inequality for the maximum degree in z of the HOMFLYPT polynomial $P_L(v, z)$ of a link L and its relation to the canonical genus of Whitehead double of a knot. We also give a brief review of Brittenham and Jensen's method. In Section 4, we prove that for all integers $r \geqslant 1$, the maximum degree in z of the HOMFLYPT polynomial $P_{W_2(\hat{\beta}_r)}(v, z)$ of the doubled link $W_2(\hat{\beta}_r)$ for the closure $\hat{\beta}_r$ of a quasitoric braid $\beta_r = (\sigma_r^{-\epsilon} \sigma_{r-1}^{\epsilon} \cdots \sigma_1^{(-1)^r \epsilon})^3$ with $\epsilon = \pm 1$ is equal to $6r - 1$ (Theorem 4.5). In Section 5, we give a family $\mathcal{K}^3 = \bigcup_{r=1}^{\infty} \mathcal{K}_r$ of alternating knots, where $\mathcal{K}_1$ contains all $(2, n)$-torus knots, 2-bridge knots and alternating pretzel knots and $\mathcal{K}_i \neq \mathcal{K}_j$ if $i \neq j$, and show that the minimal crossing number of any alternating knot in $\mathcal{K}^3$ coincides with the canonical genus of its Whitehead double (Theorem 5.2). Consequently, we give a new infinite family of alternating knots for which Conjecture 1.1 holds. The final Section 6 is devoted to prove a key Lemma 4.4, which has an essential role to prove Theorem 4.5.

2. Canonical genus and Whitehead double of a knot

Let T be a knot embedded in the unknotted solid torus $V = S^1 \times D^2$, which is essential in the sense that it meets every meridional disc in V. Let K be an arbitrary given knot in S^3 and let $N(K)$ be a tubular neighborhood of K in S^3. Suppose that $h : V = S^1 \times D^2 \rightarrow N(K)$ is a homeomorphism. Then the image $h(T) = S_T(K)$ is a new knot, which is called a *satellite* (*knot*) with *companion* K and *pattern* T. Note that if K is a non-trivial knot, then the satellite $S_T(K)$ is also a non-trivial knot [2].

Now let W_+, W_- and U denote the positive Whitehead-clasp, negative Whitehead-clasp and the doubled link embedded in V with orientations as shown in Fig. 1. Let K be an oriented knot and let $h : V = S^1 \times D^2 \rightarrow N(K)$ be an orientation preserving homeomorphism which takes the disk $\{1\} \times D^2$ to a meridian disk of $N(K)$, and the core $S^1 \times \{0\}$ of V onto the knot K. Let ℓ be the preferred longitude of V. We choose an orientation for the image $h(\ell)$ so that it is parallel to K. If the linking number of the image $h(\ell)$ and K is equal to m, then the satellite $S_{W_+}(K)$ (respectively $S_{W_-}(K)$) with companion K and pattern W_+ (respectively W_-) is called the *m-twisted positive* (respectively *negative*) *Whitehead double* of K, denoted by $W_+(K, m)$ (respectively $W_-(K, m)$), and the satellite $S_U(K)$ with companion K and pattern U is called the *m-twisted doubled link* of K, denoted by $W_2(K, m)$. The 0-twisted positive (respectively negative) Whitehead double of K is sometimes called the *untwisted* positive (respectively negative) Whitehead double of K. In what follows, we use the notation $W_{\pm}(K, m)$ to refer to the m-twisted positive/negative Whitehead double of K, respectively.

Let D be an oriented diagram of an oriented knot K and let $w(D)$ denote the writhe of D, that is, the sum of the signs of all crossings in D defined by $\text{sign}(\nearrow\!\!\!\nwarrow) = 1$ and $\text{sign}(\nwarrow\!\!\!\nearrow) = -1$. Recall that for an oriented diagram $D = D_1 \cup D_2$ of an oriented two component link $L = K_1 \cup K_2$, the *linking number* $lk(L)$ of L is defined to be the half of the sum of the signs of all crossings between D_1 and D_2. The m-twisted positive (respectively negative) Whitehead double $W_+(K, m)$ (respectively $W_-(K, m)$) has the *canonical diagram*, denoted by $W_+(D, m)$ (respectively $W_-(D, m)$), associated with D, which is the doubled link diagram of D with $(m - w(D))$ full-twists (see Fig. 2) and a positive Whitehead-clasp W_+ (respectively negative Whitehead-clasp W_-) as illustrated in (b) and (c) of Fig. 3. Also, the m-twisted doubled link $W_2(K, m)$
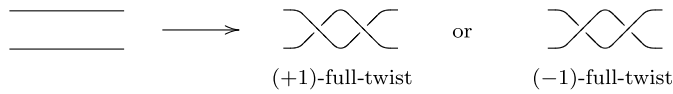
Fig. 2. ($\pm$)-full-twist.

(a) D

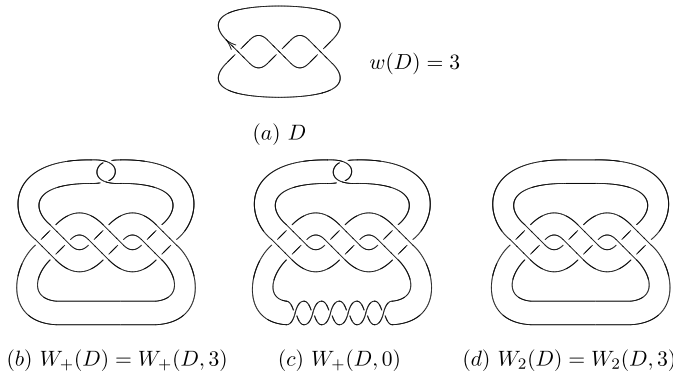(b) $W_+(D) = W_+(D, 3)$ (c) $W_+(D, 0)$ (d) $W_2(D) = W_2(D, 3)$

Fig. 3. Canonical diagram.

of K has the canonical diagram $W_2(D, m)$ associated with D, which is the doubled link diagram of D with $(m - w(D))$ full-twists without Whitehead-clasp.

In particular, the canonical diagram $W_+(D, w(D))$ (respectively $W_-(D, w(D))$) of the $w(D)$-twisted positive (respectively negative) Whitehead double $W_+(K, w(D))$ (respectively $W_-(K, w(D))$) is called the *standard diagram* of Whitehead double of K associated with the diagram D and is denoted by simply $W_+(D)$ (respectively $W_-(D)$). Likewise, the canonical diagram $W_2(D, w(D))$ of the $w(D)$-twisted doubled link $W_2(K, w(D))$ is called the *standard diagram* of the doubled link of K associated with the diagram D and is denoted by simply $W_2(D)$ (for example, see Fig. 3(d)).

Frankel and Pontrjagin [4] and Seifert [16] introduced a method to construct a compact orientable surface having a given link as its boundary. A *Seifert surface* for a link L in S^3 is a compact, connected, and orientable surface Σ in S^3 such that the boundary $\partial\Sigma$ of Σ is ambient isotopic to L, that is, $\partial\Sigma = L$. The *genus* of an oriented link L, denoted by $g(L)$, is the minimum genus of any Seifert surface of L. The genus of an unoriented link L is the minimum taken over all possible choices of orientation for L. For a diagram D of a link L, it is well known that a Seifert surface for L can always be obtained from D by applying Seifert's algorithm [16]. A Seifert surface for a link L constructed via Seifert's algorithm for a diagram D is called the *canonical Seifert surface* associated with D and denoted by $\Sigma(D)$. In what follows, we denote the genus $g(\Sigma(D))$ of the canonical Seifert surface $\Sigma(D)$ by $g_c(D)$. Then the minimum genus over all canonical Seifert surfaces for L is called the *canonical genus* of L and denoted by $g_c(L)$, i.e.,

$$g_c(L) = \min_{D \text{ a diagram of } L} g_c(D).$$

Seifert [16] showed that

$$\frac{1}{2}\deg\Delta_K(t) \leqslant g(K), \tag{2.1}$$

where $\deg\Delta_K(t)$ is the degree of the Alexander polynomial $\Delta_K(t)$ of K. If K is a torus knot, then the equality in (2.1) holds, but there are also cases where the equality does not hold. In fact, the trivial knot is the only knot with genus zero and there are many non-trivial knots whose Alexander polynomials are equal to 1. Note that Seifert's algorithm applied to a knot or link diagram might not produce a minimal genus Seifert surface and so the following inequality holds

$$g(K) \leqslant g_c(K). \tag{2.2}$$

Up to now, many authors have explored knots and links for which this inequality is strict or equal, for example, see [7–10, 12,15,18] and therein. On the other hand, Murasugi [14] proved that if K is an alternating knot, then the equality in (2.1) holds and $g(K) = g_c(K)$ in (2.2). Also we have the following:

Proposition 2.1. *([1,15,18]) Let K be a non-trivial knot and let D be an oriented diagram of K with $c(D) = c(K)$, where $c(K)$ denotes the minimal crossing number of K. Then for any integer m,*

(1) $g_c(W_\pm(D, m)) = g_c(W_\pm(D, w(D)))$.
(2) $g_c(W_\pm(K, m)) \leqslant g_c(W_\pm(D, m)) = c(K)$.
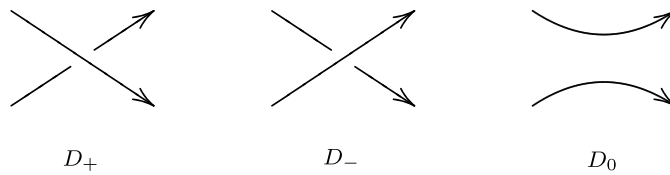
Fig. 4. Skein triple.

