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Homotopy analysis method for higher-order fractional integro-differential equations ☆

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ABSTRACT

In this paper, we present the homotopy analysis method (shortly HAM) for obtaining the numerical solutions of higher-order fractional integro-differential equations with boundary conditions. The series solution is developed and the recurrence relations are given explicitly. The initial approximation can be freely chosen with possible unknown constants which can be determined by imposing the boundary conditions. The comparison of the results obtained by the HAM with the exact solutions is made, the results reveal that the HAM is very effective and simple. The HAM contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of series solution.

1. Introduction

The HAM is developed in 1992 by Liao in [1]. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. The HAM contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of series solution for the given equation. Therefore, the HAM handles linear and nonlinear problems without any assumption and restriction. The HAM has been successfully applied to solve many types of linear and nonlinear problems in science and engineering by many authors [2–9] and also been used to solve fractional differential equations yet all in the Caputo sense [10–12].

Many physical phenomena [13–16] can be modeled by fractional differential equations which have diverse applications in various physical processes such as acoustics, electromagnetism, control theory, robotics, viscoelastic materials, diffusion, edge detection, turbulence, signal processing, anomalous diffusion and fractured media. Momani and Aslam Noor [17] established the implementation of ADM to derive analytic approximate solutions of the linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations. Nawaz [18] studied the fourth-order fractional integro-differential equations by VIM and HPM. Elsaid [19] studied the HAM for solving a class of fractional partial differential equations.

We aim in this paper to effectively employ the HAM to establish numerical solutions for the following higher-order fractional integro-differential equations with boundary conditions,

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$$\begin{cases} D^\alpha u(x) = f(x) + \beta u(x) + \int_0^x [g(t)u(t)]\mathrm{d}t, \quad 0 < x < b,\ m-1 < \alpha \le m,\ m \in Z^+, \\ u(0) = \gamma_0, \qquad u^{(i)}(0) = \gamma_i, \\ u(b) = \theta_0, \qquad u^{(i)}(b) = \theta_i, \end{cases} \tag{1}$$

where D^α is the fractional derivative in the Caputo sense, $\gamma_0, \gamma_i, \theta_0, \theta_i \left(i = 2k \left(k \in Z^+, 1 \le k < \lfloor \frac{m}{2} \rfloor\right)\right)$ and β are real constants. $u^{(i)}(*)$ denotes the value for the i-order derivative of $u(x)$ at $*, f$ and g are given and can be approximated by Taylor polynomials. By the HAM, numerical results can be obtained within a few iterations.

This paper is organized as follows. In Section 2, some basic definitions and properties of fractional calculus theory are given. In Section 3, the basic idea of the HAM is introduced. In Section 4, we obtain the solution of higher-order fractional integro-differential equations with boundary conditions, and make a comparison of approximate solution obtained by the 2-term HAM with exact solution.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this paper.

Definition 2.1. A real function $f(x),\ x > 0$ is said to be in space C_μ ($\mu \in R$) if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^n if $f^n \in C_\mu,\ n \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \ge 0$ of a function $f \in C_\mu$, $\mu \ge -1$, is defined as

$$\begin{cases} J^\alpha f(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-t)^{\alpha-1}\mathrm{d}t, \quad \alpha > 0,\ t > 0, \\ J^0 f(x) = f(x). \end{cases} \tag{2}$$

For $f(x) \in C_\mu, \mu \ge -1, \alpha, \beta \ge 0$ and $\gamma \ge -1$, some properties of the operator J^α, which are needed here, are as follows:

$$\text{(i) } J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x); \qquad \text{(ii) } J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x); \qquad \text{(iii) } J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t)\mathrm{d}t, \tag{3}$$

for $m - 1 < \alpha \le m, m \in N, t > 0, f \in C_{-1}^m$.

Lemma 2.4. *If $m - 1 < \alpha \le m, m \in N, f \in C_\mu^m, \mu \ge -1$, then the following two properties hold:*

$$\text{(i) } D^\alpha [J^\alpha f(x)] = f(x); \qquad \text{(ii) } J^\alpha [D^\alpha f(x)] = f(x) - \sum_{k=1}^{m-1} f^k(0) \frac{x^k}{k!}. \tag{4}$$

3. Basic idea of HAM

We apply the HAM [1–4] to the fractional integro-differential equation with boundary conditions. We consider the following equation

$$\mathrm{T}[u(x)] = 0, \tag{5}$$

where T is an operator, x denotes independent variable, $u(x)$ is an unknown function. For simplicity, we ignore all boundary conditions, which can be treated in the similar way. By means of the HAM, one first constructs zero-order deformation equation

$$(1-p)L[\phi(x; p)] = ph\mathrm{T}[\phi(x; p)], \tag{6}$$

where L is an auxiliary linear operator ($L[u(x)] = 0$ if and only if $u(x) = l$, where l is a real constant), $h \ne 0$ is an auxiliary parameter and $p \in [0, 1]$ is the embedding parameter. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(x; 0) = l, \phi(x; 1) = u(x), \tag{7}$$

