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Re-embedding of Projective-Planar Graphs

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In this paper, we characterize those projective-plane 3-connected graphs which admit re-embeddings in a projective plane different from their original one. The results will be applied for analysis of the uniqueness and faithfulness of the embedding of 5-connected projective-planar graphs. © 1988 Academic Press, Inc.

1. INTRODUCTION

Our graphs are finite, undirected, simple ones combinatorially and have underlying spaces with canonical topology as 1-complexes. Let G be a graph and F^2 a surface. Two embeddings $f_1, f_2: G \to F^2$ are *equivalent* if there exists a homeomorphism $h: F^2 \to F^2$ and an automorphism $\sigma: G \to G$ such that $h \circ f_1 = f_2 \circ \sigma$. A graph G is said to be *uniquely embeddable* in F^2 if there is precisely one equivalence class of embeddings of G into F^2. An automorphism $\sigma: G \to G$ is called a *symmetry* of an embedding $f: G \to F^2$ if there is a homeomorphism $h: F^2 \to F^2$ such that $h \circ f = f \circ \sigma$. The collection of symmetries of f is a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G and is denoted by $\mathrm{Sym}(f)$. A graph G is said to be *faithfully embeddable* in F^2 if there is an embedding $f: G \to F^2$ for which $\mathrm{Sym}(f) = \mathrm{Aut}(G)$.

These concepts, the uniqueness and faithfulness of embedding, were defined by the author in [1] where those for toroidal graphs were discussed. The uniqueness of duals of 3-connected planar graphs, proved by Whitney [5], implies that every 3-connected planar graph is uniquely and faithfully embeddable in a sphere.

A graph which is embeddable in a projective plane is called a *projective-planar* graph. Recently, the author has found two large classes of 5-connected projective-planar graphs which are uniquely and faithfully embeddable in a projective plane [2, 3]. Actually, he proved the following two theorems:

THEOREM 1.1 (S. Negami [2]). *A 5-connected projective-planar graph*

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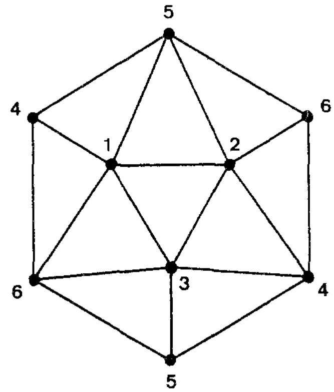


FIG. 1. The unique embedding of K_6 in P^2.

which contains a subdivision of K_6 as its proper subgraph is uniquely and faithfully embeddable in a projective plane.

THEOREM 1.2 (S. Negami [3]). *A 5-connected projective-planar graph which triangulates a projective plane is uniquely and faithfully embeddable in a projective plane unless it is isomorphic to K_6.*

The complete graph K_6 with six vertices is uniquely but not faithfully embeddable in a projective plane. The unique embedding of K_6 in a projective plane P^2 is given by Fig. 1. (To obtain a projective plane, identify each pair of vertices and edges with the same labels.) This is a *triangular* embedding; that is, each region or *face* is bounded by precisely three edges. In general, we shall call a graph a *triangulation* of a closed surface F^2 if it admits a triangular embedding in F^2.

At that time, the author could not find a non-uniquely embeddable 5-connected projective-planar graph and he asked whether such a graph exists. Our goal in this paper is to show when a 5-connected projective-planar graph is uniquely or faithfully embeddable in a projective plane and to construct an infinite number of non-uniquely and non-faithfully embeddable 5-connected projective-planar graphs.

Let $f: G \to F^2$ be an embedding of a graph G into a surface F^2 and let Γ be a simple closed curve on F^2 such that $\Gamma \cap f(G)$ consists of precisely n vertices of $f(G)$. Then Γ is called an *n-compressing curve* for the embedding f or for the graph $f(G)$ if either no component of $F^2 - \Gamma$ is an open 2-cell or each 2-cell component of $F^2 - \Gamma$ contains at least one vertex of $f(G)$. If there is no m-compressing curve $(m < n)$ for f on F^2, then f is called an *n-incompressible embedding*, and if G admits such an embedding, then G is said to be *n-incompressibly embeddable* in F^2. For example, the unique projective-planar embedding of K_6 is 3-incompressible.

In Section 6, we prove the following:

THEOREM 1.3. *Every 5-connected, 3-incompressibly embeddable, projective-planar graph is uniquely embeddable in a projective plane. Furthermore, it is faithfully embeddable in a projective plane unless it is isomorphic to* K_6.

It is easy to see that if a 3-connected projective-planar graph G contains a subgraph H contractible to K_6, then any projective-planar embedding of G is 3-incompressible. In particular, if G contains a subdivision of K_6, then G is 3-incompressibly embeddable in a projective plane. In general, any triangular embedding of a graph in a closed surface F^2 is 3-incompressible. Thus, both Theorems 1.1 and 1.2 are implications of this theorem.

Observe that a graph G is uniquely and faithfully embeddable in a surface F^2 if and only if for any two embeddings $f_1, f_2: G \to F^2$, there is a homeomorphism $h: F^2 \to F^2$ such that $h \circ f_1 = f_2$. In particular when G is already embedded in F^2, it is equivalent to the condition that any embedding $f: G \to F^2$ extends to a homeomorphism $h: F^2 \to F^2$ so that $h|_G = f$. Thus, the existence of an embedding which cannot extend to a self-homeomorphism of F^2 destroys the uniqueness or the faithfulness of G in F^2. We call such an embedding a *re-embedding* of G in F^2.

If $f: G \to P^2$ is a re-embedding, then there is a face A of G with boundary cycle C such that $f(C)$ does not bound a face of $f(G)$ in P^2. In Sections 4 and 5, we describe the details of re-embeddings of 3-connected projective-planar graphs, analyzing the behavior of each complementary piece for C, called a *bridge*. (See Section 3 for definition.)

It is easy to see that a re-embedding of a graph in a sphere or the plane is only turning over some local parts of graphs and hence such a re-embeddable planar graph is not 3-connected. (The combinatorial result corresponding to this fact can be found in [6].) In the case of a projective plane, it is possible to alter the embedding in more complicated ways. If a re-embedding $f: G \to P^2$ sends the boundary cycle C of a face to a cycle bounding a 2-cell, at least one of the bridges for C must be mapped into the 2-cell. Otherwise, more global alteration of the embedding of G will arise. Classifying such phenomena, we show the following theorem:

THEOREM 1.4 (Re-Embedding Theorem). *Let G be a non-planar 3-connected graph embedded in a projective plane P^2 and let $f: G \to P^2$ be a re-embedding of G. Then f is either a throwing-in or -out of some bridge for a cycle, or one of the re-embeddings of types* I *to* IV.

Each type of re-embedding in the theorem will be defined later in the lemmas throughout Sections 4 and 5.

This theorem restricts the structure of a graph which is not uniquely or not faithfully embeddable in a projective plane and asserts that such a graph has many vertex-cuts in most cases. In particular, we can conclude Theorem 1.3 from Theorem 1.4 with the assumption of a graph being 5-connected and we can also construct infinitely many examples for the non-uniqueness and non-faithfulness of the embedding of 5-connected projective-planar graphs, which is presented in Section 7.

2. Graphs in a Projective Plane

The projective plane is a closed surface defined as the quotient space of the unit sphere S^2 in $\mathbf{R}^3$ by identifying each pair of antipodal points x and $-x \in S^2$ to a single point. This space is denoted by P^2 throughout this paper. In this section, we discuss the topology of P^2 and consider embeddings of some typical projective-planar graphs.

The projective plane P^2 has only two types of simple closed curves. The first type bounds a 2-cell in P^2 and is called a *trivial curve*, and the other is called an *essential curve*. Any two simple closed curves of the same type can be mapped onto each other by a self-homeomorphism of P^2. Thus, the same type of curves have the completely same properties topologically. We will use no more than the following three facts in our arguments on closed curves in P^2:

(i) Any trivial curve Γ has an annular neighborhood $U(\Gamma)$ in P^2 and decomposes P^2 into a 2-cell (or a disk) and a Möbius band. So P^2 can be obtained from a disk and a Möbius band by sewing them back along their boundary curves.