3. Maximum z-degree of HOMFLYPT polynomials

The *HOMFLYPT polynomial* $P_L(v, z)$ (or $P(L)$ for short) of an oriented link L in S^3 is defined by the following three axioms:

(1) $P_L(v, z)$ is invariant under ambient isotopy of L.
(2) If O is the trivial knot, then $P_O(v, z) = 1$.
(3) If L_+, L_- and L_0 have diagrams D_+, D_- and D_0 which differ as shown in Fig. 4, then $v^{-1}P_{L_+}(v, z) - vP_{L_-}(v, z) = zP_{L_0}(v, z)$.

Let L be an oriented link and let D be its oriented diagram. Then $P_L(v, z)$ can be computed recursively by using a skein tree, switching and smoothing crossings of D until the terminal nodes are labeled with trivial links. Observe that

$$P_{L_+}(v, z) = v^2 P_{L_-}(v, z) + vz P_{L_0}(v, z), \tag{3.3}$$

$$P_{L_-}(v, z) = v^{-2} P_{L_+}(v, z) - v^{-1}z P_{L_0}(v, z). \tag{3.4}$$

Set $\delta = (v^{-1} - v)z^{-1}$. If $L_1 \sqcup L_2$ denotes the disjoint union of oriented links L_1 and L_2, then $P_{L_1 \sqcup L_2}(v, z) = \delta P_{L_1}(v, z)P_{L_2}(v, z)$ [3,6].

For the HOMFLYPT polynomial $P_L(v, z)$ of a link L, we denote the maximum degree in z of $P_L(v, z)$ by $\max \deg_z P_L(v, z)$ or $M(L)$ for short. Let L_+, L_- and L_0 denote the links with the diagrams D_+, D_- and D_0, respectively, as shown in Fig. 4. Note that the degree of the sum of two polynomials cannot exceed the larger of their two degrees and is equal to the maximum of them if the two degrees are distinct. Hence it follows from (3.3) and (3.4) that

$$M(L_+) \leqslant \max\{M(L_-), M(L_0) + 1\},$$
$$M(L_-) \leqslant \max\{M(L_+), M(L_0) + 1\},$$
$$M(L_0) \leqslant \max\{M(L_+), M(L_-)\} - 1.$$

Here, the equality holds if the two terms in the right-hand side of the inequality are distinct.

Proposition 3.1. *Let K be an oriented knot and let D be an oriented diagram of K.*

(1) *For any integer m and $\epsilon = +$ or $-$,*

$$M(W_2(D, m)) \leqslant \max\{M(W_\epsilon(D, m)), 0\} - 1.$$

In particular, if $M(W_\epsilon(K, m)) > 0$, then the equality holds, i.e.,

$$M(W_2(D, m)) = M(W_\epsilon(D, m)) - 1. \tag{3.5}$$

(2) *For any integer m, $M(W_2(D, w(D))) \leqslant \max\{M(W_2(D, m)), 1\}$.*
   In particular, if $M(W_2(D, w(D))) \neq 1$, then the equality holds, i.e.,

$$M(W_2(D, w(D))) = M(W_2(D, m)). \tag{3.6}$$

Proof. (1) Switching one of the two crossings in the clasp of $W_+(D, m)$, we get

$$v^{-1}P_{\,\substack{\raisebox{2pt}{$\nearrow$}\\ \bigcirc}}(v, z) - vP_{\,\substack{\raisebox{2pt}{$\nearrow$}\\ \bigcirc}}(v, z) = zP_{\,\substack{\raisebox{2pt}{$\nearrow$}\\ \bigcirc}}(v, z),$$

$$v^{-1}P_{W_+(D,m)}(v, z) - vP_{\,\bigcirc}(v, z) = zP_{W_2(D,m)}(v, z),$$

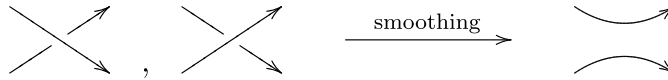$$P_{W_2(D,m)}(v, z) = v^{-1}z^{-1}P_{W_+(D,m)}(v, z) - vz^{-1}.$$

Fig. 5. Crossing smoothing.

This gives the inequality $M(W_2(D, m)) \leqslant \max\{M(W_+(D, m)), 0\} - 1$. Similarly, we obtain the inequality $M(W_2(D, m)) \leqslant \max\{M(W_-(D, m)), 0\} - 1$. It is obvious that the equality holds if $M(W_{\pm}(D, m)) > 0$.

(2) Let K be a non-trivial oriented knot and let D be an oriented diagram of K. Let $W_2(D, m)$ be the canonical diagram of the m-twisted doubled link $W_2(K, m)$ associated with D. We remind the reader that $W_2(D, m)$ is the 2-parallel link diagram of D with $m - w(D)$ full-twists. Let $n = m - w(D)$. The proof proceeds by induction on $|n|$.

If $n = 0$, then the assertion is obvious. Assume that $|n| \geqslant 1$ and the assertion holds for all $k < |n|$. Switching one of the $2n$ crossings among the n full-twists in $W_2(D, m)$ yields $W_2(D, w(D) + \frac{n}{|n|}(|n| - 1))$ (after isotopy), while smoothing the crossing yields the unknot $\bigcirc$, and so

$$P_{W_2(D, w(D)+n-1)}(v, z) = v^2 P_{W_2(D, m)}(v, z) + vz P_{\bigcirc}(v, z), \quad \text{if } n \geqslant 0,$$

$$P_{W_2(D, w(D)+n+1)}(v, z) = v^{-2} P_{W_2(D, m)}(v, z) - v^{-1} z P_{\bigcirc}(v, z), \quad \text{if } n < 0.$$

Since $P_{\bigcirc}(v, z) = 1$, if follows that

$$M\left(W_2\left(D, w(D) + \frac{n}{|n|}(|n| - 1)\right)\right) \leqslant \max\{M(W_2(D, m)), 1\}, \tag{3.7}$$

where the equality holds when $M(W_2(D, m)) \neq 1$. By induction hypothesis, it follows that

$$M\left(W_2\left(D, w(D)\right)\right) \leqslant \max\left\{M\left(W_2\left(D, w(D) + \frac{n}{|n|}(|n| - 1)\right)\right), 1\right\}, \tag{3.8}$$

where the equality holds when $M(W_2(D, w(D) + \frac{n}{|n|}(|n| - 1))) \neq 1$. Combining (3.7) and (3.8), we obtain the assertion and complete the proof. $\square$

Let D be an oriented link diagram. The *Seifert circles* of D are simple closed curves obtained from D by smoothing each crossing as illustrated in Fig. 5. We denote by $s(D)$ the number of the Seifert circles of D.

Theorem 3.2. *([13, Theorem 2]) For any oriented diagram D of an oriented knot or link L,*

$$\max \deg_z P_L(v, z) \leqslant c(D) - s(D) + 1, \tag{3.9}$$

where $c(D)$ is the number of crossings of the diagram D and $s(D)$ is the number of the Seifert circles of D.

We note that the equality in (3.9) holds for alternating links, positive links, and many other links.

Let D be an oriented diagram of an oriented knot or link L, let μ denote the number of components of L. Then the Euler characteristic $\chi(\Sigma(D))$ of the canonical Seifert surface $\Sigma(D)$ associated with D is given by

$$\chi\left(\Sigma(D)\right) = s(D) - c(D) = 2 - 2g\left(\Sigma(D)\right) - \mu.$$

Then it follows from (3.9) that for every canonical Seifert surface $\Sigma(D)$ for L, we have

$$\max \deg_z P_L(v, z) \leqslant c(D) - s(D) + 1 = 1 - \chi\left(\Sigma(D)\right) = 2g\left(\Sigma(D)\right) + \mu - 1.$$

Therefore, for a knot K, we obtain

$$\frac{1}{2} \max \deg_z P_K(v, z) \leqslant g_c(K). \tag{3.10}$$

Proposition 3.3. *Let K be a knot in S^3 with minimal crossing number $c(K)$ and let $W_{\pm}(K, m)$ be the m-twisted positive/negative Whitehead double of K. If D is an oriented diagram of K with $c(D) = c(K)$, then*

$$\frac{1}{2} \max \deg_z P_{W_{\pm}(K, m)}(v, z) \leqslant g_c\left(W_{\pm}(K, m)\right)$$

$$\leqslant g_c\left(W_{\pm}(D, m)\right) = c(K). \tag{3.11}$$
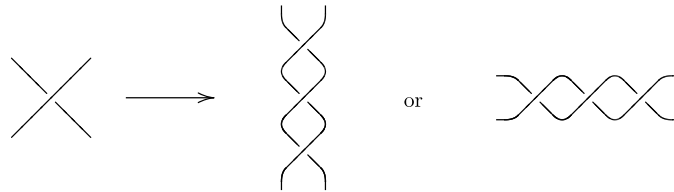
Fig. 6. Three half-twist.

Proof. This follows from Proposition 2.1 and the inequality (3.10) at once. $\square$

In the rest of this section, we briefly review Tripp's conjecture for the canonical genus of Whitehead doubles of knots. For more details, see [1,15,18]. In [18], Tripp proved that the canonical genus of an m-twisted Whitehead double $W_\pm(T(2,n),m)$ of the torus knot $T(2,n)$ is equal to its crossing number, that is, $g_c(W_\pm(T(2,n),m)) = n = c(T(2,n))$. The main part of the proof is to show that the maximum z-degree of HOMFLYPT polynomial of Whitehead doubles of $T(2,n)$ is equal to $2c(T(2,n))$. Then he made the following:

Conjecture 3.4. *(J.J. Tripp [18]) Let K be any knot with the crossing number $c(K)$. Then for any integer m,*

$$g_c\big(W_\pm(K,m)\big) = c(K). \tag{3.12}$$

In [15], Nakamura has extended Tripp's argument to show that for 2-bridge knot K, Conjecture 3.4 holds. He also observed that the torus knot $T(4,3)$, which is not an alternating knot, does not satisfy the equality (3.12) and modified Tripp's conjecture to Conjecture 1.1 in Section 1. In [1], Brittenham and Jensen showed that Conjecture 1.1 holds for alternating pretzel knots $P(k_1,\ldots,k_n)$, $k_1,\ldots,k_n \geqslant 1$ [1, Theorem 1]. The main tool of the proof is the following, Proposition 3.5, obtained by applying Proposition 3.6 twice, which give a method for building new knots K satisfying $\max\deg_z P_{W_\pm(K,m)}(v,z) = 2c(K)$.

