A note on diagonally dominant matrices

Geir Dahl*

Department of Mathematics and Department of Informatics, University of Oslo, P.O. Box 1080, Blindern, 0316 Oslo, Norway

Received 29 April 1999; accepted 18 May 2000

Submitted by R.A. Brualdi

Abstract

The set $D_n$ of diagonally dominant symmetric real matrices of order $n$ with nonnegative diagonal elements is a polyhedral convex cone. Based on its extreme rays, we derive a description of the kernel of each matrix in $D_n$ in terms of a certain support graph. Moreover, the doubly stochastic matrices in $D_n$ are studied. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Diagonally dominant matrices; Convex cones; Graphs and matrices

1. Generators and kernels

We recall (see, e.g., [5]) that a real matrix $A$ of order $n$ is called diagonally dominant if $|a_{i,i}| \geq \sum_{j \neq i} |a_{i,j}|$ for $i = 1, \ldots, n$. If all these inequalities are strict, $A$ is strictly diagonally dominant. A matrix is called nonnegative if all its elements are nonnegative. Let $D'_n$ denote the set of all diagonally dominant symmetric real matrices of order $n$ and let

$D_n = \{ A = [a_{i,j}] \in D'_n : a_{i,i} \geq 0 \text{ for } i = 1, \ldots, n \}$.

Observe that $A \in D_n$ iff $a_{i,j} = a_{j,i}$ for each $i, j$ and $a_{i,i} \geq \sum_{j \neq i} \epsilon_{i,j} a_{i,j}$ for $i = 1, \ldots, n$ and $\epsilon_{i,j} \in \{-1, 1\}$ for each $i, j$. It follows that $D_n$ is a polyhedral (convex) cone in the vector space $\mathbb{R}^{n,n}$ of real matrices of order $n$. Moreover, this cone is pointed. (Similar results are true without the symmetry assumption, but we concentrate on the symmetric case below.) Note that $D'_n$ is also a cone, but it is nonconvex.

* Tel.: 47-22-85-2425; fax: 47-22-85-2401.
E-mail address: geird@ifi.uio.no (G. Dahl).
The relative interior of $\mathcal{D}_n$ consists of the strictly diagonally dominant symmetric matrices with positive elements.

Let $e_i$ denote the $i$th unit vector in $\mathbb{R}^n$ and define the following $(-1, 0, 1)$-matrices of order $n$:

(i) $A_i = e_i e_i^T$ for $i = 1, \ldots, n$;
(ii) $A^+_{i,j} = (e_i + e_j)(e_i + e_j)^T$ for $1 \leq i < j \leq n$;
(iii) $A^-_{i,j} = (e_i - e_j)(e_i - e_j)^T$ for $1 \leq i < j \leq n$.

Let $\mathcal{S}_n$ be the set of matrices in (1). All these matrices lie in $\mathcal{D}_n$. Moreover, one can check that no matrix in $\mathcal{S}_n$ generates an extreme ray of $\mathcal{S}_n$. The following result, and some related ones, may be found in [1].

**Proposition 1.** $\mathcal{D}_n = \text{cone}(\mathcal{S}_n)$ and each matrix in $\mathcal{S}_n$ generates an extreme ray of $\mathcal{D}_n$.

**Proof.** As noted $\mathcal{S}_n \subseteq \mathcal{D}_n$ and therefore $\text{cone}(\mathcal{S}_n) \subseteq \mathcal{D}_n$. To see the converse inclusion, consider a matrix $A \in \mathcal{D}_n$. Define $P = \{(i, j) : i < j, a_{i,j} > 0\}$ and $N = \{(i, j) : i < j, a_{i,j} < 0\}$. It is easy to check that

$$A = \sum_{i=1}^{n} \left( a_{i,j} - \sum_{j \neq i} [a_{i,j}] \right) A_i + \sum_{(i,j) \in P} a_{i,j} A^+_{i,j} + \sum_{(i,j) \in N} (-a_{i,j}) A^-_{i,j}. \tag{2}$$

