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Deflating quadratic matrix polynomials with structure preserving transformations

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ABSTRACT

Given a pair of distinct eigenvalues (λ_1, λ_2) of an $n \times n$ quadratic matrix polynomial $Q(\lambda)$ with nonsingular leading coefficient and their corresponding eigenvectors, we show how to transform $Q(\lambda)$ into a quadratic of the form $\begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix}$ having the same eigenvalues as $Q(\lambda)$, with $Q_d(\lambda)$ an $(n-1) \times (n-1)$ quadratic matrix polynomial and $q(\lambda)$ a scalar quadratic polynomial with roots λ_1 and λ_2. This block diagonalization cannot be achieved by a similarity transformation applied directly to $Q(\lambda)$ unless the eigenvectors corresponding to λ_1 and λ_2 are parallel. We identify conditions under which we can construct a family of $2n \times 2n$ elementary similarity transformations that (a) are rank-two modifications of the identity matrix, (b) act on linearizations of $Q(\lambda)$, (c) preserve the block structure of a large class of block symmetric linearizations of $Q(\lambda)$, thereby defining new quadratic matrix polynomials $Q_1(\lambda)$ that have the same eigenvalues as $Q(\lambda)$, (d) yield quadratics $Q_1(\lambda)$ with the property that their eigenvectors associated with λ_1 and λ_2 are parallel and hence can subsequently be deflated by a similarity applied directly to $Q_1(\lambda)$. This is the first attempt at building elementary transformations that preserve the block structure of widely used linearizations and which have a specific action.

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1. Introduction

Consider the quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, where $M, C, K \in \mathbb{R}^{n\times n}$ with M nonsingular, and the associated quadratic eigenvalue problem

$$Q(\lambda)x_R = 0, \quad x_L^* Q(\lambda) = 0, \tag{1}$$

where λ is an eigenvalue and x_R and x_L are corresponding right and left eigenvectors, respectively. Throughout, we use the subscript R to denote right eigenvectors or when referring to transformations applied to the right, and the subscript L for left eigenvectors and transformations applied to the left. We also denote by $\Lambda(Q)$ the spectrum of Q.

Given two eigentriples $(\lambda_j, x_{Rj}, x_{Lj}), j = 1, 2$ satisfying appropriate conditions, we propose a deflation procedure that decouples $Q(\lambda)$ into a quadratic $Q_d(\lambda) = \lambda^2 M_d + \lambda C_d + K_d$ of dimension $n-1$ and a scalar quadratic $q(\lambda) = \lambda^2 m + \lambda c + k = m(\lambda - \lambda_1)(\lambda - \lambda_2)$ such that

$$\Lambda(Q) = \Lambda(Q_d) \cup \{\lambda_1, \lambda_2\}$$

and there exist well-defined relations between the eigenvectors of $Q(\lambda)$ and those of the decoupled quadratic

$$\widetilde{Q}(\lambda) = \begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix}. \tag{2}$$

This is termed “strong deflation" in the engineering community, as opposed to “weak deflation", which is achieved by introducing zeros in the trailing rows or columns of the matrices. Weak and strong deflations differ from Wielandt deflation, in which a low rank modification is applied to the eigenproblem so as to displace one or more targeted eigenvalues and leave the others unchanged. This technique is employed when solving eigenproblems with iterative methods such as Jacobi–Davidson: in this way, eigenvalues found can be displaced to zero or infinity [6,14,17].

Unlike for linear polynomials $A - \lambda B$, we cannot in general construct an $n \times n$ equivalence transformation with nonsingular matrices P_L and P_R such $P_L^T Q(\lambda) P_R = \widetilde{Q}(\lambda)$, where $\widetilde{Q}(\lambda)$ is the decoupled quadratic in (2) [19]. The standard way of treating quadratic matrix polynomials, both theoretically and numerically, is to convert them into equivalent linear matrix pencils of twice the dimension, a process called linearization [11]. For example, when M is nonsingular the block symmetric pencil

$$L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}$$

is a linearization of $Q(\lambda)$ in the sense that $L_2(\lambda)$ satisfies

$$E(\lambda) L_2(\lambda) F(\lambda) = \begin{bmatrix} Q(\lambda) & 0 \\ 0 & I_n \end{bmatrix}$$

for some unimodular $E(\lambda)$ and $F(\lambda)$, where I_n is the $n \times n$ identity matrix [11,24]. This implies that $c \cdot \det(L_2(\lambda)) = \det(Q(\lambda))$ for some nonzero constant c, so that L_2 and Q have the same eigenvalues. Deflation procedures for matrix pencils ignore the block structure of linearizations such as $L_2(\lambda)$. They produce a deflated pencil that is not in general a linearization of a quadratic matrix polynomial [18].

Garvey et al. [8] and later Chu and Xu [7] showed that for quadratics with symmetric coefficients and semisimple eigenvalues (i.e., each eigenvalue λ appears only in 1×1 Jordan blocks in a Jordan triple for Q [11]), there exists a real nonsingular matrix $W \in \mathbb{R}^{2n\times 2n}$ such that

$$W^T L_2(\lambda) W = \lambda \begin{bmatrix} 0 & D_M \\ D_M & D_C \end{bmatrix} + \begin{bmatrix} -D_M & 0 \\ 0 & D_K \end{bmatrix} =: L_D(\lambda), \tag{3}$$

with D_M, D_C, D_K diagonal. The pencil $L_D(\lambda)$ is a linearization of the diagonal quadratic $Q_D(\lambda) = \lambda^2 D_M + \lambda D_C + D_K$, which clearly has the same eigenvalues as $Q(\lambda)$. The proof of the diagonalization of the blocks of $L_2(\lambda)$ in (3) is constructive and requires the knowledge of all the eigenvalues and eigenvectors of Q. Most importantly it shows that by increasing the dimension of the transformations from $n \times n$ when working directly on Q to $2n \times 2n$ by working on a pencil of twice the dimension of Q, total decoupling of the underlying second order system can be achieved. The congruence in (3) is an example of a *structure preserving transformation* (SPT). More generally, we say that a pair (W_L, W_R) of $2n \times 2n$ real nonsingular matrices defines a structure preserving transformation for an $n \times n$ quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$ with M nonsingular if

$$W_L^T \left(\begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \right) W_R = \left(\begin{bmatrix} 0 & M_1 \\ M_1 & C_1 \end{bmatrix}, \begin{bmatrix} -M_1 & 0 \\ 0 & K_1 \end{bmatrix} \right), \tag{4}$$

where M_1, C_1, and K_1 are $n \times n$ matrices [23] that define a new quadratic $Q_1(\lambda) = \lambda^2 M_1 + \lambda C_1 + K_1$ having the same eigenvalues as $Q(\lambda)$.

Because the problem is quadratic, we need to deflate two eigenvalues at a time. For a given pair of eigenvalues λ_1, λ_2 and their associated left and right eigenvectors $x_{Lj}, x_{Rj}, j = 1, 2$, we identify conditions under which there exist elementary SPTs (W_L, W_R) that are rank-two modifications of the $2n \times 2n$ identity matrix and transform $Q(\lambda)$ into a new quadratic $Q_1(\lambda)$ for which λ_1 and λ_2 share the same left eigenvector z_L and same right eigenvector z_R, that is,

$$z_L^* Q_1(\lambda_j) = 0, \quad Q_1(\lambda_j) z_R = 0, \quad j = 1, 2. \tag{5}$$

In particular we find that λ_1 and λ_2 must be semisimple and distinct and that, if they are both real, they must also satisfy

$$\operatorname{sign}\left(\frac{x_{L2}^T Q'(\lambda_2) x_{R2}}{x_{L1}^T Q'(\lambda_1) x_{R1}} \right) = \operatorname{sign}\left(\frac{x_{L2}^T Q'(\lambda_1) x_{R1}}{x_{L1}^T Q'(\lambda_2) x_{R2}} \right),$$

which for symmetric quadratics Q means that λ_1 and λ_2 must have opposite type [3]. Under these conditions we characterize a family of elementary SPTs that transform $Q(\lambda)$ with eigentriples $(\lambda_j, x_{Rj}, x_{Lj})$ to a new quadratic $Q_1(\lambda)$ with eigentriples $(\lambda_j, z_R, z_L), j = 1, 2$. Since our transformations are structure preserving we never work with the $2n \times 2n$ matrices in (4). Indeed the matrix coefficients of $Q_1(\lambda)$ turn out to be low rank modifications of M, C and K and are therefore not expensive to compute. When (5) holds we then show how to construct two nonsingular matrices G_L, G_R such that $G_L^T Q_1(\lambda) G_R = \widetilde{Q}(\lambda)$ with $\widetilde{Q}(\lambda)$ block diagonal as in (2), that is, the pair (G_L, G_R) deflates the two eigenvalues λ_1, λ_2.