(ii) Any essential curve Γ has no annular neighborhood and its regular neighborhood $U(\Gamma)$ is a Möbius band whose center line Γ lies along. If one cuts P^2 along Γ with knife, it open out into a disk D^2 and each point on Γ splits into two points on ∂D^2. (We denote the boundary of a surface E^2 by ∂E^2.) Conversely, P^2 can be obtained from D^2 by identifying each antipodal pair of points on ∂D^2.

(iii) No two essential curves are disjoint from each other. Thus if a simple closed curve Γ does not meet an essential curve Γ', then Γ is trivial and bounds a 2-cell disjoint from Γ'. (For the 2-cell $P^2 - U(\Gamma')$ contains Γ.)

Lemma 2.1. *Each face of a non-planar 2-connected graph embedded in a projective plane is bounded by a cycle. In other words, any embedding of a non-planar 2-connected graph in a projective plane is 2-incompressible.*

Proof. Let G be a connected graph embedded in a projective plane and let A be a face whose boundary is not a cycle. Then there is a 1-compressing curve Γ which passes through a vertex v on the boundary of A. If Γ bounds a 2-cell, then v separates G into the two parts inside and outside the 2-cell; that is, v is a cut vertex of G. Thus, G is not 2-connected. If Γ is essential, then there will be obtained by cutting P^2 along Γ an embedding of G in a disk where v splits into two vertices v_1 and v_2. Contract one of arcs joining v_1 and v_2 along the boundary of the disk and contact v_1 with v_2. The resulting graph is nothing but G and it is embedded in a disk. Thus, G is planar. ▌

By this lemma and (i), the removal of any face of a 3-connected non-planar graph G embedded in P^2 yields a Möbius band which contains the whole of G. So we shall often draw only such a Möbius band to present a projective-planar embedding of G. In other cases, we shall cut open P^2 into a disk along an essential curve which either crosses G transversely or is a cycle of G. Do not forget that each antipodal pair of points on the boundary of the disk comes from one point on the cutting line at that time.

We define O_n $(n \geqslant 3)$ as the graph obtained from a cycle of length $2n$, given as a cyclic sequence $\{v_1, \ldots, v_{2n}\}$ of $2n$ vertices, by adding edges $v_i v_{i+n}$ $(i = 1, \ldots, n)$, and call it a *Möbius ladder* with *n spokes* $v_i v_{i+n}$ $(i = 1, \ldots, n)$. The cycle of length $2n$ is denoted by ∂O_n. Note that O_n contains a subdivision of the complete bipartite graph $K_{3,3}$ and hence it is not planar. In particular, O_3 is isomorphic to $K_{3,3}$.

Every Möbius ladder O_n has an embedding in P^2 as shown in Fig. 2, so it is a projective-planar graph. (Attach an extra 2-cell to the Möbius band along its boundary curve to obtain the whole of P^2.) We call this embedding the *canonical embedding* of O_n. The canonical embedding can be characterized clearly such that ∂O_n bounds a face in P^2. If we draw it with the face omitted, we will get the same picture as that in Fig. 2.

The Möbius ladder O_n $(n \geqslant 4)$ is not uniquely embeddable in a projective plane. (Note that O_3 $(\cong K_{3,3})$ is uniquely embeddable. We shall omit the details.) Delete one spoke from O_n canonically embedded in P^2 and re-

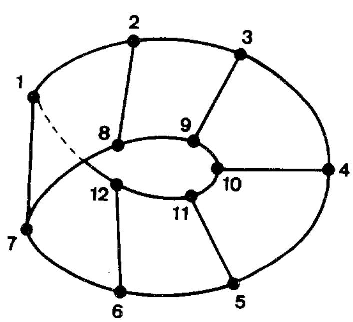


FIG. 2. Möbius ladder $(n = 6)$.

embed it in the face bounded by ∂O_n. The resulting embedding has two $(n+1)$-gons, one hexagon, and $n-2$ squares as faces while all faces but one $2n$-gon are squares in the canonical embedding. Thus they are not equivalent to each other. This kind of re-embedding will be generalized as a *throwing-in* of a bridge.

LEMMA 2.2. *Let $f\colon O_n \to P^2$ be any projective-planar embedding of the Möbius ladder with n spokes. If $n \geqslant 4$ then $f(\partial O_n)$ is a trivial curve in P^2.*

Proof. We identify O_n with $f(O_n)$. Suppose that ∂O_n is an essential curve in P^2 and cut open P^2 into a disk D^2 along ∂O_n. Then each vertex v_i occurs twice along the boundary of D^2 and each spoke $v_i v_{i+n}$ runs from one of v_i's to one of v_{i+n}'s. The end vertices separate the boundary cycle of D^2 into two paths of length n and $3n$. Let Q_i be the shorter one. Since Q_i does not contain both v_j and v_{j+n} $(j \neq i)$ together, no spoke of O_n starts at any inner vertex of Q_i. Thus, Q_i's $(i = 1,..., n)$ are mutually disjoint. Since each Q_i contains precisely $n+1$ vertices, we have the inequality

$$4n \geqslant n(n+1).$$

This implies that $0 \leqslant n \leqslant 3$, a contradiction. ∎

Note that there is an embedding of O_3 which embeds ∂O_3 onto an essential curve of P^2, as shown in Fig. 3. (Identify each pair of antipodal points on the boundary cycle. The vertex v_i is indicated simply with the label i in Fig. 3.) This embedding is however equivalent to the canonical one. For the bent hexagon 145236 and edges 12, 34, and 56 can be regarded as ∂O_3 and three spokes, respectively, in the canonical form. The automorphism of O_3 which sends the trivial cycle 145236 to the essential cycle 123456 is an example of re-embedding of global type.

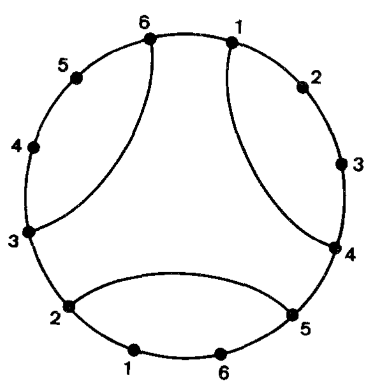


FIG. 3. An embedding of O_3 in P^2.

3. Bridges for Face Boundaries

Let G be a non-planar 3-connected graph already embedded in a projective plane P^2 and let $f: G \to P^2$ be an embedding of G into P^2. By Lemma 2.1, each face A of G is bounded by a cycle of G. We denote this cycle by ∂A. A face A of G is said to be *extendable* for f if $f(\partial A)$ bounds a face of $f(G)$. If all faces of G are extendable for f, then we can define an extension $h: P^2 \to P^2$ of f by mapping each face A onto the face bounded by $f(\partial A)$.

Therefore, given a re-embedding $f: G \to P^2$, there is a face A of G which is not extendable for f. Since ∂A is just a simple closed curve in P^2 topologically, $f(\partial A)$ is either a trivial curve or an essential curve. If $f(\partial A)$ is trivial, then it bounds a 2-cell Δ but Δ is not a face of $f(G)$ since A is not extendable for f and hence f carries some subgraph of G into Δ. On the other hand, if $f(\partial A)$ is essential, then $G - \partial A$ is mapped into the disk obtained from P^2 by cutting it open along $f(\partial A)$.

As above, we must discuss where and how the complementary part of G, except for A, is mapped by a re-embedding in order to establish the re-embedding theorem. In this section, we prepare our terminology to describe such a situation.