respectively. The solution $\phi(x; p)$ varies from l to $u(x)$. Liao [1–4] expanded $\phi(x; p)$ in Taylor series about the embedding parameter

$$\phi(x; p) = \sum_{m=0}^{\infty} u_m(x)p^m, \tag{8}$$

where

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x; p)}{\partial p^m}\bigg|_{p=0}, \quad (m \geq 1). \tag{9}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter h are quite properly chosen, the series (8) converges at $p = 1$, one has

$$u(x) = \sum_{m=0}^{\infty} u_m(x). \tag{10}$$

Define the vectors $\overrightarrow{u}_n = \{u_0(x), u_1(x), \dots, u_n(x)\}$. Differentiating the zero-order deformation equation (6) m-times with respect to p and then dividing them by $m!$ and finally setting $p = 0$, we get the following mth-order deformation equation

$$L[u_m(x)] = h\Re_m(\overrightarrow{u}_{m-1}), \tag{11}$$

where

$$\Re_m(\overrightarrow{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} T[\phi(x; p)]}{\partial p^{m-1}}\bigg|_{p=0}. \tag{12}$$

It should be emphasized that $u_m(x)$ for $m \geq 2$ is governed by the Eq. (11) with the boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

4. Applications

We will apply the HAM to higher-order fractional integro-differential equations with a known exact solution at $\alpha = 4$ and $\alpha = 6$.

Example 1. Consider the following fourth-order fractional integro-differential equation:

$$\begin{cases} D^\alpha u(x) = x(1 + xe^x) + 3e^x + u(x) - \displaystyle\int_0^x u(t)\mathrm{d}t, \quad 0 < x < 1,\ 3 < \alpha \leq 4, \\ u(0) = 1, \qquad u''(0) = 2, \\ u(1) = 1 + e, \qquad u''(1) = 3e. \end{cases} \tag{13}$$

For $\alpha = 4$, the exact solution of Eq. (13) is $u(x) = 1 + xe^x$. By the boundary conditions, the initial approximation can be chosen in the following way: $u_0(x) = u(0) + u'(0)x + u''(0)\frac{x^2}{2} + u'''(0)\frac{x^3}{6}$. In order to avoid difficult fractional integration, we can take the truncated Taylor expansion for the exponential term in (13): e.g., $e^x \sim 1 + x + x^2/2 + x^3/6$, and let

$$G(u(x)) = x(1 + xe^x) + 3e^x + u(x) - \int_0^x u(t)\mathrm{d}t = 3 + 5x + \frac{5}{2}x^2 + x^3 + \frac{1}{6}x^4 + u(x) - \int_0^x u(t)\mathrm{d}t.$$

We choose the operator

$$L[\phi(x; p)] = D^\alpha(\phi(x; p)), \tag{14}$$

and define another operator as

$$T[\phi(x; p)] = D^\alpha(\phi(x; p)) - G(\phi(x; p)). \tag{15}$$

By the analysis in Section 3, we construct the zero-order deformation equation

$$(1 - p)L[\phi(x; p)] = ph\mathrm{T}[\phi(x; p)], \tag{16}$$

for $p = 0$ and $p = 1$, we can write as $\phi(x; 0) = l$ and $\phi(x; 1) = u(x)$. Thus, we obtain the mth-order deformation equation

$$L[u_m(x)] = h\Re_m(\overrightarrow{u}_{m-1}), \tag{17}$$

where $\Re_m(\overrightarrow{u}_{m-1}) = D^\alpha(u_{m-1}(x)) - G(u_{m-1}(x))$.

Table 1
Values of A and B for different values of α at $h = -0.8$.

	$\alpha = 3.5$	$\alpha = 3.8$	$\alpha = 4.0$
A	1.06172444793295	1.00713751259838	0.96461853025138
B	1.60468681403168	2.78251445794985	3.43572350417823

Table 2
Values of A and B for different values of h at $\alpha = 3.8$.

	$h = -0.8$	$h = -1.0$	$h = -1.2$
A	1.00713751259838	1.02294249793449	1.01357181099270
B	2.78251445794985	2.65514630861582	2.81546932478833

By applying the inverse integral operator, the solution of the mth-order deformation equation (17) for $m \geq 2$ will become

$$u_m(x) = hJ^{\alpha}[D^{\alpha}(u_{m-1}(x)) - G(u_{m-1}(x))]. \tag{18}$$

In order to satisfy the boundary conditions and convenient calculation, we will chose $u_0(x), u_1(x)$ and $u_m(x)$ ($m = 2, 3, 4, \ldots$) as follows:

$$\begin{cases} u_0(x) = 1, \\ u_1(x) = Ax + x^2 + \dfrac{B}{6}x^3 + hJ^{\alpha}[D^{\alpha}(u_0(x)) - G(u_0(x))], \\ u_m(x) = hJ^{\alpha}[D^{\alpha}(u_{m-1}(x)) - G(u_{m-1}(x))], \quad m = 2, 3, 4, \ldots, \end{cases} \tag{19}$$

