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The volume and foundation of star trades

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Abstract

An x-star trade consists of two disjoint decompositions of some simple graph H into copies of $K_{1,x}$, the graph known as the x-star. The number of vertices of H is referred to as the *foundation* of the trade, while the number of copies of $K_{1,x}$ in each of the decompositions is called the *volume* of the trade. We determine all values of x, v and s for which there exists a $K_{1,x}$-trade of volume s and foundation v.

Keywords: Trade; Trade spectrum; Graphical trade; Star

1. Introduction

A *decomposition* of a graph H consists of a set of edge-disjoint subgraphs of H, which partition the edges of H. If each of the subgraphs in the decomposition is isomorphic to some graph G, then the decomposition is called a G-decomposition of H, or a decomposition of H into copies of G.

Given a simple graph G, let T_1 and T_2 be two different decompositions of some graph H on v vertices, with the properties that H has no isolated vertices and that the copies of G in T_1 are distinct from the copies of G in T_2; that is, $T_1 \cap T_2 = \emptyset$. Then the pair $\{T_1, T_2\}$ is a G-*trade* of volume $s = |T_1| = |T_2|$ and foundation v, with *underlying graph* H. The trade is *Steiner* provided that H is simple; we are interested only in Steiner trades here. We let $T_G(s; H)$ denote a Steiner G-trade trade of volume s with underlying graph H, and similarly we let $T_G(s; v)$ denote a Steiner G-trade trade of volume s and foundation v. If H has v vertices then any $T_G(s; H)$ is a $T_G(s; v)$; usually we are not interested in the form of the underlying graph and so we use the more general form. The copies of G in T_1 and T_2 are referred to as *blocks*. We call such a G-trade a *graphical trade* to distinguish it from trades based on other combinatorial objects, such as blocks designs and Latin squares. The various forms of combinatorial trades are surveyed in [8,1].

For integers $x \geqslant 0$, the graph $K_{1,x}$ is called the x-*star*. We let $[a_0 : a_1, a_2, a_3, \ldots, a_x]$ denote a copy of $K_{1,x}$ with vertex set $\{a_i \mid 0 \leqslant i \leqslant x\}$ and edge set $\{a_0 a_i \mid 1 \leqslant i \leqslant x\}$. The vertex a_0 is known as the *centre* vertex.

In this paper we determine the $K_{1,x}$ trade spectrum for every possible value of x and every possible foundation; that is, we determine the set of triples (x, s, v) for which there exists a $T_{K_{1,x}}(s; v)$. For various small graphs, including the cycles C_3, C_4, C_5, C_6, and also the graph $K_4 - e$, the trade spectrum has been determined for each possible foundation (see [3,4,11,10,9], respectively). Here we solve this problem for an infinite family of graphs, the star graphs. The results are summarised in the following theorem.

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Theorem 1.1. *There exists a* $T_{K_{1,x}}(s; v)$ *if and only if* $x \geqslant 2$, $v \geqslant x+2$, *and* $m_x(v) \leqslant s \leqslant M_x(v)$, *where* $m_x(v)$ *and* $M_x(v)$ *are defined by*

$$m_x(v) = \begin{cases} 3 & \text{if } v \leqslant 2x \text{ and } x \geqslant 3, \\ \left\lceil \dfrac{2v}{2x+1} \right\rceil & \text{otherwise}, \end{cases}$$

$$M_x(v) = \begin{cases} 2(v-x)-1 & \text{if } v \leqslant 2x, \\ \left\lfloor \dfrac{v(v-1)}{2x} \right\rfloor & \text{if } v \geqslant 2x+1. \end{cases}$$

The design spectrum and design intersection problems have been completed for stars of arbitrary size. The design spectrum problem is to determine, for a given x, the values of v for which there exists a decomposition of K_v into copies of $K_{1,x}$ (that is, an x-star design of order v); the obvious necessary condition is that $2x \mid v(v-1)$. In [5], this problem was solved for $v = rx$ and $v = rx+1$, $r \geqslant 2$; the obvious necessary conditions imply that either r is even or x is odd, and these conditions were proved sufficient. But this is only a partial solution, except when x is prime. The problem was later completed, independently, by Huang [7] and Tarsi [13]. Tarsi in fact completed the λ-fold star design problem, determining, for a given x, all v and λ for which there exists a decomposition of λK_v into copies of $K_{1,x}$; in the case $\lambda = 1$, the only conditions are that $2x \mid v(v-1)$ and $v \geqslant 2x$.

