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Product integration methods for second-kind Abel integral equations

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Abstract: The construction and convergence of high-order product integration methods for the second-kind Abel equation are discussed and the results of De Hoog and Weiss are generalised. Backward difference methods are introduced, and numerical results are presented which verify the theoretical rates of convergence.

Keywords: Product integration, second-kind Abel equation, convergence.

1. Introduction

This paper will be concerned with the construction and analysis of convergence of high-order product integration methods for the second-kind Abel equation

$$y(t)+\int_0^t p(t,s)k(t,s,y(s))\,\mathrm{d}s=f(t), \tag{1.1}$$

where $y(t)$ is the unknown function whose value is to be determined in the interval $0 \leqslant t \leqslant T < \infty$ and the kernel $k(t, s, y(s))$ is Lipschitz continuous in its third variable. The methods developed are suitable for the case when $p(t, s)$ is unbounded in the range of integration but integrable over $[0, T]$. Such equations arise in a number of practical applications, e.g. Levinson [11], Mann and Wolf [16], Roberts and Mann [19] and Handelsman and Olmstead [8]. These authors consider equations arising from the theory of superfluidity and heat transfer between solids and gases. Some typical forms of $p(t, s)$ are

$$p(t,s)=(t-s)^{-\alpha} \quad \text{and} \quad p(t,s)=(t^2-s^2)^{-1/2}, \quad 0 \leqslant s \leqslant t \leqslant T.$$

This paper will be concerned with the first of these although much of this paper is easily extendable (see e.g. [5]).

There have only been a few papers dealing with numerical methods, see e.g. those of Wagner [21] and Levinson [11]. None of these early papers provided a theoretical justification of the algorithms used. Linz [12] discussed the convergence of the product integration Simpson's rule and showed that third-order convergence was possible. Later De Hoog and Weiss [6] by sharper analysis were able to show that for $p(t, s) = (t-s)^{-1/2}$ convergence of order $3\frac{1}{2}$ was actually attained. It is the main purpose of this paper to extend the results of De Hoog and Weiss to methods using three-, four- and five-point formulae for solving equation (1.1) with $p(t, s) = (t-s)^{-\alpha}$, $0 \leqslant \alpha < 1$. Indeed it will be shown that the three- and five-point methods for $p(t, s) = (t-s)^{-\alpha}$ are convergent of order $4-\alpha$ and $6-\alpha$, respectively, while the four-point method converges of order 4 only. Finally backward difference methods are also introduced.

It is important to note that the results of this paper are only valid if $y(t)$ has sufficient continuity. Clearly, if $y(t)$ has a discontinuous first derivative at $t = 0$, as does frequently occur in practice, high-accuracy convergence is lost, and it would be necessary to develop special starting formulae. De Hoog and Weiss [6] discuss this question but most recently Brunner [3], extending results of te Riele [20], has shown definitely how to get over this problem for recursive collocation methods. Other authors active in this field include Brunner and Norsett [4], Garey [7] and Logan [13]. Other relevant work which appeared after submission of this paper include Abdalkhani [1], Kershaw [9] and Lubich [14].

2. Product integration methods

On the interval $[0, T]$ we define the grid $\{t_j = jh, j = 0, 1, \ldots, N;\ Nh = T\}$ and consider the discretised form of (1.1) with $p(t, s) = (t-s)^{-\alpha}$, i.e.

$$y(t_i) + \int_0^{t_i} \frac{k(t_i, s, y(s))}{(t_i - s)^{\alpha}} \mathrm{d}s = f(t_i), \quad i = 1, 2, \ldots. \tag{2.1}$$