where $A = u'(0)$ and $B = u'''(0)$ are to be determined. Then we will get that $u_0(x) = 1$ and

$$u_1(x) = Ax + x^2 + \frac{B}{6}x^3 - \left(\frac{4h}{\Gamma(\alpha+1)} + \frac{4h}{\Gamma(\alpha+2)}x + \frac{5h}{\Gamma(\alpha+3)}x^2 + \frac{6h}{\Gamma(\alpha+4)}x^3 + \frac{4h}{\Gamma(\alpha+5)}x^4\right)x^{\alpha},$$

$$\begin{aligned} u_2(x) \;=\; & -\left(\frac{(4h+3)h}{\Gamma(\alpha+1)} + \frac{(4h+5+A)h}{\Gamma(\alpha+2)}x + \frac{(5h+7-A)h}{\Gamma(\alpha+3)}x^2 + \frac{(6h+4+B)h}{\Gamma(\alpha+4)}x^3 + \frac{(4h+4-B)h}{\Gamma(\alpha+5)}x^4\right)x^{\alpha} \\ & + \left(\frac{4h^2}{\Gamma(2\alpha+1)} + \frac{h^2}{\Gamma(2\alpha+3)}x^2 + \frac{h^2}{\Gamma(2\alpha+4)}x^3 - \frac{2h^2}{\Gamma(2\alpha+5)}x^4 - \frac{4h^2}{\Gamma(2\alpha+6)}x^5\right)x^{2\alpha}, \end{aligned}$$

$$\vdots$$

Now, we can form the 2-term approximation solution for Eq. (13)

$$\begin{aligned} u(x) \approx \Phi_2(x) = u_0 + u_1 + u_2 = {} & 1 + Ax + x^2 + \frac{B}{6}x^3 - \frac{(4h+7)h}{\Gamma(\alpha+1)}x^{\alpha} - \frac{(4h+9+A)h}{\Gamma(\alpha+2)}x^{\alpha+1} \\ & - \frac{(5h+12-A)h}{\Gamma(\alpha+3)}x^{\alpha+2} - \frac{(6h+10-B)h}{\Gamma(\alpha+4)}x^{\alpha+3} - \frac{(4h+8-B)h}{\Gamma(\alpha+5)}x^{\alpha+4} + \frac{4h^2x^{2\alpha}}{\Gamma(2\alpha+1)} \\ & + \frac{h^2x^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{h^2x^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{2h^2x^{2\alpha+4}}{\Gamma(2\alpha+5)} - \frac{4h^2x^{2\alpha+5}}{\Gamma(2\alpha+6)}, \end{aligned}$$

where A and B can be determined by imposing boundary conditions of Eq. (13) on $\Phi_2(x)$.

Table 1 shows the values of A and B for different values of α when $h = -0.8$, and Table 2 shows the values of A and B for different values of h when $\alpha = 3.8$.

In Fig. 1, we compare the approximate solutions obtained by HAM with the exact solution at $h = -0.8$, $h = -1.0$ and $h = -1.2$, respectively, for $\alpha = 4.0$. In Fig. 2, we draw absolute error functions $E_i(x) = |(1 + xe^x) - \Phi_2^{(\alpha,-0.8)}|$, $E_{i+1}(x) = |(1 + xe^x) - \Phi_2^{(\alpha,-1.0)}|$ and $E_{i+2}(x) = |(1 + xe^x) - \Phi_2^{(\alpha,-1.2)}|$ ($i = 1, 4, 7$), where $1 + xe^x$ is an exact solution of Eq. (13). For given α, $\Phi_2^{(\alpha,-0.8)}$, $\Phi_2^{(\alpha,-1.0)}$ and $\Phi_2^{(\alpha,-1.2)}$ represent the values of $\Phi_2(x)$ at $h = -0.8$, $h = -1.0$ and $h = -1.2$, respectively. It is clear from Fig. 2(a) that the approximate solutions are in good agreement with an exact solution of Eq. (13) at $h = -1.0$.

In Fig. 3, we compare the approximate solutions obtained by HAM with an exact solution at $\alpha = 3.5$, $\alpha = 3.8$ and $\alpha = 4.0$, respectively, for $h = -1.0$. In Fig. 4, we draw absolute error functions $E_{i+9}(x) = |(1 + xe^x) - \Phi_2^{(3.5,h)}|$, $E_{i+10}(x) = |(1+xe^x) - \Phi_2^{(3.8,h)}|$ and $E_{i+11}(x) = |(1+xe^x) - \Phi_2^{(4.0,h)}|$ ($i = 1, 4, 7$), where $1+xe^x$ is an exact solution of Eq. (13). For given h, $\Phi_2^{(3.5,h)}$, $\Phi_2^{(3.8,h)}$ and $\Phi_2^{(4.0,h)}$ represent the values of $\Phi_2(x)$ at $\alpha = 3.5$, $\alpha = 3.8$ and $\alpha = 4.0$, respectively. It is clear from Fig. 4(a) that the approximate solutions are in good agreement with an exact solution of Eq. (13) at $\alpha = 4.0$ and $\alpha = 3.8$. It is to be noted that the accuracy can be improved by computing more terms of approximated solutions and/or by taking more terms in the Taylor expansion for the exponential term.