Proposition 3.5. *([1, Proposition 2]) If K' is a knot satisfying*

$$\max\deg_z P_{W_\pm(K',m)}(v,z) = 2c\big(K'\big),$$

and if for a $c(K')$-minimizing diagram D' for K' we replace a crossing of D', thought of as a half-twist, with three half-twists as shown in Fig. 6, producing a knot K, then

$$\max\deg_z P_{W_\pm(K,m)}(v,z) = 2c(K),$$

and therefore $g_c(W_\pm(K,m)) = c(K)$.

Proposition 3.6. *([1, Proposition 4]) If L' is a non-split link with a diagram D' satisfying $c(D') = c(L')$ and*

$$\max\deg_z P_{W_2(D')}(v,z) = 2c\big(D'\big) - 1,$$

and L is a link having diagram D obtained from D' by replacing a crossing in the diagram D' with a full-twist (so that $c(D) = c(D')+1$), then

$$\max\deg_z P_{W_2(D)}(v,z) = 2c(D) - 1 = \max\deg_z P_{W_2(D')}(v,z) + 2.$$

In fact, Brittenham and Jensen proved that Conjecture 1.1 holds for a larger class of alternating knots, including $(2,n)$-torus knots, 2-bridge knots, and alternating pretzel knots, as in the following Proposition 3.7:

Proposition 3.7. *([1, Proposition 3]) Let $\mathcal{K}$ be the class of knots having diagrams which can be obtained from the standard diagram of the left- or right-handed trefoil knot $T(2,3)$, the $(2,3)$-torus knot, by repeatedly replacing a crossing, thought of as a half-twist, by a full-twist. Then for every $K \in \mathcal{K}$,*

$$\max\deg_z P_{W_\pm(K,m)}(v,z) = 2c(K),$$

and so $g_c(W_\pm(K,m)) = c(K)$.

The remaining part of this paper will be devoted to enlarge the class $\mathcal{K}$ in Proposition 3.7 by applying Brittenham and Jensen's argument starting with a certain class of closed quasitoric braids.
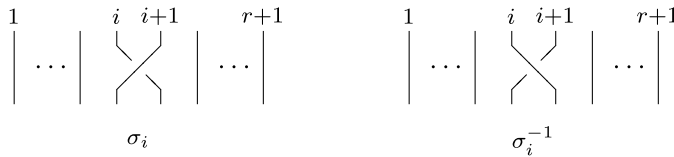
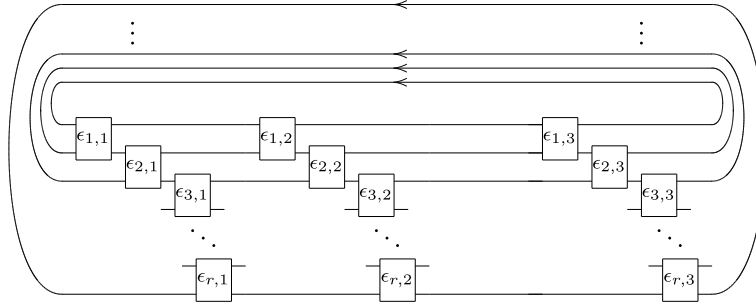**Fig. 7.** σ_i and σ_i^{-1}.

Fig. 8. Oriented closed braid $\hat{\beta}_r$.

4. Maximum z-degree of HOMFLYPT polynomials for doubled links of closed quasitoric braids $T(r+1, 3)$

Let $r \geqslant 1$ be an arbitrary given integer and let B_{r+1} be the $(r+1)$-strand braid group with the standard generators $\sigma_1, \sigma_2, \ldots, \sigma_r$ as shown in Fig. 7.

We recall that a toric braid $T(p, q)$ of type (p, q) is a p-strand braid given by the following formula:

$$T(p, q) = (\sigma_1 \cdots \sigma_{p-1})^q.$$

The closures of toric braids yield all torus knots and links. In 2002, Manturov showed that all knots and links can be represented by the closures of a small class of braids, called quasitoric braids. We briefly review here the quasitoric braids; for more details, see [11].

Let $m \geqslant 1$ and $n \geqslant 1$ be two integers. A braid β is said to be a *quasitoric braid* of type (m, n) if it can be expressed as an $(m + 1)$-braid of the form

$$\beta = \left(\sigma_1^{\epsilon_{1,1}} \sigma_2^{\epsilon_{2,1}} \cdots \sigma_m^{\epsilon_{m,1}}\right)\left(\sigma_1^{\epsilon_{1,2}} \sigma_2^{\epsilon_{2,2}} \cdots \sigma_m^{\epsilon_{m,2}}\right) \cdots \left(\sigma_1^{\varepsilon_{1,n}} \sigma_2^{\epsilon_{2,n}} \cdots \sigma_m^{\epsilon_{m,n}}\right),$$

where $\epsilon_{i,j} = \pm 1$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In other words, a quasitoric braid of type (m, n) is a braid obtained from the standard diagram of the toric braid $T(m, n)$ by switching some crossing types. It is worth noting that the quasitoric m-braids form a proper subgroup of the m-braid group B_m (see [11, Proposition 1]). One of the particular utilities of the quasitoric braids is the following:

Theorem 4.1. *([11]) Any link can be obtained as a closure of some quasitoric braid.*

In this section we consider a special class of quasitoric braids β_r of type $(r+1, 3)$ for all integers $r \geqslant 1$, which is an $(r+1)$-braid of the form:

$$\beta_r = \left(\sigma_r^{\epsilon_{1,1}} \sigma_{r-1}^{\epsilon_{2,1}} \cdots \sigma_1^{\epsilon_{r,1}}\right)\left(\sigma_r^{\epsilon_{1,2}} \sigma_{r-1}^{\epsilon_{2,2}} \cdots \sigma_1^{\epsilon_{r,2}}\right)\left(\sigma_r^{\epsilon_{1,3}} \sigma_{r-1}^{\epsilon_{2,3}} \cdots \sigma_1^{\epsilon_{r,3}}\right), \tag{4.13}$$

where

$$\epsilon_{i,j} = \pm 1 \quad (1 \leqslant i \leqslant r, \ 1 \leqslant j \leqslant 3),$$

$$\epsilon_{i,j}\epsilon_{i,j+1} > 0 \quad (1 \leqslant i \leqslant r, \ 1 \leqslant j \leqslant 2),$$

$$\epsilon_{i,j}\epsilon_{i+1,j} < 0 \quad (1 \leqslant i \leqslant r - 1, \ 1 \leqslant j \leqslant 3). \tag{4.14}$$

Let $w(\beta_r)$ denote the exponent sum of β_r, i.e., $w(\beta_r) = \sum_{i=1}^{r} \sum_{j=1}^{3} \epsilon_{i,j}$. Note that $w(\beta_r)$ is just the writhe of the oriented link $\hat{\beta}_r$, the closure of β_r.

Remark 4.2. Let $\hat{\beta}_r$ denote the closure of β_r with the orientation as shown in Fig. 8. Then:

(1) $\hat{\beta}_1$ is the right-handed trefoil knot $T(2, 3)$ or the left-handed trefoil knot $T(2, 3)^*$ according as $\epsilon_{1,1} = 1$ or $\epsilon_{1,1} = -1$. And, $\hat{\beta}_2$ is the Borromean ring (see Fig. 12).
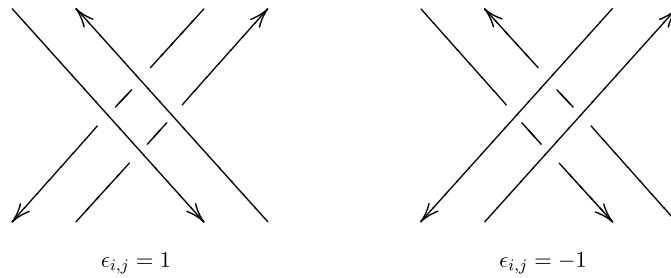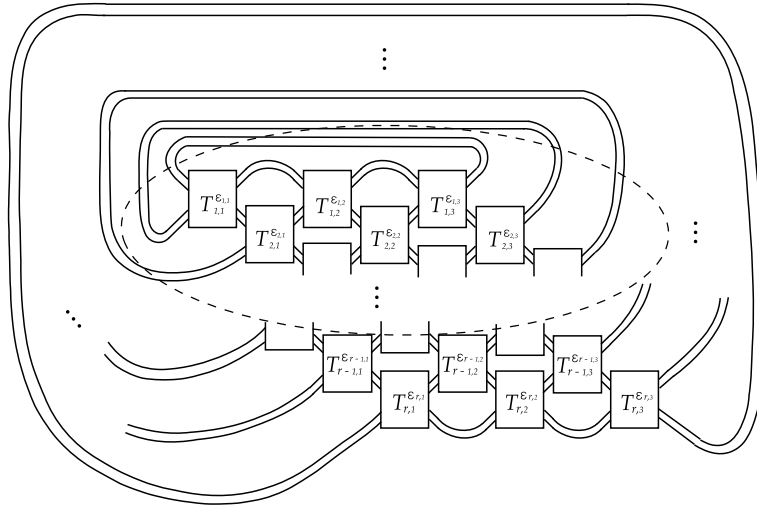
Fig. 9. $T_{i,j}^{\epsilon_{i,j}}$.

Fig. 10. $W_2(\hat{\beta}_r)$.

