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Nonoscillatory Solutions of Higher Order Differential Equations

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1. INTRODUCTION

We want to consider the existence and growth of nonoscillatory solutions of the differential equation

$$x^{(n)} + f(t, x) = 0,$$

where f is a continuous real valued function for $t \geqslant 0$ and $x \in R$ such that $f(t, x)$ is nondecreasing in x for fixed t, and $xf(t, x) > 0$ if $x \neq 0$. We will show that under certain conditions, there exist solutions for this equation that grow like a polynomial of given degree.

It is known (cf. [8]) that if a nonoscillatory solution $x(t)$ does exist, then it may be continued to $+\infty$ and satisfies the property that for some t_0, there is an integer ℓ, $0 \leqslant \ell < n$, which is odd if n is even and even if n is odd, such that for $t \geqslant t_0$, $x(t)\, x^{(i)}(t) > 0$ for $i = 0, 1,..., \ell$, and $(-1)^{n+i}\, x(t)\, x^{(i)}(t) < 0$ for $i = \ell + 1, \ell + 2,..., n$.

Such a solution is subsequently referred to as a function of degree ℓ for $t \geqslant t_0$. (In [7], Kartsatos denotes this property for $x(t)$ by saying that $x(t)$ belongs to the class $B(t_0, \ell)$, while Lovelady [10, 11] uses the notation $\ell = j_x$.)

A solution $x(t)$ of the above nth order equation having degree ℓ for $t \geqslant t_0$ is bounded below by a polynomial of degree $(\ell - 1)$ as $t \to \infty$ since $x^{(\ell)} > 0$, $x^{(\ell+1)} < 0$, and $x' > 0$ for $t \geqslant t_0$. Also, such a solution is bounded above by a polynomial of degree ℓ, and consequently a solution of degree ℓ is bounded between two polynomials of degrees $\ell - 1$ and ℓ. The question is whether a solution of degree ℓ exists that grows like a polynomial of degree $\ell - 1$ in the sense that it is bounded between two polynomials of degree $\ell - 1$, or a solution exists that grows like a polynomial of degree ℓ in that sense.

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Answers to this question have been given in a number of papers on the subject of oscillation and nonoscillation for various forms of the above equation. Most of these studies have considered a second order equation. In his 1955 paper [1], F.V. Atkinson gave a condition which guarantees the existence of nonoscillatory solutions of the equation $x'' + p(t)\,x^{2k+1} = 0$, and it was noted by Moore and Nehari [12] that Atkinson's proof shows that the nonoscillatory solution in fact converges to a constant. They then proved that another condition yields solutions that grow like t. Nehari provided similar results [13] for an equation that may be written in the form $x'' + f(t, x) = 0$, and an extension of this result to a somewhat more general equation was given in a survey paper [15] on second order equations by Wong. An overview of such results regarding bounded and asymptotically linear or unbounded and asymptotically linear solutions of a second order equation was provided by Coffman and Wong in [3]. Finally, for the second order equation, Heidel has proved results on solutions that grow like fractional powers of t in [6].

Analogous theorems for the nth order equation must concern the existence of nonoscillatory solutions that are like a constant, like a linear function, a second degree polynomial, and so on, and should perhaps determine if such solutions are asymptotic to a polynomial of some degree. Note that no nonoscillatory solution of $x^{(n)} + f(t, x) = 0$ can grow faster than a polynomial of degree $n - 1$ due to the sign condition on the function $f(t, x)$. As already stated, a nonoscillatory solution must have degree ℓ where ℓ is even if n is odd and odd if n is even. Recently, Lovelady [11] has given a condition for the existence of solutions having all odd degrees up to and including ℓ of the even order superlinear equation $x^{(n)} + q(t) \mid x^{\gamma} \mid \operatorname{sgn}(x) = 0$, and he also gives an alternative condition implying that all nonoscillatory solutions grow no faster than a polynomial of degree $\ell - 2$.

In this paper we study the more general nth order equation $x^{(n)} + f(t, x) = 0$ with the assumptions on f given above, but where n may be either even or odd. Applying the results for this equation to the equation $x^{(n)} + p(t)\,x^{\gamma} = 0$, where for simplicity we assume γ is the quotient of odd integers, in the case $\gamma > 1$, we find conditions for the existence of nonoscillatory solutions of degrees (in steps of two) up to and including a given ℓ, just as Lovelady found. In contrast, the case $0 < \gamma < 1$ yields solutions of degrees down to ℓ, that is, of degrees ℓ, $\ell + 2, \ldots$, and $n - 1$. This verifies the intuitive notion that for the superlinear equation, nonoscillatory solutions of lowest degree are the first to occur, while for the sublinear equation nonoscillatory solutions of highest degrees are first to occur.

More in fact will be established than just the existence of nonoscillatory solutions of degree ℓ. One condition (I_ℓ) is shown to imply the existence of a nonoscillatory solution $x(t)$ of degree ℓ that has its $(\ell - 1)$st derivative bounded above by a constant. This means that $x(t)$ is bounded above by a polynomial of degree $\ell - 1$, while it is also bounded below by a polynomial of degree $\ell - 1$

since $x(t)$ has degree ℓ. Thus $x(t)$ grows like a polynomial of degree $\ell - 1$. Another condition $(I_{\ell+1})$ yields a solution $x(t)$ of degree ℓ with $x^{(\ell)}(t)$ bounded below by a positive constant, and thus $x(t)$ is seen to be bounded between two polynomials of degree ℓ. It is now clear that to say a nonoscillatory solution has degree ℓ is not being sufficiently specific since such a solution may have one of two, or perhaps more, different rates of growth. However, we shall use this intuitive terminology for describing nonoscillatory solutions, and then additionally specify that the solution grows like a polynomial of degree $\ell - 1$ or of degree ℓ.

2. Preliminary Results

We begin by extending to odd order equations a result that was proved by Kartsatos [7, Lemma 2.1] for even order equations which thus provides a complete generalization of the initial result stated by Atkinson [2, Lemma 1]. Another preliminary result then is proved which is again an nth order analogue of a standard theorem for second order equations (cf. [15, Theorem 3]) on the existence of nonoscillatory solutions. The mian results then follow with the application to the special equation mentioned above.