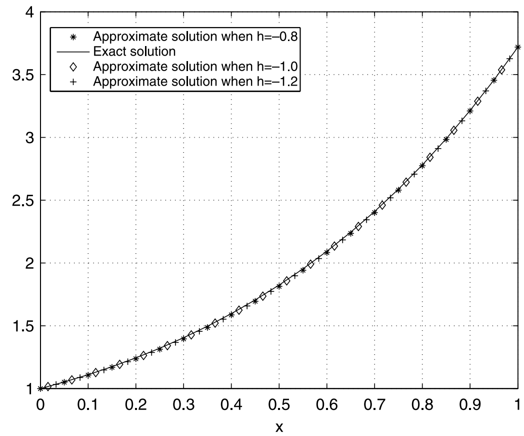
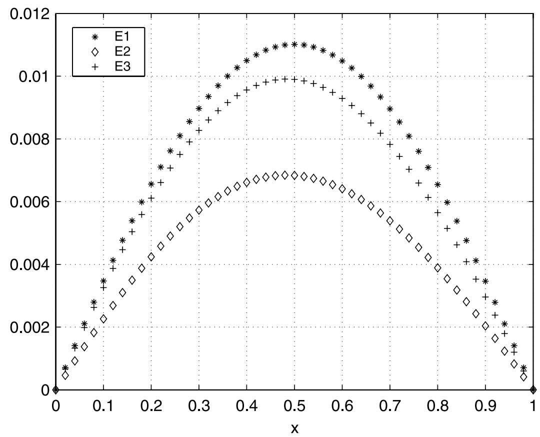
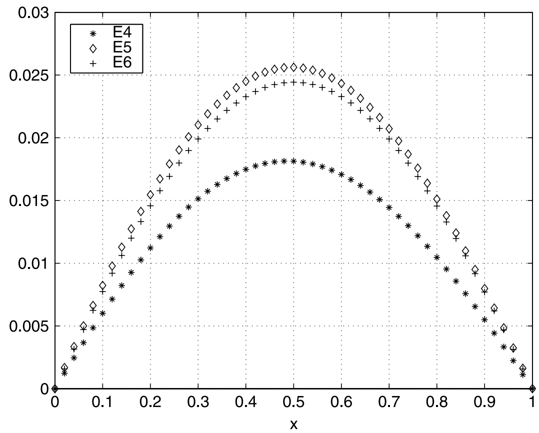


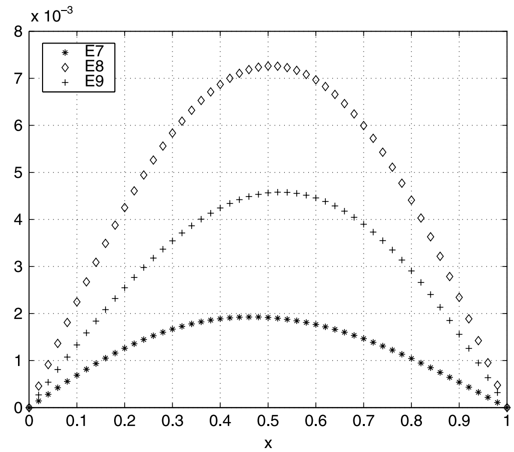
Fig. 1. Comparison of approximate solutions obtained by 2-term HAM for different values of h with exact solution at $\alpha = 4.0$.



(a) $\alpha = 4.0$.



(b) $\alpha = 3.5$.



(c) $\alpha = 3.8$.

Fig. 2. Absolute error functions $E_1(x) - E_9(x)$ obtained by 2-term HAM for given α at different values of h.

Example 2. Consider the following sixth-order fractional integro-differential equation:

$$\begin{cases} D^{\alpha}u(x) = 1 + (5+x)\mathrm{e}^{x} + u(x) - \int_0^x u(t)\mathrm{d}t, \quad 0 < x < 1,\ 5 < \alpha \le 6, \\ u(0) = 0, \qquad u''(0) = 2, \qquad u^{(4)}(0) = 4, \\ u(1) = e, \qquad u''(1) = 3e, \qquad u^{(4)}(1) = 5e. \end{cases} \tag{20}$$