In [2], Billington and Hoffman solved the intersection problem for star designs. The case $x = 1$ is trivial, so we assume that $x \geqslant 2$. It is easily seen that, for $x \geqslant 3$, a $K_{1,x}$-trade of volume 2 must have foundation $2x+1$; this result is included here as Lemma 2.3. It follows that if $x \geqslant 3$ and $v = 2x$, it is not possible for two x-star designs of order v to intersect in $v(v-1)/2x - 2 = 2x - 3$ blocks. Otherwise, all intersection sizes up to $v(v-1)/2x$ are possible, except for $v(v-1)/2x - 1$ (which would imply a trade of volume 1). To prove the sufficiency of these conditions, Billington and Hoffman noted that an x-star design of order v can be regarded as a copy of K_v with directed edges, with the property that the outdegree of each vertex is divisible by x (Tarsi also used this approach). Given a directed n-cycle on this digraph, a second design can be produced by reversing the direction of each edge on the cycle. This second design will have n blocks which are different from the original design, and any other blocks with centre vertex on the cycle may also be permuted. The existence of the necessary directed cycles was proved with the help of Moon's theorem on tournaments [12].

In the following section we prove the necessity of the conditions of Theorem 1.1, and then in Section 3 we complete the proof of Theorem 1.1 by constructing the necessary trades.

2. Necessary conditions

If $x \leqslant 1$, then $K_{1,x}$ is either an isolated vertex or a single edge, and thus there is no $K_{1,x}$-trade. Therefore we assume from here on that $x \geqslant 2$. We begin the section with some simple bounds on the possible trade volume s.

Lemma 2.1. *Let* x, s *and* v *be integers such that there exists a* $T_{K_{1,x}}(s; v)$. *Then*

$$\left\lceil \frac{2v}{2x+1} \right\rceil \leqslant s \leqslant \left\lfloor \frac{v(v-1)}{2x} \right\rfloor.$$

Proof. The upper bound on s comes from a simple edge count, since there are sx edges in both halves of the trade, and $v(v-1)/2$ edges in K_v.

Suppose that G is a simple graph on n vertices which contains a cut-vertex. It was shown in [10] that a necessary condition for the existence of a $T_G(s; v)$ is that $s \geqslant 2v/(2n-1)$. Since the x-star contains $x+1$ vertices and has a cut-vertex (the centre vertex) provided that $x \geqslant 2$, the lower bound on s follows from this general result. □

In the case $v \leqslant 2x$, we can improve both the upper bound and the lower bound.

Lemma 2.2. *Let* x, s *and* v *be integers such that there exists a* $T_{K_{1,x}}(s; v)$. *If* $v \leqslant 2x$ *then* $s \leqslant 2(v-x)-1$.

Proof. By Lemma 2.1, we have $s \leqslant v(v-1)/2x$, and by assumption $v \leqslant 2x$; therefore $s \leqslant v-1$.

Let the trade be $\{T_1, T_2\}$, with underlying graph H. Every edge used in the trade must be incident with the centre vertex of at least one block of T_1 (and similarly for T_2). There are s blocks of T_1, and hence at most s vertices which are the centre of one or more blocks. Therefore, there are at least $v - s$ vertices which are not centre points. There can be no edges joining pairs of these vertices, and hence there are at most $\binom{s}{2} + s(v - s)$ edges in H. Therefore $sx \leqslant s(s-1)/2 + s(v-s)$. The result follows. □

Lemma 2.3. *Let x, s and v be integers such that there exists a $T_{K_{1,x}}(s; v)$. If $x \geqslant 3$ and $v \leqslant 2x$, then $s \geqslant 3$.*

Proof. We need to prove that there is no $T_{K_{1,x}}(2; v)$ with $x \geqslant 3$ and $v \leqslant 2x$; we assume such a trade exists and seek a contradiction.