Lemma (Kiguradze). *Let $f(t)$ be a function such that it and each of its derivatives up to order $(n - 1)$ inclusive is absolutely continuous and of constant sign in an interval (t_0, ∞). If n is even {odd} and $f^{(n)}(t) f(t) \leqslant 0$ for $t \geqslant t_0$, then there is an odd {even} integer ℓ, $0 \leqslant \ell \leqslant n - 1$, such that for $t \geqslant t_0$*

(i) $f^{(k)}(t) f(t) \geqslant 0$ *for* $k = 0, 1,..., \ell$;

(ii) $(-1)^{n+k} f^{(k)}(t) f(t) \leqslant 0$ *for* $k = \ell + 1, \ell + 2,..., n$;

(iii) $(t - t_0) \,|f^{(\ell-k)}(t)| \leqslant (1 + k)\, |f^{(\ell-k-1)}(t)|$ *for* $k = 0, 1,..., \ell - 1$.

The next lemma establishes the equivalence of the existence of nonoscillatory solutions of degree ℓ for the nth order equation (1) and the corresponding nth order differential ineqiality (2),

$$x^{(n)} + f(t, x) = 0, \tag{1}$$

$$z^{(n)} + f(t, z) \leqslant 0. \tag{2}$$

Lemma 1. *Let $z(t)$ be a solution of (2) that is positive for all large t. Then for some t_0, $z(t)$ has degree ℓ for $t \geqslant t_0$ and for some integer ℓ where $0 \leqslant \ell \leqslant n - 1$, ℓ is odd if n is even, and ℓ is even if n is odd.*

If $1 \leqslant \ell \leqslant n - 1$ *and if x_0 is such that $0 < x_0 \leqslant z(t_0)$, then there exists a solution $x(t)$ of (1) with $x(t_0) = x_0$ and satisfying for $t \geqslant t_0$ that*

$$0 < x^{(k)}(t) \leqslant z^{(k)}(t) \qquad \textit{for } k = 0, 1,..., \ell,$$

and

$$0 > (-1)^{n+k} x^{(k)}(t) \geqslant (-1)^{n+k} z^{(k)}(t) \qquad \text{for } k = \ell + 1, \ell + 2, \dots, n.$$

If $\ell = 0$ *when* n *is odd and if* x_∞ *satisfies* $0 < x_\infty \leqslant z(\infty)$ *then there exists a solution* $x(t)$ *of* (1) *with* $\lim_{t\to\infty} x(t) = x_\infty$ *and satisfying* $0 < (-1)^k x^{(k)}(t) \leqslant (-1)^k z^{(k)}(t)$ *for* $k = 0, 1, \dots, n$.

Proof. If $z(t)$ is a positive solution of (2) for all large t, then $zz^{(n)} \leqslant 0$ and the first part of the lemma follows from Kiguradze's Lemma. The second part of this lemma was proven for the case that n is even by Kartsatos [7, Lemma 1], and we now outline the proof for the case n = odd using essentially his argument.

Assume $z(t)$ is a solution of (2) of degree ℓ for some ℓ even (since n is odd) and for all $t \geqslant t_0$ where t_0 is sufficiently large. Following the proof given in [7, Lemma 1], after k integrations from t to ∞ we would get that for $k = 1, 2, \dots, n - (\ell + 1)$,

$$(-1)^{k-1} z^{(n-k)}(t) \geqslant \int_t^\infty [(s-t)^{k-1} f(s, z(s))/(k-1)!]\, ds$$

since for these values of k, $z^{(n-k)}(\infty) = 0$. Letting $k = n - (\ell + 1)$ and after another integration, we find that

$$\begin{aligned} z^{(\ell)}(t) &\geqslant \int_t^\infty [(s-t)^{n-\ell-1} f(s, z(s))/(n-\ell-1)!]\, ds \\ &\equiv \Phi_\ell(t, z). \end{aligned}$$

Now if $\ell > 0$ then integrating this inequality ℓ-times from t_0 to t, we get

$$\begin{aligned} z(t) &\geqslant z(t_0) + \int_{t_0}^t \int_{t_0}^{v_{\ell-1}} \cdots \int_{t_0}^{v_1} \Phi_\ell(v, z)\, dx\, dv_1 \cdots dv_{\ell-1} \\ &\equiv z(t_0) + \Psi_\ell(t, z). \end{aligned}$$

On the other hand, if $\ell = 0$, then letting $k = n - 1$ and integrating once we have that

$$z(t) \geqslant z(\infty) + \int_t^\infty [(s-t)^{n-1} f(s, z(s))/(n-1)!]\, ds$$

Now if $\ell > 0$, then by defininig the sequence $\{x_n(t)\}$ by

$$x_0(t) = z(t), \qquad x_{n+1}(t) = x_0 + \Psi_\ell(t, x_n), \qquad n = 0, 1, \dots$$

we see, as in [7, Lemma 1] that lim $x_n(t) = x(t)$ exists and satisfies the integral equation

$$x(t) = x_0 + \Psi_\ell(t, x), \qquad \text{for } t \geqslant t_0,$$

and it follows that $x(t)$ has the properties stated in the lemma.

On the other hand, if $\ell = 0$, then by defininig $\{x_n(t)\}$ by

$$x_0(t) = z(t), \qquad x_{n+1}(t) = x_\infty + \Psi_0(t, x_n),$$

it follows in a similar fashion that lim $x_n(t) = x(t)$ exists and is a solution of the integral equation

$$x(t) = x_\infty + \int_t^\infty [(s-t)^{n-1} f(s, x(s))/(n-1)!]\, ds.$$

We may conclude that $x(t)$ has the properties stated in the lemma, thus completing the proof.

The next lemma is a generalization to higher order equations of a standard result for second order equations. The proof given is patterned after the proof of Wong [15, Theorem 3].

LEMMA 2. (i) *If n is even, then for each t_0 sufficiently large,* (2) *has a positive solution bounded above by a constant for $t \geqslant t_0$, if and only if, for some $a \geqslant 0$, and some $c > 0$*

$$\int_a^\infty t^{n-1} f(t, c)\, dt < \infty. \tag{I$_1$}$$

(ii) *If n is odd, then for each t_0 sufficiently large,* (2) *has a positive nonincreasing solution bounded below by a positive constant for $t \geqslant t_0$, if and only if,* (I_1) *holds for some $a \geqslant 0$ and some $c > 0$.*

Proof. (i) If n is even and $z(t)$ is a bounded positive solution of (2) for all large t, then $z(t)$ has degree 1 for large t, and hence $(-1)^{k-1} z^{(k)}(t) \geqslant 0$ for $k = 1,..., n$ and t large. Now by Taylor's theorem and (2) it follows that

$$z(t) \geqslant z(t_0) + \sum_{j=1}^{n-1} (-1)^{j+1} (t-t_0)^j z^{(j)}(t)/j! + \int_{t_0}^t [(s-t_0)^{n-1} f(s, z(s))/(n-1)!]\, ds,$$

and since the sign alternations of the derivatives of z imply that the summation term on the right is nonnegative, we may conclude that

$$z(t) \geqslant z(t_0) + \int_{t_0}^t [(s-t_0)^{n-1} f(s, z(s))/(n-1)!]\, ds.$$

The hypotheses on $z(t)$ also allow that $z(\infty) = \lim_{t\to\infty} z(t)$ exists and is finite, and thus, letting $t \to \infty$ in the last inequality, we get that

$$z(\infty) \geqslant \int_{t_0}^{\infty} [(s - t_0)^{n-1} f(s, z(t_0))/(n-1)!]\, ds$$

which establishes the necessity of (I_1).