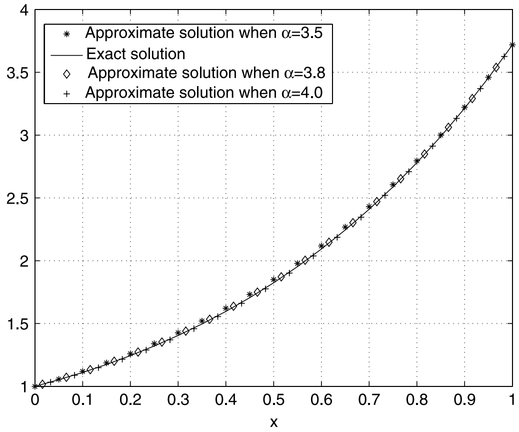
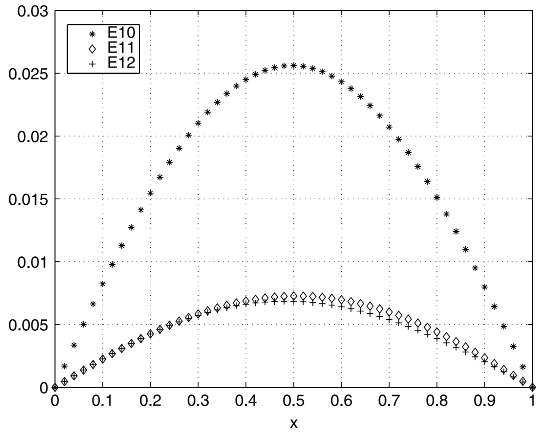
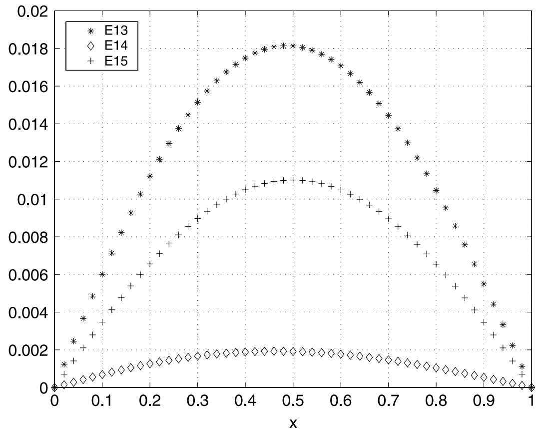


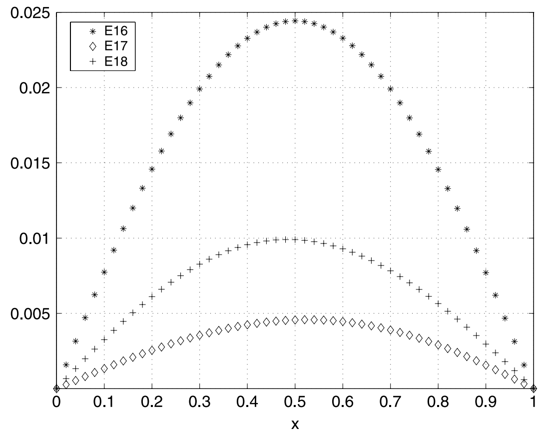
Fig. 3. Comparison of approximate solutions obtained by 2-term HAM for different values of α with exact solution at $h = -1.0$.



(a) $h = -1.0$.



(b) $h = -0.8$.



(c) $h = -1.2$.

Fig. 4. Absolute error functions $E_{10}(x) - E_{18}(x)$ obtained by 2-term HAM for given h at different values of α.

For $\alpha = 6$, the exact solution of Eq. (20) is $u(x) = xe^x$. By the boundary conditions, the initial approximation can be chosen in the following way: $u_0(x) = u(0) + u'(0)x + u''(0)\frac{x^2}{2} + u'''(0)\frac{x^3}{6} + u^{(4)}(0)\frac{x^4}{24} + u^{(5)}(0)\frac{x^5}{120}$. In order to avoid difficult fractional integration, we can take the truncated Taylor expansion for the exponential term in Eq. (20): e.g., $e^x \sim 1 + x + x^2/2 + x^3/6$, and let

$$F(u(x)) = 1 + (5 + x)e^x + u(x) - \int_0^x u(t)\mathrm{d}t = 6 + 6x + \frac{7}{2}x^2 + \frac{8}{6}x^3 + \frac{1}{6}x^4 + u(x) - \int_0^x u(t)\mathrm{d}t.$$

Table 3
Values of A, B and C for different values of α at $h = -1.0$.

	$\alpha = 5.5$	$\alpha = 5.75$	$\alpha = 5.95$
A	0.98838222923558	0.99316340171805	0.99865843551750
B	3.12327836901348	3.07167229136778	3.01409171243346
C	2.09504310397868	3.67087396408289	4.76072346012555

Table 4
Values of A, B and C for different values of h at $\alpha = 5.75$.