This paper is organized as follows. After some preliminary results in Section 2 on structure preserving transformation s, we explain in Section 3 how to deflate eigenvalues of symmetric quadratic matrix polynomials. We then extend in the following section the symmetric deflation procedure to quadratics with nonsymmetric coefficient matrices. We present in Section 5 some numerical examples that illustrate our deflation procedure. To the best of our knowledge, this work is the first attempt at constructing a family of *nontrivial elementary SPTs that have a specific action of practical use: that of "mapping" two linearly independent eigenvectors to a set of linearly dependent eigenvectors*.

2. Structure preserving transformations

In this section, we recall some necessary results from [9,23]. SPTs, defined in (4), have a number of important and useful properties that we begin by summarizing.

Lemma 1 [23]. *Let (W_L, W_R) be an SPT transforming $Q(\lambda) = \lambda^2 M + \lambda C + K$ with M nonsingular into $\widetilde{Q}(\lambda) = \lambda^2 \widetilde{M} + \lambda \widetilde{C} + \widetilde{K}$. Then*

(i) *$Q(\lambda)$ and $\widetilde{Q}(\lambda)$ share the same eigenvalues.*
(ii) *$\widetilde{M}$ is nonsingular.*
(iii) *If (λ, x, y) is an eigentriple of $Q(\lambda)$ then*

$$W_R^{-1} \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \begin{bmatrix} \lambda \widetilde{x} \\ \widetilde{x} \end{bmatrix}, \quad W_L^{-1} \begin{bmatrix} \bar{\lambda} y \\ y \end{bmatrix} = \begin{bmatrix} \bar{\lambda} \widetilde{y} \\ \widetilde{y} \end{bmatrix},$$

for some nonzero $\widetilde{x}, \widetilde{y} \in \mathbb{C}^n$ such that $\widetilde{Q}(\lambda)\widetilde{x} = 0$ and $\widetilde{y}^* \widetilde{Q}(\lambda) = 0$.
(iv) *If $L(\lambda)$ belongs to the vector space of pencils* [15,20]

$$\mathbb{DL}(Q) = \left\{ \lambda \begin{bmatrix} v_1 M & v_2 M \\ v_2 M & v_2 C - v_1 K \end{bmatrix} + \begin{bmatrix} v_1 C - v_2 M & v_1 K \\ v_1 K & v_2 K \end{bmatrix} : \ v \in \mathbb{R}^2 \right\},$$

with vector v then $\widetilde{L}(\lambda) = W_L^T L(\lambda) W_R \in \mathbb{DL}(\widetilde{Q})$ with vector v. In other words, the SPT (W_L, W_R) preserves the block structure of $\mathbb{DL}(Q)$. Moreover if $L(\lambda)$ is a linearization of Q then $\widetilde{L}(\lambda)$ is a linearization of $\widetilde{Q}(\lambda)$.

(v) *If $W_L = W_R$ and $Q(\lambda)$ is symmetric (i.e., M, C and K are symmetric) then $\widetilde{Q}(\lambda)$ is symmetric.*

Matrix pairs (G_L, G_R) of the form

$$G_S = \begin{bmatrix} \widetilde{G}_S & 0 \\ 0 & \widetilde{G}_S \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad \det(\widetilde{G}_S) \neq 0, \quad S = L, R$$

always define an SPT for any $n \times n$ quadratic Q. They have the property that if (G_L, G_R) transforms $Q(\lambda)$ into $\widetilde{Q}(\lambda)$ then $\widetilde{Q}(\lambda) = \widetilde{G}_L^T Q(\lambda) \widetilde{G}_R$. The pair (G_L, G_R) is called a *class one elementary SPT* when $\widetilde{G}_S = I - m_S n_S^T$ for some nonzero vectors $m_S, n_S \in \mathbb{R}^n$, $S = L, R$ [9].

The key elementary SPT used in our deflation procedure has the form

$$T_S = \begin{bmatrix} I + a_S b_S^T & a_S d_S^T \\ a_S f_S^T & I + a_S h_S^T \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \tag{6}$$

where $a_S, b_S, d_S, f_S, h_S \in \mathbb{R}^n$ with a_S, d_S, f_S nonzero. The matrix T_S differs from the identity matrix by a matrix of rank at most two and it is nonsingular if [5,23]

$$\det(T_S) = \left(1 + a_S^T b_S\right)\left(1 + a_S^T h_S\right) - (a_S^T d_S)(a_S^T f_S) \neq 0.$$

With the notation

$$\alpha_M := a_L^T M a_R, \quad \alpha_C := a_L^T C a_R, \quad \alpha_K := a_L^T K a_R,$$

a pair (T_L, T_R) of nonsingular matrices with T_S, $S = L, R$, as in (6) forms a class two elementary SPT if [9,23]

$$\alpha_C = a_L^T C a_R \neq 0 \tag{7}$$

and

$$\frac{1}{2}\alpha_C f_L + \alpha_M b_L = -M a_R, \tag{8}$$

$$\alpha_K f_L + \frac{1}{2}\alpha_C (b_L + h_L) + \alpha_M d_L = -C a_R, \tag{9}$$

$$\alpha_K h_L + \frac{1}{2}\alpha_C d_L = -K a_R, \tag{10}$$

$$\frac{1}{2}\alpha_C f_R + \alpha_M b_R = -M^T a_L, \tag{11}$$

$$\alpha_K f_R + \frac{1}{2}\alpha_C (b_R + h_R) + \alpha_M d_R = -C^T a_L, \tag{12}$$

$$\alpha_K h_R + \frac{1}{2}\alpha_C d_R = -K^T a_L. \tag{13}$$

The constraints (8)–(13) force preservation of structure. Multiplying the constraints (8) and (10) on the left by a_L^T and the constraints (11) and (13) on the left by a_R^T allows us to rewrite the determinant of T_L and T_R as

$$\det(T_S) = \alpha_C^{-2}\left(1 + a_S^T b_S\right)\left(1 + a_S^T h_S\right)\left(\alpha_C^2 - 4\alpha_K \alpha_M\right), \quad S = L, R$$

which shows that

$$\alpha_C^2 - 4\alpha_K\alpha_M \neq 0 \tag{14}$$

is a necessary condition for (T_L, T_S) to be an SPT.

From (8)–(13) we have that if (T_L, T_R) transforms $Q(\lambda)$ to $\widetilde{Q}(\lambda)$ then

$$\begin{aligned}
\widetilde{K} &= K - \alpha_K h_L h_R^T - \frac{1}{2}\alpha_C \left(h_L d_R^T + d_L h_R^T\right) - \alpha_M d_L d_R^T, \\
\widetilde{C} &= C - \alpha_K \left(h_L f_R^T + f_L h_R^T\right) - \frac{1}{2}\alpha_C \left(h_L b_R^T + b_L h_R^T + d_L f_R^T + f_L d_R^T\right) - \alpha_M \left(d_L b_R^T + b_L d_R^T\right), \\
\widetilde{M} &= M - \alpha_K f_L f_R^T - \frac{1}{2}\alpha_C \left(b_L f_R^T + f_L b_R^T\right) - \alpha_M b_L b_R^T,
\end{aligned}$$

which shows that $\widetilde{M}$, $\widetilde{C}$, and $\widetilde{K}$ are low rank modifications of M, C, and K.