Let $\{\{A_1, B_1\}, \{A_2, B_2\}\}$ be such a trade, and let a_1, b_1, a_2 and b_2 be the centre vertices of A_1, B_1, A_2 and B_2, respectively. The vertices a_1 and b_1 have degree 3 or greater in the underlying graph H (recall that $H = A_1 \cup B_1 = A_2 \cup B_2$), while all other vertices have degree 2 or less in H. This is also true of a_2 and b_2, therefore $\{a_1, b_1\} = \{a_2, b_2\}$; say $a_1 = a_2 = a$ and $b_1 = b_2 = b$.

If $a = b$, this vertex is the centre of both A_1 and B_1, implying that it is adjacent to $2x$ vertices in H. Since $v \leqslant 2x$ this is a contradiction, and we are done.

This leaves the case $a \neq b$. Apart from the edge ab, all edges of H incident with a must occur in both A_1 and A_2, while all edges of H incident with b must occur in both B_1 and B_2. This leaves only the edge ab. If ab occurs in A_1 it must also occur in A_2, or else A_1 will have a different number of edges to A_2 (which is a contradiction). Similarly, if ab occurs in B_1 it must also occur in B_2, and if ab occurs in neither A_1 nor B_1 then it cannot occur in A_2 or B_2 either. Thus $A_1 = A_2$ and $B_1 = B_2$, again giving a contradiction. □

In fact, for $x \geqslant 3$ the only $K_{1,x}$-trade of volume 2 is a $T_{K_{1,x}}(2; K_{1,2x})$, and so has the same centre vertex in both its blocks (the construction for this easy trade is covered in Lemma 3.1). Lemma 2.3 is equivalent to Lemma 6 of [2].

We conclude the necessary conditions by noting the implied lower bound on v.

Corollary 2.1. *Let x, s and v be integers such that there exists a $T_{K_{1,x}}(s; v)$. Then $v \geqslant x + 2$.*

Proof. If $v \geqslant 2x + 1$, then since $x \geqslant 2$ the result follows immediately. Therefore we assume that $v \leqslant 2x$. If $x \geqslant 3$, then by Lemmas 2.2 and 2.3 we have $3 \leqslant 2(v - x) - 1$, and the result follows. Finally, if $x = 2$, then by Lemma 2.2 we have $2 \leqslant s \leqslant 2(v - x) - 1 = 2v - 5$. Since v is an integer, it follows that $v \geqslant 4$, and so $v \geqslant x + 2$ in this case also, and we are done. □

Lemmas 2.1,2.2,2.3 and Corollary 2.1 prove the necessity of the conditions of Theorem 1.1.

3. Constructions

We begin with an easy family of trades.

Lemma 3.1. *For every x and s satisfying $x \geqslant 2$ and $s \geqslant 2$, there exists a $T_{K_{1,x}}(s; K_{1,sx})$.*

Proof. Let

$$V = \{a_\infty\} \cup \{a_i \mid i \in \mathbb{Z}_{sx}\},$$

$$T_1 = \{[a_\infty : a_{ix}, a_{ix+1}, a_{ix+2}, \ldots, a_{ix+x-1}] \mid 0 \leqslant i \leqslant s-1\},$$

$$T_2 = \{[a_\infty : a_{ix+1}, a_{ix+2}, a_{ix+3}, \ldots, a_{ix+x}] \mid 0 \leqslant i \leqslant s-1\},$$

with the subscripts of a taken modulo sx. Then $\{T_1, T_2\}$ is a $T_{K_{1,x}}(s; K_{1,sx})$, with vertex set V and independent vertex set $\{a_i \mid i \in \mathbb{Z}_{sx}\}$. □

We now give the main construction for the case $s \leqslant v$.

Lemma 3.2. *If* x, v *and* s *satisfy* $x \geqslant 2$ *and*

$$\lceil \max(3, v/x) \rceil \leqslant s \leqslant \min(v, 2(v-x)-1),$$

then there exists a $T_{K_{1,x}}(s; v)$.