A (*non-singular*) *bridge* for a cycle C in a graph G is a subgraph B obtained from one component of $G - C$ by adding the edges which join B to C. Also a subgraph consisting of a single edge uv is a bridge if $u, v \in V(C)$ but $uv \notin E(C)$ and is said to be *singular*. A path Q is said to be *C-avoiding* if none of its inner vertices belongs to C. Any two vertices in a bridge B can be joined by a C-avoiding path in B. A vertex of a bridge B which belongs to C is called a *foot* of B. The set of feet of B is denoted by $F(B)$. We shall often use the same notation $F(H)$ for any subgraph H of G, meaning $V(H) \cap V(C)$ by it. Any two bridges have no edge in common and meet each other only in their feet if they do at all.

An edge $e \in E(B)$ is called a *leg* of a bridge B if it is incident to a foot of B. The set of legs of B is denoted by $L(B)$. If B is not singular then $B - F(B)$ is a non-empty subgraph of B without legs. We call $B - F(B)$ the *body* of B and denote it $\bar{B}$. In our figures, we shall often draw a shaded ellipse for the body $\bar{B}$.

Two subsets X, Y of $V(C)$ are said to be *mixed* on C if there are four distinct vertices $x, x' \in X$ and $y, y' \in Y$ such that x, y, x', y' lie along C in this cyclic order. Observe that if the feet $F(B)$, $F(B')$ of two bridges B, B' are mixed on C, then $C \cup B \cup B'$ cannot be embedded in the plane so that both B and B' are placed outside of C.

Now suppose that G is embedded in P^2 and that C bounds a face A of G which is an open 2-cell, that is, $C = \partial A$. Then $M^2 = P^2 - A$ is a Möbius band with boundary $C = \partial M^2$ and all the bridges for C are contained in

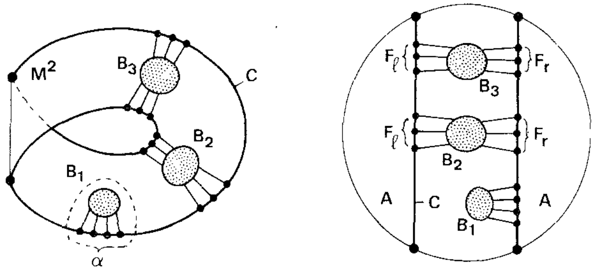


FIG. 4. Local and ladder type bridges.

M^2. In the series of figures hereafter, the projective plane is cut open into a disk and the rectangular region between two vertical parallel lines (or nearly parallel curves) corresponds to the Möbius band M^2. The two lines and the semicircular regions, right and left, are joined to form the cycle C and the face A, respectively, in P^2.

We classify bridges for C into three types topologically, considering how they are embedded in P^2. The first type, B_1 in Fig. 4, is contained in a 2-cell Δ^2 which meets C in an arc α on $\partial\Delta^2$, and is said to be *local*. This type can be characterized as a bridge B_1 such that $C \cup B_1$ contains no essential cycle. The arc α contains all feet of B_1 and the two feet at the ends form a vertex-cut of G. So there is no local bridge for C if G is 3-connected.

The second type is a bridge like B_2 or B_3 in Fig. 4, is called a *ladder type*, and is defined as a bridge, say B_2, such that $C \cup B_2$ contains an essential cycle but $\bar{B}_2$ does not. Such a bridge B_2 joins two segments of C, crossing the center line of M^2, and hence there is no 0-compressing essential curve for $C \cup B_2$. For example, each spoke of O_n is a ladder type bridge for ∂O_n in the canonical embedding (Fig. 2) but they are singular.

When B_2 is a non-singular bridge, the center line of M^2 separates $L(B_2)$ into two non-empty disjoint subsets $L_r(B_2)$ and $L_l(B_2)$, called *rights legs* and *left legs*. A *right foot* (or a *left foot*) is a foot of B_2 incident to a right leg (or a left leg) and the collections of right and left feet are denoted by $F_r(B_2)$ and $F_l(B_2)$, respectively. These namings above however lose their meaning globally since a Möbius band is non-orientable.

Note that $F_r(B_2)$ and $F_l(B_2)$ might not be disjoint and that any essential cycle in $C \cup B_2$ passes through B_2 exactly once, running along a path between right and left feet. When B_2 is singular, it consists of a unique leg e with two feet, right and left, which cuts the Möbius band into a rectangle and $C \cup B_2$ consists of a union of two essential cycles which contain e in common. Every ladder type bridge shrinks into this kind of a singular bridge on the projective plane.

The third type is a *global* bridge, like B shown in Fig. 5, whose body $\bar{B}$

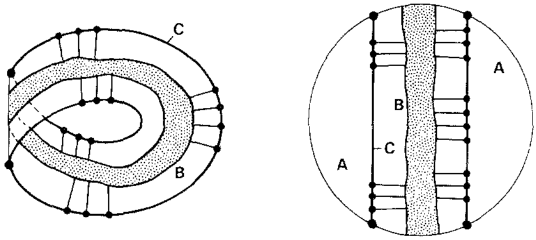


FIG. 5. Global bridges.

contains an essential cycle along the center line of M^2. If there is a global bridge in a 3-connected graph G, then there are no other bridges at all. When G is not 3-connected, there may be some local bridges.

4. THROWING-IN AND -OUT OF BRIDGES

Let G be a 3-connected non-planar graph embedded in P^2 and $f\colon G \to P^2$ a re-embedding of G in P^2. Then there is a face A of G, with boundary cycle $C = \partial A$, which is not extendable for f and $M^2 = P^2 - A$ is a Möbius band. These assumptions and the usage of symbols G, f, A, C, and M^2 will be unchanged throughout Sections 4 and 5. In this section, we characterize such a re-embedding f that $f(C)$ is a trivial curve in P^2.

A ladder type bridge B for C in G is said to be *squeezed* if there are no two disjoint C-avoiding paths which join $L_r(B)$ to $L_l(B)$ (that is, whose end edges belong to $L_r(B)$ and $L_l(B)$, one each.) A union of ladder type bridges $B_1 \cup \cdots \cup B_n$ $(n \geqslant 2)$ is called a *squeezed union* if all $F_r(B_i)$ (or $F_l(B_i)$) consist of only a common single vertex v. When $n = 1$, this definition is compatible with the definition of a squeezed bridge. So we shall often also call a single squeezed bridge a squeezed union.

By Menger's theorem, there is a vertex v in a squeezed bridge B which splits B into the right and left such that any pair of paths between $L_r(B)$ and $L_l(B)$ meet at v. Then we call v a *squeezing point* of B and say that B is *squeezed at* v. If there were another squeezing point u of B not adjacent to v, then $\{u, v\}$ would be a 2-vertex-cut of G, contrary to our assumption of G being 3-connected throughout. Thus, there is either a unique squeezing point or a unique pair of adjacent squeezing points. For a squeezed union $B = B_1 \cup \cdots \cup B_n$ with $F_r(B_i) = \{v\}$ $(i = 1, ..., n)$, we call the common foot v the *squeezing point* of B.

See Figs. 6a–6c. The first two, (a) and (b), illustrate a squeezed union and a squeezed bridge B but (c) is not a squeezed type. We identify the antipodal pairs of points along each circle to obtain a projective plane. Each pair of vertical parallel lines forms a cycle C in the projective plane

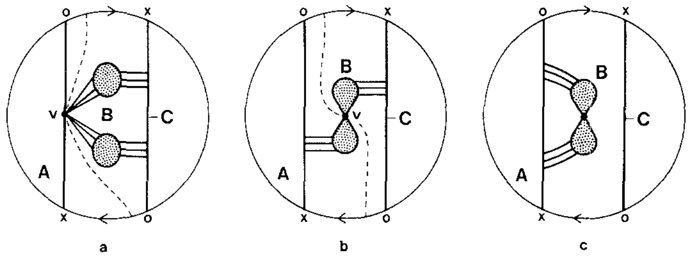


FIGURE 6

and it bounds a Möbius band M^2 derived from the middle rectangular part of each disk.