Here all coefficients are nonnegative, so $A \in \text{cone}(\mathcal{S}_n)$ as desired. Moreover, one can check that no matrix in $\mathcal{S}_n$ is the sum of other matrices in $\mathcal{S}_n$, so each matrix in $\mathcal{S}_n$ generates an extreme ray of $\mathcal{D}_n$. □

A variation of this result concerns sign-restrictions on the matrix elements. Let $T^+$ and $T^-$ be a partition of the index set $\{(i, j) : 1 \leq i < j \leq n\}$. Consider the set $\mathcal{D}(T^+, T^-)$ of matrices in $\mathcal{D}_n$ satisfying $a_{i,j} \geq 0$ for all $(i, j) \in T^+$ and $a_{i,j} \leq 0$ for all $(i, j) \in T^-$. For instance, if $T^-$ is empty, we obtain the nonnegative matrices in $\mathcal{D}_n$, or if $T^+$ is empty, we get the matrices in $\mathcal{D}_n$ with nonpositive off-diagonal elements. As above, it is easy to see that $\mathcal{D}(T^+, T^-)$ is spanned by the matrices $A_i$ for $i = 1, \ldots, n$, $A^+_{i,j}$ for $(i, j) \in T^+$ and $A^-_{i,j}$ for $(i, j) \in T^-$. Moreover, $\mathcal{D}(T^+, T^-)$ is a simplex cone, meaning that the mentioned matrices (spanning the cone) are linearly independent. We also remark that a similar result to Proposition 1 holds for the possibly nonsymmetric matrices that are diagonally dominant and have nonnegative diagonal. The dimension of $\mathcal{D}_n$ is $n(n+1)/2$.

Let $A \in \mathcal{D}_n$. We shall determine the kernel of $A$ and to this end some graph notation is introduced. Define the node set $V = \{v_1, \ldots, v_n\}$ and the following three edge sets. $L$ consists of the loops $[v_i, v_i]$ when $a_{i,j} > \sum_{j \neq i} |a_{i,j}|$ for $i = 1, \ldots, n$, and $P$ (resp. $N$) consists of the edges $[v_i, v_j]$ when $a_{i,j} > 0$ (resp. $a_{i,j} < 0$) for $1 \leq i < j \leq n$ (this notation is consistent with $P$ and $N$ defined before (2)). Let $V_1, \ldots, V_t \subseteq V$ be the connected components of the graph $(V, E)$. Let $H$ denote the graph with
node set $\{V_1, \ldots, V_t\}$ and edges $\{V_i, V_j\}$ whenever $[v_i, v_j] \in P$ for some $v_i, v_j \in V_i$ and $v_j \in V_j$ (when $i = j$ this is a loop). Moreover, $H$ contains a loop $\{V_i, V_j\}$ whenever $[v_i, v_j] \in L$ for some $v_i, v_j \in V_i$. We call $H$ the support graph of $A$. Let $C_1, \ldots, C_p$ denote the components of $H$ that are bipartite and without any loop ($p = 0$ means that no such component exists.) Note that $C_1, \ldots, C_p$ correspond to a coarser partition of $V$ than $V_1, \ldots, V_t$ does. Thus, we may write $v_i \in C_j$ (meaning that $v_i \in V_i$ and $V_i \in C_j$ for some $V_i$). For each $j \leq p$, let $C_j^+$ and $C_j^-$ be the two color classes of $C_j$. Thus, $C_j^+$, $C_j^-$ is a partition of $C_j$ and each edge of $H$ joins a node in $C_j^+$ and a node in $C_j^-$. Let $x^j \in \mathbb{R}^p$ be a vector whose support is $C_j$, and $z^j = 1$ if $v_i \in C_j^+$ and $z^j = -1$ if $v_i \in C_j^-$. (If the color classes change role, we obtain the negative of $x^j$, but this ambiguity will not matter below.) Note that we allow the component $C_j$ to be trivial, i.e., with a single node $V_j$ (but no loop), and then $z^j = 1$ if $v_i \in V_j$ and $z^j = 0$ otherwise.

With this notation, we have the following result on the kernel $\text{Ker}(A)$ of a matrix $A \in \mathcal{S}_n$.