Conversely, if (I_1) holds for $a \geqslant 0$ and $c > 0$, then it follows as in the proof given by Wong [15, Theorem 3] that there exists a bounded positive solution $x(t)$ of

$$x(t) = c - \int_{t}^{\infty} [(s - t)^{n-1} f(s, x(s))/(n-1)!]\, ds$$

for all t sufficiently large. A solution of this integral equation is also a solution of (1), thus establishing the sufficiency of (I_1).

(ii) If n is odd and $z(t)$ is a positive nonincreasing solution of (2) that is bounded below by a positive constant, then $z(t)$ must have degree zero and thus satisfy $(-1)^k z^{(k)}(t) \geqslant 0$ for each $k = 0, 1,..., n$ and all large t. Arguing as in the proof of (i) we would find that

$$z(t) \leqslant z(t_0) - \int_{t_0}^{t} [(s - t_0)^{n-1} f(s, x(s))/(n-1)!]\, ds.$$

Now the hypotheses that $z \geqslant c$ for some positive c, and $z' \geqslant 0$ imply that $z(\infty) = \lim_{t\to\infty} z(t)$ exists and is positive with $z(\infty) \leqslant z(t_0)$. Taking the above integral inequality to the limit, we get that

$$z(t_0) \geqslant \int_{t_0}^{\infty} [(s - t_0)^{n-1} f(s, z(s))/(n-1)!]\, ds$$

$$\geqslant \int_{2t_0}^{\infty} [(s/2)^{n-1} f(s, c)/(n-1)!]\, ds$$

thus verifying the necessity of (I_1).

The converse may be proven in the same way as in part (i) except that (I_1) would be shown to imply the existence of a positive nonincreasing solution $x(t)$ of the integral equation

$$x(t) = (c - \epsilon) + \int_{t}^{\infty} [(s - t)^{n-1} f(s, x(s))/(n-1)!]\, ds$$

where $\epsilon > 0$ is chosen so that $c - \epsilon > 0$, and then t is such that $\int_t^\infty s^{n-1} f(s, c)\, ds < \epsilon$. It follows that $x(t)$ is a solution of (1) satisfying $c - \epsilon \leqslant x(t) \leqslant c$ for all sufficiently large t. This proves the sufficiency of (I_1) and completes the proof of the lemma.

COROLLARY 1. (i) *If n is even, then for t_0 large there exists a positive solution of* (1) *that is bounded above by a constant for $t \geqslant t_0$, if and only if,* (I_1) *holds.*

(ii) *If n is odd, then for t_0 large there exists a positive solution of* (1) *that is nonincreasing and bounded below by a positive constant for $t \geqslant t_0$, if and only if,* (I_1) *holds.*

Proof. Lemma 2 establishes this result for (2) and the existence of such solutions for (2) implies the existence of such solutions for (1) via Lemma 1, proving this corollary.

Indeed, there are many such nonoscillatory solutions of (1) {or (2)} if condition (I_1) holds, and this is the content of the next proposition.

COROLLARY 2. (i) *Let n be even. If* (I_1) *holds for $c > 0$ and some $a \geqslant 0$, then for each c' with $0 < c' < c$, there exists a $t_0 \geqslant 0$ such that* (2) {(1)} *has a solution $x(t)$ with $0 < x(t) \leqslant c'$ for $t \geqslant t_0$.*

(ii) *Let n be odd. If* (I_1) *holds for $c > 0$ and $a \geqslant 0$, then for each c', $0 < c' < c$, there is a $t_0 \geqslant 0$ such that* (2) {(1)} *has a solution $x(t)$ that is nonincreasing and such that $x(t) \geqslant c'$ fot $t \geqslant t_0$.*

Proof. The proofs of (i) and (ii) of Lemma 2 establish that when (I_1) holds for $c > 0$ and $a \geqslant 0$, then for $\epsilon > 0$ with $c - \epsilon > 0$, if t_0 is sufficiently large so that $\int_{t_0}^{\infty} t^{n-1} f(t, c)\, dt < \epsilon$, then there is a solution $x(t)$ of (2) with the desired properties and satisfying $c - \epsilon \leqslant x(t) \leqslant c$ for $t \geqslant t_0$. Part (i) of this corollary follows by letting c' take the role of c, while part (ii) follows by choosing ϵ so that $c' = c - \epsilon$.

Remark. The referee has noted that in both parts (i) and (ii) of Corollary 2, the solution $x(t)$ can be chosen so that $x(t) \to c'$ as $t \to \infty$. This follows from the fact that $x(t)$ is a solution of an integral equation of the form $x(t) = c' \pm \int_t^{\infty} F(t, s, x(s))\, ds$.

3. MAIN RESULTS

THEOREM 1. *Given $t_0 \geqslant 0$, there exists a positive solution of* (1) *having degree $= \ell$ for all $t \geqslant t_0$ where $\ell \geqslant 1$, if and only if, there exists a positive solution having degree 1 for $t \geqslant t_0$ of*

$$y^{(n-\ell+1)} + f(t, (t - t_0)^{\ell-1} y/\ell!) = 0. \qquad (E'_\ell)$$

Proof. Let $x(t)$ be a positive solution of (1) and assume that $x(t)$ has degree ℓ for $t \geqslant t_0$, where ℓ is an odd {even} integer with $0 \leqslant \ell \leqslant n - 1$ when n is even {odd}. Now when the ($\ell - 1$)-inequalities from part (iii) of Kiguradze's

Lemma corresponding to the values $k = \ell - 1, \ell - 2, \ldots, 1$ are chained together, then we get that

$$x(t) \geqslant (t - t_0)^{\ell-1} x^{(\ell-1)}(t)/\ell!, \qquad \text{for } t \geqslant t_0 .$$

Combining this inequality with (1) and using the hypothesis that $f(t, x)$ is nondecreasing in x, we get that $x(t)$ is a solution of

$$x^{(n)}(t) + f(t, (t - t_0)^{\ell-1} x^{(\ell-1)}(t)/\ell!) \leqslant 0, \qquad t \geqslant t_0 .$$

Therefore $z(t) \equiv x^{(\ell-1)}(t)$ is a solution of

$$z^{(n-\ell+1)} + f(t, (t - t_0)^{\ell-1} z/\ell!) \leqslant 0, \qquad t \geqslant t_0 ,$$

and since $x(t)$ has degree ℓ, $z(t)$ has degree 1. Now by Lemma 1 the existence of the solution $z(t)$ of degree 1 of the above differential inequality of even order guarantees the existence of a positive solution $y(t)$ of (E_ℓ') such that $y(t)$ has degree 1 for $t \geqslant t_0$.