(2) $\hat{\beta}_r$ is a non-split alternating link without nugatory crossings and so is a minimal crossing diagram. Hence it follows that the minimal crossing number $c(\hat{\beta}_r)$ of $\hat{\beta}_r$ is given by

$$c(\hat{\beta}_r) = \sum_{i=1}^{r} \sum_{j=1}^{3} |\epsilon_{i,j}| = 3r. \tag{4.15}$$

(3) If $r = 3k - 1$ for some integer $k \geqslant 1$, then the closed braid $\hat{\beta}_r$ is an oriented link of three components, otherwise it is always an oriented knot.

For a given oriented knot or link diagram D, let $W_2(D)$ denote the doubled link represented by the oriented link diagram obtained from D as follows: Draw a parallel copy of D pushed off of D to the left according to the orientation of D, and then orient the parallel copy in the opposite direction. Notice that if D is a knot diagram, then $W_2(D) = W_2(D, w(D))$.

Now we consider the doubled link $W_2(\hat{\beta}_r)$ of the closed quasitoric braid $\hat{\beta}_r$. Notice that the link $W_2(\hat{\beta}_r)$ has no full-twists of two parallel strands and each crossing $\epsilon_{i,j}$ of the closed braid diagram $\hat{\beta}_r$ as shown in Fig. 8 produces a tangle $T_{i,j}^{\epsilon_{i,j}}$ as shown in Fig. 9 in the standard diagram of $W_2(\hat{\beta}_r)$ associated with $\hat{\beta}_r$ according as $\epsilon_{i,j} = 1$ or $\epsilon_{i,j} = -1$. The standard diagram of $W_2(\hat{\beta}_r)$ is equivalent to the diagram shown in Fig. 10 in which each rectangle labeled $T_{i,j}^{\epsilon_{i,j}}$ corresponds to the crossing $\epsilon_{i,j}$ of $\hat{\beta}_r$.

In order to state the main result, we first define some notation. For our convenience, we represent the standard diagram $W_2(\hat{\beta}_r)$ in Fig. 10 the $r \times 3$ matrix Q_r with the entries $T_{i,j}^{\epsilon_{i,j}}$:

$$Q_r = \begin{pmatrix} T_{1,1}^{\epsilon_{1,1}} & T_{1,2}^{\epsilon_{1,2}} & T_{1,3}^{\epsilon_{1,3}} \\ T_{2,1}^{\epsilon_{2,1}} & T_{2,2}^{\epsilon_{2,2}} & T_{2,3}^{\epsilon_{2,3}} \\ \vdots & \vdots & \vdots \\ T_{r-1,1}^{\epsilon_{r-1,1}} & T_{r-1,2}^{\epsilon_{r-1,2}} & T_{r-1,3}^{\epsilon_{r-1,3}} \\ T_{r,1}^{\epsilon_{r,1}} & T_{r,2}^{\epsilon_{r,2}} & T_{r,3}^{\epsilon_{r,3}} \end{pmatrix}.$$
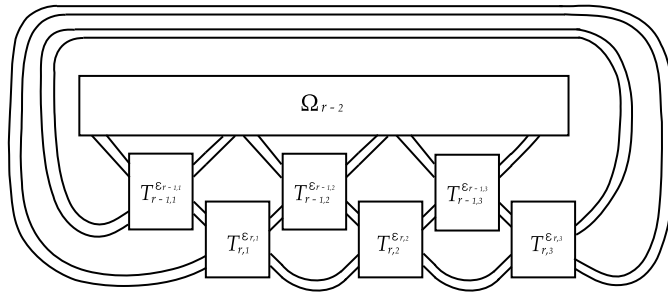
Fig. 11. $D_r = W_2(\hat{\beta}_r)$ with $\epsilon_{r,1} = 1$.

In the case that $\epsilon_{r,1} = 1$ (and hence $\epsilon_{r,2} = \epsilon_{r,3} = 1$), we will denote the diagram $W_2(\hat{\beta}_r)$ simply by D_r and let N_r denote the integer given by

$$N_r = c(D_r) - s(D_r) + 1 = 6r - 1 \quad (r \geqslant 1). \tag{4.16}$$

In what follows, instead of the diagram D_r illustrated in Fig. 10, we use a shortcut diagram shown in Fig. 11 for D_r for the sake of simplicity.

Example 4.3. Let β_2 be the quasitoric braid of type $(3, 3)$, i.e.,

$$\beta_2 = (\sigma_2 \sigma_1^{-1})(\sigma_2 \sigma_1^{-1})(\sigma_2 \sigma_1^{-1}).$$

Then the closed braid $\hat{\beta}_2$ is the Borromean ring (see Fig. 12) and the 2-parallel link $D_2 = W_2(\hat{\beta}_2)$ is represented by 2×3 matrix Q_2:

$$Q_2 = \begin{pmatrix} T_{1,1}^1 & T_{1,2}^1 & T_{1,3}^1 \\ T_{2,1}^{-1} & T_{2,2}^{-1} & T_{2,3}^{-1} \end{pmatrix}.$$

By a direct computation, we obtain

$$\begin{aligned}
P_{W_2(\hat{\beta}_2)}(v, z) = {}& z^{-5}\bigl(-v^5 + 5v^3 - 10v + 10v^{-1} - 5v^{-3} + v^{-5}\bigr) \\
&+ z^{-1}\bigl(8v^5 - 40v^3 + 80v - 80v^{-1} + 40v^{-3} - 8v^{-5}\bigr) \\
&+ z\bigl(12v^5 - 68v^3 + 144v - 144v^{-1} + 68v^{-3} + 12v^{-5}\bigr) \\
&+ z^3\bigl(2v^5 - 22v^3 + 56v - 56v^{-1} + 22v^{-3} - 2v^{-5}\bigr) \\
&+ z^5\bigl(-v^7 - 5v^5 + 13v^3 - 7v + 7v^{-1} - 13v^{-3} + 5v^{-5} + v^{-7}\bigr) \\
&+ z^7\bigl(-2v^5 + 8v^3 + 10v - 10v^{-1} - 8v^{-3} + 2v^{-5}\bigr) \\
&+ z^9\bigl(v^3 + 11v - 11v^{-1} - v^{-3}\bigr) + z^{11}\bigl(2v - 2v^{-1}\bigr).
\end{aligned}$$

Hence the maximal z-degree of the HOMFLYPT polynomial $P_{W_2(\hat{\beta}_2)}(v, z)$ of the doubled link $W_2(\hat{\beta}_2)$ is given by

$$\max \deg_z P_{W_2(\hat{\beta}_2)}(v, z) = 11 = 2 \cdot 6 - 1 = 2c(\hat{\beta}_2) - 1.$$

On the other hand, let $\hat{\beta}_2^*$ denote the mirror image of $\hat{\beta}_2$. Then we also have

$$\begin{aligned}
\max \deg_z P_{W_2(\hat{\beta}_2^*)}(v, z) &= \max \deg_z P_{W_2(\hat{\beta}_2)}\bigl(v^{-1}, z\bigr) \\
&= 11 = 2 \cdot 6 - 1 = 2c\bigl(\hat{\beta}_2^*\bigr) - 1.
\end{aligned}$$

Now we construct a partial skein tree as shown in Fig. 13 for the tangle $T_{r,3}^1$ in D_r of the left-hand side of Fig. 9. We label all nodes in the skein tree with A, B, E_1, F_1, F_2, F_3, F_4, and G as shown in Fig. 13. Now let D_r^i $(1 \leqslant i \leqslant 8)$ denote the link diagram represented by the $r \times 3$ matrix:

$$D_r^i = \begin{pmatrix} T_{1,1}^{\epsilon_{1,1}} & T_{1,2}^{\epsilon_{1,2}} & T_{1,3}^{\epsilon_{1,3}} \\ T_{2,1}^{\epsilon_{2,1}} & T_{2,2}^{\epsilon_{2,2}} & T_{2,3}^{\epsilon_{2,3}} \\ \vdots & \vdots & \vdots \\ T_{r-1,1}^{\epsilon_{r-1,1}} & T_{r-1,2}^{\epsilon_{r-1,2}} & T_{r-1,3}^{\epsilon_{r-1,3}} \\ T_{r,1}^1 & T_{r,2}^1 & T_i \end{pmatrix}.$$
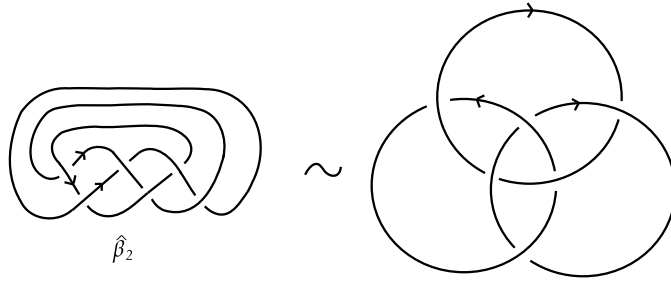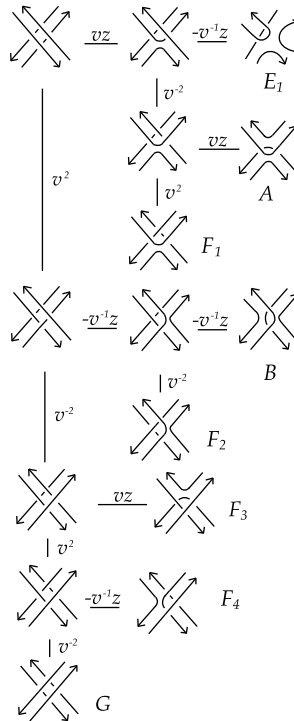
Fig. 12. Borromean ring.

Fig. 13. A partial skein tree for $T_{r,3}^1$.