**Theorem 2.** Let $A \in \mathcal{S}_n$. Then $\text{Ker}(A) = \text{span}([x^1, \ldots, x^p])$ and therefore $\text{rank}(A) = n - p$.

**Proof.** Consider the conical representation of $A$ given in (2) and define $\lambda_i = a_{i,i} - \sum_{j \neq i} |a_{i,j}|$ for $i = 1, \ldots, n$. Let $x \in \mathbb{R}^n$. From the simple structure of the matrices $A_i, A_{i,j}^+$ and $A_{i,j}^-$, we obtain the following identities:

(i) $Ax = \sum_{i=1}^n \lambda_i x_i e_i + \sum_{(i,j) \in P} a_{i,j} (x_i + x_j)(e_i + e_j)$
\[ + \sum_{(i,j) \in N} (-a_{i,j})(x_i - x_j)(e_i - e_j); \]

(ii) $x^T A x = \sum_{i=1}^n \lambda_i x_i^2 + \sum_{(i,j) \in P} a_{i,j} (x_i + x_j)^2$
\[ + \sum_{(i,j) \in N} (-a_{i,j})(x_i - x_j)^2. \]

Moreover, in (3) it suffices to sum over those $i$ for which $[v_i, v_j] \in L$ (i.e., $\lambda_i > 0$). Note that $x^T A x \geq 0$ ($A$ is positive semidefinite). Let now $x \in \text{Ker}(A)$. Then $Ax = 0$ and therefore $x^T A x = 0$. Thus, (3)(ii) implies that (a) $x_i = 0$ whenever $[v_i, v_j] \in L$, and (b) $x_i = -x_j$ whenever $[v_i, v_j] \in P$ and (c) $x_i = x_j$ whenever $[v_i, v_j] \in N$. From (c), we see that for each component $V_j$ of $(V, N)$ (i.e., each node of $H$) there is a number $\alpha_j$ such that $x_i = \alpha_j$ for all $v_i \in V_j$. If there is a loop $[v_i, v_j] \in L$ for some $v_i \in V_j$ or if $[v_i, v_k] \in P$ for some $v_i, v_k \in V_j$, then it follows that $\alpha_j = 0$. Moreover, if $H$ contains an edge between $V_i$ and $V_j$, then we conclude from (b) that $\alpha_i = -\alpha_j$. It follows that $\alpha_i = 0$ for all nodes $V_i$ in a component of $H$ that contains a loop or an odd cycle. Consider one of the components $C_j$, $j \leq p$ (which
has no loop and is bipartite). Then there is some number \( \alpha \) (depending on \( j \)) such that 
\[ x_i = \alpha \] for each \( v_i \in C_j^+ \) and 
\[ x_i = -\alpha \] for each \( v_i \in C_j^- \). Thus, the restriction of \( x \)
to the nodes in \( C_j \) lies in \( \text{span}(z^j) \). This holds for every \( j \leq p \) and we conclude that 
\( x \in \text{span}(x^1, \ldots, x^p) \).

Conversely, assume that \( x \in \text{span}(x^1, \ldots, x^p) \). To prove that \( Ax = 0 \), it is sufficient to prove that \( A z^k = 0 \) for each \( k \leq p \). First, observe that \( (A z^k)_{ij} = 0 \) for each \( v_i \not\in C_k \) as \( z^k \) has its support in \( C_k \). Recall that \( z_i^k = 1 \) if \( v_i \in C_k \) and \( z_i^k = -1 \) if \( v_i \not\in C_k \). It follows that \( \lambda_i = 0 \) for each \( v_i \in C_k \) (as \( C_k \) has no loop), 
\[ z_i = -z_i^k \] for each \( \{v_i, v_j\} \in E \) (with \( v_i, v_j \in C_k \)), and \( z_i = z_i^k \) for each \( \{v_i, v_j\} \in N \) (with \( v_i, v_j \in C_k \)). Therefore, the restriction of \( A z^k \) to the components in \( C_k \) must also be zero which proves that \( A z^k = 0 \) and \( x \) lies in \( \text{Ker}(A) \). Finally, we see that \( z^1, \ldots, z^p \) are linearly independent (they are nonzero and have disjoint supports), so the kernel has dimension \( p \) and \( \text{rank}(A) = n - p \). \( \square \)