Conversely, assume that for $t \geqslant t_0$, $y(t)$ is a solution of (E_ℓ') having degree 1 and $y(t) > 0$ for $t \geqslant t_0$. Then

$$w(t) \equiv \frac{1}{\ell} \int_{t_0}^{t} \int_{t_0}^{s_{\ell-1}} \cdots \int_{t_0}^{s_2} y(s_1)\, ds_1 \cdots ds_{\ell-1} . \tag{3}$$

has the properties that $w^{(k)}(t) \geqslant 0$ for $k = 0, 1, \ldots, \ell$ and $(-1)^{n+k} w^{(k)}(t) \leqslant 0$ for $k = \ell + 1, \ldots, n$ when $t \geqslant t_0$, since y is positive, has degree 1 and $w^{(\ell-1)}(t) = y(t)/\ell$. Hence, $w(t)$ is a function having degree ℓ for $t > t_0$, and moreover $y'(t) \geqslant 0$ implies

$$w(t) \leqslant \frac{1}{\ell} \int_{t_0}^{t} \cdots \int_{t_0}^{s_2} y(t)\, ds_1 \cdots ds_{\ell-1} = y(t)\,(t - t_0)^{\ell-1}/\ell!$$

Now $w^{(n)} = y^{(n-\ell+1)}/\ell \leqslant y^{(n-\ell+1)}$, and so for $t \geqslant t_0$,

$$w^{(n)}(t) + f(t, w(t)) \leqslant y^{(n-\ell+1)}(t) + f(t, (t - t_0)^{\ell-1} y(t)/\ell!)$$

showing that $w(t)$ is a solution of (2) for $t \geqslant t_0$ since y is a solution of (E_ℓ'). Finally, letting $t_1 > t_0$ then $w(t)$ is a positive solution of (2) having degree ℓ for $t \geqslant t_1$ and so by Lemma 1, there exists a positive solution $x(t)$ of (1) of degree ℓ with initial value $x(t_1) = w(t_1)$. This completes the proof of the theorem.

THEOREM 2. *If n is even {odd} and ℓ is odd {even} with $1 \leqslant \ell \leqslant n - 1$, then there exists a positive nonoscillatory solution $x(t)$ of degree ℓ of* (1) *such that $x^{(\ell-1)}(t)$ is bounded above by a constant, if and only if, for some $a' \geqslant 0$, $c' > 0$,*

$$\int_{a'}^{\infty} (t - t_0)^{n-\ell} f(t, c'(t - t_0)^{\ell-1})\, dt < \infty \tag{I_ℓ'}$$

or equivalently, for some $a \geqslant 0$, $c > 0$

$$\int_a^\infty t^{n-\ell} f(t, ct^{\ell-1})\, dt < \infty. \tag{I_ℓ}$$

Proof. The equivalence of the integral conditions (I_ℓ) and (I'_ℓ) follows from the hypothesis that $f(t, x)$ is nondecreasing in x. Thus, on one hand (I_ℓ) implies (I'_ℓ) since $t \geqslant t - t_0$ where $t_0 \geqslant 0$, while on the other (I'_ℓ) implies (I_ℓ) since $\frac{1}{2}t \leqslant t - t_0$ if $t \geqslant 2t_0$.

First assume (I'_ℓ) holds for some $a' \geqslant 0$ and some $c' > 0$. Then, since $(n - \ell + 1)$ is even, by the Corollary 1(i), there is a solution $y(t)$ of (E'_ℓ) which is positive, bounded, and of degree 1 for all $t \geqslant$ some t_0. Define the function $w(t)$ by (3). Then it follows as in the proof of the converse of Theorem 1 that $w(t)$ is a positive solution of degree ℓ of (2) for $t > t_0$, and that for $t_1 > t_0$, there exists a solution $x(t)$ of (1) with $x(t_1) = w(t_1)$ such that $x(t)$ has degree ℓ and satisfying for $t \geqslant t_1$

$$x^{(\ell-1)}(t) \leqslant w^{(\ell-1)}(t) = y(t)/\ell \leqslant y(t).$$

Thus, $x^{(\ell-1)}(t)$ is bounded above since $y(t)$ is bounded above by a constant. This proves the sufficiency of the integral conditions.

Conversely, assume that (1) has a positive solution with degree ℓ for $t \geqslant t_0$ and such that $x^{(\ell-1)}$ is bounded above by a constant. Then, as in the proof of Theorem 1, it follows that for $t \geqslant t_0$ $z(t) \equiv x^{(\ell-1)}(t)$ is a solution of $z^{(n-\ell+1)} + f(t, (t - t_0)^{\ell-1} z/\ell!) \leqslant 0$. Now since $(n - l + 1)$ is even and $x^{(\ell-1)}$ is assumed bounded, then $z(t)$ is a bounded positive solution of this differential inequality of even order and hence by Lemma 2(i) there exists $a' \geqslant 0$ and $c' > 0$ such that (I'_ℓ) holds. This completes the proof of the converse and thus the proof of the theorem.

Corollary 3. *There exists a positive solution of* (1) *having degree* $\ell \geqslant 1$ *and such that its* $(\ell - 1)$st *derivative is bounded by a constant for all large* t, *if and only if, there exists a positive bounded solution of degree* 1 *of*

$$y^{(n-\ell+1)} + f(t, t^{\ell-1}y) = 0. \tag{E_ℓ}$$

Proof. By Theorem 2 the existence of such a solution $x(t)$ of (1) is equivalent to the integral condition (I_ℓ), and by Corollary 1(i) this is equivalent to the existence of a bounded positive solution of (E_ℓ).