That is, D_r^i is the link diagram obtained from the link diagram D_r by replacing the tangle $T_{r,3}^1$ with the tangle T_i, where

$$T_1 = A, \qquad T_2 = B, \qquad T_3 = E_1, \qquad T_4 = F_1, \qquad T_5 = F_2, \qquad T_6 = F_3, \qquad T_7 = F_4, \qquad T_8 = G.$$

Hence two diagrams D_r and D_r^i are identical except for the tangle corresponding to the $(r, 3)$-entry of the matrix notations. In these terminologies, we have the following Lemma 4.4 that will play an essential role in the proof of Theorem 4.5 below.

Lemma 4.4.

(1) $\max \deg_z P_{D_r^4}(v, z) \leqslant N_r - 3$ *if* $r \geqslant 3$.
(2) $\max \deg_z P_{D_r^5}(v, z) \leqslant N_r - 3$ *if* $r \geqslant 3$.
(3) $\max \deg_z P_{D_r^6}(v, z) \leqslant N_r - 3$ *if* $r \geqslant 3$.
(4) $\max \deg_z P_{D_r^7}(v, z) \leqslant N_r - 3$ *if* $r \geqslant 3$.
(5) $\max \deg_z P_{D_r^8}(v, z) \leqslant N_r - 4$ *if* $r \geqslant 3$.

The proof of this Lemma 4.4 will be given in the final Section 6. Now, let us state our main theorem of this section.

Theorem 4.5. *Let* β_r *(*$r \geqslant 1$*) be a quasitoric braid of type* $(r + 1, 3)$ *as in* (4.13) *and let* $W_2(\hat{\beta}_r)$ *be the doubled link of* $\hat{\beta}_r$. *Then*

$$\max \deg_z P_{W_2(\hat{\beta}_r)}(v, z) = 2c(\hat{\beta}_r) - 1 = 6r - 1. \tag{4.17}$$
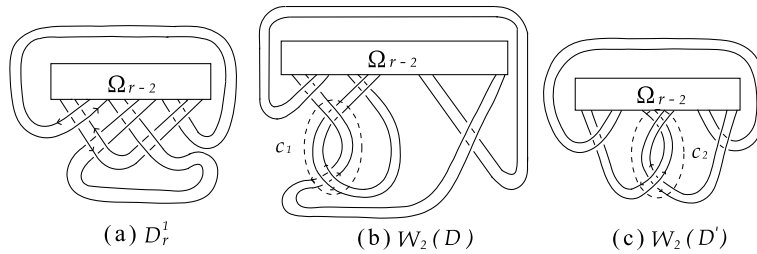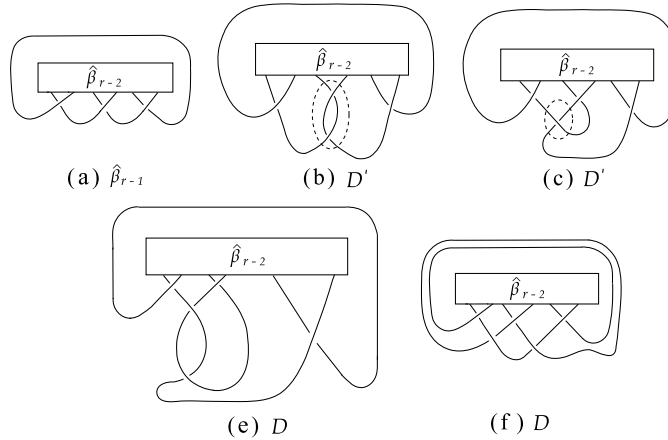
Fig. 14. D_r^1.

Fig. 15. L'.

Proof. We prove the assertion (4.17) by induction on r. If $r = 1$, then $\beta_1 = \sigma_1^3$ or σ_1^{-3}, and so $\hat{\beta}_1$ is the right-handed trefoil knot or the left-handed trefoil knot. In either cases, it is immediate from direct calculations that

$$\max\deg_z P_{W_2(\hat{\beta}_1)}(v,z) = \max\deg_z P_{D_1}(v,z) = 5 = 2 \cdot 3 - 1 = 2c(\hat{\beta}_1) - 1.$$

(In the case that $r = 2$, it follows from Example 4.3 that the assertion (4.17) also holds.)

Now we assume that $r \geqslant 3$ and the assertion (4.17) holds for every integer $\leqslant r - 1$. We consider two cases separately.

Case I. $\epsilon_{r,3} = 1$. First we observe from (4.14) that $\epsilon_{r,1} = \epsilon_{r,2} = 1$. In this case, we have $W_2(\hat{\beta}_r) = D_r$ by the notational convention above.

Claim. $\max\deg_z P_{D_r}(v,z) = 2c(\hat{\beta}_r) - 1 = 6r - 1$.

Proof. From the skein relation for the HOMFLYPT polynomial and a partial skein tree for $T_{r,3}^1$ in Fig. 13, we obtain

$$P_{D_r}(v,z) = \left(P_{D_r^1}(v,z) + P_{D_r^2}(v,z) - P_{D_r^3}(v,z)\right)z^2 + \left(vP_{D_r^4}(v,z) - v^{-1}P_{D_r^5}(v,z) + vP_{D_r^6}(v,z) - vP_{D_r^7}(v,z)\right)z$$
$$+ P_{D_r^8}(v,z). \tag{4.18}$$

We observe that the link diagram D_r^1 is isotopic to the link diagram (a) of Fig. 14, which is isotopic to the diagram (b) in Fig. 14.

Now let L' be an oriented link having diagram D' obtained from the standard closed braid diagram of a non-split alternating link $\hat{\beta}_{r-1}$ by replacing the crossing $\sigma_1^{\epsilon_{r-1,2}}$ in $\hat{\beta}_{r-1}$ with a full-twist (so that $c(D') = c(\hat{\beta}_{r-1}) + 1$) as illustrated in (a) and (b) of Fig. 15. By induction hypothesis, we have

$$\max\deg_z P_{W_2(\hat{\beta}_{r-1})}(v,z) = 2c(\hat{\beta}_{r-1}) - 1 = 6(r-1) - 1 \quad (r \geqslant 2). \tag{4.19}$$

By Proposition 3.6, we then obtain

$$\max\deg_z P_{W_2(L')}(v,z) = 2c(D') - 1$$
$$= \max\deg_z P_{W_2(\hat{\beta}_{r-1})}(v,z) + 2. \tag{4.20}$$
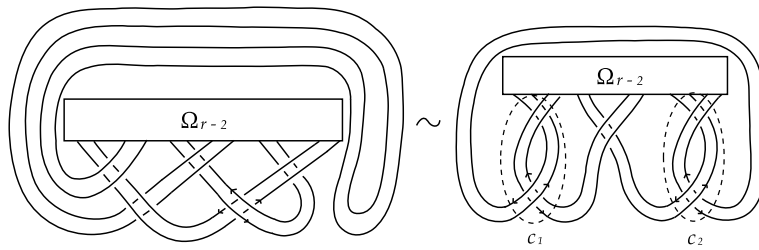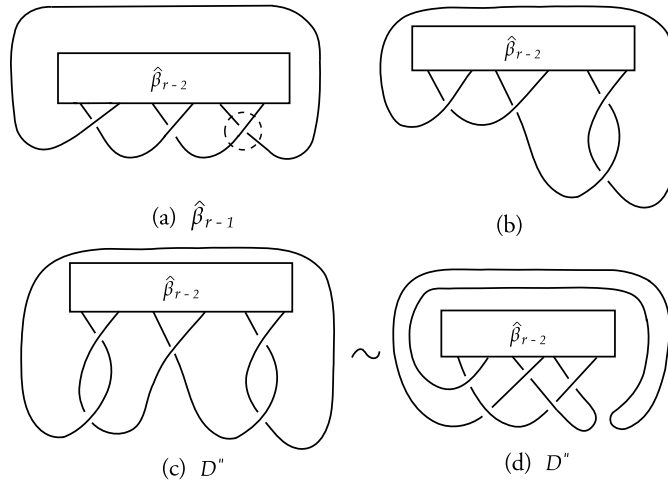
Fig. 16. D_r^2.

Fig. 17. L''.

It is obvious that L' is a non-split alternating link satisfying $c(L') = c(D')$ and the doubled link $W_2(L')$ has a diagram $W_2(D')$ in (c) of Fig. 14. Now let L be an oriented link having diagram D obtained from D' by replacing a crossing in D' with a full-twist as illustrated in (c), (e) and (f) of Fig. 15 so that $c(D) = c(D') + 1$. Then the doubled link $W_2(L)$ has a diagram $W_2(D)$ in (b) of Fig. 14. By Proposition 3.6 again, we have

$$\max \deg_z P_{W_2(L)}(v, z) = 2c(D) - 1$$
$$= \max \deg_z P_{W_2(L')}(v, z) + 2. \tag{4.21}$$

Then we obtain from (4.20) and (4.21) that

$$\max \deg_z P_{D_r^1}(v, z) = \max \deg_z P_{W_2(L)}(v, z)$$
$$= \max \deg_z P_{W_2(\hat{\beta}_{r-1})}(v, z) + 4$$
$$= \max \deg_z P_{D_{r-1}}(v, z) + 4. \tag{4.22}$$

Similarly, we observe that the link diagram D_r^2 is isotopic to the link diagram in the left-hand side of Fig. 16, which is isotopic to the diagram in the right-hand side of Fig. 16.