Note that once the two vectors a_L and a_R are chosen such that (7) and (14) hold, the structure preserving constraints (8)–(13) are linear in the remaining unknown vectors. They can be rewritten in matrix form as

$$VA = B \iff V_L A = B_R, \quad V_R A = B_L, \tag{15}$$

where $A \in \mathbb{R}^{4\times 3}$ and $B = \begin{bmatrix} B_R \\ B_L \end{bmatrix} \in \mathbb{R}^{2n\times 3}$ are given by

$$A = \begin{bmatrix} \alpha_M & \frac{1}{2}\alpha_C & 0 \\ 0 & \alpha_M & \frac{1}{2}\alpha_C \\ \frac{1}{2}\alpha_C & \alpha_K & 0 \\ 0 & \frac{1}{2}\alpha_C & \alpha_K \end{bmatrix}, \quad B = -\begin{bmatrix} Ma_R & Ca_R & Ka_R \\ M^T a_L & C^T a_L & K^T a_L \end{bmatrix} \tag{16}$$

and $V = \begin{bmatrix} V_L \\ V_R \end{bmatrix} \in \mathbb{R}^{2n\times 4}$ with $V_S = \begin{bmatrix} b_S & d_S & f_S & h_S \end{bmatrix} \in \mathbb{R}^{n\times 4}$ for $S = L, R$ contains the remaining unknown vectors. Some calculations show that

$$\det(A^T A) = \frac{1}{4}\left(\alpha_C^2 - 4\alpha_M\alpha_K\right)^2 \left(\alpha_C^2 + \alpha_M^2 + \alpha_K^2\right)$$

which is nonzero by (7) and (14), so that A has full rank and all solutions to (15) are given by

$$V = BA^+ + U(I - AA^+) \iff \begin{cases} V_L = B_R A^+ + U_L(I - AA^+), \\ V_R = B_L A^+ + U_R(I - AA^+), \end{cases}$$

for some arbitrary $U = \begin{bmatrix} U_L \\ U_R \end{bmatrix} \in \mathbb{R}^{2n\times 4}$. Here A^+ is the pseudoinverse of A, which is given by $A^+ = (A^T A)^{-1} A^T$ since A has full rank (see Stewart and Sun [22, Section 3.1]).

The transformation T_S used in our deflation procedure performs a specific action: that of mapping a quadratic matrix polynomial with two nonparallel eigenvectors associated with a pair of eigenvalues to a quadratic whose eigenvectors associated to that pair of eigenvalues are now parallel. This results in an additional constraint of the form $z_S^T V_S = w_S^T$ for some given z_S and w_S that the solutions V_L and V_R of (15) must satisfy. The next result will then be useful.

Theorem 2. *Let $A \in \mathbb{R}^{r\times k}$, $r \geqslant k$ have full rank, $B \in \mathbb{R}^{n\times k}$, $w \in \mathbb{R}^r$, and nonzero $z \in \mathbb{R}^n$ be given. The problem of finding $V \in \mathbb{R}^{n\times r}$ such that*

$$VA = B, \quad z^T V = w^T, \tag{17}$$

has a solution if and only if $w^T A = z^T B$. In this case the general solution is

$$V = (I - zz^+)BA^+ + U(I - AA^+) + z\left(z^T z\right)^{-1} w^T, \tag{18}$$

where $U \in \mathbb{R}^{n\times r}$ is any matrix such that $z^T U = 0$.

Proof. If V is a solution to (17) then $z^TB = z^TVA = w^TA$. Conversely, if $z^TB = w^TA$ then since $A^+A = I$ multiplying V in (18) on the right by A yields $VA = B$ and since $z^TU = 0$ we have that $z^TV = w^T$ so that V in (18) is a solution to (17). Now every solution V to (17) can be rewritten as

$$\begin{aligned} V &= (I - zz^+)VAA^+ - (I - zz^+)VAA^+ + V - zz^+V + zz^+V \\ &= (I - zz^+)VAA^+ + (I - zz^+)V(I - AA^+) + zz^+V \\ &= (I - zz^+)BA^+ + (I - zz^+)V(I - AA^+) + z\left(z^Tz\right)^{-1} w^T, \end{aligned}$$

which is of the form (18) with $U := \left(I - zz^+\right) V$ satisfying $z^TU = 0$. □

3. Deflation for symmetric quadratics

Symmetric quadratics have the property that if x is a right eigenvector associated with the eigenvalue λ then $y = \bar{x}$ is the corresponding left eigenvector. So if we use congruence transformations to preserve the symmetry of the quadratic we just need to consider the deflation of eigenpairs rather than eigentriples. We denote by (λ_1, x_1) and (λ_2, x_2) the two eigenpairs to be deflated. First we show that when x_1 and x_2 are parallel there exists an $n \times n$ congruence transformation which, when applied directly to Q, deflates λ_1 and λ_2. When x_1 and x_2 are linearly independent, we show how to construct a class two elementary SPT that transforms Q to a new quadratic Q_1 for which λ_1 and λ_2 share the same eigenvector. In other words, the SPT allows us to transform the original deflation problem into one we know how to handle.

3.1. Linearly dependent eigenvectors

We first treat the case where the eigenvalues λ_1 and λ_2 have a common eigenvector $z \in \mathbb{R}^n$. The next lemma is crucial to proving the existence of a congruence transformation that deflates these two eigenvalues. Some relations in this lemma have already been observed by Chu et al. [6].

Lemma 3. *Consider the $n \times n$ symmetric quadratic $Q(\lambda) = \lambda^2M + \lambda C + K$.*

(i) *If $Q(\lambda_j)z = 0, j = 1, 2$ with $z \in \mathbb{R}^n\backslash\{0\}$ and $\lambda_1 \neq \lambda_2$ then $Cz = c\,Mz$ and $Kz = k\,Mz$ with $c = -(\lambda_1 + \lambda_2)$ and $k = \lambda_1\lambda_2$. Moreover, $z^TMz \neq 0$ if and only if $z^TQ'(\lambda_j)z \neq 0, j = 1, 2$.*
(ii) *If $Cz = c\,Mz$ and $Kz = k\,Mz$ for some nonzero $z \in \mathbb{R}^n$ and $c, k \in \mathbb{C}$ then $Q(\lambda_j)z = 0, j = 1, 2$ with $\lambda_{1,2} = -(c \pm \sqrt{c^2 - 4k})/2$.*

Proof

(i) It follows from $\lambda_j^2Mz + \lambda_jCz + Kz = 0$, $j = 1, 2$ that when $\lambda_1 \neq \lambda_2$, $Cz = -(\lambda_1 + \lambda_2)Mz = cMz$ and then $Kz = -\lambda_1^2Mz + \lambda_1(\lambda_1 + \lambda_2)Mz = \lambda_1\lambda_2Mz = kMz$. The last part of statement (i) follows from $z^TQ'(\lambda_j)z = (2\lambda_j + c)z^TMz$ and the fact that $\lambda_1 \neq \lambda_2$. Note that here $Q'(\lambda)$ is the first derivative of Q with respect to λ, that is, $Q'(\lambda) = 2\lambda M + C$.
(ii) If $Cz = c\,Mz$ and $Kz = k\,Mz$ then $Q(\lambda_j)z = (\lambda_j^2 + \lambda_jc + k)Mz = 0, j = 1, 2$, from which the formula for $\lambda_{1,2}$ follows. □

Assume there exists a nonsingular matrix G such that

$$Ge_n = z, \quad G^T(Mz) = me_n, \quad m = z^TMz, \tag{19}$$

where e_n is the last column of the $n \times n$ identity matrix. Since G and M are nonsingular we must have $m \neq 0$, which by Lemma 3(i) holds when λ_1 and λ_2 are distinct and $z^TQ'(\lambda_j)z \neq 0, j = 1, 2$. Then we have that

$$G^TMGe_n = G^TMz = me_n.$$

Now if λ_1 and λ_2 are distinct then by Lemma 3(i), $Cz = cMz$ and $Kz = kMz$, so that

$$G^T(\lambda^2M + \lambda C + K)G = \lambda^2\begin{bmatrix}\widetilde{M} & 0\\ 0 & m\end{bmatrix} + \lambda\begin{bmatrix}\widetilde{C} & 0\\ 0 & mc\end{bmatrix} + \begin{bmatrix}\widetilde{K} & 0\\ 0 & mk\end{bmatrix}, \tag{20}$$

where $c = -(\lambda_1 + \lambda_2)$ and $k = \lambda_1\lambda_2$; thus G deflates the two eigenvalues λ_1 and λ_2. Note that if $\lambda_1 = \lambda_2$ and, Cz and Kz are multiples of Mz then, as long as $z^TMz \neq 0$, G in (19) deflates λ_1 and λ_2 from Q. It is easily seen from (20) that in this case λ_1 $(= \lambda_2)$ must be a defective eigenvalue with partial multiplicity 2.