Note that if B is a squeezed union for C, then a 1-compressing essential curve Γ for $C \cup B$, nearly parallel to the center line of the Möbius band M^2, passes through the squeezing point v. In Figs. 6a and 6b such a 1-compressing essential curve Γ is drawn by a dashed line and it cuts B into two subgraphs H_r and H_l which contact at v and which contain $F_r(B)$ and $F_l(B)$, respectively. Possibly, either H_r or H_l might be a trivial subgraph consisting only of v.

The bridge B as shown in Fig. 6c is very similar to the second one which splits at one vertex, but it is not squeezed since it is a local bridge. Our assumption of G being 3-connected however excludes this type.

LEMMA 4.1. *Let B be a union of bridges for the boundary cycle C of a face A in G. Then there is an embedding $f: G \to P^2$ such that $f(C)$ bounds a 2-cell Δ^2 in P^2 and $f(B)$ is contained in Δ^2 if and only if B is squeezed.*

We call such a re-embedding f a *throwing-in* of a bridge B (into a face A) and its inverse f^{-1} a *throwing-out* of a bridge B.

Proof. Assume that f is a throwing-in of B into A. Since G is non-planar and is 3-connected, there is another bridge B' such that $f(B')$ lies in the Möbius band $P^2 - \Delta^2$ and both bridges B and B' are ladder type bridges in the original embedding of G. Going along C in one direction, we encounter the members of $F_r(B)$, $F_r(B')$, $F_l(B)$, and $F_l(B')$ in order.

Suppose that there are two disjoint C-avoiding paths Q_1 and Q_2 in B joining $L_r(B)$ to $L_l(B)$. Let $s_i \in F_r(B)$ and $t_i \in F_l(B)$ be the end vertices of Q_i $(i = 1, 2)$. Then the segments $\overline{s_1 s_2}$ and $\overline{t_1 t_2}$ look parallel in $P^2 - A^2$ and form a rectangle R together with Q_1 and Q_2. (The union $C \cup Q_1 \cup Q_2$ may be regarded as a Möbius ladder with two spokes canonically embedded in P^2, up to homeomorphism, which is however excluded from our definition since it is the planar graph K_4.)

The re-embedding f must map homeomorphically this rectangle R (only

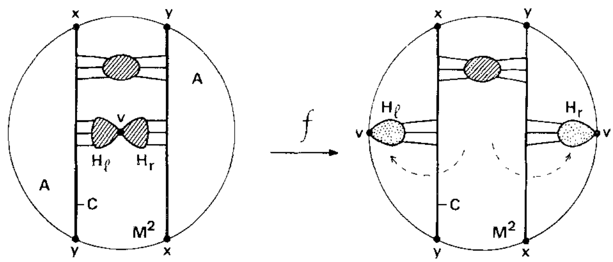


FIG. 7. Throwing-in and -out.

its boundary) into Δ^2, but it is impossible because $f(s_1)$, $f(s_2)$, $f(t_1)$, and $f(t_2)$ lie along $f(C) = \partial\Delta^2$ in this order. The two paths $f(Q_1)$ and $f(Q_2)$ would have to cross each other so as to join $f(s_1)$ to $f(t_1)$ and $f(s_2)$ to $f(t_2)$, respectively, a contradiction. Therefore, B contains no two disjoint paths between $L_r(B)$ and $L_l(B)$, so B is squeezed.

Conversely assume that B is squueezed at v and splits into subgraphs H_r and H_l which contact at v (Fig. 6b). Cut off H_r and H_l at v, and flip them out of $P^2 - A$, leaving $F_r(B)$ and $F_l(B)$ fixed. Then H_r and H_l are reversed and put into right and left semicircular regions, namely the halves of the face A. We can join the two v's in the projective plane P^2, pulling them to the peripheral circle. This procedure derives a throwing-in of B into A. (See Fig. 7.) ▌

A simple example of a throwing-in of a bridge has already been shown in Section 2. Since each spoke of a Möbius ladder O_n canonically embedded in P^2 is a ladder type bridge for ∂O_n, it can be thrown into the face bounded by ∂O_n.

Note that the union of all bridges for C is not squeezed; if it were, then G would have an embedding in a disk and would be planar. It is impossible to throw two or more maximal squeezed unions of bridges into A since their feet are mixed on C. So we conclude that:

LEMMA 4.2. *If a face A with boundary cycle C is not extendable for f and if $f(C)$ is trivial in P^2, then there are at least two bridges for C in G and f throws exactly one squeezed union of bridges into A.*

5. GLOBAL ALTERATIONS OF EMBEDDINGS

In this section, we assume that neither a throwing-in nor a throwing-out of bridges for any cycle arises in f and that $f(C)$ is an essential curve in P^2. So all bridges for the cycle C in the Möbius band M^2 are mapped inside the disk D^2, bounded by the double of $f(C)$, which is obtained from P^2 by cutting along $f(C)$. Note that if the boundary cycle of a face of G is sent to a trivial curve by f then it is now extendable for f and in particular that any face which does not meet C is extendable for f.

LEMMA 5.1. *Under our assumption, if C has only one bridge B in G, then f is equivalent to one of the following two re-embeddings*:

(i) *Re-embedding of type I*:

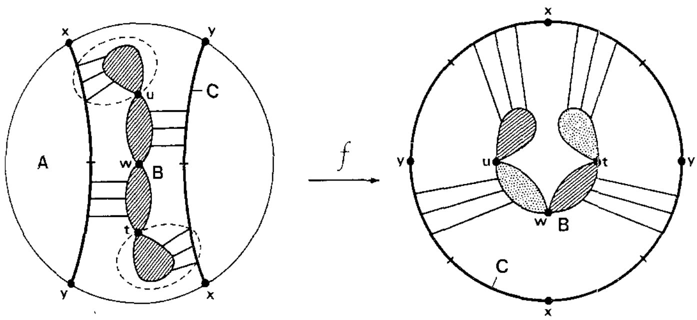


FIGURE 8

(ii) *Re-embedding of type II*:

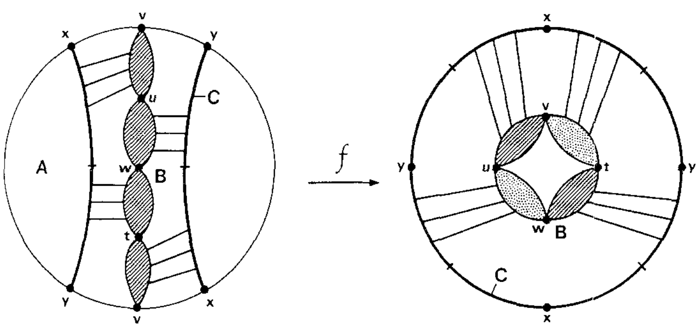


FIGURE 9

Either re-embedding twists and turns over the body of B partially. Such a reversed part is drawn in a tone different from the original in each right-hand figure of D^2. (To obtain a projective plane, identify each pair of antipodal points of the peripheral circle of each disk.) The detailed descriptions of the above two re-embeddings will be obtained in the proof below. Several degenerate types are allowed as long as the non-planarity of G is assured; each round part in B may consist of a single vertex or a single edge and each part surrounded by a dotted circle may shrink into a foot of B.

Proof. First we shall find a 2- or 3-compressing essential curve Γ for G which passes through the face A. Let $S(B)$ be the union of $\bar{B}$ with all faces bounded by only edges of $\bar{B}$. The boundary of any face S contained in $S(B)$ is necessarily a cycle in $\bar{B}$. Otherwise, we could find a 1-compressing curve for G in S which passes through a point on ∂S, contrary to Lemma 2.1. If B is a ladder type bridge, then there is clearly such a 2-compressing essential curve. (In fact, the right-hand disk in Fig. 4 is obtained as P^2 is cut open along it.) On the other hand, if B is global, then its body $\bar{B}$ contains an essential cycle J which runs along the center line of the Möbius band M^2. Choose a regular neighborhood $U(J)$ of J in M^2 and consider the situation around each vertex v of J.