We remark that the expression for \( \text{rank}(A) \) also may be derived from Taussky’s theorem. This theorem says that a square matrix \( A \) is nonsingular if it is irreducibly diagonally dominant, i.e., if \( A \) is irreducible and diagonally dominant and \( |ai;i| > \sum_{j \neq i} |ai;j| \) for at least one \( i \); see [5].

From Theorem 2, we see the interesting fact that the kernel, and therefore the rank, of a matrix \( A \in S_n \) depends only on the support graph \( H \), i.e., the structure (and sign) of the nonzero entries in \( A \) and the set \( \{i : ai;i > \sum_{j \neq i} |ai;j| \} \); otherwise the magnitudes of all these numbers are irrelevant. The reduced row-echelon form of \( A \) has a similar feature. It is also interesting to note that the kernel has a basis consisting of orthogonal \((-1, 0, 1)\)-vectors. Finally, we see that the calculation of \( \text{rank}(A) \) and \( \text{Ker}(A) \) is easily done using breadth-first-search (to determine (bipartite) components), so no numerical calculation is required. Theorem 2 and its proof is related to a result of [7] saying that each nonnegative matrix \( A \in S_n \) is completely positive, i.e., \( A = B^T B \) for some nonnegative \( n \times m \) matrix \( B \). In the proof of this result [7] considered, the graph \( G_A = (V, L \cup P) \) and defined its weighted incidence matrix \( B \) as follows. \( B \) has a row for each node of \( G_A \) and a column for each edge in \( E_A \), and \( b_{v_i, [v_i, v_j]} = b_{v_j, [v_i, v_j]} = a_{i;j}^{1/2} \) when \( [v_i, v_j] \in E_A \) and \( i \neq j \), \( b_{v_i, [v_i, v_i]} = (ai;i - \sum_{j \neq i} |ai;j|)^{1/2} \) while all other entries are zero. Then one can check that \( A = B^T B \).

In connection with the proof of Theorem 2, we note that \( \text{Ker}(B^T) = \text{Ker}(BB^T) = \text{Ker}(A) \), and that \( B^T x = 0 \) is just conditions (a) and (b) in our proof.

It also follows from Theorem 2 that the positive definite matrices in \( S_n \) may be characterized in terms of the support graph \( H \) in the following way: a matrix \( A \in S_n \) is positive definite (and therefore nonsingular) if and only if each component of \( H \) contains a loop or an odd cycle. For instance, consider the matrices

\[
A = \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}, \quad 
B = \begin{bmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 1
\end{bmatrix}, \quad 
C = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}.
\]
A is positive definite because $H$ is an odd cycle (a triangle), $B$ is singular ($H$ is a path) and $C$ is singular ($H$ is a single node). Note that these matrices are not strictly diagonally dominant. As another example let $A \in \mathcal{P}_n$ be tridiagonal, i.e., $a_{i,j} = 0$ when $|i - j| > 1$, and assume that $a_{i,i} > 0$ for each $i$ and $a_{i,i+1} = a_{i+1,i} \neq 0$ for $i = 1, \ldots, n - 1$. It is easy to see that $H$ is a path, possibly with some loops. Thus, $A$ is nonsingular (and positive definite) if and only if $H$ contains a loop, i.e., $a_{i,i} > |a_{i,j+1}| + |a_{i,i-1}|$ for some $i$ (here we define $a_{1,0} = a_{n,n+1} = 0$). Such matrices are of interest in connection with cubic splines.