The kernel $k(t_i, s, y(s))$ is approximated by some $(n+1)$-point formula over $[t_j, t_{j+n}]$ and the coefficients of the quadrature expression are calculated analytically. The resulting quadrature scheme will generally consist of a main repeated rule alternatively ended by itself or a series of end rules. Thus

$$k(t_i, s, y(s)) \simeq \sum_{k=0}^{n} l_k(s) k(t_i, t_{k+j}, y(t_{k+j})), \tag{2.2}$$

where the $l_k(s)$ are the Lagrangian polynomials of degree n. Hence

$$\int_{t_j}^{t_{j+n}} \frac{k(t_i, s, y(s))}{(t_i - s)^{\alpha}} \mathrm{d}s \simeq \sum_{k=0}^{n} k(t_i, t_{k+j}, y(t_{k+j})) \int_{t_j}^{t_{j+n}} \frac{l_k(s)}{(t_i - s)^{\alpha}} \mathrm{d}s.$$

Note that the integrals (which constitute the quadrature weights) can be written, using the transformation $s = t_j + ph$, as

$$h^{1-\alpha} \int_0^n \frac{\rho(p)\, \mathrm{d}p}{(i-j-p)^{\alpha}} \tag{2.3}$$

where $\rho(p)$ is a polynomial of degree n, i.e. $h^{1-\alpha}$ times a term independent of h. Thus the discretisation method can be written as

$$\Phi_h y = 0,$$

where

$$[\Phi_h y]_i = \begin{cases} y_i - \tilde{y}_i, & i = 0, 1, \ldots, r-1, \\ y_i + h^{1-\alpha} \sum_{j=0}^{i} w_{ij} k(t_i, t_j, y_j) - f_i, & i = r, r+1, \ldots, N, \end{cases} \tag{2.4}$$

and y_i, $i = 0, 1, \ldots, N$ are approximate values of $y(t_i)$ and $\tilde{y}_i$, $i = 0, 1, \ldots, r-1$ are given starting values. If $w_{ii} \neq 0$, the scheme is implicit and requires the solution of a nonlinear equation at each step. When the kernel is linear, (2.4) can be written in matrix notation as

$$\Phi_h y = (I + h^{1-\alpha} A_N) y - g$$

where

$$y = (y_0, y_1, \ldots, y_N)^{\mathrm{T}} \quad \text{and} \quad g = (\tilde{y}_0, \ldots, \tilde{y}_{r-1}, f_r, f_{r+1}, \ldots, f_N)^{\mathrm{T}}.$$

Example 2.1. When $n = 2$ a three-point formula is used as the main rule with a four-point formula as an end rule. Following Atkinson [2] these are termed the product Simpson's rule and the product three-eighths rule. This case was considered by Linz [12] and De Hoog and Weiss [6]. The product Simpson's rule is used over $[0, t_i]$ when i is even, and when i is odd it will be used over $[0, t_{i-3}]$ and the product three-eighths rule will be used over $[t_{i-3}, t_i]$. Hence when i is even

$$\int_{t_j}^{t_{j+2}} \frac{\phi(s)\,\mathrm{d}s}{(t_i - s)^\alpha} = h^{1-\alpha}\{\phi(t_j) b_0(i-j) + \phi(t_{j+1}) b_1(i-j) + \phi(t_{j+2}) b_2(i-j)\},$$

where

$$\phi(s) = k(t_i, s, y(s))$$

and

$$h^{1-\alpha} b_0(i-j) = \int_{t_j}^{t_{j+2}} \frac{l_0(s)\,\mathrm{d}s}{(t_i - s)^\alpha},$$

giving

$$b_0(i-j) = \tfrac{1}{2} \int_0^2 \frac{(p-1)(p-2)}{(i-j-p)^\alpha}\,\mathrm{d}p,$$

$$h^{1-\alpha} b_1(i-j) = \int_{t_j}^{t_{j+2}} \frac{l_1(s)\,\mathrm{d}s}{(t_i - s)^\alpha},$$

giving

$$b_1(i-j) = -\int_0^2 \frac{p(p-2)}{(i-j-p)^\alpha}\,\mathrm{d}p,$$

$$h^{1-\alpha} b_2(i-j) = \int_{t_j}^{t_{j+2}} \frac{l_2(s)\,\mathrm{d}s}{(t_i - s)^\alpha},$$