Theorem 3. *If* n *is even* {*odd*} *and* ℓ *is odd* {*even*} *where* $0 \leqslant \ell \leqslant n - 1$, *then there exists a positive nonoscillatory solution* $x(t)$ *of* (1) *having degree* ℓ *and such* *that $x^{(\ell)}(t)$ is bounded below by a positive constant for all large t, if and only if, for some $a \geqslant 0$, $c > 0$,*

$$\int_a^\infty t^{n-\ell-1} f(t, ct^\ell)\, dt < \infty. \tag{$\mathrm{I}_{\ell+1}$}$$

Proof. The case where $\ell = 0$ when n is odd was proven in the Corollary 1(ii). Thus for $\ell \geqslant 1$, assume a solution $x(t)$ of (1) exists which has degree ℓ, and has $x^{(\ell)}(t) \geqslant \alpha > 0$ for some α and all $t \geqslant t_0$. By the Lemma of Kiguradze, if t_0 is sufficiently large, then

$$x(t) \geqslant x^{(\ell)}(t)\,(t - t_0)^\ell/\ell! \qquad \text{for } t \geqslant t_0,$$

and then $z(t) = x^{(\ell)}(t)$ satisfies for $t \geqslant t_0$,

$$z^{(n-\ell)} + f(t, z(t)\,(t - t_0)^\ell/\ell!) \leqslant 0.$$

Therefore, by Lemma 2(ii), since $(n - \ell)$ is odd and $z(t)$ is a positive nonincreasing function that is bounded below by a positive constant, then there exists a $c' > 0$ and $a \geqslant 0$ such that

$$\int_a^\infty t^{n-\ell-1} f(t, c'(t - t_0)^\ell/\ell!)\, dt < \infty.$$

This is equivalent to $(\mathrm{I}_{\ell+1})$ for some $c > 0$.

Conversely assume that $(\mathrm{I}_{\ell+1})$ holds for some $a' \geqslant 0$, $c' > 0$. Then for c such that $0 < c < c'/2$, choose ϵ so that $0 < \epsilon < c$ and $T = T(\epsilon)$ so that for $t_0 \geqslant T$

$$\int_{t_0}^\infty t^{n-\ell-1} f(t, c't^\ell)\, dt < \epsilon.$$

Define the sequence of functions $\{x_n(t)\}$ for $t \geqslant t_0$ by

$$x_0(t) = c(t - t_0)^\ell$$
$$x_{n+1}(t) = c(t - t_0)^\ell + \Psi_\ell(t, x_n)$$

where $\Psi_\ell(t, x)$ is the same functional defined in the proof of Lemma 1, that is

$$\psi_\ell(t, x) = \int_{t_0}^t \frac{(t - v)^{\ell-1}}{(\ell - 1)!} \int_v^\infty \frac{(s - v)^{n-\ell-1}}{(n - \ell - 1)!} f(s, x(s))\, ds\, dv.$$

This is an increasing sequence of functions since for $n = 0$

$$x_1(t) = c(t - t_0)^\ell + \Psi(t, x_0(t)) \geqslant c(t - t_0)^\ell = x_0(t) \qquad \text{for } t \geqslant t_0,$$

while by induction, if $x_n(t) \geqslant x_{n-1}(t)$ for $t \geqslant t_0$, then

$$x_{n+1}(t) - x_n(t) = \Psi_\ell(t, x_n) - \Psi_\ell(t, x_{n-1}).$$

Now, the right hand side of this equation depends for its sign through the quantity $f(s, x_n(s)) - f(s, x_{n-1}(s))$, and since $f(t, x)$ is nondecreasing in x, then $x_n(s) \geqslant x_{n-1}(s)$ implies this quantity is nonnegative for all $s \geqslant t_0$. Therefore, $x_{n+1}(t) \geqslant x_n(t)$ for all $t \geqslant t_0$.

Furthermore, this is a sequence of functions each of which is bounded above by the function $g(t) = c'(t - t_0)^\ell$. To see this, recall that T was chosen so that $\int_T^\infty t^{n-\ell-1} f(t, c't^\ell)\, dt < \epsilon$. Proceeding by induction, we have by definition that $x_0(t) = c(t - t_0)^\ell \leqslant c'(- t_0)^\ell$ since $c < c'/2$. Assume that $x_n(t) \leqslant c'(t - t_0)^\ell$. Then $x_n(t) \leqslant c't^\ell$ for $t \geqslant t_0$, and so

$$\begin{aligned} x_{n+1}(t) &= c(t - t_0)^\ell + \Psi_\ell(t, x_n) \\ &\leqslant c(t - t_0)^\ell + \int_{t_0}^{t} \frac{(t - v)^{\ell-1}}{(\ell - 1)!} \int_v^\infty s^{n-\ell-1} f(s, c's^\ell)\, ds\, dv \\ &\leqslant c(t - t_0)^\ell + \epsilon \int_{t_0}^{t} [(t - v)^{\ell-1}/(\ell - 1)!]\, dv \\ &= (c + \epsilon)\, (t - t_0)^\ell. \end{aligned}$$

Finally, $0 < \epsilon < c < c'/2$ implies that $c + \epsilon < c'$ and hence that $x_{n+1}(t) \leqslant c'(t - t_0)^\ell$.

It also follows that $\{x_n(t)\}$ forms an equicontinuous family since the derivative of each $x_n(t)$ is bounded by a function of the form $c''(t - t_0)^{\ell-1}$. In fact

$$\begin{aligned} x'_{n+1}(t) &= c\ell(t - t_0)^{\ell-1} + \Psi'_\ell(t, x_n) \\ &\leqslant c\ell(t - t_0)^{\ell-1} + \epsilon[(t - t_0)^{\ell-1}/(\ell - 1)!] \\ &= (c\ell + (\epsilon/(\ell - 1)!))\, (t - t_0)^{\ell-1}. \end{aligned}$$

Therefore on any compact interval $[t_0, t_1]$ both sequences $\{x_n\}$ and $\{x'_n\}$ are uniformly bounded, the first by $c'(t_1 - t_0)^\ell$ and the second by $c''(t_1 - t_0)^{\ell-1}$, and so $\{x_n(t)\}$ is a uniformly bounded and equicontinuous family on a given interval $[t_0, t_1]$. Therefore the limit $x(t) = \lim_{x\to\infty} x_n(t)$ exists for $t \geqslant t_0$ and by the Ascoli–Arzela theorem, $x_n(t)$ converges uniformly to $x(t)$ on each $[t_0, t_1]$ showing that $x(t)$ is continuous since by induction each $x_n(t)$ is continuous. Moreover by the Lebesque Monotone convergence theorem, $x(t)$ satisfies

$$x(t) = c(t - t_0)^\ell + \Psi_\ell(t, x).$$

As in the proof of Lemma 1, this implies that $x(t)$ is an n-times differentiable function which satisfies (1) and has degree ℓ for $t > t_0$ with $x^{(\ell)} \geqslant c\ell!$. This completes the proof of the theorem.