	$h = -0.8$	$h = -1.0$	$h = -1.2$
A	0.99585806184052	0.99316340171805	0.99434655177876
B	3.04542033195779	3.07167229136778	3.05971655641827
C	2.55229179610577	3.67087396408289	8.51909873303597

As analysis in Example 1, the mth-order deformation equation for $m \geq 2$ will become

$$u_m(x) = hJ^{\alpha}[D^{\alpha}(u_{m-1}(x)) - F(u_{m-1}(x))]. \tag{21}$$

In order to satisfy the boundary conditions and convenient calculation, we will chose $u_0(x)$, $u_1(x)$ and $u_m(x)$ $(m \geq 2)$ as follows:

$$\begin{aligned} u_0(x) &= 0, \\ u_1(x) &= Ax + x^2 + \frac{B}{6}x^3 + \frac{1}{6}x^4 + \frac{C}{120}x^5 + hJ^{\alpha}[D^{\alpha}(u_0(x)) - F(u_0(x))], \\ u_m(x) &= hJ^{\alpha}[D^{\alpha}(u_{m-1}(x)) - F(u_{m-1}(x))], \quad m = 2, 3, 4, \ldots, \end{aligned} \tag{22}$$

where $A = u'(0)$, $B = u'''(0)$ and $C = u^{(5)}(0)$ are to be determined. Then we will get that $u_0(x) = 0$ and

$$\begin{aligned} u_1(x) &= Ax + x^2 + \frac{B}{6}x^3 + \frac{1}{6}x^4 + \frac{C}{120}x^5 - \frac{6hx^{\alpha}}{\Gamma(\alpha+1)} - \frac{6hx^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{7hx^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{8hx^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{4hx^{\alpha+4}}{\Gamma(\alpha+5)}, \\ u_2(x) &= -\frac{(6h+6)h}{\Gamma(\alpha+1)}x^{\alpha} - \frac{(6h+6+A)h}{\Gamma(\alpha+2)}x^{\alpha+1} - \frac{(7h+9-A)h}{\Gamma(\alpha+3)}x^{\alpha+2} - \frac{(8h+6+B)h}{\Gamma(\alpha+4)}x^{\alpha+3} \\ &\quad - \frac{(4h+8-B)h}{\Gamma(\alpha+5)}x^{\alpha+4} - \frac{(C-4)h}{\Gamma(\alpha+6)}x^{\alpha+5} + \frac{Ch}{\Gamma(\alpha+7)}x^{\alpha+6} + \frac{6h^2x^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad + \frac{h^2x^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{h^2x^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{4h^2x^{2\alpha+4}}{\Gamma(2\alpha+5)} - \frac{4h^2x^{2\alpha+5}}{\Gamma(2\alpha+6)}, \\ &\vdots \end{aligned}$$

Now, we can form the 2-term approximation solution for Eq. (20),

$$\begin{aligned} u(x) \approx \Psi_2(x) &= u_0 + u_1 + u_2 = Ax + x^2 + \frac{B}{6}x^3 + \frac{1}{6}x^4 + \frac{C}{120}x^5 - \frac{(6h+12)h}{\Gamma(\alpha+1)}x^{\alpha} - \frac{(6h+12+A)h}{\Gamma(\alpha+2)}x^{\alpha+1} \\ &\quad - \frac{(7h+16-A)h}{\Gamma(\alpha+3)}x^{\alpha+2} - \frac{(8h+14+B)h}{\Gamma(\alpha+4)}x^{\alpha+3} - \frac{(4h+12-B)h}{\Gamma(\alpha+5)}x^{\alpha+4} - \frac{(C-4)h}{\Gamma(\alpha+6)}x^{\alpha+5} \\ &\quad + \frac{Ch}{\Gamma(\alpha+7)}x^{\alpha+6} + \frac{6h^2x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{h^2x^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{h^2x^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{4h^2x^{2\alpha+4}}{\Gamma(2\alpha+5)} - \frac{4h^2x^{2\alpha+5}}{\Gamma(2\alpha+6)}; \end{aligned}$$

where A, B and C can be determined by the imposing boundary conditions of Eq. (20) on $\Psi_2(x)$.

Table 3 shows the values of A, B and C for different values of α when $h = -1.0$, and Table 4 shows the values of A, B and C for different values of h when $\alpha = 5.75$.

In Fig. 5, we compare the approximate solutions obtained by HAM with the exact solution at $h = -0.8$, $h = -1.0$ and $h = -1.2$, respectively, for $\alpha = 5.95$. In Fig. 6, we draw absolute error functions $E_{i+18}(x) = |xe^x - \Psi_2^{(\alpha,-0.8)}|$, $E_{i+19}(x) = |xe^x - \Psi_2^{(\alpha,-1.0)}|$ and $E_{i+20}(x) = |xe^x - \Psi_2^{(\alpha,-1.2)}|$ $(i = 1, 4, 7)$, where xe^x is an exact solution of Eq. (20). For given α, $\Psi_2^{(\alpha,-0.8)}$, $\Psi_2^{(\alpha,-1.0)}$ and $\Psi_2^{(\alpha,-1.2)}$ represent the values of $\Psi_2(x)$ at $h = -0.8$, $h = -1.0$ and $h = -1.2$, respectively. It is clear from Fig. 6(a) that the approximate solutions are in good agreement with an exact solution of Eq. (20) at $h = -1.2$.