Proof. Let $V = \{a_i \mid i \in \mathbb{Z}_s\} \cup \{b_i \mid i \in \mathbb{Z}_{v-s}\}$, and let $\alpha = \min(x-1, v-s)$. For $i \in \mathbb{Z}_s$, define

$$K_{1,x}^i = [a_i : a_{i+1}, a_{i+2}, \ldots, a_{i+x-\alpha}, b_{i\alpha}, b_{i\alpha+1}, b_{i\alpha+2}, \ldots, b_{i\alpha+\alpha-1}],$$

$$K_{1,x}^{i*} = [a_i : a_{i-1}, a_{i-2}, \ldots, a_{i-(x-\alpha)}, b_{i\alpha}, b_{i\alpha+1}, b_{i\alpha+2}, \ldots, b_{i\alpha+\alpha-1}],$$

with the subscripts of b taken modulo $v-s$. Then $\{\{K_{1,x}^i \mid i \in \mathbb{Z}_s\}, \{K_{1,x}^{i*} \mid i \in \mathbb{Z}_s\}\}$ is a $T_{K_{1,x}}(s; v)$.

Note that $x - \alpha = \max(1, x-v+s)$. Thus the blocks $K_{1,x}^i$ and $K_{1,x}^{i*}$ always contain the edges $a_i a_{i+1}$ and $a_i a_{i-1}$, respectively (but not vice versa), ensuring the trade is proper (that is, $K_{1,x}^i \neq K_{1,x}^{i*}$). Since $3 \leqslant s \leqslant 2(v-x)-1$, we have $1 \leqslant (s-1)/2$ and $x - v + s \leqslant (s-1)/2$, and hence $x - \alpha \leqslant (s-1)/2$. This ensures that no edge $a_i a_j$ is repeated.

Since $\alpha \leqslant v-s$, no vertex b_j is repeated in any block. If $\alpha = v-s$, every vertex of $\{b_i \mid i \in \mathbb{Z}_{v-s}\}$ occurs in every block (in this case the $i\alpha$ terms in the subscripts of b can be replaced by 0). If $\alpha = x-1$, then by the condition $v/x \leqslant s$ we have $s\alpha = s(x-1) \geqslant v-s$, and hence every vertex of $\{b_i \mid i \in \mathbb{Z}_{v-s}\}$ occurs in at least one block. □

Corollary 3.1. *The conditions of Theorem* 1.1 *are sufficient for* $v \leqslant 2x$.

Proof. When $v \leqslant 2x$, $\lceil \max(3, v/x) \rceil = 3$ and $\min(v, 2(v-x)-1) = 2(v-x)-1$. Thus the result follows from Lemma 3.2, except when $s = 2$. In the case $s = 2$ we have $x = 2$ by Lemma 2.3; hence $v \leqslant 2x = 4$ by assumption and $v \geqslant 4$ by Corollary 2.1. It follows that $v = 4$; a suitable $T_{K_{1,2}}(2; 4)$ is $\{\{[3:1,2],[4:1,2]\}, \{[1:3,4],[2:3,4]\}\}$. □

Corollary 3.2. *For integers* x, v, s *satisfying* $x \geqslant 2$, $v \geqslant 2x+1$, *and* $\lceil v/x \rceil \leqslant s \leqslant v$, *there exists a* $T_{K_{1,x}}(s; v)$; *that is, the conditions of Theorem* 1.1 *are sufficient for* $v \geqslant 2x+1$ *and* $\lceil v/x \rceil \leqslant s \leqslant v$.

Proof. When $v \geqslant 2x+1$, $\lceil \max(3, v/x) \rceil = \lceil v/x \rceil$ and $\min(v, 2(v-x)-1) = v$. Thus the result follows from Lemma 3.2. □

The problem is complete for $v \leqslant 2x$, but for $v \geqslant 2x+1$ we still require trades of volumes satisfying $v+1 \leqslant s \leqslant \lfloor v(v-1)/2x \rfloor$ and $\lceil 2v/(2x+1) \rceil \leqslant s < v/x$. We now give the most complex construction, dealing with the case $v \leqslant s$.

Lemma 3.3. *Let* x, v *and* s *be integers satisfying* $x \geqslant 2$, $v \leqslant 4x-1$ *and* $v \leqslant s \leqslant v(v-1)/2x$. *Then there exists a* $T_{K_{1,x}}(s; v)$.