The segment of J separates locally the Möbius band $U(J)$ into two sides. If two legs of B are incident to v at different sides, then there is a 3-compressing essential curve Γ which runs along the two legs, passing through their feet and v. Otherwise, at least one side of the local part of $U(J)$ is contained in $S(B)$. This implies that if there is no 3-compressing essential curve for G then $S(B) \cap U(J)$ is a Möbius band.

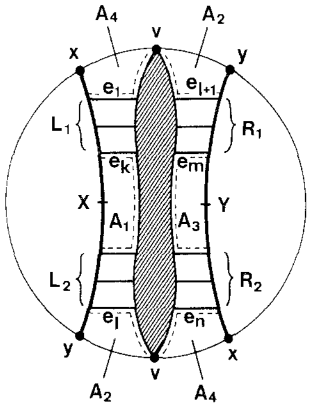
Recall that our assumption excludes a throwing-in and -out, so any face S in $S(B)$ is extendable for f since $f(\partial S)$ is a trivial curve in the 2-cell $P^2 - f(C)$. Thus f sends the Möbius band $S(B) \cap U(J)$ with $S(B)$ homeomorphically into the 2-cell $P^2 - f(C)$. It is however impossible since a 2-cell contains no Möbius band. Therefore, there is the required 3-compressing essential curve Γ for G when B is a global bridge.

In either case, the essential curve Γ cuts P^2 into a 2-cell where the face A separates into two semicircular regions which meet the rectangular region derived from M^2 at the right and left. (See Fig. 10a. Note that the boundary circle corresponds to Γ.) We shall however assume that Γ intersects G in two vertices x, y on C and in a vertex v of $\bar{B}$. Neglecting all occurrences of v, we can regard the text below as a proof for the case in which Γ is 2-compressing.

Let D^2 be the 2-cell obtained from P^2 by cutting it along $f(C)$. The boundary cycle $\tilde{C}$ of D^2 is twice as long as C. Then $f(x)$ and $f(y)$ split into $\{x_1, x_2\}$ and $\{y_1, y_2\}$, respectively, on ∂D^2 so that these pairs separate each other. The four vertices x_1, y_1, x_2, and y_2 cut $\tilde{C}$ into four arcs X_1, Y_1, X_2, and Y_2 which lie along $\tilde{C}$ in this order. Let X and Y be the two arcs on C corresponding to $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$, respectively. (See Fig. 10b.)

Let $e_1, ..., e_n$ be the legs of B numbered so that one encounters them in this order, starting at x and tracing C first along X and next along Y. Define L_j as the set of edges e_i such that $f(e_i)$ meet X_j $(j = 1, 2)$ in D^2 and likewise R_j for Y_j. These four sets are mutually disjoint.

Suppose that e_i and e_{i+2} belong to L_1 but e_{i+1} belongs to L_2. Then there is a C-avoiding path Q which has e_i and e_{i+2} as its end edges and a path P



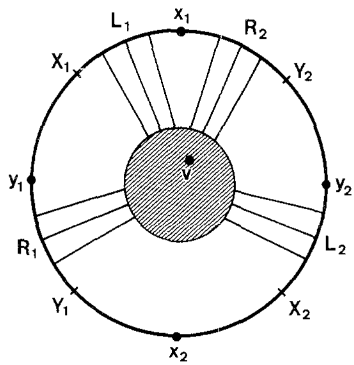


FIGURE 10

in X which joins the two feet u_i and u_{i+2} incident to e_i and e_{i+2}, respectively. The path $f(Q)$ starts at X_1 and comes back to X_1 in D^2. The end vertices $f(u_i)$ and $f(u_{i+2})$ bound one of the two copies of $f(P)$ in the arc X_1 on the boundary of D^2. (The other copy of $f(P)$ is contained in X_2.) Thus the cycle $f(Q \cup P)$ bounds a 2-cell Δ^2 in D^2; that is, it is a trivial curve in P^2. Since $f(e_{i+1})$ is incident to X_2, it does not lie in Δ^2. This implies that the bridge for $Q \cup P$ in G which includes e_{i+1} were thrown out, contrary to our assumption. The same argument works for the other cases. Thus, edges of L_1 and L_2 and edges of R_1 and R_2 are placed separately along X and Y, respectively.

After renaming and renumbering, we may assume that L_1, L_2, R_1, and R_2 lie along C in this order, as shown in Fig. 10a. Set

$$L_1 = \{e_1, ..., e_k\},$$
$$L_2 = \{e_{k+1}, ..., e_l\},$$
$$R_1 = \{e_{l+1}, ..., e_m\},$$
$$R_2 = \{e_{m+1}, ..., e_n\} \qquad (1 \leqslant k \leqslant l \leqslant m \leqslant n),$$

and let A_1, A_2, A_3, and A_4 be the faces whose boundary cycles contain $\{e_k, e_{k+1}\}$, $\{e_l, e_{l+1}\}$, $\{e_m, e_{m+1}\}$, and $\{e_n, e_1\}$, respectively. (In Fig. 10a, the two faces A_2 and A_4 split up and down.)

First we assume that all L_1, L_2, R_1, and R_2 are non-empty. Let Q_1 be the C-avoiding path in B, with end edges e_k and e_{k+1}, running along the boundary of A_1, and let Q_3 be the similar C-avoiding path in B, with end edges e_m and e_{m+1}, running around A_3. Let Q_2 be the C-avoiding path in B, with end edges e_1 and e_{l+1}, which runs first along ∂A_4 and next along

∂A_2 after passing through v, and let Q_4 be the similar C-avoiding path in B from e_l through v and to e_n. (In Fig. 10a, these four paths Q_1 to Q_4 are drawn by dotted lines without their labels.)

The path $f(Q_1)$ joins X_1 to X_2 while $f(Q_3)$ joins Y_1 to Y_2, so $f(Q_1)$ and $f(Q_3)$ cross each other in D^2. Since an embedding is a one-to-one mapping, Q_1 and Q_3 must have at least one common vertex w which is not a foot of B. Thus, B decomposes into two subgraphs H_1 and H_2 so that $H_1 \cap H_2$ consists of $\{v, w\}$ and $f(F(H_1)) \subset X_1 \cup Y_1$, $f(F(H_2)) \subset X_2 \cup Y_2$.

Consider the pair of Q_1 and Q_2 similarly. Their images $f(Q_1)$ and $f(Q_2)$ both start from X_1 but reach X_2 and Y_1, respectively. Since the starting point of $f(Q_2)$ is nearer x_1 than that of $f(Q_1)$, $f(Q_1)$ and $f(Q_2)$ must also cross each other in D^2. So we can conclude that H_1 decomposes to H_{11} and H_{12} so that $H_{11} \cap H_{12}$ consists of a single vertex u and $f(F(H_{11})) \subset X_1$, $f(F(H_{12})) \subset Y_1$. The situation is illustrated in Figs. 11a or 11b, according to whether v lies on H_{11} or not. (This distinction is meaningless when Γ is 2-compressing.) We can however reduce the case (b) to (a), considering how B is mapped in D^2. If there were not a crossing point of Q_1 and Q_2 between v and e_{l+1}, then the two v's at the top and bottom could not be mapped to the same point in D^2.

By the symmetrical arguments, H_2 also decomposes into H_{21} and H_{22} so that $H_{21} \cap H_{22}$ consists of a single vertex t and $f(F(H_{21})) \subset X_2$, $f(F(H_{22})) \subset Y_2$, and H_{21}, H_{22} contains w, v, respectively. That is the conclusion of the theorem.

Now assume that L_1, L_2, R_1, or R_2 is empty. If either L_2 (or R_1) were empty, then there would be a 1-compressing curve for $f(G)$ in P^2 starting at the middle point in Y_1 (or X_1) to that of Y_2 (or X_2), contrary to Lemma 2.1. So we may assume that R_2 is empty, up to symmetry. Then A_3 and A_4 are identical faces and $m = n$.