We build the matrix G in two steps. First, we construct a Householder reflector $H = I - 2vv^T/(v^Tv)$ [12] such that

$$H(Mz) = \|Mz\|_2e_n.$$

Second, we form $L = I_n + rs^T$, where $s^Te_n = 1$ and $r = \frac{\|Mz\|_2}{m}\,Hz - e_n$, so that

$$Le_n = \frac{\|Mz\|_2}{m}Hz, \quad L^Te_n = e_n$$

since $r^Te_n = \frac{\|Mz\|_2}{m}z^THe_n - 1 = \frac{z^TMz}{m} - 1 = 0$. Hence

$$G = \frac{m}{\|Mz\|_2}HL \tag{21}$$

satisfies (19). It is shown in [10] that taking

$$s = e_n - \frac{1 + \sqrt{1 + r^Tr}}{r^Tr}r$$

minimizes the condition number $\kappa(L)$ of L and that with this choice,

$$\kappa_2(G)^2 = \kappa_2(L)^2 = \frac{\sqrt{1 + \|r\|_2^2} + \|r\|_2}{\sqrt{1 + \|r\|_2^2} - \|r\|_2},$$

which is reasonably small as long as $\|r\|_2$ is not much larger than 1. Using $\|Mz\|_2He_n = Mz$ and the definition of r we have that

$$\|r\|_2^2 = r^Tr = \left(z^TM^2z\right)\left(z^Tz\right)\Big/\left(z^TMz\right)^2 - 1$$

showing that $\|r\|_2$ does not depend on the norm of z or M.

Note that G in (21) depends on $2n$ parameters: the Householder vector $v \in \mathbb{R}^n$ and $r \in \mathbb{R}^n$ which is consistent with the $2n$ constraints in (19).

3.2. Linearly independent eigenvectors

When x_1 and x_2 are linearly independent there is clearly no nonsingular transformation mapping the full rank matrix $[x_1 \quad x_2]$ to the rank-one matrix $[e_n \quad e_n]$. The idea in this case is to build an SPT T that transforms $Q(\lambda)$ with eigenpairs $(\lambda_j, x_j), j = 1, 2$ to $Q_1(\lambda)$ with eigenpairs $(\lambda_j, z), j = 1, 2$ that can then be deflated using the procedure described in Section 3.1. We only consider the case where $\lambda_1 \neq \lambda_2$. Indeed when the two eigenvalues are equal and x_1 is not parallel to x_2, λ_1 and λ_2 belong to two distinct Jordan blocks. In this case, the decoupling (20) cannot be achieved.

Since we aim to treat the deflation of real eigenpairs together with that of complex conjugate eigenpairs, we introduce the real matrices $\Lambda \in \mathbb{R}^{2\times 2}$ and $X \in \mathbb{R}^{n\times 2}$ defined by

$$\Lambda = \begin{cases}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} & \text{if } \lambda_1 \text{ and } \lambda_2 \text{ are real,}\\ \begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix} & \text{if } \lambda_1 = \bar{\lambda}_2 = \alpha + i\beta \text{ with } \beta \neq 0,\end{cases} \tag{22}$$

and

$$X = \begin{cases} [x_1 \quad x_2] & \text{for real eigenpairs,} \\ [u \quad v] & \text{for complex eigenpairs with } \ x_1 = \bar{x}_2 = u + iv. \end{cases} \tag{23}$$

We want to construct a class two elementary SPT $T = I_{2n} + \begin{bmatrix} ab^T & ad^T \\ af^T & ah^T \end{bmatrix}$ with $a, b, d, f, h \in \mathbb{R}^n$ and a nonzero vector $z \in \mathbb{R}^n$ such that

$$T^{-1} \begin{bmatrix} X\Lambda \\ X \end{bmatrix} = \begin{bmatrix} ze^T\Lambda \\ ze^T \end{bmatrix}, \tag{24}$$

where $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This constraint means that $T^{-1} \begin{bmatrix} \lambda_j x_j \\ x_j \end{bmatrix} = \begin{bmatrix} \lambda_j \delta_j z \\ \delta_j z \end{bmatrix}$, for some nonzero $\delta_j, j = 1, 2$. Hence if T transforms $Q(\lambda)$ to $Q_1(\lambda)$ then by Lemma 1(iii), $Q_1(\lambda_j)z = 0, j = 1, 2$. We rewrite (24) in terms of the $6n$ unknown vectors a, b, d, f, h, z as

$$ze^T\Lambda + \left(b^Tz\right)ae^T\Lambda + (d^Tz)ae^T = X\Lambda, \tag{25}$$

$$ze^T + \left(f^Tz\right)ae^T\Lambda + \left(h^Tz\right)ae^T = X, \tag{26}$$

and solve (25) and (26) for a, z and the scalars b^Tz, d^Tz, f^Tz, h^Tz as follows.

Let nonzero $p, q \in \mathbb{R}^2$ be such that

$$e^Tp = 0, \quad e^T\Lambda p = 1, \quad e^Tq = 1, \quad e^T\Lambda q = 0.$$

Since $\lambda_1 \neq \lambda_2$, it is easily seen that

$$p = \gamma(\lambda_1 - \lambda_2)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad q = \Lambda p - (\lambda_1 + \lambda_2)p, \quad \Lambda q = -\lambda_1\lambda_2 p,$$

with $\gamma = 1$ for real eigenpairs and $\gamma = i$ for complex eigenpairs. Multiplying (26) on the right by p yields $(f^Tz)a = Xp$. Since the columns of X are linearly independent, we have that $f^Tz \neq 0$. Now without loss of generality, we normalize a such that $a^Ta = 1$. It follows that

$$a = \left(f^Tz\right)^{-1} Xp, \quad f^Tz = \|Xp\|_2 \neq 0. \tag{27}$$

Multiplying (25) on the right by p yields $z + (b^Tz)a = X\Lambda p$. If we choose to normalize z such that $e_\ell^Tz = 1$, where we let ℓ be such that $|e_\ell^Ta| = \|a\|_\infty$ then

$$b^Tz = \left(e_\ell^TX\Lambda p - 1\right) \Big/ \left(e_\ell^Ta\right), \quad z = X\Lambda p - \left(b^Tz\right)a. \tag{28}$$

Multiplying (25) and (26) on the right by q and on the left by e_ℓ^T gives

$$d^Tz = \left(e_\ell^TX\Lambda q\right) \Big/ \left(e_\ell^Ta\right), \quad h^Tz = \left(e_\ell^TXq - 1\right) \Big/ \left(e_\ell^Ta\right). \tag{29}$$

What is now left is the construction of $V := [b \quad d \quad f \quad h]$ such that $z^TV = w^T$, where $w^T = \left[b^Tz \quad d^Tz \quad f^Tz \quad h^Tz\right]$, and $VA = B$, since T is structure preserving (see Section 2), where $B = -[Ma \quad Ca \quad Ka]$ and A is as in (16) with $\alpha_M = a^TMa, \alpha_C = a^TCa \neq 0$ and $\alpha_K = a^TKa$. We know from Theorem 2 that a solution V to $VA = B, z^TV = w^T$ exists if and only if

$$w^TA = z^TB. \tag{30}$$

The next lemma, crucial for the deflation process, provides a necessary and sufficient condition on the eigenpairs $(\lambda_j, x_j), j = 1, 2$ for (30) to hold.