4. An Example

These results will now be applied to the equation

$$x^{(n)} + p(t)\, x^{\gamma} = 0$$

where $p(t)$ is positive and continuous and γ is the quotient of odd integers.

Theorem 4. *Consider that for some integer* ℓ, $0 \leqslant \ell \leqslant n-1$, *where* ℓ *is odd {even} when n is even {odd},*

$$\int^{\infty} t^{n-\ell+\gamma(\ell-1)} p(t)\, dt < \infty.$$

(i) *If* $\gamma > 1$, *then* (5_ℓ) *is necessary and sufficient for* (4) *to have positive solutions* $x_j(t)$, *of degree* $= j$ *and such that* $x_j^{(j-1)}$ *is bounded above by a constant for all large t where* $j = 1, 3, 5, \ldots, \ell$ *if* ℓ *is odd and n is even, and* $j = 2, 4, \ldots, \ell$ *if* ℓ *is even when n is odd, while for the case n odd and* $\ell = 0$, *there exists a solution that is bounded below by a positive constant having degree* $= 0$.

(ii) *If* $0 < \gamma < 1$, *then* (5_ℓ) *is necessary and sufficient for* (4) *to have positive solutions* $x_j(t)$ *of degree* $= j$ *and such that* $x_j^{(j-1)}$ *is bounded above for all large t where* $j = \ell, \ell + 2, \ldots, n-1$ *(whether* ℓ *is odd or even) while for* $\ell = 0$, $x_0(t)$ *will be a solution of degree* $= 0$ *which is bounded below by a positive constant.*

Remark. Lovelady [11] has shown for n even and $\gamma > 1$ that (5_ℓ) is necessary and sufficient for (4) to have a solution of degree ℓ. It follows then from (i) above then that if (4) has a solution of degree ℓ, it must have a solution of degree ℓ with $x^{(\ell-1)}$ bounded. These may not be the same solution, e.g. $x'' + (1/4t^3)\, x^3 = 0$ has $x(t) = t^{1/2}$ as a solution where $n = 2$ and $\ell = 1$.

Proof. Since (4) has the form of (1) when $f(t, x) = p(t)\, x^{\gamma}$ then the integral in condition (I_ℓ) becomes

$$\int_a^{\infty} t^{n-\ell} f(t, ct^{\ell-1})\, dt = \int_a^{\infty} t^{n-\ell} p(t)\, [ct^{\ell-1}]^{\gamma}\, dt$$

$$= c^{\gamma} \int_a^{\infty} t^{n-\ell+\gamma(\ell-1)} p(t)\, dt.$$

Therefore, (I_ℓ) is equivalent to (5_ℓ) for (4).

Now if $\gamma > 1$, since $t^{n-\ell+\gamma(\ell-1)} = t^{n-\gamma+\ell(\gamma-1)} \geqslant t^{n-\gamma+j(\gamma-1)} = t^{n-j+\gamma(j-1)}$ for $j \leqslant \ell$, and in particular for $j = \ell, \ell - 2, \ell - 4, \ldots, \ell - 2k$, as long as $\ell - 2k$ is positive. This means that (I_ℓ) implies the integral conditions (I_j) also hold for these values of j. Therefore, part (i) follows from Theorem 2 for $\ell \geqslant 1$, and from the Corollary 1(ii) for $l = 0$ (n odd).

If $0 < \gamma < 1$, then $t^{n-\ell+\gamma(\ell-1)} \geqslant t^{n-j+\gamma(j-1)}$ provided $j \geqslant \ell$, and so in particular for $j = \ell, \ell + 2,..., n - 1$ we see that (I_ℓ) implies (I_j). Part (ii) of this theorem then follows from Theorem 2 and the Corollary 1(ii) for the cases $\ell \geqslant 1$, and $\ell = 0$, respectively. This proves the theorem.

THEOREM 5. *Consider that for some integer ℓ, $0 \leqslant \ell \leqslant n - 1$, where ℓ is odd $\{$even$\}$ when n is even $\{$odd$\}$,*

$$\int^{\infty} t^{n-\ell-1+\ell\gamma} p(t)\, dt < \infty. \tag{$6_{\ell+1}$}$$

(i) *If $\gamma > 1$, then $(6_{\ell+1})$ is necessary and sufficient for (4) to have positive solutions $x_j(t)$ having degree $= j$ and such that $x_j^{(j)}(t)$ is bounded below by a positive constant for all large t where j has each of the nonnegative values of the form $j = \ell, \ell - 2,..., \ell - 2k$.*

(ii) *If $0 < \gamma < 1$, then $(6_{\ell+1})$ is necessary and sufficient for (4) to have positive solutions $x_j(t)$ having degree $= j$ such that $x^{(j)}(t)$ is bounded below by a positive constant for all large t where $j = \ell, \ell + 2,..., n - 1$.*

Proof. (i) Arguing in the same way as in the proof of Theorem 4(i), we see that the integral conditions (I_{j+1}) will hold for each of the nonnegative values of $j = \ell - 2k$, $k = 0, 1,....$ Therefore part (i) follows from Theorem 3 and the Corollary 1(ii).

(ii) Again, arguing as in the proof of Theorem 4(ii), we see that (I_{j+1}) holds for $j = \ell, \ell + 2,..., n - 1$ and so part (ii) follows from Theorem 3 and the Corollary 1(ii). This proves the theorem.

Remark. A form of the sublinear result Theorem 5(ii) was given by Lovelady in [10] for even order equations. However, his result provides only for a solution of degree ℓ and does not indicate that such a solution has its ℓth derivative bounded below, that is, that the solution itself is bounded below by a polynomial of degree ℓ. In a more recent paper [11] Lovelady has treated the superlinear equation of even order deriving the results of Theorem 5(i) and the second portion of Theorem 6(i) that follows.