In Fig. 7, we compare the approximate solutions obtained by HAM with the exact solution at $\alpha = 5.5$, $\alpha = 5.75$ and $\alpha = 5.95$, respectively, for $h = -0.8$. In Fig. 8, we draw absolute error functions $E_{i+27}(x) = |xe^x - \Psi_2^{(5.50,h)}|$, $E_{i+28}(x) = |xe^x - \Psi_2^{(5.75,h)}|$ and $E_{i+29}(x) = |xe^x - \Psi_2^{(5.95,h)}|$ $(i = 1, 4, 7)$, where xe^x is an exact solution of Eq. (20). For given h,

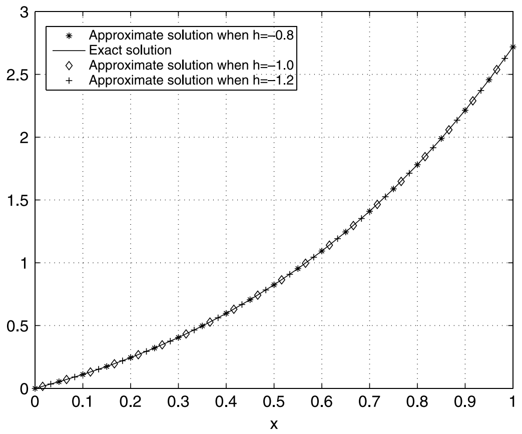
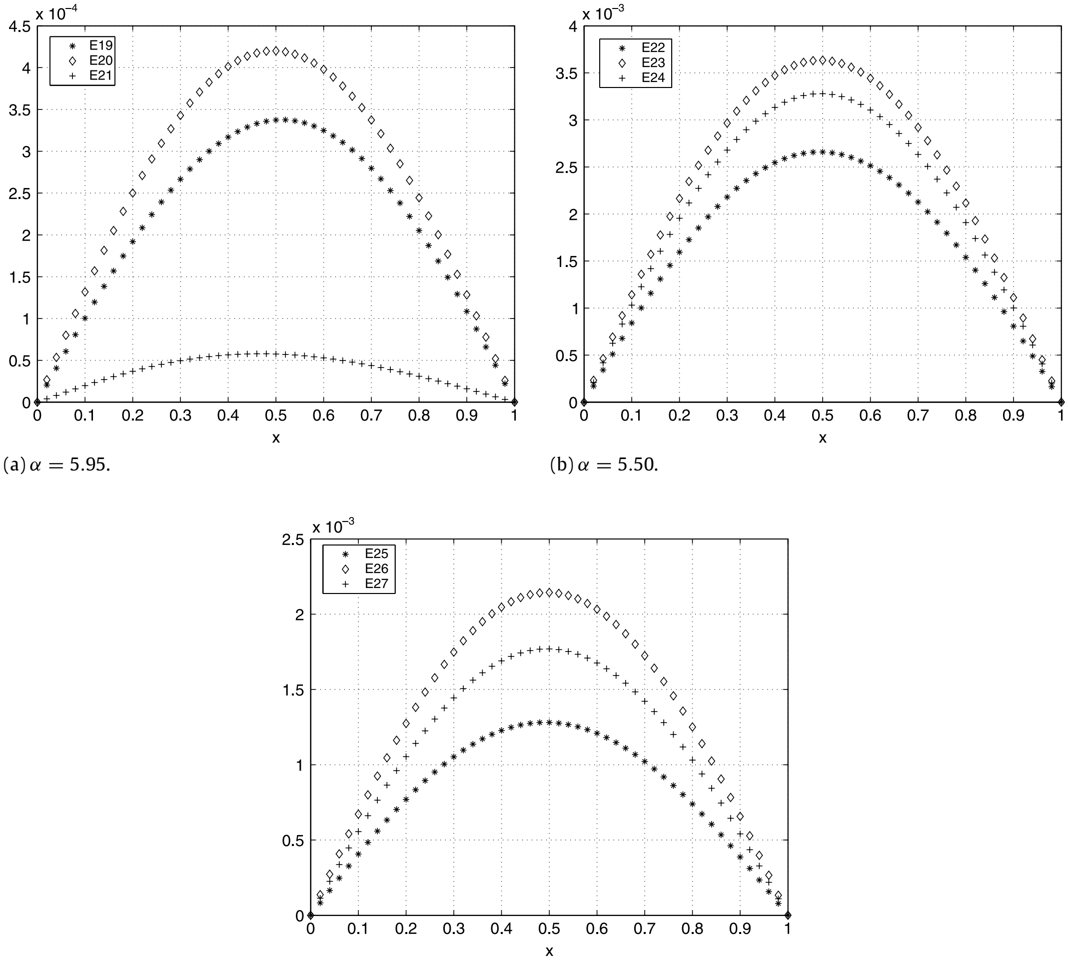


Fig. 5. Comparison of approximate solutions obtained by 2-term HAM for different values of h with exact solution at $\alpha = 5.95$.



(a) $\alpha = 5.95$.

(b) $\alpha = 5.50$.

(c) $\alpha = 5.75$.

Fig. 6. Absolute error functions $E_{19}(x) - E_{27}(x)$ obtained by 2-term HAM for given α at different values of h.