Lemma 4. *The relation $w^TA = z^TB$ holds if and only if the eigenpairs (λ_1, x_1) and (λ_2, x_2) of $Q(\lambda)$ satisfy*

$$x_1^T Q'(\lambda_1)x_1 = \epsilon x_2^T Q'(\lambda_2)x_2 \tag{31}$$

with $\epsilon = -1$ *for real eigenpairs and* $\epsilon = 1$ *for complex conjugate eigenpairs.*

Proof. Tedious calculations left to Appendix A show that the row vector $g^T = w^T A - z^T B$ has the form

$$g^T = \gamma_g \left(x_1^T Q'(\lambda_1)x_1 - \epsilon x_2^T Q'(\lambda_2)x_2\right)[1 \quad c \quad k],$$

where γ_g is a nonzero scalar, $c = -(\lambda_1 + \lambda_2)$, $k = \lambda_1\lambda_2$, $\epsilon = -1$ for real eigenpairs and $\epsilon = 1$ for complex eigenpairs. □

For real eigenpairs such that $x_j^T Q'(\lambda_j)x_j \neq 0$ (this latter condition holds for simple eigenvalues [1, Theorem 3.2] and for semisimple eigenvalues for some x_j in the null space of $Q(\lambda_j)$), the condition (31) implies that λ_1 and λ_2 must have opposite type, the type of a real eigenvalue λ of $Q(\lambda)$ with associated eigenvector x being the sign of $x^T Q'(\lambda)x = 2\lambda x^T Mx + x^T Cx$. Note that this is to be expected from the theory of Hermitian matrix polynomials since for a symmetric quadratic with $2r$ distinct real eigenvalues, r of them are of positive type and r of them are of negative type (see [11]). Hence when deflating two real eigenpairs, one must be of positive type and the other of negative type. Now under this condition, (31) is achieved with the scaling

$$x_1 \leftarrow x_1/\sqrt{|x_1^T Q'(\lambda_1)x_1|}, \quad x_2 \leftarrow x_2/\sqrt{|x_2^T Q'(\lambda_2)x_2|}.$$

For complex conjugate eigenpairs such that $x_j^T Q'(\lambda_j)x_j \neq 0$, (31) is achieved with the scaling

$$x_1 \leftarrow x_1/\sqrt{x_1^T Q'(\lambda_1)x_1}, \quad x_2 = \bar{x}_1.$$

(Note here the use of "T" rather than "$*$".)

With the above scaling, Lemma 4 together with Theorem 2 tells us that the equations $VA = B$ and $z^T V = w^T$ have the solutions

$$V = \left(I - \frac{zz^T}{z^T z}\right)BA^+ + U(I - AA^+) + \frac{z}{z^T z}w^T, \tag{32}$$

where $U \in \mathbb{R}^{n\times 4}$ is any matrix such that $z^T U = 0$. It follows that (27)–(29) and (32) define a family of class two elementary SPTs T transforming $Q(\lambda)$ with eigenpairs (λ_j, x_j) to $Q_1(\lambda)$ with eigenpairs $(\lambda_j, z), j = 1, 2$. Identifying which solution minimizes the condition number $\kappa_2(T) = \|T\|_2\|T^{-1}\|_2$ remains an open problem.

4. Deflation for nonsymmetric quadratics

The deflation procedure described in Section 3 extends to the case where M, C, and K are nonsymmetric. We denote by $(\lambda_j, x_{Rj}, x_{Lj}), j = 1, 2$ the two eigentriples to be deflated from $Q(\lambda)$ with $(\lambda_2, x_{R2}, x_{L2}) = (\bar{\lambda}_1, \bar{x}_{R1}, \bar{x}_{L1})$ when $\mathrm{Im}(\lambda_1) \neq 0$. In contrast with the symmetric deflation procedure we use equivalence transformations rather than congruence transformations since we do not need to preserve symmetry. Three situations must be considered.

4.1. Parallel left eigenvectors and parallel right eigenvectors

Without loss of generality let us assume in this case that $x_{L1} = x_{L2} \equiv z_L$ and $x_{R1} = x_{R2} \equiv z_R$ with $z_L, z_R \in \mathbb{R}^n$ so that

$$z_L^T Q_1(\lambda_j) = 0, \quad Q_1(\lambda_j)z_R = 0, \quad j = 1, 2. \tag{33}$$

As in Lemma 3 it is easily shown that if (33) holds with $\lambda_1 \neq \lambda_2$ then

$$C_1 z_R = cM_1 z_R, \quad K_1 z_R = kM_1 z_R, \tag{34}$$

$$z_L^T C_1 = cz_L^T M_1, \quad z_L^T K_1 = kz_L^T M_1, \tag{35}$$

where $c = -(\lambda_1 + \lambda_2)$ and $k = \lambda_1\lambda_2$. Suppose there exist nonsingular matrices G_L and G_R such that

$$G_L^T M z_R = m e_n, \quad G_L e_n = z_L, \tag{36}$$
$$G_R^T M^T z_L = m e_n, \quad G_R e_n = z_R, \tag{37}$$

where $m = z_L^T M z_R$. (Note that the left (right) transformation G_L (G_R) depends on the right (left) eigenvector.) Since M, G_L, and G_R are nonsingular we must have $m \neq 0$ which is guaranteed when λ_1 and λ_2 are distinct and $z_L Q'(\lambda_j) z_R \neq 0$. With G_L and G_R satisfying (36) and (37) we have

$$G_L^T M G_R e_n = G_L^T M z_R = m e_n, \quad e_n^T G_L^T M G_R = z_L^T M G_R = m e_n^T$$

and on using (34)–(37) it follows that

$$G_L^T (M, C, K) G_R = \left(\begin{bmatrix} \widetilde{M} & 0 \\ 0 & m \end{bmatrix}, \begin{bmatrix} \widetilde{C} & 0 \\ 0 & mc \end{bmatrix}, \begin{bmatrix} \widetilde{K} & 0 \\ 0 & mk \end{bmatrix} \right). \tag{38}$$

If we let $u_L = M z_R$ and $u_R = M^T z_L$, the matrices G_L and G_R can be taken in the form

$$G_S = \frac{m}{\|u_S\|_2} H_S L_S, \quad S = L, R,$$

where H_S is a Householder reflector such that $H_S u_S = \|u_S\|_2 e_n$ and $L_S = I_n - r_S s_S^T$ with

$$r_S = \frac{\|u_S\|_2}{m} H_S z_S - e_n, \quad s_S = e_n - \frac{1 + \sqrt{1 + r_S^T r_S}}{r_S^T r_S} r_S$$

so that

$$L_S e_n = \frac{\|u_S\|_2}{m} H_S z_S, \quad L_S^T e_n = e_n.$$

Then it is easy to check that the pair (G_L, G_R) satisfies (34) and (35) and therefore deflates λ_1 and λ_2 from Q.

4.2. Nonparallel left eigenvectors and nonparallel right eigenvectors

As for the symmetric case our aim is to build a class two elementary SPT (T_L, T_R), with T_L not necessarily equal to T_R, that transforms $Q(\lambda)$ to a new quadratic $Q_1(\lambda)$ for which λ_1 and λ_2 share the same left eigenvector z_L and the same right eigenvector z_R. In order to apply the deflation process of Section 4.1, we assume that λ_1 and λ_2 are distinct and $x_{Lj}^* Q'(\lambda_j) x_{Rj} \neq 0, j = 1, 2$. When $\lambda_1 = \lambda_2$ with lineary independent eigenvectors then λ_1 and λ_2 belong to two distinct Jordan blocks and the decoupling (38) cannot be achieved.

Let T_S be such that

$$T_S^{-1} \begin{bmatrix} X_S \Lambda_S \\ X_S \end{bmatrix} = \begin{bmatrix} z_S e^T \Lambda_S \\ z_S e^T \end{bmatrix}, \tag{39}$$

with $\Lambda_L = \Lambda^T$ and $\Lambda_R = \Lambda$ where Λ, X_L and X_R are formed as in (22) and (23), and $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If the pair (T_L, T_R) is structure preserving and transforms $Q(\lambda)$ to $Q_1(\lambda)$ then the constraint (39) for $S = L$ and $S = R$ together with Lemma 1(iv) implies that $z_L^T Q_1(\lambda_j) = 0$ and $Q_1(\lambda_j) z_R = 0, j = 1, 2$.