COROLLARY 4. (i) *If $\gamma > 1$, then (4) has positive solutions $x_j(t)$ of degree j for $j = \ell, \ell - 2,...$ and $j \geqslant 0$ and $y_k(t)$ of degree k for $k = \ell - 2, \ell - 4,..., k \geqslant 0$ such that $x_j^{(j-1)}(t)$ is bounded above by a constant and $y_k^{(k)}(t)$ is bounded below by a positive constant, if and only if, (5_ℓ) holds.*

(ii) *If $0 < \gamma < 1$, then (4) has positive solutions $x_j(t)$ of degree j, $j = \ell, \ell + 2,..., n - 1$ and $y_k(t)$ of degree k, $k = \ell, \ell + 2,..., n - 1$ such that $x^{(j-1)}$ is bounded above and $y_k^{(k)}$ is bounded below by positive constants, if and only if, (5_ℓ) holds.*

Proof. (i) If $\gamma < 1$ then by Theorem 4, (5_ℓ) is necessary and sufficient for the existence of positive solutions x_j, $j = \ell, \ell - 2, \ldots, j \geqslant 0$ such that $x_j^{(j-1)} \leqslant c_j$ for some constants $\{c_j\}$. On the other hand, Theorem 5 states that $(6_{\ell-1})$ is n.a.s. for the existence of solutions y_k, for $k = \ell - 2, \ell - 4, \ldots, k \geqslant 0$. Now (5_ℓ) implies $(6_{\ell-1})$ since for $k = \ell - 2m$, $t^{n-k-1+\gamma k} \leqslant t^{n-\ell+\gamma(\ell-1)}$ if (and only if) $2m(1-\gamma) - 1 \leqslant -\gamma$, hence if $2m \geqslant 1$, since $1 - \gamma < 0$. Therefore $m = 1$ or $k = \ell - 2$ suffices, and so (5_ℓ) implies (6_{k+1}) when $k = \ell - 2$. This proves part (i).

(ii) If $0 < \gamma < 1$, then (5_ℓ) is necessary and sufficient for the existence of the solutions x_j, $j = \ell, \ell + 2, \ldots, n - 1$ with $x_j^{(j-1)} \leqslant c_j$ for some constants $\{c_j\}$ by Theorem 4. Now condition (5_ℓ) implies that $(6_{\ell+1})$ holds since $\gamma < 1$ implies $n - \ell - 1 + \gamma\ell \leqslant n - \ell - \gamma + \gamma\ell = n - \ell + \gamma(\ell - 1)$. Therefore by Theorem 5, $(6_{\ell+1})$ is necessary and sufficient for the existence of solution y_k, $k = \ell, \ell + 2, \ldots, n - 1$, with $y_k^{(k)}(t) \geqslant d_k > 0$ for all large t. The corollary is proved.

COROLLARY 5. (i) *If $\gamma > 1$, then there exist positive solutions $x_j(t)$ and $y_k(t)$ of degrees j and k respectively of* (4) *for $j = \ell, \ell - 2, \ldots, j \geqslant 0$ and $k = \ell, \ell - 2, \ldots, k \geqslant 0$ such that $x_j^{(j-1)} \leqslant c_j$ and $y_k^{(k)} \geqslant d_k$ for some positive constants $\{c_j\}$ and $\{d_k\}$, if and only if, $(6_{\ell+1})$ holds.*

(ii) *If $0 < \gamma < 1$, then there exist positive solution x_j and y_k of degrees j and k respectively for $j = \ell + 2, \ell + 4, \ldots, n - 1$ and $k = \ell, \ell + 2, \ldots, n - 1$ such that $x_j^{(j-1)} \leqslant c_j$, $y_k^{(k)} \geqslant d_k$ for some positive constants $\{c_j\}$ and $\{d_k\}$ and all large t, if and only if, $(6_{\ell+1})$ holds.*

Proof. If $\gamma > 1$ then $n - \ell - 1 + \gamma\ell \geqslant n - \ell + \gamma(\ell - 1)$ and so $(6_{\ell+1})$ implies (5_ℓ). Part (i) follows from Theorems 4 and 5.

If $0 < \gamma < 1$ then $n - \ell - 1 + \gamma\ell \geqslant n - (\ell + 2m) + \gamma(\ell + 2m - 1)$ if $2m \geqslant 1$ and so $m = 1$ suffices implying that if $(6_{\ell+1})$ holds then (5_k) holds for $k = \ell + 2$. Thus part (ii) follows from Thzorems 4 and 5 also, proving the corollary.

THEOREM 6. (i) *If $\gamma > 1$, then* (4) *has a positive solution $x(t)$ of highest degree ℓ with $x^{(\ell-1)}(t)$ bounded above by a constant, if and only if,*

$$\int^\infty t^{n-\ell+\gamma(\ell-1)} p(t)\, dt < \infty \qquad \text{and} \qquad \int^\infty t^{n-\ell-1+\gamma\ell} p(t)\, dt = \infty,$$

while (4) *has a positive solution of highest degree ℓ such that $x^{(\ell)}$ is bounded below by a positive constant, if and only if,*

$$\int^\infty t^{n-\ell-1+\gamma\ell} p(t)\, dt < \infty \qquad \text{and} \qquad \int^\infty t^{n-\ell-2+\gamma(\ell+1)} p(t)\, dt = \infty.$$

(ii) *If* $0 < \gamma < 1$, *then* (4) *has a positive solution* $x(t)$ *with lowest degree* ℓ *with* $x^{(\ell-1)}$ *bounded above by a constant, if and only if,*

$$\int^{\infty} t^{n-\ell+\gamma(\ell-1)}p(t)\,dt < \infty \qquad \text{and} \qquad \int^{\infty} t^{n-\ell+1+\gamma(\ell-2)}p(t)\,dt = \infty,$$

while (4) *has a solution of lowest degree* ℓ *with* $x^{(\ell)}$ *bounded below by a positive constant, if and only if,*

$$\int^{\infty} t^{n-\ell-1+\gamma\ell}p(t)\,dt < \infty \qquad \text{and} \qquad \int^{\infty} t^{n-\ell+\gamma(\ell-1)}p(t)\,dt = \infty.$$

Proof. (i) There exists a positive solution of (4) of degree ℓ with $x^{(\ell-1)}(t) \leqslant c_\ell$ for all large t, if and only if, (5_ℓ) holds, and there is no solution of degree ℓ with $x^{(\ell)}$ bounded below by a positive constant if and only if $(6_{\ell+1})$ does not hold, that is, $\int^\infty t^{n-\ell-1+\gamma\ell}p\,dt = \infty$. Also there exists a solution $x(t)$ of (4) with $x^{(\ell)}(t) \geqslant d_\ell > 0$ iff $(6_{\ell+1})$ holds, but no solution of higher degree $(\ell + 2k)$ if and only if $(5_{\ell+2k})$ does not hold, in particular $(5_{\ell+2})$: $\int^\infty t^{n-(\ell+2)+\gamma((\ell+2)-1)}p\,dt = \infty$. This proves the two claims in part (i).