Consider $A \in \mathcal{P}_n$ and let $F(A)$ denote the smallest face of $\mathcal{P}_n$ that contains $A$. (Since $\mathcal{P}_n$ is polyhedral, each face is the intersection between $\mathcal{P}_n$ and one of its supporting hyperplanes.) Then $\dim(F(A)) = |L| + |P| + |N|$ and $F(A)$ is the simplex cone spanned by the matrices $\Lambda_i$ for $[v_i, v_i] \in L$, $\Lambda_{i,j}^+$ for $[v_i, v_j] \in P$ and $\Lambda_{i,j}^-$ for $[v_i, v_j] \in N$. It follows from Theorem 2 that the maximum rank among the matrices in $F(A)$ is $n - p$ where $p$ is the number of components of $H$ that are bipartite and without any loop. This maximum rank is achieved for all matrices in the relative interior of $F(A)$.

Define $\mathcal{P}_n = \{ A = [a_{i,j}] \in \mathcal{P}_n; \sum_{i=1}^n a_{i,i} = 2 \}$ which is a polytope in $\mathbb{R}^{n,n}$. It follows from Proposition 1 that the vertices of $\mathcal{P}_n$ are $\hat{A}_i := 2 \cdot \Lambda_i$ for $i \leqslant n$ and $\Lambda_{i,j}^+$, $\Lambda_{i,j}^-$ for $i < j$. Recall that two vertices $A_1$ and $A_2$ are adjacent on $\mathcal{P}_n$ if $\text{conv}(\{A_1, A_2\})$ is a one-dimensional face (edge) of $\mathcal{P}_n$. We may now describe the one-skeleton of $\mathcal{P}_n$ (the graph with a node for each vertex of $\mathcal{P}_n$ and where two nodes are adjacent iff the corresponding vertices span a one-dimensional face (edge) of $\mathcal{P}_n$).

**Proposition 3.** $\hat{A}_i$ and $\hat{A}_j$ are not adjacent when $i \neq j$. $\Lambda_{i,j}^+$ and $\Lambda_{i,j}^-$ are not adjacent for $i < j$. Any other pair of vertices of $\mathcal{P}_n$ is adjacent. In particular, $\mathcal{P}_n$ has diameter 2.

**Proof (sketch).** We use the following adjacency characterization: two vertices $A_1$ and $A_2$ are adjacent if and only if the line through $A_1$ and $A_2$ does not intersect the convex hull of the remaining vertices. The first two statements follow from the relation

\[
\Lambda_{i,j}^+ + \Lambda_{i,j}^- = \hat{A}_i + \hat{A}_j.
\]

Consider the vertices $\Lambda_{i,j}^+$ and $\Lambda_{i,k}^+$, where $i < j < k$. Any matrix of the form $A = (1 - \lambda)\Lambda_{i,j}^+ + \lambda \Lambda_{i,k}^+$ will have a nonzero in one of the entries $(i, j)$ and $(i, k)$. Assume that $A$ is a convex combination of the remaining vertices. In this combination, the coefficient of $\Lambda_{u,v}^+$ and $\Lambda_{u,v}^-$ must be equal for every $(u, v)$ with $u < v$ and $(u, v) \neq (i, j)$, $(i, k)$ (in order to produce a zero in position $(u, v)$). Therefore (confer the relation (\ref{convex})), we can replace $\Lambda_{u,v}^+$ and $\Lambda_{u,v}^-$ by $\hat{A}_u$ and $\hat{A}_v$ in our convex combination.

Thus, $A$ is convex combination of the matrices $\hat{A}_i$, $i \leqslant n$, which contradicts that $a_{i,j}$ or $a_{i,k}$ is nonzero. Therefore $\Lambda_{i,j}^+$ and $\Lambda_{i,k}^+$ are adjacent. The remaining adjacency relations are proved with the same technique. \qed
2. Doubly stochastic diagonally dominant matrices

A matrix is doubly stochastic if it is nonnegative and each row and column sum is 1. We let $B_n$ denote the set of doubly stochastic matrices of order $n$. The set $B_n$ is a convex polytope in $\mathbb{R}^{n,n}$, often called the Birkhoff polytope. The classical Birkhoff–von Neumann theorem states that $B_n$ is the convex hull of all permutation matrices of order $n$. For more information about this theorem and doubly stochastic matrices, we refer to [2,5].