Now if we choose T_S to have the form (6) then with the following normalizations of a_S and z_S,

$$a_S^T a_S = 1, \quad e_{\ell_S}^T z_S = 1, \quad |e_{\ell_S}^T a_S| = \|a_S\|_\infty, \tag{40}$$

we obtain in a similar way to the symmetric case described in Section 3.2, that under the constraint (39),

$$\begin{aligned} &f_S^T z_S = \|X_S p_S\|_2 \neq 0, && a_S = (f_S^T z_S)^{-1} X_S p_S, \\ &b_S^T z_S = \left(e_{\ell_S}^T X_S \Lambda_S p_S - 1\right) \Big/ \left(e_{\ell_S}^T a_S\right), && z_S = X_S \Lambda_S p_S - (b_S^T z_S) a_S, \\ &d_S^T z_S = \left(e_{\ell_S}^T X_S \Lambda_S q_S\right) \Big/ \left(e_{\ell_S}^T a_S\right), && h_S^T z_S = \left(e_{\ell_S}^T X_S q_S - 1\right) \Big/ \left(e_{\ell_S}^T a_S\right), \end{aligned} \tag{41}$$

where $p_S, q_S \in \mathbb{R}^2$ are such that

$$e^T p_S = 0, \quad e^T \Lambda_S p_S = 1, \quad e^T q_S = 1, \quad e^T \Lambda_S q_S = 0.$$

Assuming that $a_L^T C a_R \neq 0$, the class two elementary SPT (T_L, T_R) is completely determined if we can find two matrices $V_L, V_R \in \mathbb{R}^{n\times 4}$ of the form $\begin{bmatrix} b_S & d_S & f_S & h_S \end{bmatrix}$ with $S = L, R$ such that

$$V_L A = B_R, \quad z_L^T V_L = w_L^T, \tag{42}$$

$$V_R A = B_L, \quad z_R^T V_R = w_R^T, \tag{43}$$

where $A \in \mathbb{R}^{4\times 3}$ and $B \in \mathbb{R}^{2n\times 3}$ are as in (16) and $w_S^T = \begin{bmatrix} b_S^T z_S & d_S^T z_S & f_S^T z_S & h_S^T z_S \end{bmatrix}$, $S = L, R$. From Theorem 2, a solution V_L to (42) and a solution V_R to (43) exist if and only if $w_L^T A = Z_L^T B_R$ and $w_R^T A = Z_R^T B_L$.

Lemma 5. *The relations*

$$w_L^T A - Z_L^T B_R = 0, \quad w_R^T A - Z_R^T B_L = 0$$

hold if and only if the eigentriples $(\lambda_1, x_{R1}, x_{L1})$ *and* $(\lambda_2, x_{R2}, x_{L2})$ *of* $Q(\lambda)$ *satisfy*

$$x_{L1}^T Q'(\lambda_1) x_{R1} = \epsilon x_{L2}^T Q'(\lambda_2) x_{R2}, \quad x_{L1}^T Q'(\lambda_2) x_{R2} = -x_{L2}^T Q'(\lambda_1) x_{R1} \tag{44}$$

with $\epsilon = -1$ *for real eigentriples and* $\epsilon = 1$ *for complex conjugate eigentriples.*

Proof. Let $g_L^T = w_L^T A - Z_L^T B_R$ and $g_R^T = w_R^T A - Z_R^T B_L$. Calculations along the same lines as those presented in Appendix Appendix A for the symmetric case show that for real eigentriples,

$$g_L^T = \gamma_L (\xi_1 + \xi_2 - \xi_3 - \xi_4) \begin{bmatrix} 1 & c & k \end{bmatrix},$$
$$g_R^T = \gamma_R (\xi_1 + \xi_2 - \xi_5 - \xi_6) \begin{bmatrix} 1 & c & k \end{bmatrix},$$

where γ_L and γ_R are nonzero scalars, $c = -(\lambda_1 + \lambda_2)$, $k = \lambda_1\lambda_2$ and

$$\begin{aligned} \xi_1 &= x_{L1}^T Q'(\lambda_1) x_{R1}, \quad & \xi_3 &= x_{L1}^T Q'(\lambda_1) x_{R2}, \quad & \xi_5 &= x_{L1}^T Q'(\lambda_2) x_{R2}, \\ \xi_2 &= x_{L2}^T Q'(\lambda_2) x_{R2}, \quad & \xi_4 &= x_{L2}^T Q'(\lambda_2) x_{R1}, \quad & \xi_6 &= x_{L2}^T Q'(\lambda_1) x_{R1}. \end{aligned} \tag{45}$$

From $x_{L1}^T Q(\lambda_j) x_{R2} = 0, j = 1, 2$ we find that $x_{L1}^T C x_{R2} = -(\lambda_1 + \lambda_2) x_{L1}^T M x_{R2}$, from which it follows that $x_{L1}^T Q'(\lambda_1) x_{R2} = -x_{L1}^T Q'(\lambda_2) x_{R2}$, that is, $\xi_3 = -\xi_5$. In an analogous way we find that $x_{L2}^T Q'(\lambda_1) x_{R1} = -x_{L2}^T Q'(\lambda_2) x_{R1}$, that is, $\xi_4 = -\xi_6$. Hence, $g_L = g_R = 0$ if and only if $\xi_1 + \xi_2 = 0$ and $\xi_5 + \xi_6 = 0$.

For complex conjugate eigentriples, we find that

$$g_L^T = \widetilde{\gamma}_L (i\xi_7 + i\xi_8 + \xi_5 + \xi_6) \begin{bmatrix} 1 & c & k \end{bmatrix},$$
$$g_R^T = \widetilde{\gamma}_R (i\xi_1 + i\xi_2 + \xi_5 + \xi_6) \begin{bmatrix} 1 & c & k \end{bmatrix},$$

where $\widetilde{\gamma}_L$ and $\widetilde{\gamma}_R$ are nonzero complex scalars, $\xi_j, j = 1, 2, 5, 6$ are defined in (45) and $\xi_7 = x_{L1}^T Q'(\lambda_2) x_{R1}$, $\xi_8 = x_{L2}^T Q'(\lambda_1) x_{R2}$. Using $x_{L1}^* Q(\lambda_j) x_{R2} = 0, j = 1, 2$ it is easily shown that $x_{L1}^* Q'(\lambda_1) x_{R2} = -x_{L1}^* Q'(\lambda_2) x_{R2}$ which, by taking the conjugate, becomes $\xi_7 = -\xi_1$. We show similarly that $\xi_8 = -\xi_2$. Hence, $g_L = g_R = 0$ if and only if $\xi_1 - \xi_2 = 0$ and $\xi_5 + \xi_6 = 0$ which completes the proof. $\square$

The assumption that λ_1 and λ_2 are distinct and $x_{Lj}^* Q'(\lambda_j) x_{Rj} \neq 0, j = 1, 2$ implies that in (44) the terms on the left-hand side relation for real eigentriples and the terms on the right-hand side relation for complex conjugate eigentriples are nonzero. If $x_{Lj}^T Q'(\lambda_j) x_{Rj} = 0$ or $x_{Lj}^T Q'(\lambda_k) x_{Rk} = 0, j \neq k$, then a scaling similar to that described after Lemma 4 can be applied to ensure that (44) holds. When both $x_{L1}^T Q'(\lambda_1) x_{R1}$ and $x_{L1}^T Q'(\lambda_2) x_{R2}$ are nonzero, we let

$$\rho_1 = \frac{x_{L2}^T Q'(\lambda_2) x_{R2}}{x_{L1}^T Q'(\lambda_1) x_{R1}}, \quad \rho_2 = \frac{x_{L2}^T Q'(\lambda_1) x_{R1}}{x_{L1}^T Q'(\lambda_2) x_{R2}}.$$

Then for real eigentriples, the relations (44) hold after an appropriate scaling of the eigenvectors only if $\mathrm{sign}(\rho_1) = \mathrm{sign}(\rho_2)$, in which case we can apply the scaling

$$\begin{aligned} x_{L1} &\leftarrow |\rho_1|^{1/2} x_{L1}, & x_{R1} &\leftarrow |\rho_1|^{1/2} x_{R1}, \\ x_{L2} &\leftarrow |\rho_2|^{-1/2} x_{L2}, & x_{R2} &\leftarrow |\rho_2|^{1/2} x_{R2}. \end{aligned} \tag{46}$$

For complex eigentriples, $(\lambda_1, x_{R1}, x_{L1}) = (\overline{\lambda_2, x_{R2}, x_{L2}}) = (\lambda, x, y)$ and (44) holds when x and y are scaled such that $y^T Q'(\lambda) x$ is real and $y^* Q'(\lambda) x$ is purely imaginary.