Proof. By assumption we have

$$v \leqslant s, \tag{1}$$

$$s \leqslant \frac{v(v-1)}{2x}, \tag{2}$$

$$v \leqslant 4x - 1. \tag{3}$$

Define $m = s - v$ and $n = 2v - s$, so that

$$m + n = v \tag{4}$$

and

$$2m + n = s. \tag{5}$$

By (2) and (3), we have $s \leqslant (4x-1)(v-1)/2x < 2(v-1)$, and thus $n \geqslant 3$. By (1), we have $m \geqslant 0$.

Let $M = \{a_i \mid i \in \mathbb{Z}_m\}$, $N = \{b_i \mid i \in \mathbb{Z}_n\}$, and $V = M \cup N$. On the vertex set V we construct a set of m copies of $K_{1,2x}$ and n copies of $K_{1,x}$, which we call blocks, which are pairwise edge-disjoint. We say that these $m+n$ blocks form a *packing* of the complete graph on V. We label the blocks $K^i_{1,2x}$, $i \in \mathbb{Z}_m$, and $K^j_{1,x}$, $j \in \mathbb{Z}_n$, respectively. We let a_i be the centre vertex of $K^i_{1,2x}$ and let b_i be the centre vertex of $K^i_{1,x}$. Thus the $K_{1,2x}$ blocks are centred on vertices of M, while the $K_{1,x}$ blocks are centred on vertices of N.

We first consider the $K_{1,2x}$ blocks. As far as possible, we will complete these blocks with vertices of M. The m copies of $K_{1,2x}$ contain a total of $2mx$ edges, and there are $\binom{m}{2}$ edges in M. Therefore the m copies of $K_{1,2x}$ must contain a total of at least $2mx - m(m-1)/2$ edges between M and N; thus, on average, each $K_{1,2x}$ block must contain at least $2x - (m-1)/2$ vertices of N.

We define $\alpha = 2x - (m-1)/2$. By (2) and (3), we have

$$m = s - v \leqslant (4x-1)(v-1)/2x - v < v - 2 \leqslant 4x - 3;$$

it follows that α is positive (thus the $K_{1,2x}$ blocks cannot be completed entirely with edges and vertices of M).

If m is odd, α is an integer. We will construct the $K_{1,2x}$ blocks so that each one will contain exactly α vertices of N. For $i \in \mathbb{Z}_m$, we define

$$K^i_{1,2x} = [a_i : a_{i+1}, \ldots, a_{i+\frac{m-1}{2}}, b_{\alpha i}, b_{\alpha i+1}, \ldots, b_{\alpha i+\alpha-1}],$$

with the subscripts of b taken modulo n.

If m is even the construction is similar though necessarily a little more complex, since α is not an integer. Note that $\lfloor\alpha\rfloor = 2x - m/2$ and $\lceil\alpha\rceil = 2x - (m-2)/2$.

For $i \in \mathbb{Z}_m$, we let

$$K^i_{1,2x} = \begin{cases} [a_i : a_{i+1}, \ldots, a_{i+\frac{m}{2}}, b_{\lfloor\alpha\rfloor i}, b_{\lfloor\alpha\rfloor i+1}, \ldots, b_{\lfloor\alpha\rfloor i+\lfloor\alpha\rfloor-1}] & \text{if } 0 \leqslant i \leqslant \dfrac{m}{2} - 1, \\ [a_i : a_{i+1}, \ldots, a_{i+\frac{m}{2}-1}, b_{\lceil\alpha\rceil i-\frac{m}{2}}, b_{\lceil\alpha\rceil i-\frac{m}{2}+1}, \ldots, b_{\lceil\alpha\rceil i-\frac{m}{2}+\lceil\alpha\rceil-1}] & \text{if } \dfrac{m}{2} \leqslant i \leqslant m-1, \end{cases}$$

with the subscripts of b taken modulo n.