Let L'' be an oriented link having diagram D'' obtained from the standard closed braid diagram of a non-split alternating link $\hat{\beta}_{r-1}$ by replacing two crossings $\sigma_1^{\epsilon_{r-1,1}}$ and $\sigma_1^{\epsilon_{r-1,3}}$ in $\hat{\beta}_{r-1}$ with full-twists, respectively, as illustrated in Fig. 17. So $c(D'') = c(\hat{\beta}_{r-1}) + 2$. It is obvious that the doubled link $W_2(L'')$ has a diagram in the right-side of Fig. 16. By induction hypothesis and Proposition 3.6, we then have

$$\max \deg_z P_{D_r^2}(v, z) = \max \deg_z P_{W_2(L'')}(v, z)$$
$$= 2c(D'') - 1$$
$$= \max \deg_z P_{W_2(\hat{\beta}_{r-1})}(v, z) + 4$$
$$= \max \deg_z P_{D_{r-1}}(v, z) + 4. \tag{4.23}$$

Since max $\deg_z P_{D_r^3}(v, z)$ is too low to interfere with our main calculation by applying Morton's inequality, we see that maximal degree in z for $P_{D_r^3}(v, z)$ does not contribute anything to max $\deg_z P_{D_r}(v, z)$. From (4.18), (4.22), (4.23) and Lemma 4.4, it is easily seen that

$$\max \deg_z P_{D_r}(v, z) = \max\{M(D_{r-1}) + 6, N_r - 2\}. \tag{4.24}$$

On the other hand, we see from (4.16) and (4.19) that

$$\begin{aligned} M(D_{r-1}) + 6 &= \max \deg_z P_{D_{r-1}}(v, z) + 6 \\ &= \max \deg_z P_{W_2(\hat{\beta}_{r-1})}(v, z) + 6 \\ &= \big(2c(\hat{\beta}_{r-1}) - 1\big) + 6 \\ &= 6r - 1 \\ &= N_r \quad (r \geqslant 2). \end{aligned} \tag{4.25}$$

Hence it follows from (4.24) and (4.25) that

$$\max \deg_z P_{D_r}(v, z) = N_r = \max \deg_z P_{D_{r-1}}(v, z) + 6. \tag{4.26}$$

Combining (4.19) and (4.26), we finally obtain

$$\begin{aligned} \max \deg_z P_{D_r}(v, z) &= \max \deg_z P_{D_{r-1}}(v, z) + 6 \\ &= 2c(\hat{\beta}_{r-1}) - 1 + 6 \\ &= 2\big(c(\hat{\beta}_{r-1}) + 3\big) - 1 \\ &= 2c(\hat{\beta}_r) - 1. \end{aligned}$$

Case II. $\epsilon_{r,3} = -1$.

In this case, it follows from the condition (4.14) that $\epsilon_{r,1} = \epsilon_{r,2} = -1$. Then it is easily seen that the corresponding link diagram $W_2(\hat{\beta}_r)$ is just the mirror image of the diagram D_r for which the assertion has already been established in the previous Case I. On the other hand, it is well known that if L^* is the mirror image of an oriented link L, then $P_{L^*}(v, z) = P_L(v^{-1}, z)$. This fact implies that $P_{W_2(\hat{\beta}_r)}(v, z) = P_{D_r}(v^{-1}, z)$. Hence

$$\begin{aligned} \max \deg_z P_{W_2(\hat{\beta}_r)}(v, z) &= \max \deg_z P_{D_r}\big(v^{-1}, z\big) \\ &= \max \deg_z P_{D_r}(v, z) \\ &= 2c(\hat{\beta}_r) - 1. \end{aligned}$$

Finally, it is straightforward from (4.15) that $2c(\hat{\beta}_r) - 1 = 6r - 1$ for each $r \geqslant 1$. This completes the proof of Theorem 4.5. ☐

5. A family of alternating knots for which Tripp's conjecture holds

Let us begin this section with the following:

Lemma 5.1. *Let* β_r $(r \geqslant 1)$ *be a quasitoric braid of type* $(r + 1, 3)$ *as in* (4.13). *If* L *is a link having diagram* D *obtained from the standard closed braid diagram of* $\hat{\beta}_r$ *as shown in Fig.* 8 *by replacing a crossing with a full-twist (so that* $c(D) = c(\hat{\beta}_r) + 1$), *then*

$$\max \deg_z P_{W_2(D)}(v, z) = 2c(D) - 1.$$

Proof. Let L' be the link represented by a quasitoric braid β_r. It is obvious that L' is a non-split alternating link with a diagram $D' = \hat{\beta}_r$ satisfying $c(L') = c(D') = 3r$. By Theorem 4.5, max $\deg_z P_{W_2(D')}(v, z) = 2c(D') - 1$. Hence the assertion follows from Proposition 3.6. ☐

Theorem 5.2. *Let* β_r $(r \geqslant 1)$ *be a quasitoric braid of type* $(r + 1, 3)$ *as in* (4.13) *and let* $\mathcal{K}_r$ *be the class consisting of the alternating knot* $\hat{\beta}_r$ *itself (if it is a knot) and all alternating knots having diagrams which can be obtained from the standard diagram of the closed braid* $\hat{\beta}_r$ *as shown in Fig.* 8, *by repeatedly replacing a crossing by a full-twist. Then for every* $K \in \mathcal{K}_r$ *and any integer* m,

$$\max \deg_z P_{W_\pm(K,m)}(v, z) = 2c(K), \tag{5.27}$$

and therefore

$$g_c\big(W_\pm(K, m)\big) = c(K).$$

Proof. Let K be an alternating knot in K_r. Then K has a diagram D which is obtained from the standard diagram of the closed braid $\hat{\beta}_r$ by repeatedly replacing a crossing by a full-twist. By Lemma 5.1 and repeatedly applying Proposition 3.6, we obtain

$$\max \deg_z P_{W_2(D)}(v, z) = 2c(D) - 1. \tag{5.28}$$

Now, for any given integer m, let $W_\pm(K, m)$ be the m-twisted positive/negative Whitehead double of K and let $W_\pm(D, m)$ be the canonical diagram for $W_\pm(K, m)$ associated with D. Since $c(D) > 3$, it follows from (5.28) and Proposition 3.1 that $\max \deg_z P_{W_\pm(K,m)}(v, z) > 0$ and hence $\max \deg_z P_{W_2(D, w(D))}(v, z) \neq 1$. By (3.5) and (3.6), we have

$$\begin{aligned}
\max \deg_z P_{W_\pm(K,m)}(v, z) &= \max \deg_z P_{W_\pm(D,m)}(v, z) \\
&= \max \deg_z P_{W_2(D,m)}(v, z) + 1 \\
&= \max \deg_z P_{W_2(D,w(D))}(v, z) + 1 \\
&= \max \deg_z P_{W_2(D)}(v, z) + 1 \\
&= 2c(D) - 1 + 1 \\
&= 2c(D) = 2c(K).
\end{aligned}$$

This establishes the desired identity (5.27).

Finally, it follows from (3.11) and (5.27) that

$$\begin{aligned}
c(K) &= \frac{1}{2} \max \deg_z P_{W_\pm(K,m)}(v, z) \leqslant g_c(W_\pm(K, m)) \\
&\leqslant g_c(W_\pm(D, m)) = c(K).
\end{aligned}$$

This gives $g_c(W_\pm(K, m)) = c(K)$ and competes the proof. □

Remark 5.3.

(1) The closure $\hat{\beta}_1$ of the quasitoric braid $\beta_1 = (\sigma^{\epsilon_{1,1}})^3$ is the right-handed trefoil or left-handed trefoil knot (see Remark 4.2(1)) and so the class $\mathcal{K}_1$ in Theorem 5.2 is just the class $\mathcal{K}$ in Proposition 3.7. So, in case of $r = 1$, Theorem 5.2 is the same as Proposition 3.7. Hence $\mathcal{K}_1$ contains all $(2, n)$-torus knots, all the 2-bridge knots, and all alternating pretzel knots.

(2) In [1], Brittenham and Jensen noticed that the Borromean ring L, the closure of the quasitoric braid β_2, satisfies $\max \deg_z P_{W_2(L)}(v, z) = 2c(L) - 1$ (see Example 4.3), which gives rise, using Proposition 3.6, to a family, it is indeed the family $\mathcal{K}_2$ in Theorem 5.2, of alternating knots satisfying the equality (3.12), different from the family $\mathcal{K}$ given by Proposition 3.7. On the other hand, it is clear that $\hat{\beta}_2 \notin \mathcal{K}_3$ and so $\mathcal{K}_3$ is also a family of alternating knots satisfying the equality (3.12), different from $\mathcal{K}_2$, and so on. Therefore, Theorem 5.2 provides an infinite sequence

$$\mathcal{K}_1(= \mathcal{K}), \mathcal{K}_2, \mathcal{K}_3, \ldots, \mathcal{K}_i, \ldots$$

of infinite families $\mathcal{K}_i$ of alternating knots satisfying Tripp–Nakamura's Conjecture. We define

$$\mathcal{K}^3 = \bigcup_{r=1}^{\infty} \mathcal{K}_r.$$

Then the infinite family $\mathcal{K}^3$ of alternating knots is an extension of the previous results of Tripp [18], Nakamura [15] and Brittenham and Jensen [1].

Example 5.4. Let $A = (n_{i,j})_{1 \leqslant i \leqslant r; 1 \leqslant j \leqslant 3}$ be an arbitrary given $r \times 3$ integral matrix, i.e.,

$$A = \begin{pmatrix} n_{1,1} & n_{1,2} & n_{1,3} \\ n_{2,1} & n_{2,2} & n_{2,3} \\ \vdots & \vdots & \vdots \\ n_{r,1} & n_{r,2} & n_{r,3} \end{pmatrix}.$$

Let K_A denote an oriented link in S^3 having a diagram D_A as shown in Fig. 18(a) in which each tangle labeled a non-zero integer $n_{i,j}$ denotes a vertical $n_{i,j}$ half-twists as shown in Fig. 18(b) or a horizontal $n_{i,j}$ half-twists.