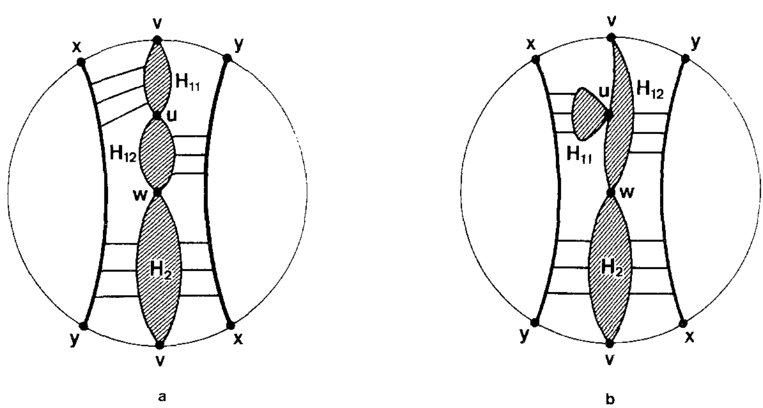


FIGURE 11

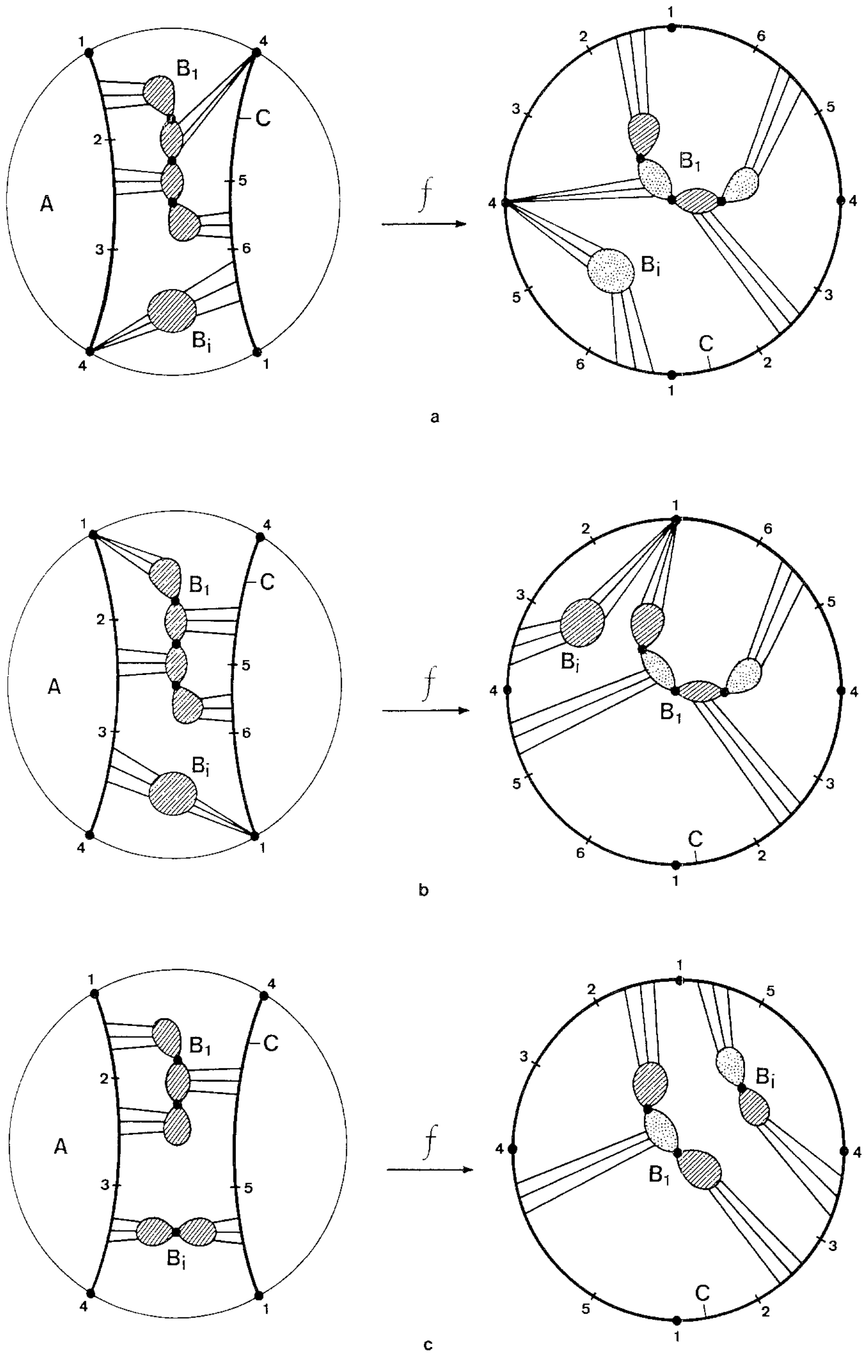


FIGURE 12

In this case, similar arguments for the pairs $\{Q_1, Q_2\}$ and $\{Q_1, Q_4\}$ conclude that B has the same structure as above where H_{22} degenerates into a single vertex $v = t$. Note that L_1 and R_2 cannot be empty simultaneously; if they could, then there would be a 1-compressing curve for $f(G)$ in P^2 through $f(y) = y_1 = y_2$. ∎

Now we shall consider the case when C has two or more bridges in G. The difference from the previous case is that each bridge may be squeezed or may be able to be thrown into A.

LEMMA 5.2. *Under our assumption, if C has at least two bridges in G, then f is equivalent to one of the following re-embeddings.*

(i) *Re-embedding of type III: There exists precisely one bridge B_1, not squeezed, and the other bridges separate into at most three squeezed unions, B_2, B_3, and B_4, at most one of which is disjoint from B_1. The restriction $f|_{C \cup B_1}$ is a re-embedding of type I and $f|_{C \cup B_1 \cup B_i}$ $(i = 2, 3, 4)$ is equivalent to one of the three figures, Fig.* 12a, 12b, *or* 12c.

(ii) *Re-embedding of type IV: The bridges for C separate into precisely three squeezed unions B_1, B_2, and B_3 each containing a path such that the three paths are pairwise disjoint.* (See Fig. 13.)

Proof. Assume that there is a bridge B_1 which is not squeezed. If $f|_{C \cup B_1}$ were a re-embedding of type II, then the bridge B_1 would be a global bridge and there could be no other bridges for C, contrary to our hypothesis. Thus $f|_{C \cup B_1}$ is a re-embedding of type I. If there were another non-squeezed bridge B', then $C \cup B'$ would have the same structure as $C \cup B_1$ shown on the left-hand side of Fig. 8. It is however impossible to draw B' together with B_1 on the right-hand side of Fig. 8 although it is possible on the left-hand side. Therefore, all the bridges but B_1 must be

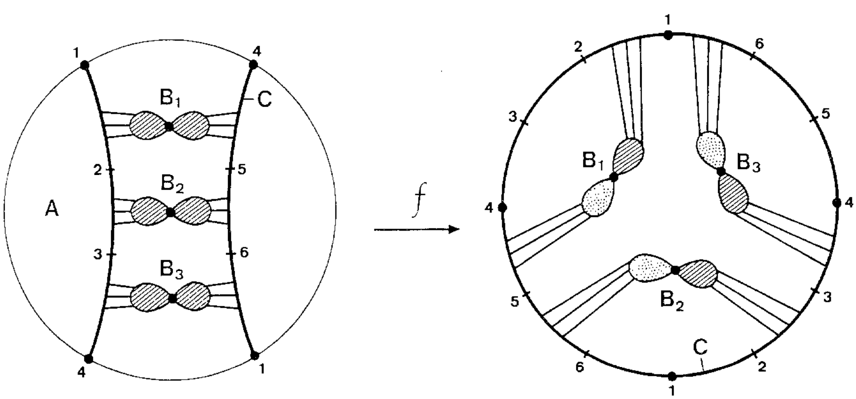


FIGURE 13

squeezed. Then we can see that the type for $f|_{C \cup B_1 \cup B_i}$ for any squeezed union B_i ($i \geqslant 2$) is one of three as given in Figs. 12a, 12b, and 12c, adding B_i and $f(B_i)$ to Fig. 8.