We need to check that $\alpha \leqslant n$, to ensure that no vertices of N are repeated in any $K_{1,2x}$ block (when either m is odd or m is even). If $n \geqslant 2x$ this is clearly true, so we assume that $n \leqslant 2x-1$, giving $x \geqslant (n+1)/2 > (n-1)/2$, and hence

$$nx > \binom{n}{2}. \tag{6}$$

By (5) and (2) we have

$$nx + 2mx = sx \leqslant \binom{v}{2}. \tag{7}$$

By (4), $m + n = v$, and hence by considering the set of pairs on a v-set we have

$$\binom{v}{2} = \binom{m}{2} + \binom{n}{2} + mn. \tag{8}$$

Combining Eqs. (6), (7) and (8), we have

$$2mx < \binom{m}{2} + mn.$$

We can divide through by m, since m is not zero, and then rearrange to obtain $\alpha = 2x - (m-1)/2 < n$, as required.

We begin the $K_{1,x}$ blocks in a similar way to the $K_{1,2x}$ blocks (centred on vertices of N rather than M), except that they do not necessarily contain any vertices from M. If vertices of M are required in the $K_{1,x}$ blocks, we leave them unspecified at this point. Using an asterisk to denote an unspecified entry, for $i \in \mathbb{Z}_n$ we define

$$K_{1,x}^i = \begin{cases} [b_i : b_{i+1}, b_{i+2}, \ldots, b_{i+x}], & x \leqslant \dfrac{n-1}{2}, \\ [b_i : b_{i+1}, b_{i+2}, \ldots, b_{i+(n-1)/2}, *, *, \ldots, *], & x \geqslant \dfrac{n}{2}, n \text{ odd}, \\ \quad \left(x - \dfrac{n-1}{2} \text{ unspecified entries}\right), & \\ [b_i : b_{i+1}, b_{i+2}, \ldots, b_{i+n/2}, *, *, \ldots, *], & x \geqslant \dfrac{n}{2}, n \text{ even}, \quad 0 \leqslant i \leqslant \dfrac{n}{2} - 1, \\ \quad \left(x - \dfrac{n}{2} \text{ unspecified entries}\right), & \\ [b_i : b_{i+1}, b_{i+2}, \ldots, b_{i+n/2-1}, *, *, \ldots, *], & x \geqslant \dfrac{n}{2}, n \text{ even}, \quad \dfrac{n}{2} \leqslant i \leqslant n-1. \\ \quad \left(x - \dfrac{n}{2} + 1 \text{ unspecified entries}\right) & . \end{cases}$$

If $x \leqslant (n-1)/2$, the packing of K_v with m copies of $K_{1,2x}$ and n copies of $K_{1,x}$ is complete. Otherwise we must complete the $K_{1,x}$ blocks, using vertices from M; we now prove that this is possible.

Let $x \geqslant n/2$, and, for $i \in \mathbb{Z}_n$, let l_i be the number of vertices of M to which the vertex n_i is not adjacent in any $K_{1,2x}$ block; that is, l_i is the number of edges available to complete the block $K_{1,x}^i$.

The $K_{1,2x}$ blocks use a total of $2xm$ edges, including all the edges between vertices of M and some of the edges between M and N. So by counting edges we have

$$\binom{v}{2} = 2xm + \binom{n}{2} + \sum_{i \in \mathbb{Z}_n} l_i. \tag{9}$$

For each $i \in \mathbb{Z}_n$, we let l_i^* be the number of unspecified vertices in the $K_{1,x}^i$ construction given above; that is (since $x \geqslant n/2$),

$$l_i^* = \begin{cases} x - \dfrac{n-1}{2}, & n \text{ odd}, \\ x - \dfrac{n}{2}, & n \text{ even}, \quad 0 \leqslant i \leqslant \dfrac{n}{2} - 1, \\ x - \dfrac{n}{2} + 1, & n \text{ even}, \quad \dfrac{n}{2} \leqslant i \leqslant n-1. \end{cases}$$

To complete the packing we require, for each $i \in \mathbb{Z}_n$, that $l_i \geqslant l_i^*$.

Given the construction of the $K_{1,2x}$ blocks (in which the vertices of N were used in order from b_0 to b_{n-1}, then back to b_0, repeating as many times as required), for $i, j \in \mathbb{Z}_n$ with $0 \leqslant i < j \leqslant n-1$ we have $l_i = l_j$ or $l_i = l_j - 1$; from the definition of l_i^*, we likewise have $l_i^* = l_j^*$ or $l_i^* = l_j^* - 1$.