(ii) Equation (4) has a positive solution $x(t)$ of degree ℓ with $x^{(\ell-1)} \leqslant c_{\ell-1}$, if and only if, (5_ℓ) holds but there can be no solution of degree $< \ell$, if and only if, there is no solution of degree $(\ell - 2)$ that has $x^{(\ell-2)}$ bounded below and this is equivalent to $(6_{(\ell-2)+1}) \equiv (6_{\ell-1})$ which is $(6_{\ell+1})$ with ℓ replaced by $(\ell - 2)$. On the other hand, (4) has a solution of degree ℓ with $x^{(\ell)} \geqslant c_\ell > 0$, if and only if, $(6_{\ell+1})$ holds while there is no solution of degree ℓ with $x^{(\ell-1)}$ bounded above, if and only if, (5_ℓ) does not hold. This proves the theorem.

Remark. The above results for the superlinear and sublinear cases of (4) can be generalized to equation (1) which is considered to be superlinear if

for some $\epsilon > 1$,

$$x^{-\epsilon}f(t, x) \geqslant y^{-\epsilon}f(t, y) \qquad \text{for each } t \geqslant 0, \quad x \geqslant y$$

and is sublinear if

for some ϵ, $0 < \epsilon < 1$,

$$x^{-\epsilon}f(t, x) \leqslant y^{-\epsilon}f(t, y) \qquad \text{for } t \geqslant 0, \quad x \geqslant y.$$

The generalizations of Theorems 4 and 5 are:

THEOREM 7. (i) *If* (1) *is superlinear, then* (I_ℓ) *is a* n.a.s.c. *for* (1) *to have positive solutions* $x_j(t)$ *of degree* j *with* $x_j^{(j-1)} \leqslant c_j$ *for* $j = \ell, \ell - 2,\ldots,$ *and* $j > 0$ *and* $x_0(t)$ *a solution of degree* 0 *with* $x_0(t) \geqslant c_0 > 0$ *if* $j = 0$.

(ii) *If* (1) *is sublinear, then* (I_ℓ) *is a* n.a.s.c. *for* (1) *to have positive solutions*

$x_j(t)$ of degree j, $j = \ell, \ell + 2, \ldots, n - 1$, with $x_j^{(j-1)} \leqslant c_j$ and when $\ell = 0$, $x_0(t)$ is a solution of degree 0 *with $x_0(t) \geqslant c_0 > 0$.*

THEOREM 8. (i) *If* (1) *is superlinear, then* $(\mathrm{I}_{\ell+1})$ *is a* n.a.s.c. *for* (1) *to have positive solutions $x_j(t)$ of degree j with $x_j^{(j)}(t) \geqslant c_j > 0$ for $j = \ell, \ell - 2, \ldots, \ell - 2k$ and $j \geqslant 0$.*

(ii) *If* (1) *is sublinear, then* $(\mathrm{I}_{\ell+1})$ *is a* n.a.s.c. *for* (1) *to have positive solutions $x_j(t)$ of degree j with $x_j^{(j)}(t) \geqslant c_j > 0$ for $j = \ell, \ell + 2, \ldots, n - 1$.*

The proofs of these theorems are similar to the proofs of Theorems 4 and 5. To see this for the superlinear case, note that if $f(t, x)$ is superlinear, then for some $\epsilon > 1$, if $x \geqslant y > 0$,

$$f(t, x) \geqslant (x/y)^{\epsilon} f(t, y).$$

Thus to show that (I_ℓ) implies (I_j) for $j = \ell - 2k$ and $j \geqslant 0$, it suffices to show that $t^{n-\ell} f(t, ct^{\ell-1}) \geqslant t^{n-j} f(t, ct^{j-1})$. Now letting $x = ct^{\ell-1}$, $y = ct^{j-1}$ then $j = \ell - 2k$ implies $x \geqslant y$ and so

$$\begin{aligned} t^{n-\ell} f(t, ct^{\ell-1}) &\geqslant t^{n-\ell}[(ct^{\ell-1}/ct^{j-1})^{\epsilon} f(t, ct^{j-1})] \\ &= t^{n-\ell}[t^{(\ell-j)\epsilon} f(t, ct^{j-1})] \geqslant t^{n-\ell+2k} f(t, ct^{j-1}) \\ &= t^{n-j} f(t, ct^{j-1}) \end{aligned}$$

since $\epsilon > 1$ and $\ell - j = \ell - (\ell - 2k) = 2k$.

A similar argument for the sublinear case will prove that (I_ℓ) implies (I_j) for $j = \ell + 2k$, $j \leqslant n - 1$. Likewise it can be shown that $(\mathrm{I}_{\ell+1})$ implies (I_{j+1}) for the appropriate j values in the two different cases.

The two corollaries of Theorems 4 and 5 may now be stated for equation (1) instead of (4) when $f(t, x)$ is (i) superlinear or (ii) sublinear, and replacing (5_ℓ) and $(6_{\ell+1})$ by (I_ℓ) and $(\mathrm{I}_{\ell+1})$, respectively. Finally we would get the generalization of Theorem 6.

THEOREM 9. (i) *If* (1) *is superlinear then* (1) *has a positive solution $x(t)$ with highest degree ℓ and such that*

$$x^{(\ell-1)}(t) \leqslant c_\ell < \infty \Leftrightarrow (\mathrm{I}_\ell) \text{ holds and } (\mathrm{I}_{\ell+1}) \text{ does not;}$$

and

$$x^{(\ell)}(t) \geqslant c_{\ell+1} > 0 \quad \Leftrightarrow (\mathrm{I}_{\ell+1}) \text{ holds but } (\mathrm{I}_{(\ell+2)}) \text{ does not.}$$

(ii) *If $f(t, x)$ is sublinear, then* (1) *has a positive solution $x(t)$ with lowest degree ℓ and such that*

$$x^{(\ell-1)} \leqslant c_\ell < \infty \Leftrightarrow (\mathrm{I}_\ell) \text{ holds but } (\mathrm{I}_{(\ell-2)+1}) \text{ does not;}$$

and

$$x^{(\ell)} \geqslant c_{\ell+1} > 0 \quad \Leftrightarrow (\mathrm{I}_{\ell+1}) \textit{ holds but } (\mathrm{I}_\ell) \textit{ does not.}$$

Remark. Throughout we have considered only existence of positive solutions and stated results for such solutions. However, all statements are valid for negative solutions also due to the hypothesis that $f(t, x)$ has the same sign as x. Thus, whenever a n.a.s. condition is given for the existence of a positive solution, it is also n.a.s. for the existence of a negative solution having the analogous properties.

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