Suppose that $n_{i,j}n_{i+1,j} < 0$ and $n_{i,j}n_{i,j+1} > 0$ for each $i = 1, 2, \ldots, r - 1$ and $j = 1, 2, 3$ and K_A is a knot (eventually, an alternating knot). Let $A' = (\epsilon_{i,j})_{1 \leqslant i \leqslant r; 1 \leqslant j \leqslant 3}$ be the integral matrix obtained from A by defining $\epsilon_{i,j} = \frac{n_{i,j}}{|n_{i,j}|}$
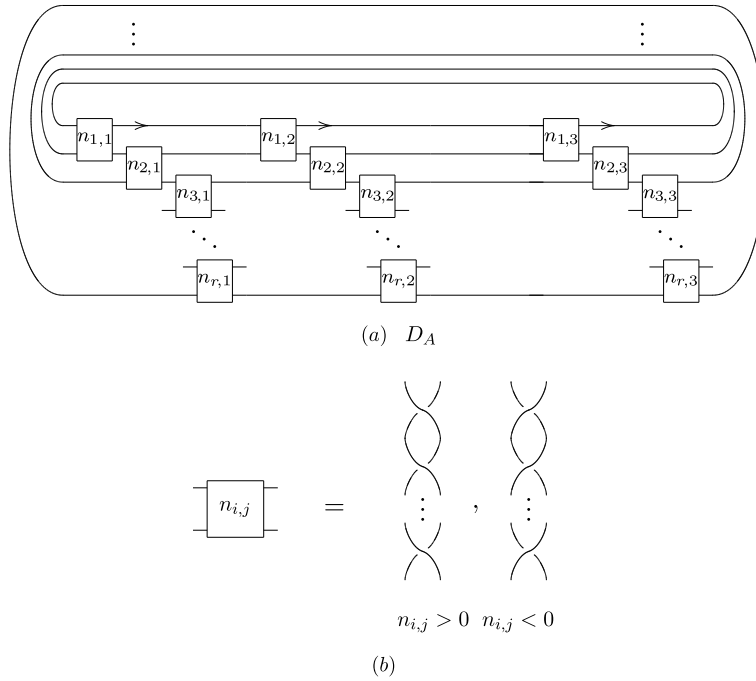
(a) D_A

$n_{i,j} > 0$ $n_{i,j} < 0$

(b)

Fig. 18. K_A.

$(1 \leqslant i \leqslant r; \ 1 \leqslant j \leqslant 3)$ and let $K_{A'}$ be the oriented alternating link having a diagram $D_{A'}$. Then $K_{A'}$ is the closure of a quasitoric braid β_r in (4.13). Then it follows from Theorem 5.2 that $K_A \in \mathcal{K}_r$ and so

$$\max \deg_z P_{W_2(K_A)}(v, z) = \max \deg_z P_{W_2(K_{A'})}(v, z) + 2 \sum_{i=1}^{r} \sum_{j=1}^{3} \big(|n_{i,j}| - 1 \big).$$

Consequently, for every integer m,

$$g_c \big(W_{\pm}(K_A, m) \big) = \sum_{i=1}^{r} \sum_{j=1}^{3} |n_{i,j}| = c(K_A).$$

6. Proof of Lemma 4.4

In this section, we prove Lemma 4.4. For this purpose, we first remind the reader Lemma 4.4. Recall that D_r denotes the doubled link $W_2(\hat{\beta}_r)$ corresponding to the matrix notation Q_r with $\epsilon_{r,3} = 1$ and D_r^i $(4 \leqslant i \leqslant 8)$ denotes the link diagram obtained from D_r by replacing $T_{r,3}^1$ with T_i, where $T_4 = F_1$, $T_5 = F_2$, $T_6 = F_3$, $T_7 = F_4$, $T_8 = G$ (cf. Section 4).

Lemma 4.4.

(1) $\max \deg_z P_{D_r^4}(v, z) \leqslant N_r - 3$ if $r \geqslant 3$.
(2) $\max \deg_z P_{D_r^5}(v, z) \leqslant N_r - 3$ if $r \geqslant 3$.
(3) $\max \deg_z P_{D_r^6}(v, z) \leqslant N_r - 3$ if $r \geqslant 3$.
(4) $\max \deg_z P_{D_r^7}(v, z) \leqslant N_r - 3$ if $r \geqslant 3$.
(5) $\max \deg_z P_{D_r^8}(v, z) \leqslant N_r - 4$ if $r \geqslant 3$.

Proof. (1) Consider a partial skein tree for D_r^4 $(r \geqslant 3)$ and isotopy deformations as shown in Fig. 19, which yields the identity:

$$P_{D_r^4}(v, z) = v^{-2} P_{a_6}(v, z) + v^{-3} z P_{a_5}(v, z) - v^{-2} z^2 P_{a_4}(v, z)$$
$$+ v^{-2} z^2 P_{a_3}(v, z) - v^{-3} z P_{a_2}(v, z) - v^{-1} z P_{a_1}(v, z). \tag{6.29}$$
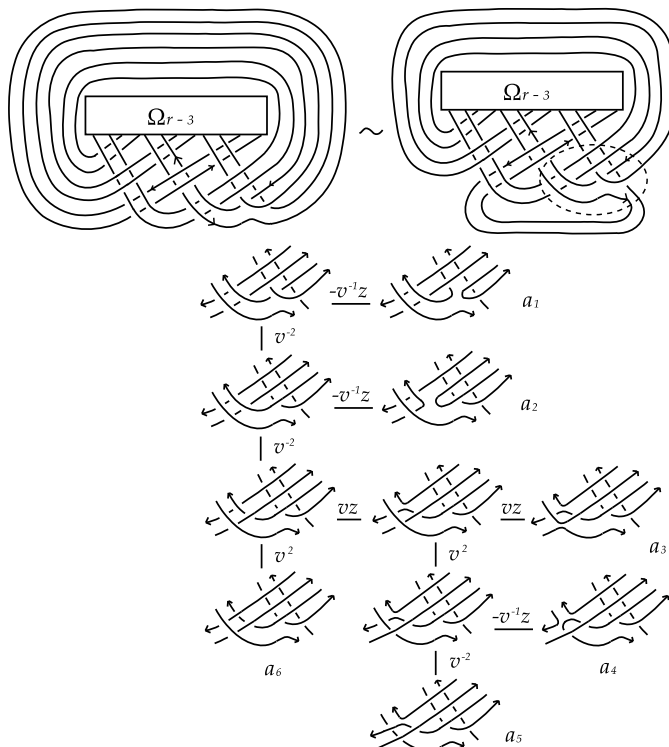
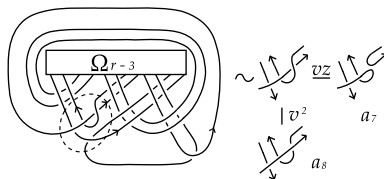**Fig. 19.** A partial skein tree for D_r^4.

Fig. 20. A partial skein tree for a_3.

It is clear from Fig. 19 that the link a_1 does not contribute anything to $\max\deg_z P_{D_r^4}(v, z)$. For the links a_2, a_4 and a_5, it follows from Morton's inequality in (3.9) that

$$\max\deg_z P_{a_2}(v, z) \leqslant c(a_2) - s(a_2) + 1$$
$$\leqslant \big(c(D_r) - 6\big) - \big(s(D_r) - 2\big) + 1$$
$$= N_r - 4, \tag{6.30}$$

$$\max\deg_z P_{a_4}(v, z) \leqslant c(a_4) - s(a_4) + 1$$
$$\leqslant \big(c(D_r) - 7\big) - \big(s(D_r) - 2\big) + 1$$
$$= N_r - 5, \tag{6.31}$$

$$\max\deg_z P_{a_5}(v, z) \leqslant c(a_5) - s(a_5) + 1$$
$$\leqslant \big(c(D_r) - 11\big) - \big(s(D_r) - 5\big) + 1$$
$$= N_r - 6. \tag{6.32}$$

For the link a_3, we obtain from Fig. 20 that

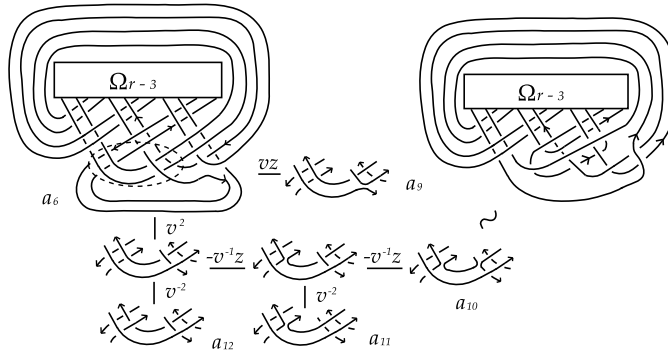$$P_{a_3}(v, z) = v^2 P_{a_8}(v, z) + vz P_{a_7}(v, z).$$

Fig. 21. A partial skein tree for a_6.

Clearly, the link a_7 does not contribute anything to $\max\deg_z P_{a_3}(v,z)$ and so by Morton's inequality,

$$
\begin{aligned}
\max\deg_z P_{a_3}(v,z) = \max\deg_z P_{a_8}(v,z) &\leqslant c(a_8) - s(a_8) + 1 \\
&\leqslant \big(c(D_r) - 13\big) - \big(s(D_r) - 6\big) + 1 \\
&= N_r - 7.
\end{aligned}
\tag{6.33}
$$

From (6.29)–(6.33) and Claim 1 below, we obtain

$$
\begin{aligned}
\max\deg_z P_{D_r^4}(v,z) &\leqslant \max\big\{M(a_6), M(a_5) + 1, M(a_4) + 2, M(a_3) + 2, M(a_2) + 1\big\} \\
&\leqslant \max\big\{M(a_6), N_r - 5, N_r - 3, N_r - 5, N_r - 3\big\} \\
&= N_r - 3.
\end{aligned}
$$

This establishes (1), as desired.

Claim 1. $M(a_6) = \max\deg_z P_{a_6}(v,z) \leqslant N_r - 3$ $(r \geqslant 3)$.