Assume that $l_j < l_j^*$ for some $i \in \mathbb{Z}_n$ (we seek a contradiction). Then by the above properties of l_i and l_i^* we have $l_i \leqslant l_i^*$ for all $i \in \mathbb{Z}_n$, and hence $\sum_{i \in \mathbb{Z}_n} l_i < \sum_{i \in \mathbb{Z}_n} l_i^*$. But by the definition of l_i^* we have $\sum_{i \in \mathbb{Z}_n} l_i^* = xn - \binom{n}{2}$, so by (9) we have $\binom{v}{2} < 2xm + \binom{n}{2} + [xn - \binom{n}{2}] = x(2m+n)$. This is a contradiction by Eqs. (5) and (2); thus $l_i \geqslant l_i^*$ for all $i \in \mathbb{Z}_n$, and hence we can complete the packing of K_v with m copies of $K_{1,2x}$ and n copies of $K_{1,x}$.

Now we use this packing to construct a $T_{K_{1,x}}(s; v)$.

For each $i \in \mathbb{Z}_n$, we construct the block $K_{1,x}^{i*}$ by starting with the block $K_{1,x}^i$, and replacing each vertex b_{i+d}, $1 \leqslant d \leqslant n/2$, with the vertex b_{i-d}. We have $n \geqslant 3$, so $\{\{K_{1,x}^i \mid i \in \mathbb{Z}_n\}, \{K_{1,x}^{i*} \mid i \in \mathbb{Z}_n\}\}$ is a trade.

By Lemma 3.1, we can place a $K_{1,x}$ trade of volume 2 on each block $K^i_{1,2x}$, $i \in \mathbb{Z}_m$. Let this trade be $\{\{B^i_1, B^i_2\}, \{B^i_3, B^i_4\}\}$. Define

$$T_1 = \{B^i_1, B^i_2 \mid i \in \mathbb{Z}_m\} \cup \{K^i_{1,x} \mid i \in \mathbb{Z}_n\},$$

$$T_2 = \{B^i_3, B^i_4 \mid i \in \mathbb{Z}_m\} \cup \{K^{i*}_{1,x} \mid i \in \mathbb{Z}_n\}.$$

Then $\{T_1, T_2\}$ is a $T_{K_{1,x}}(s; v)$. □

We now extend this result inductively to arbitrarily large foundation sizes.

Lemma 3.4. *For integers x, v, s satisfying* $x \geqslant 2$, $v \geqslant 2x + 1$, *and* $v \leqslant s \leqslant \lfloor v(v-1)/2x \rfloor$, *there exists a* $T_{K_{1,x}}(s; v)$; *that is, the conditions of Theorem* 1.1 *are sufficient for* $v \geqslant 2x + 1$ *and* $s \geqslant v$.

Proof. For $2x + 1 \leqslant v \leqslant 4x - 1$, the result follows from Lemma 3.3.

Regarding x as a constant ($x \geqslant 2$), assume that the result holds for a given v; that is, for some $v \geqslant 2x + 1$, there exists a $T_{K_{1,x}}(s; v)$ for every s satisfying $v \leqslant s \leqslant \lfloor v(v-1)/2x \rfloor$.

Take disjoint vertex sets A and B such that $|A| = v$ and $|B| = 2x$. By assumption we can place a $K_{1,x}$-trade of any volume between v and $\lfloor v(v-1)/2x \rfloor$ on A, and by Lemma 3.2 we may place a trade of any volume between 3 and $2x - 1$ on B. The bipartite graph $K_{v,2x}$ with vertex set $A \cup B$ may be decomposed into v copies of $K_{1,2x}$, and by Lemma 3.1 (with $s = 2$) we may (optionally) place a $K_{1,x}$-trade of volume 2 on some or all of these copies of $K_{1,2x}$. The combined trade has foundation $v + 2x$ and any volume between $v + 3$ and $\lfloor v(v-1)/2x \rfloor + 2v + 2x - 1$. Since $v + 3 < v + 2x$ and $\lfloor v(v-1)/2x \rfloor + 2v + 2x - 1 = \lfloor (v + 2x)(v + 2x - 1)/2x \rfloor$, the result holds for foundation $v + 2x$.