$\Psi_2^{(5.50,h)}$, $\Psi_2^{(5.75,h)}$ and $\Psi_2^{(5.95,h)}$ represent the values of $\Psi_2(x)$ at $\alpha = 5.50, \alpha = 5.75$ and $\alpha = 5.95$, respectively. It is clear from Fig. 8(a) that the approximate solutions are in good agreement with an exact solution of Eq. (20) at $\alpha = 5.95$.

Also it is to be noted that the accuracy can be improved by computing more terms of approximated solutions and/or by taking more terms in the Taylor expansion for the exponential term.

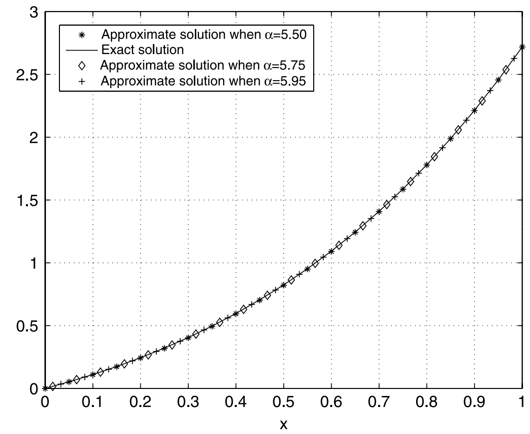
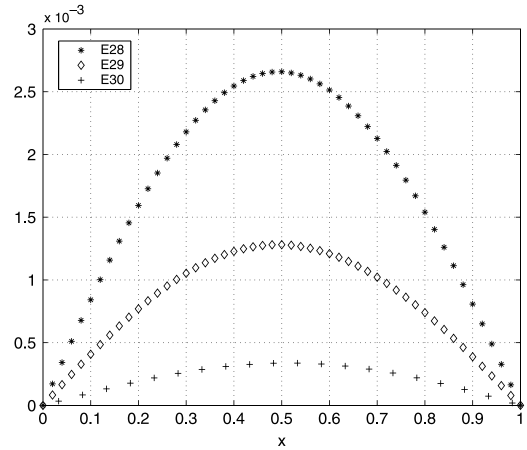
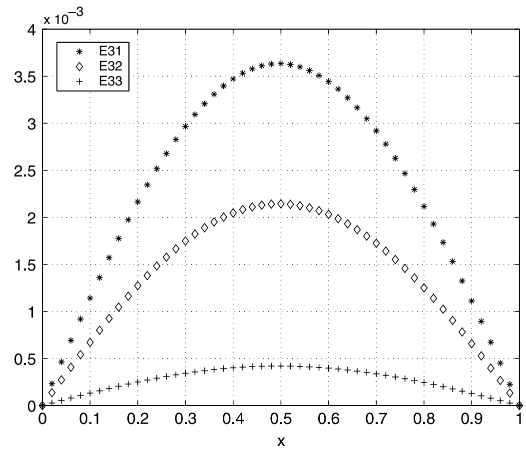


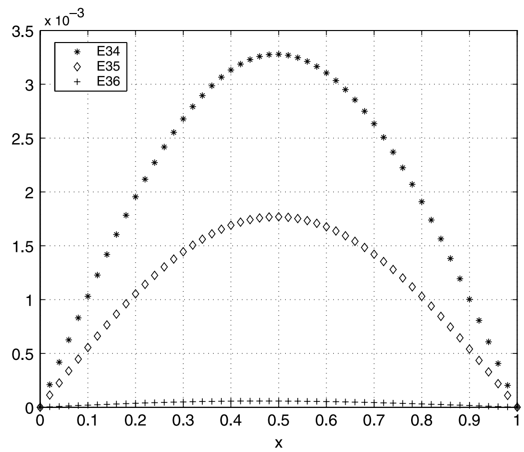
Fig. 7. Comparison of approximate solutions obtained by 2-term HAM for different values of α with exact solution at $h = -0.8$.



(a) $h = -0.8$.



(b) $h = -1.0$.



(c) $h = -1.2$.

Fig. 8. Absolute error functions $E_{28}(x) - E_{36}(x)$ obtained by 2-term HAM for given h at different values of α.

5. Conclusion

Liao [4] showed that whatever a solution series converges it will be one of the solutions of considered problem and presented that the rate of convergence of approximate solutions obtained by the HAM can be controlled by the auxiliary parameter h. In this paper, the HAM was used for obtaining approximate solutions of the higher-order integro-differential equations with boundary conditions. Two examples are presented to illustrate the accuracy of the present scheme of HAM. The approximate solutions were almost identical to analytic solutions of the considered equations. The numerical results showed that this method has very accuracy and reductions of the size of calculations. It may be concluded that this methodology is very powerful and efficient technique in finding approximate solutions for wide classes of problems. Also it is to be noted that the accuracy can be improved by computing more terms of approximated solutions and/or by taking more terms in the Taylor expansion for the exponential term. This shows that the HAM is a very useful method to get high-precision numerical solutions for many problems.

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