If there were four or more squeezed unions any pair of which could not be regarded as one squeezed union, then we could find a subdivision of the Möbius ladder O_4 with C corresponding to ∂O_4 in G and $f(C)$ would be a trivial curve in P^2 by Lemma 2.2, contrary to the assumption of $f(C)$ being essential in P^2. Thus, there are at most three such squeezed unions B_2, B_3, and B_4 for C. If both $f|_{C \cup B_1 \cup B_i}$ and $f|_{C \cup B_1 \cup B_j}$ ($i \neq j$) had type (c) simultaneously, then B_i and B_j could be placed in parallel, like ladder steps, on the left-hand side of Fig. 12c but they would be mapped into the right-hand side by f so that they cross each other. Therefore, $f|_{C \cup B_1 \cup B_i}$ is of type (c) for at most one of B_i ($i = 2, 3, 4$) and the other B_i's have a foot in common with B_1. We have observed all the conditions for f to be of type III.

Now assume that all of the bridges are squeezed. If G has four disjoint C-avoiding paths for any pair of which their end vertices are mixed on C, then there is an embedding from the Möbius ladder O_4 which maps ∂O_4 onto C and the four spokes to these paths. By Lemma 2.2, $f(C)$ could not be essential in P^2, contrary to our hypothesis. If bridges separated into only two squeezed unions, then there would be obtained by throwing one of the unions into A an embedding of G which admits a 1-compressing curve passing through the other union, now contrary to Lemma 2.1. Thus, it is possible to divide bridges into precisely three squeezed unions B_1, B_2, and B_3. If they shrink to three edges, then G deforms into O_3. So f has an appearance similar to the embedding in Fig. 3, and is of type IV. ∎

Re-embeddings of types III and IV do not play essentially in our later discussion, so we shall analyze their details no more. Note that if G admits a re-embedding of type III or IV, then it admits also a throwing-in of a bridge.

Gathering the lemmas in Sections 4 and 5, we obtain Theorem 1.4. Note that if G admits one of the re-embeddings in the lemmas, then there is a 2- or 3-compressing curve for G in P^2. From this fact, it follows that:

THEOREM 5.3. *Every 4-incompressibly embeddable, projective-planar graph is uniquely and faithfully embeddable in a projective plane.*

By Lemma 4 in [4], every n-incompressible embeddable graph with $n+1$ vertices is n-connected. So the above theorem covers a subset of 4-connected projective-planar graphs, but not the total set. In fact, there are infinitely many 4-connected projective-planar graphs which are not uniquely embeddable and ones which are not faithfully embeddable in a projective plane. Such examples are shown in Section 7.

6. 5-Connected Projective-Planar Graphs

This chapter is devoted to the proof of Theorem 1.3. Our arguments in the previous two sections have already determined roughly the structure of projective-planar graphs which are not uniquely or faithfully embeddable in a projective plane. They seem to have many 3- or 4-vertex-cuts. Hence if they are 5-connected, then most of their parts will degenerate to a vertex or an edge.

Lemma 6.1. *There is precisely one non-singular bridge for the boundary cycle C of a face A in a 5-connected projective-plane graph G. The unique non-singular bridge B spans G; that is, $V(B) = V(G)$, and it is not squeezed.*

Proof. Recall the arguments in Section 3. Under our hypothesis, each bridge for C is either a ladder type or a global one. If there is a global bridge for C, then it is not squeezed and there are no other bridges for C. So it is sufficient to prove the lemma when all bridges are of ladder type. Then their legs and feet are separated right and left.

First suppose that all bridges $B_1, ..., B_g$ are singular; that is, each bridge B_i consists of a single edge $v_i u_i$. Since each vertex has degree at least 5, we may assume that $v_1 = v_2 = v_3$ and that u_1, u_2, and u_3 lie along C in this order. The bridge $v_2 u_2$ cuts the Möbius band $P^2 - A$ into a rectangle R which is homeomorphic to a 2-cell. Let α be the segment of C which joins u_1 and u_2, not passing through u_3. Then the cycle consisting of α and the path $u_1 v_1 u_2$ can be regarded naturally as one in the rectangle R, so it bounds a 2-cell in R and also in $P^2 - A$. If α contained another vertex u different from u_1 and u_2, then the singular bridge incident to u would be a local one. Thus $\alpha = u_1 u_2$ and $u_1 v_1 u_2$ is a cycle which bounds a triangular face in $P^2 - A$. Similarly, $u_3 v_1 u_2$ is the boundary cycle of a triangular face which meets the one bounded by $u_1 v_1 u_2$ in $v_1 u_2$. This implies that u_2 would have degree 3, contrary to G being 5-connected. Therefore, there is at least one-singular bridge, say B_1.

Let x_1 and y_1 be the first and last right feet of B_1 lying along C and let x_2 and y_2 be the respective left ones. If $V(G) \neq V(B_1)$, the removal of $\{x_1, x_2, y_1, y_2\}$ would separate a vertex of B_1 and one not belonging to B_1. Thus, B_1 spans G and necessarily any other bridge is singular.

Suppose that B_1 is squeezed at v and decomposes into H_1 and H_2 which contact at v, numbered so that $\{x_i, y_i\} \subset H_i$ $(i = 1, 2)$. Since the removal of $\{v, x_1, y_1\}$ cannot disconnect G, either H_1 spans G and $v = x_2 = y_2$ or $V(H_1) = \{v, x_1, y_1\}$. The first case however does not occur; if it did, the union of all bridges, B_1 and several singular ones, would be a squeezed union and there would be a 1-compressing curve for G which passes

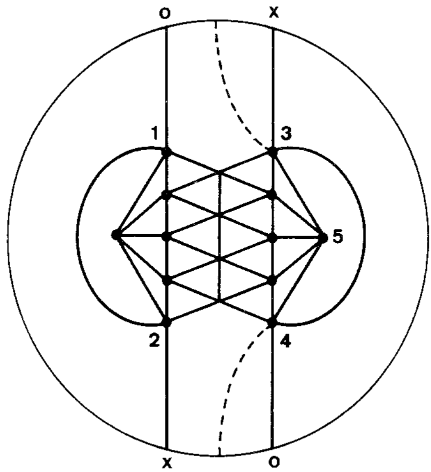


FIGURE 14

through v, contrary to Lemma 2.1. Symmetrically, $V(H_2) = \{v, x_2, y_2\}$ and the vertex v would have degree 4, now contrary to G being 5-connected. Therefore, B_1 is not squeezed. ▌

By the above, there is no squeezed non-singular bridge for C in G. Thus, any 5-connected projective-plane graph G admits no re-embedding of type IV, but other types occur. Figure 14 shows an example of a 5-connected projective-planar graph which admits a throwing-in and -out of bridges, re-embeddings of types I and III. Regard 1234 as C; then the unique non-singular bridge is one for a re-embedding of type I. Now regard 12354 as C and throw out the edge 34 onto the dashed line; then there arises the situation where a re-embedding of type III is applicable. In either case, H_{11}, H_{21}, and H_{22} degenerate into vertices in the unique non-singular bridge.

If a graph is uniquely and faithfully embedded in P^2, then any embedding $f: G \to P^2$ extends to a self-homeomorphism of P^2. So this example is either not uniquely or not faithfully embeddable in P^2. In fact, the original embedding is faithful but the one with 34 replaced is not faithful, and hence this graph is faithfully but not uniquely embeddable in P^2.