For this induction to work we require $2x$ basis cases, but we only have $2x - 1$ (namely, those v with $2x + 1 \leqslant v \leqslant 4x - 1$). However, we can include $v = 2x$ and $s = 2x - 1$ as a basis case to prove the result for $v = 4x$ (this trade exists by Lemma 3.2), and hence the result follows inductively for all $v \geqslant 2x + 1$. □

We conclude with a construction for the case $\lceil 2v/(2x+1) \rceil \leqslant s < v/x$.

Lemma 3.5. *For integers x, v, s satisfying* $x \geqslant 2$, $v \geqslant 2x + 1$, *and* $\lceil 2v/(2x+1) \rceil \leqslant s < v/x$, *there exists a* $T_{K_{1,x}}(s; v)$; *that is, the conditions of Theorem* 1.1 *are sufficient for* $v \geqslant 2x + 1$ *and* $s < v/x$.

Proof. Let x, v and s be integers satisfying the above conditions. Then $s \geqslant 2$, and $sx + 1 \leqslant v \leqslant \lfloor s(2x+1)/2 \rfloor$.

By Lemma 3.1 with $s = 2$ and with $s = 3$, there exist trades $T_{K_{1,x}}(2; K_{1,2x})$ and $T_{K_{1,x}}(3; K_{1,3x})$.

If s is even, take $s/2$ copies of $T_{K_{1,x}}(2; K_{1,2x})$, which are vertex disjoint except that one vertex is common to $s(2x+1)/2 - v + 1$ of the trades. The combined trade is a $T_{K_{1,x}}(s; v)$.

If s is odd, note that $\lfloor s(2x+1)/2 \rfloor = (s(2x+1) - 1)/2$. Take $(s-3)/2$ copies of $T_{K_{1,x}}(2; K_{1,2x})$ and one copy of $T_{K_{1,x}}(3; K_{1,3x})$, which are vertex disjoint except that one vertex is common to $(s(2x+1) - 1)/2 - v + 1$ of the trades. The combined trade is a $T_{K_{1,x}}(s; v)$. □

The sufficiency of Theorem 1.1 is established by Corollaries 3.1 and 3.2 and Lemmas 3.4 and 3.5. This completes the proof of Theorem 1.1.

References

[1] E.J. Billington, Combinatorial trades: a survey of recent results, in: W.D. Wallis (Ed.), Designs 2002: Further Computational and Constructive Design Theory, Kluwer Academic Publishers, Boston, Dordrecht, London, pp. 47–67 (Chapter 3). Book Series: Mathematics and its Applications, vol. 563.

[2] E.J. Billington, D.G. Hoffman, The intersection problem for star designs, Discrete Math. 179 (1998) 217–222.

[3] D.E. Bryant, On the volume of trades in triple systems, Austr. J. Combin. 15 (1997) 161–176.

[4] D.E. Bryant, M. Grannell, T. Griggs, B. Maenhaut, On the volume of 4-cycle trades, Graphs Combin. 19 (2003) 53–63.
[5] P. Cain, Decompositions of complete graphs into stars, Bull. Austral. Math. Soc. 10 (1974) 23–30.
[7] C. Huang, Decompositions of complete graphs into stars, unpublished manuscript.
[8] G.B. Khosrovshahi, H.R. Maimani, R. Torabi, On trades: an update, Discrete Appl. Math. 95 (1999) 361–376.
[9] J. Lefevre, Trade spectrum of $K_4 - e$, Graphs Combin. 21 (2005) 475–488.
[10] J. Lefevre, Six-cycle trades and a lower bound on the trade volumes of weakly connected graphs, Utilitas Math., to appear.
[11] B. Maenhaut, On the volume of 5-cycle trades, Graphs Combin. 17 (2001) 315–328.
[12] A.J. Moon, On subtournaments of a tournament, Can. Math. Bull. 9 (1966) 297–301.
[13] M. Tarsi, Decomposition of complete multigraphs into stars, Discrete Math. 26 (1979) 273–278.