We are here concerned with the set $DB_n$ of symmetric, diagonally dominant and doubly stochastic matrices (of order $n$), i.e., $DB_n \subset B_n$.

Note that $DB_n$ is a (convex) polytope and that the only integral matrix in $DB_n$ is the identity matrix. We shall give different representations of $DB_n$.

Let $B_n$ be the set of symmetric matrices in $B_n$. Let $\delta(v)$ denote the set of edges incident to a node $v$ in a graph (including, possibly, the loop $Tv;v$). We need the following lemma concerning the “fractional perfect matching polytope” in graphs with loops (see [3]). We use the notation $x(v) := \sum_{e \in \delta(v)} x_e$.

Lemma 4. Let $G = (V, E)$ be a connected graph, possibly with loops and define the polytope $FM(G) = \{ x \in \mathbb{R}_+^E : x \geq 0, x(\delta(v)) = 1 \text{ for all } v \in V \}$. Then $x \in FM(G)$ is a vertex of $FM(G)$ if and only if $x_e \in \{0, 1/2, 1\}$ for each $e \in E$ and the edges $e$ with $x_e = 1/2$ form node disjoint odd cycles.

Proof (sketch). Let $x$ be a vertex and define $F = \{ e \in E : 0 < x_e < 1 \}$. Using the extreme point property one can gradually deduce that $F$ cannot contain an even cycle, a cycle with a loop, a cycle with a chord or two disjoint cycles connected by a path. From this it follows that $x$ must have the mentioned form. Sufficiency: let $x'$ satisfy the properties above and define $a \in \mathbb{R}^E$ by $a_e = -1$ if $x'_e = 0$ and $a_e = 0$ if $x'_e > 0$. Then, for each $x \in FM(G) \setminus \{ x' \}$ it holds that $a^T x < 0$. But $a^T x' = 0$ which implies that $x'$ is a vertex.

This result may be reformulated in terms of matrices. A symmetric matrix $A \in \mathbb{R}^{n,n}$ may be represented by a weighted graph $G = (V, E)$ with nodes $v_1, \ldots, v_n$ and edges $[v_i, v_j]$ with associated weight $x_{i,j} := a_{i,j} = a_{j,i}$ for $1 \leq i \leq j \leq n$ (when $i = j$, we have a loop $[v_i, v_i]$). We see that $A$ is symmetric and doubly stochastic iff $x \in \mathbb{R}_+^E$ (as just defined) is nonnegative and $x(\delta(v_i)) = 1$ for each $i \leq n$. Thus, $DB_n$ and $FM(G)$ are affinely isomorphic. Let $A$ be a vertex of $DB_n$. Consider the corresponding vertex $x$ of $FM(G)$ and choose an ordering of the vertices, so that (i) the vertices of each fractional cycle (having edges with $x_e = 1/2$) occur consecutively, and (ii) the endnodes of each edge with $x_e = 1$ occur consecutively. The node ordering corresponds to simultaneous line permutations in $A$ and we see from Lemma 4 that the resulting matrix $Q^T A Q$ may be written as the direct sum of the matrices

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

C(p)
where \( p \) is odd. Here \( C^p = \{ c_{i,j} \} \in \mathbb{R}^{p \times p} \) is defined by \( c_{i,i+1} = c_{i+1,i} = \frac{1}{2} \) for \( 1 \leq i \leq p-1 \), \( c_{1,p} = c_{p,1} = \frac{1}{2} \) and \( c_{i,j} = 0 \) otherwise. In (\( \ast \)) the first and the second matrix corresponds to a loop and an edge with \( x_e = 1 \), respectively, while \( C^p \) corresponds to a fractional cycle of odd length \( p \). Note that \( C^p = \frac{1}{2}(P^p + (P^p)^T) \), where \( P^p \) is the permutation matrix of order \( p \) having ones in positions \( (i, i+1) \) for \( 1 \leq i \leq p-1 \) and \( (p, 1) \). Let \( \mathcal{P} \) denote the set of all permutation matrices that may be obtained by simultaneous line permutations from a direct sum of the following matrices: the first two matrices in (\( \ast \)) and \( P^p \), where \( p \) is odd. We therefore obtain the following result due to Katz [6] (see also [4]).