When (44) holds, Lemma 5 and Theorem 2 tell us that the set of solutions to (42) and (43) is given by

$$V_L = \left(I - \frac{z_L z_L^T}{z_L^T z_L} \right) B_R A^+ + U_L \left(I - AA^+ \right) + \frac{z_L}{z_L^T z_L} w_L^T,$$

$$V_R = \left(I - \frac{z_R z_R^T}{z_R^T z_R} \right) B_L A^+ + U_R \left(I - AA^+ \right) + \frac{z_R}{z_R^T z_R} w_R^T,$$

where $U_L, U_R \in \mathbb{R}^{n \times m}$ are any matrices such that $z_S^T U_S = 0, S = L, R$.

The matrices V_L and V_R together with a_L and a_R in (41) define an SPT (T_L, T_R) that transforms $Q(\lambda)$ into $Q_1(\lambda)$ such that (33) holds.

4.3. Nonparallel left (right) eigenvectors and parallel right (left) eigenvectors

When for example $\mathrm{rank}([x_{L1}, x_{L2}]) = 1$ and $\mathrm{rank}([x_{R1}, x_{R2}]) = 2$ we might want to look for an SPT of the form (I_{2n}, T_R) with T_R a class two elementary SPT, since the left eigenvectors are already parallel to each other. Unfortunately, the pair (I_{2n}, T_R) is not structure preserving. However we can still use the procedure described in Section 4.2 to map $(\lambda_j, x_{Rj}, x_{Lj})$ to $(\lambda_j, z_R, z_L), j = 1, 2$ as long as we make sure that after the scaling (46), the vector $X_L p_L$ is nonzero so that a_L in (41) is defined. If $X_L p_L = 0$ then we replace x_{L1} by μx_{L1} and x_{R1} by μx_{R1}, where $\mu = -1$ for real eigentriples and $\mu = i$ for complex conjugate eigenpairs so that (46) still holds but $X_L p_L$ is nonzero.

5. Numerical experiments

We now describe some numerical experiments designed to give insight into our deflation procedure. It is not our aim to investigate the numerical stability properties of the procedure. This is a separate issue that will be addressed in a future paper. In all our experiments we take $U = 0$ in (18). Our computations were done in MATLAB 7.6 (R2008a) for which $u = 2^{-53} \approx 1.1 \times 10^{-16}$.

Recall that (T_L, T_R) defines a class two elementary SPT that maps a quadratic matrix polynomial with two nonparallel eigenvectors associated with a pair of eigenvalues to a quadratic whose eigenvector s associated to that pair of eigenvalues are now parallel, and that (G_L, G_R) defines a deflating transformation. We drop the subscripts L and R when the left and right transformations are equal. If $Q(\lambda)$ is $n \times n$, the cost of deflating (λ_1, λ_2) is $O(n^2)$ operations.

Experiment 1. Our first example is a 2×2 symmetric quadratic $Q(\lambda) = \lambda^2 M + \lambda C + K$ defined by

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \tag{47}$$

with $\Lambda(Q) = \{-0.34 \pm 1.84i, 0.14 \pm 0.51i\}$ to two decimal places. Note that $M^{-1}C$ does not commute with $M^{-1}K$, so $Q(\lambda)$ is not proportionally damped. Therefore the system cannot be decoupled by a 2×2 congruence transformation directly applied to $Q(\lambda)$.

Table 1
Relative magnitude of the off-diagonal elements of the deflated quadratic $Q_2(\lambda) = \lambda^2 M_2 + \lambda C_2 + K_2$ in Experiment 2 and condition number of the transformations.

Deflated e'values	off(M_2)	off(C_2)	off(K_2)	$\kappa_2(T_L)$	$\kappa_2(T_R)$	$\kappa_2(G_L)$	$\kappa_2(G_R)$
Real	3.0e−15	1.7e−13	1.6e−13	6.0e5	2.0e2	3.6e1	3.3e0
Complex	2.0e−16	1.4e−14	5.6e−14	1.8e3	4.5e1	1.0	1.1

Table 2
Condition numbers of the SPTs T and deflating transformations G for different pairs of eigenvalues for Experiment 4.

	(λ_1, λ_5)	(λ_1, λ_6)	(λ_1, λ_7)	(λ_1, λ_8)
$\kappa_2(T)$	4.62e1	1.43e3	4.41e2	7.15e1
$\kappa_2(L)$	2.09e0	6.41e0	1.61e0	4.61e0

Given the pair of complex conjugate eigenvalues $\lambda_{1,2} = -0.34 \pm 1.84i$ and their associated linearly independent eigenvectors our symmetric deflation procedure transforms $Q(\lambda)$ into

$$\lambda^2 \begin{bmatrix} 5.6 & 2.0\text{e}{-16} \\ 2.0\text{e}{-16} & -1.4\text{e}{-1} \end{bmatrix} + \lambda \begin{bmatrix} -1.6 & -9.4\text{e}{-16} \\ -9.4\text{e}{-16} & -9.3\text{e}{-2} \end{bmatrix} + \begin{bmatrix} 1.6 & -9.8\text{e}{-17} \\ -9.8\text{e}{-17} & -4.8\text{e}{-1} \end{bmatrix},$$

to two significant digits, with $\kappa_2(T) = 7.9$ and $\kappa_2(G) \approx 1$. Thus we have accomplished the decoupling (2) to within the working precision.

Experiment 2. Our second example is a 2×2 quadratic matrix polynomial arising in the study of the dynamic behaviour of a bicycle [21]. The coefficient matrices are nonsymmetric. They can be generated using the NLEVP MATLAB toolbox [4] via `nlevp('bicycle')`. This quadratic has two real eigenvalues, $\lambda_1 = -0.32$ and $\lambda_2 \approx -14$ and two complex conjugate eigenvalues $-0.78 \pm 4.5i$. Table 1 shows that the left and right transformations corresponding to the deflation of the complex conjugate eigentriples have a smaller condition number than those used for the deflation of the real eigentriples. The large condition number of T_L in the real case affects the size of the off-diagonal elements of the deflated quadratic. Here off $(E) = \|E - \text{diag}(E)\|_2/\|E\|_2$, $E = M_2, C_2, K_2$.

Experiment 3. Our next example is a 4×4 hyperbolic symmetric quadratic eigenvalue problem generated as in [13, Section 6]. The eigenvalues, real since the quadratic is hyperbolic, are uniformly distributed between 1 and 8. If we order them increasingly then $\lambda_1, \ldots, \lambda_4$ have negative type and $\lambda_5, \ldots, \lambda_8$ have positive type [2, Proof of Theorem 1]. Any pairs (λ_j, λ_k) with $1 \leqslant j \leqslant 4$ and $5 \leqslant k \leqslant 8$ can be deflated from the quadratic. Table 2 displays the condition numbers of the SPT T and deflating transformation G for different pairings. It shows that the choice of pairings affects the conditioning of the transformations.

Experiment 4. We now consider a symmetric quadratic eigenvalue problem coming from a model describing the motion of a beam simply supported at both ends and damped at the midpoint. It can be generated with the NLEVP toolbox via `nlevp('damped_beam',nele)`, where `nele` is the number of finite elements. It is shown in [16, Theorem A1] that the damped problem $Q(\lambda) = \lambda^2 M + \lambda C + K$ and the undamped problem $Q_u(\lambda) = \lambda^2 M + K$ have n eigenvalues and n eigenvectors in common: those corresponding to the anti-symmetric modes. Because M and K are positive definite, the eigenvalues of $Q_u(\lambda)$ are pure imaginary; they come in pairs $(\lambda, \bar{\lambda})$, each pair sharing the same eigenvector.