Proof. Consider a partial skein tree for a_6 and isotopy deformations as shown in Fig. 21, which gives the identity:

$$
P_{a_6}(v,z) = P_{a_{12}}(v,z) - v^{-1}z P_{a_{11}}(v,z) + z^2 P_{a_{10}}(v,z) + vz P_{a_9}(v,z).
\tag{6.34}
$$

Using Morton's inequality, we obtain

$$
\max\deg_z P_{a_{12}}(v,z) \leqslant \big(c(D_r) - 6\big) - \big(s(D_r) - 3\big) + 1 = N_r - 3,
\tag{6.35}
$$

$$
\max\deg_z P_{a_{11}}(v,z) \leqslant \big(c(D_r) - 8\big) - \big(s(D_r) - 4\big) + 1 = N_r - 4,
\tag{6.36}
$$

$$
\max\deg_z P_{a_{10}}(v,z) \leqslant \big(c(D_r) - 8\big) - \big(s(D_r) - 3\big) + 1 = N_r - 5.
\tag{6.37}
$$

By a partial skein tree for a_9 and isotopy deformations as shown in Fig. 22, we get

$$
P_{a_9}(v,z) = v^{-6} P_{a_{16}}(v,z) - v^{-5}z P_{a_{15}}(v,z) - v^{-3}z P_{a_{14}}(v,z) - v^{-1}z P_{a_{13}}(v,z).
$$

It is clear that the links a_{13}, a_{14} and a_{15} do not contribute anything to $\max\deg_z P_{a_9}(v,z)$. Then

$$
\max\deg_z P_{a_9}(v,z) = \max\deg_z P_{a_{16}}(v,z).
\tag{6.38}
$$

In the link diagram a_{16}, we consider the three crossings labeled 1, 2 and 3 in the $(r-1)$-st row as indicated in the first row of Fig. 23 according as the case (a) $r \equiv 2 \pmod 3$, (b) $r \equiv 0 \pmod 3$ and (c) $r \equiv 1 \pmod 3$. For a regular projection of $\hat{\beta}_r$ as shown in Fig. 24(a), we observe that there are three arcs, say S_1, S_2, S_3, in the dotted rectangle R in Fig. 24(a) that are obtained from the arcs in the small dotted rectangles C_1, C_2, C_3, C_4 in R as shown in Fig. 24(b) by gluing them in the obvious way, written $R = C_1 C_2 C_3 C_4$. From this, it is not difficult to see in general that

$$
\hat{\beta}_r = C_1 C_2 C_3 C_4 C_2 C_3 C_4 \cdots C_m,
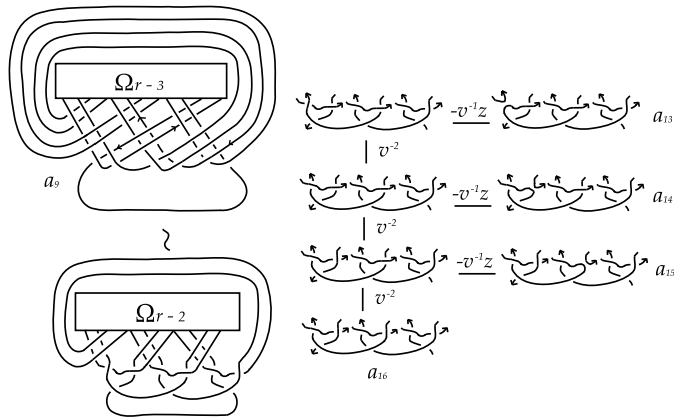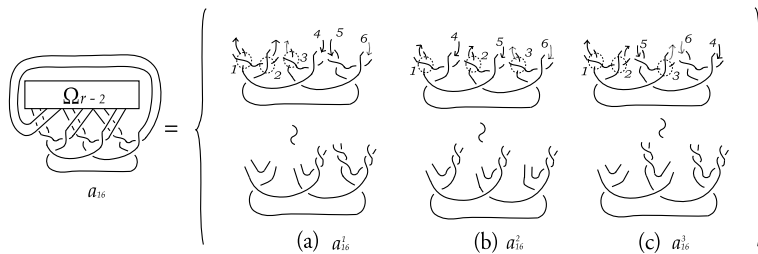\tag{6.39}
$$

where

Fig. 22. A partial skein tree for a_9.

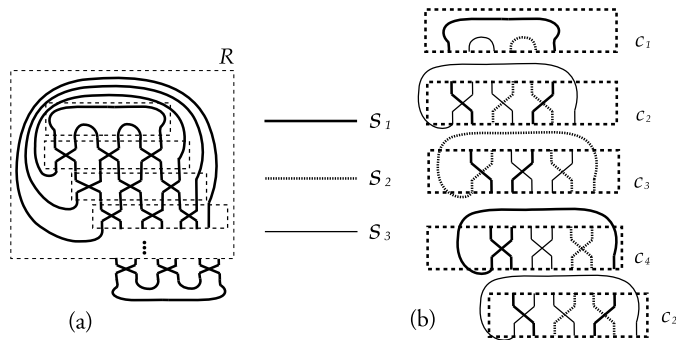**Fig. 23.** $r \equiv 2$, $r \equiv 0$, $r \equiv 1$ (mod 3).

Fig. 24. $\hat{\beta}_r$.

$$m = 2, \qquad r \equiv 2 \pmod 3,$$
$$m = 3, \qquad r \equiv 0 \pmod 3,$$
$$m = 4, \qquad r \equiv 1 \pmod 3.$$

Pushing each crossing labeled 1, 2, 3 into the part of Ω_{r-2} along the 2-parallel strings, it follows from (6.39) that it returns to the arrow labeled 4, 5, 6 in the $(r-1)$-st row, respectively, illustrated in (a), (b) and (c) of Fig. 23 according as the case $r \equiv 2$ (mod 3), $r \equiv 0$ (mod 3) and $r \equiv 1$ (mod 3).

Now, by a similar argument in the proof of Proposition 3.1(2), the full-twists in a_{16}^i can be removed from without contributing to $\max \deg_z P_{a_{16}^i}(v, z)$ for each $i = 1, 2, 3$ and so we obtain

$$\max \deg_z P_{a_{16}^i}(v, z) = \max \deg_z P_{a_{17}}(v, z),$$

where a_{17} is the link diagram as shown in Fig. 25.

On the other hand, by Morton's inequality, we obtain

$$\max \deg_z P_{a_{17}}(v, z) \leqslant \big(c(D_r) - 9\big) - \big(s(D_r) - 5\big) + 1 = N_r - 4. \tag{6.40}$$
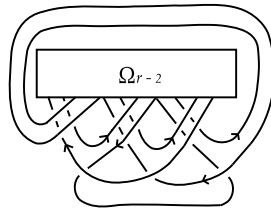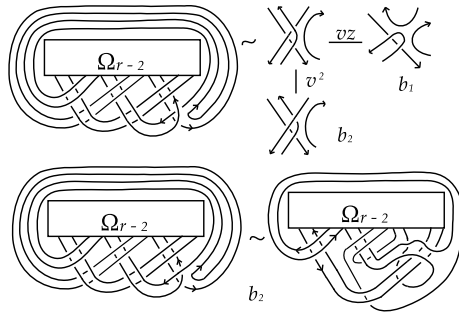
Fig. 25. a_{17}.

Fig. 26. A partial skein tree for D_r^5.

Then it is direct from (6.38) and (6.40) that

$$\max \deg_z P_{a_9}(v, z) \leqslant N_r - 4. \tag{6.41}$$

Therefore we have from (6.34)–(6.37) and (6.41) that

$$\max \deg_z P_{a_6}(v, z) \leqslant \max\big\{M(a_{12}), M(a_{11}) + 1, M(a_{10}) + 2, M(a_9) + 1\big\}$$
$$\leqslant \max\{N_r - 3, N_r - 3, N_r - 3, N_r - 3\} = N_r - 3.$$

This completes the proof of Claim 1. □

Proof of (2). From a partial skein tree for D_r^5 as shown in Fig. 26, we obtain

$$P_{D_r^5}(v, z) = v^2 P_{b_2}(v, z) + vz P_{b_1}(v, z).$$

It is quite easy to see that the link b_1 does not contribute anything to $\max \deg_z P_{D_r^5}(v, z)$. By Morton's inequality, we obtain

$$\max \deg_z P_{D_r^5}(v, z) = \max \deg_z P_{b_2}(v, z)$$
$$\leqslant \big(c(D_r) - 4\big) - \big(s(D_r) - 1\big) + 1 = N_r - 3.$$

This completes the proof of (2). □

Proof of (3). It follows from Morton's inequality that

$$\max \deg_z P_{D_r^6}(v, z) \leqslant \big(c(D_r) - 5\big) - \big(s(D_r) - 2\big) + 1 = N_r - 3.$$

This completes the proof of (3). □

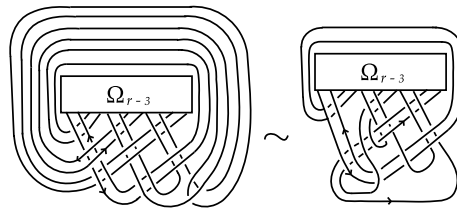*Proof of* (4). By Morton's inequality and isotopy deformations as shown in Fig. 27, we obtain

$$\max \deg_z P_{D_r^7}(v, z) \leqslant \big(c(D_r) - 4\big) - \big(s(D_r) - 1\big) + 1 = N_r - 3.$$

This completes the proof of (4). □

Proof of (5). It follows from Morton's inequality that

$$\max \deg_z P_{D_r^8}(v, z) \leqslant \big(c(D_r) - 8\big) - \big(s(D_r) - 4\big) + 1 = N_r - 4.$$

This completes the proof of (5). □

Fig. 27. D_r^7.

Acknowledgements

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