**Proposition 5.** The set \( \mathcal{B}_n^p \) of symmetric doubly stochastic matrices is the convex hull of matrices of the form \((1/2)(P + P^T)\), where \( P \in \mathcal{P}^n \).

Observe that a matrix \( A \) with \( \sum_j a_{i,j} = 1 \) satisfies \( a_{i,i} \geq \sum_{j \neq i} a_{i,j} \) if and only if it satisfies \( a_{i,j} \geq 1/2 \). It follows that \( \mathcal{D} \mathcal{B}_n^p \) consists of the matrices \( A \) satisfying the following linear system:

(i) \( \sum_{j=1}^n a_{i,j} = 1 \) for \( i = 1, \ldots, n \);
(ii) \( a_{i,j} = a_{j,i} \) for \( 1 \leq i < j \leq n \);
(iii) \( a_{i,j} \geq 1/2 \) for \( i = 1, \ldots, n \); and
(iv) \( a_{i,j} = 0 \) for \( 1 \leq i < j \leq n \).

It follows from Proposition 5 that \( \mathcal{D} \mathcal{B}_n^p = (1/2) \cdot I_n + (1/2) \cdot \mathcal{B}_n^p \) (where \( I_n \) denotes the identity matrix of order \( n \)) and that \( \mathcal{D} \mathcal{B}_n^p \) is the convex hull of the matrices \((1/2) \cdot I_n + (1/4) \cdot (P + P^T)\), where \( P \in \mathcal{P}^n \). The polytopes \( \mathcal{D} \mathcal{B}_n^p \) and \( \mathcal{B}_n^p \) are affinely isomorphic and have dimension \( n(n-1)/2 \). We get similar relations for \( \mathcal{D} \mathcal{B}_n \) and \( \mathcal{B}_n \) for diagonal dominant doubly stochastic matrices. So \( \mathcal{D} \mathcal{B}_n = (1/2) \cdot I_n + (1/2) \cdot \mathcal{B}_n \) and \( \mathcal{D} \mathcal{B}_n \) is also equal to the convex hull of the matrices \((1/2) \cdot I_n + (1/2) \cdot P\), where \( P \) is a permutation matrix of order \( n \). The polytope \( \mathcal{D} \mathcal{B}_n \) and the Birkhoff polytope \( \mathcal{B}_n \) are affinely isomorphic.

Finally, we point out that the set \( \mathcal{D} \mathcal{B}_n \) may be of interest in connection with majorization (see [8]). If \( x, y \in \mathbb{R}^n \), one says that \( y \) is majorized by \( x \), denoted by \( y \prec x \), provided that \( \sum_{j=1}^k y(j) \leq \sum_{j=1}^k x(j) \) for \( k = 1, \ldots, n-1 \) and \( \sum_{j=1}^n y(j) = \sum_{j=1}^n x(j) \). (Here \( x(j) \) denotes the \( j \)th largest number among the components of \( x \).) A well-known theorem of Hardy–Littlewood and Pólya (see [8]) says that \( y \prec x \) if and only if \( Bx = y \) for some \( B \in \mathcal{B}_n \). Consider now the stronger property that

\[
Ax = y \quad \text{for some} \ A \in \mathcal{D} \mathcal{B}_n. 
\]

so \( A \) is not just doubly stochastic, but also diagonally dominant. From the description of \( \mathcal{D} \mathcal{B}_n \) given above, we see that (4) holds if and only if \( y = (1/2)x + (1/2)x \) for some \( z \prec x \). The geometrical interpretation is that \( y \) is the midpoint of the line segment between \( x \) and a point \( z \) in the convex hull of all permutations of \( x \), or, equivalently, \( y \) is a convex combination of points of the form \((1/2)x + (1/2)Px\), where \( P \) is a permutation matrix. A similar characterization may be given when (4) holds for a matrix in \( \mathcal{D} \mathcal{B}_n^p \).
Acknowledgement

The author thanks the referees for constructive criticism and suggestions.

References