We computed the n eigenpairs corresponding to the anti-symmetric modes of $Q_u(\lambda)$ using the MATLAB function `eig` with the option '`chol`' and deflated all of them from $Q(\lambda)$ using the procedure described in Section 3.1. Let

$$\widetilde{Q}(\lambda) = G_{acc}^T Q(\lambda) G_{acc} = \lambda^2 \widetilde{M} + \lambda \widetilde{C} + \widetilde{K}$$

be the deflated quadratic, where G_{acc} is the matrix which accumulates the product of the $n/2$ deflating transformations of the form (21) and $\widetilde{M}, \widetilde{C}, \widetilde{K}$ are block 2×2 diagonal with $(n/2) \times (n/2)$

Table 3
Scaled residuals and condition numbers of the transformations used in Example 4.

n	res(M)	res(C)	res(K)	$\kappa_2(G_{acc})$	$\kappa_2(W_2)$
8	3.07e−15	4.63e−18	3.90e−16	1.69e1	1.52e1
16	5.52e−15	5.08e−17	3.59e−15	4.47e1	3.79e1
32	1.34e−13	3.15e−16	1.68e−14	9.57e1	7.84e1
64	3.22e−12	6.09e−15	3.56e−14	1.95e2	1.57e2

blocks, the lower block being diagonal. Table 3 displays the scaled residuals res(M), res(C), and res(K), where

$$\mathrm{res}(E) = \frac{\left\| G_{acc}^T E G_{acc} - \widetilde{E} \right\|_2}{\|G_{acc}\|_2^2 \|E\|_2 + \|\widetilde{E}\|_2},$$

and the 2-norm condition numbers $\kappa_2(G_{acc})$ for different values of $n = 2\times$ `nele`.

The quadratic of the beam problem can be block diagonalized as (see [16, Appendix A1])

$$W^T Q(\lambda) W = \left[\begin{array}{c|c} \lambda^2 M_1 + \lambda D_1 + K_1 & 0 \\ 0 & \lambda^2 M_2 + K_2 \end{array}\right],$$

where W is orthogonal, M_2 and K_2 are both symmetric positive definite and $\lambda^2 M_2 + K_2$ contains the anti-symmetric modes. The last column of Table 3 displays the condition number of the transformation W_2 that block diagonalizes $\lambda^2 M_2 + K_2$. As a comparison, we note that $\kappa_2(G_{acc})$ is not much larger than $\kappa_2(W_2)$.

Appendix A. Technical results for the proof of Lemma 4

We start by recalling the notation. Let (λ_1, x_1) and (λ_2, x_2) be two eigenpairs of a symmetric quadratic $Q(\lambda) = \lambda^2 M + \lambda C + K$ such that $\lambda_1 \neq \lambda_2$. For real eigenpairs let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ and let $X = [x_1\ x_2]$. For complex conjugate eigenpairs let $\Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ and $X = [u\ v]$, where $\lambda_1 = \bar{\lambda}_2 = \alpha + i\beta$, $\beta \neq 0$ and $x_1 = \bar{x}_2 = u + iv$. Let

$$p = \gamma(\lambda_1 - \lambda_2)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad q = \Lambda p - (\lambda_1 + \lambda_2) p$$

with $\gamma = 1$ for real eigenpairs and $\gamma = i$ for complex eigenpairs and let

$$\begin{aligned} f^T z &= \|Xp\|_2 \neq 0, & a &= \left(f^T z\right)^{-1} Xp, \\ b^T z &= \left(e_\ell^T X \Lambda p - 1\right) \Big/ \left(e_\ell^T a\right), & z &= X\Lambda p - \left(b^T z\right) a, \\ d^T z &= \left(e_\ell^T X \Lambda q\right) \Big/ \left(e_\ell^T a\right), & h^T z &= \left(e_\ell^T X q - 1\right) \Big/ \left(e_\ell^T a\right), \end{aligned}$$

where ℓ is such that $a_\ell = e_\ell^T a \neq 0$. Define

$$A = \begin{bmatrix} \alpha_M & \frac{1}{2}\alpha_C & 0 \\ 0 & \alpha_M & \frac{1}{2}\alpha_C \\ \frac{1}{2}\alpha_C & \alpha_K & 0 \\ 0 & \frac{1}{2}\alpha_C & \alpha_K \end{bmatrix}, \quad \begin{aligned} B &= -\left[Ma \quad Ca \quad Ka\right], \\ V &= \left[b \quad d \quad f \quad h\right], \\ w^T &= \left[b^T z \quad d^T z \quad f^T z \quad h^T z\right], \end{aligned}$$

where $\alpha_M = a^T M a$, $\alpha_C = a^T C a$ and $\alpha_K = a^T K a$. The next lemma contains useful relations.

Lemma 6. *The following relations hold:*

$$x_1^T C x_2 = c\, x_1^T M x_2, \tag{48}$$

$$x_1^T K x_2 = k\, x_1^T M x_2, \tag{49}$$

$$d^T z = -k\, f^T z, \tag{50}$$

$$h^T z - b^T z = c\, f^T z, \tag{51}$$

where $c = -(\lambda_1 + \lambda_2)$ *and* $k = \lambda_1 \lambda_2$. *Also for any symmetric matrix* E *we have*

$$a^T E a = \alpha_E = \left(f^T z\right)^{-2} p^T X^T E X p, \tag{52}$$

$$z^T E a = \left(f^T z\right)^{-1} p^T \Lambda^T X^T E X p - \left(b^T z\right)\left(f^T z\right)^{-2} p^T X^T E X p, \tag{53}$$

with

$$p^T X^T E X p = \begin{cases} \mu(x_1^T E x_1 + x_2^T E x_2 - 2x_1^T E x_2) & \text{for real eigenpairs,} \\ \frac{\mu}{4}(i x_1^T E x_1 - i x_2^T E x_2 + 2 x_1^T E x_2) & \text{otherwise,} \end{cases} \tag{54}$$

$$p^T \Lambda^T X^T E X p = \begin{cases} \mu(\lambda_1 x_1^T E x_1 + \lambda_2 x_2^T E x_2 + c x_2^T E x_1) & \text{for real eigenpairs,} \\ \frac{\mu}{4}(i\lambda_1 x_1^T E x_1 - i\lambda_2 x_2^T E x_2 - c x_2^T E x_1) & \text{otherwise,} \end{cases} \tag{55}$$

where $\mu = (\lambda_1 - \lambda_2)^{-2} \neq 0$ *is defined since* $\lambda_1 \neq \lambda_2$.

Proof. The relations (48) and (49) follow from $x_1^T Q(\lambda_1) x_2 = x_2^T Q(\lambda_1) x_1 = 0$ and $x_1^T Q(\lambda_2) x_2 = 0$. The relations (50)–(53) follow from the definition of p, q, a and z and (54) and (55) follow from the definition of Λ and X and p. $\square$

With these relations in hand we can now prove the formula for $g^T = w^T A - z^T B$ in Lemma 4. From the definition of A, B w and z we find that

$$g = \begin{bmatrix} \left(b^T z\right)\alpha_M + \frac{1}{2}\left(f^T z\right)\alpha_C + z^T M a \\ \frac{1}{2}\left(b^T z\right)\alpha_C + (d^T z)\alpha_M + \left(f^T z\right)\alpha_k + \frac{1}{2}\left(h^T z\right)\alpha_C + z^T C a \\ \frac{1}{2}(d^T z)\alpha_C + \alpha_K h^T z + z^T K a \end{bmatrix}.$$

Using (52) with $E = M$ and $E = C$ and (53) with $E = M$ we obtain that the first component of g satisfies

$$2\left(f^T z\right) g_1 = p^T X^T C X p + 2 p^T \Lambda^T X^T M X p. \tag{56}$$

In a similar way we find that the other components of g satisfy

$$2\left(f^T z\right) g_2 = c p^T X C X p - 2k p^T X M X p + 2 p^T \Lambda^T X^T C X p + 2 p^T X K X p,$$

$$2\left(f^T z\right) g_3 = -k p^T X^T C X p + 2c p^T X^T K X p + 2 p^T \Lambda^T X^T K X p.$$

Using (54) and (55) with $E = M$, C and K and the relations (48)–(51) we find that for real eigenpairs,

$$2\left(f^T z\right) g^T = \mu\left(x_1^T Q'(\lambda_1) x_1 + x_2^T Q'(\lambda_2) x_2\right)\begin{bmatrix}1 & c & k\end{bmatrix}$$

and that for complex conjugate eigenpairs,

$$2\left(f^T z\right) g^T = \frac{i}{4}\mu\left(x_1^T Q'(\lambda_1) x_1 - x_2^T Q'(\lambda_2) x_2\right)\begin{bmatrix}1 & c & k\end{bmatrix}.$$

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