To exclude a throwing-in and -out of bridges, re-embeddings of types I and III, it suffices to assume that a projective-planar graph is 3-incompressibly embedded. Although a 3-incompressibly embeddable, projective-planar graph may admit a re-embedding of type II, we shall observe that if it is 5-connected, then it does not with only one exception, as Theorem 1.3 states.

Proof of Theorem 1.3. Let G be a graph with the assumption of the theorem and suppose that there is a re-embedding $f: G \to P^2$. A graph embedded 3-incompressibly in P^2 admits only a re-embedding of type II, and so is f. Then we use the same notation as that given by Figs. 10a and 11a in the proof of Lemma 5.1 and denote by u_i the foot of B incident to e_i. The vertices v, u, w, t are joints of four ellipses of B in Fig. 9.

Consider the removal of $\{v, u, u_1, u_k\}$, which cannot disconnect G since G is 5-connected. If H_{11} contains another vertex different from these four, then H_{11} must span G and hence w and t are equal to u or v. In this case, there could be found a 2-compressing essential curve which passes through v and u, contrary to G being 3-incompressibly embedded. Thus, H_{11} consists of an edge vu or a vertex $v = u$ with at most three edges $e_1, ..., e_k$ ($k = 1, 2$, or 3). The same argument works for the other parts, H_{12}, H_{21}, and H_{22}.

Now $\{v, u, w, t\}$ forms an essential cycle of length 4 or 3. Note that each v, u, w, and t is adjacent to precisely three vertices of C, and conversely that each vertex on C is adjacent to at least three of v, u, w, and t. It is routine to see that all v, u, w, t are not distinct and to conclude that G is isomorphic to K_6. Since K_6 is uniquely embeddable in P^2, K_6 is a unique exception for the faithfulness but not for the uniqueness. ∎

7. Examples

The graph given by Fig. 14 is 5-connected but is not uniquely embeddable in a projective plane. The complete graph K_6 is also 5-connected but is not faithfully embeddable in a projective plane. They show that the assumption of being 3-incompressibly embeddable cannot be omitted from Theorem 1.3. In fact, infinitely many such examples exist:

THEOREM 7.1. *There are an infinite number of 5-connected projective-planar graphs which are not uniquely embeddable and there are ones which are not faithfully embeddable in a projective plane.*

Proof. Prepare a cycle C_{2n+1} of odd length $2n+1$ given as a cyclic sequence $\{1, ..., 2n+1\}$ of vertices, add edges $(i, i+n)$ $(i = 1, ..., 2n+1)$, and join two extra vertices x and y to $\{1, ..., n\}$ and $\{n+1, ..., 2n+1\}$, respectively. The resulting graph G_n has two embeddings shown in Figs. 15a and 15b, which are not equivalent since the numbers of their triangular faces are different. Hence G_n is not uniquely embeddable in a projective plane, and it is 5-connected if $n \geqslant 5$. Of course, G_n $(n \geqslant 3)$ is not planar since C_{2n+1} with edges $(i, i+n)$ $(i = 1, ..., n)$ forms O_n.

Now let H_n denote the graph as given in Fig. 16a. Then H_n is 5-connected if $n \geqslant 5$ and it has an automorphism $\sigma: H_n \to H_n$ which fixes x, y and $n-1$ vertices of degree 6 placed vertically and which interchanges z, 1,

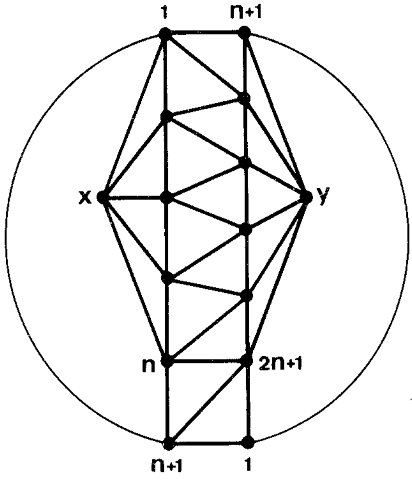


FIGURE 15

2, ..., n with z', $1'$, $2'$, ..., n', respectively. The automorphism σ is not a symmetry of the embedding of H_n since xyz is essential but $\sigma(xyz)=xyz'$ is trivial in the projective plane. Thus, the embedding of H_n is not faithful.

The subgraph of H_n indicated with bold edges in Fig. 16b is a subdivision of O_n. By Lemma 2.2, the subgraph minus xy is uniquely embedded and the uniqueness extends to the whole of H_n in order. Thus, H_n has

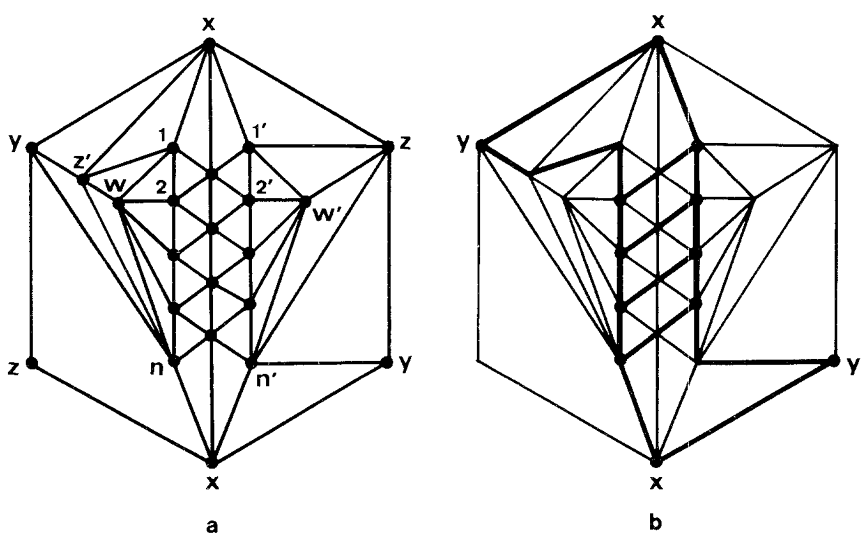


FIGURE 16

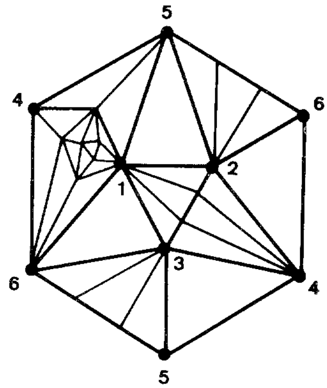


FIGURE 17

only the unique embedding in a projective plane which is not faithful and hence it is not faithfully embeddable in a projective plane.

The two infinite sequences $\{G_n : n \geqslant 5\}$ and $\{H_n : n \geqslant 5\}$ are the desired ones. ∎

To see that Theorem 1.3 is the best possible with respect to connectivity, we shall construct below 4-connected projective-planar triangulations whose embeddings are not unique or faithful and which also show that Theorems 1.1 and 1.2 are the best possible for all of their hypotheses.

THEOREM 7.2. *There are an infinite number of* 4-*connected projective-planar triangulations, with a subdivision of* K_6, *whose embeddings are not unique and there are ones which have no faithful embedding.*

Proof. One of the graphs for non-uniqueness has been already constructed in [2]. It is easy to create an infinite number of non-uniquely embeddable projective-planar triangulations, starting from that graph. Here we shall show only examples for non-faithfulness.

Figure 17 shows such an example. This triangulation has an automorphism which reverses the diamond 1234 around two vertices 1 and 4, leaving the outer vertices fixed. The automorphism sends the boundary cycle 136 of a face to the essential cycle 126, so it is not a symmetry of the embedding given in Fig. 17. This triangulation contains a subdivision of K_6 with six vertices of degree 6 labeled by 1, 2, 3, 4, 5, 6. Since the uniqueness of its embedding is derived from that of K_6, it is not faithfully embeddable in a projective plane.

An infinite number of examples will be constructed from this by inserting many vertices of degree 4 into the path between 2 and 3 so that they are adjacent to both 1 and 4. ∎

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