giving

$$b_2(i-j)=\tfrac{1}{2}\int_0^2 \frac{p(p-1)}{(i-j-p)^\alpha}\,\mathrm{d}p.$$

When i is odd the product Simpson's rule is used over $[0, t_{i-3}]$ and

$$\int_{t_{i-3}}^{t_i} \frac{\phi(s)\,\mathrm{d}s}{(t_i-s)^\alpha} \approx \sum_{j=0}^{3} \phi(t_{i+j-3}) \int_{t_{i-3}}^{t_i} \frac{\tilde{l}_j(s)\,\mathrm{d}s}{(t_i-s)^\alpha}$$

is used over $[t_{i-3}, t_i]$ where $\tilde{l}_j(s)$ are the appropriate interpolation polynomials of degree three. Rewriting we have:

$$\int_{t_{i-3}}^{t_i} \frac{\phi(s)\,\mathrm{d}s}{(t_i-s)^\alpha} = \sum_{j=0}^{3} \phi(t_{i+j-3})\,d_j(3)$$

where

$$d_0(3)=-\tfrac{1}{6}\int_0^3 \frac{(p-1)(p-2)(p-3)}{(3-p)^\alpha}\,\mathrm{d}p, \quad d_1(3)=\tfrac{1}{2}\int_0^3 \frac{p(p-2)(p-3)}{(3-p)^\alpha}\,\mathrm{d}p,$$

$$d_2(3)=-\tfrac{1}{2}\int_0^3 \frac{p(p-1)(p-3)}{(3-p)^\alpha}\,\mathrm{d}p, \qquad d_3(3)=\tfrac{1}{6}\int_0^3 \frac{p(p-1)(p-2)}{(3-p)^\alpha}\,\mathrm{d}p.$$

With $k(t, s, y(s)) = y(s)$, for clarity, the matrix A_N will have the form:

$$\begin{bmatrix}
0 & & & & & & \mathbf{0} \\
0 & 0 & & & & & \\
b_0(2) & b_1(2) & b_2(2) & & & & \\
d_0(3) & d_1(3) & d_2(3) & d_3(3) & & & \\
b_0(4) & b_1(4) & b_2(4)+b_0(2) & b_1(2) & b_2(2) & & \\
b_0(5) & b_1(5) & b_2(5)+d_0(3) & d_1(3) & d_2(3) & d_3(3) & \\
b_0(6) & b_1(6) & b_2(6)+b_0(4) & b_1(4) & b_2(4)+b_0(2) & b_1(2) & b_2(2) \\
\vdots & & & & & & \ddots
\end{bmatrix}$$

3. Consistency

When i is a multiple of n, say $i = nk$, then using the error term in Lagrangian interpolation for polynomials of degree n gives the total error over $[0, t_i]$ as a constant times

$$\sum_{j=0}^{k-1} \int_{t_{nj}}^{t_{n(j+1)}} \frac{\omega(s)\phi^{(n+1)}(t_{nj}+n\zeta_j h)\,\mathrm{d}s}{(t_{nk}-s)^\alpha}, \quad \zeta_j \in (0, 1),$$

where $\omega(s)=\prod_{l=0}^{n}(s-t_{nj+l})$. This can be written as a constant times

$$h^{n+1}\sum_{j=0}^{k-1} nh\int_0^1 \omega^*(np)\left\{\frac{\phi^{(n+1)}(t_{nj}+nph)+\mathrm{O}(h)}{(t_{nk}-t_{nj}-nph)^{\alpha}}\right\}\mathrm{d}p$$

on putting $s=t_{nj}+nph$. This is a constant times

$$h^{n+1}\int_0^1 \omega^*(np)nh\sum_{j=0}^{k-1}\frac{\phi^{(n+1)}(t_{nj}+nph)\,\mathrm{d}p}{(t_{nk}-t_{nj}-nph)^{\alpha}}+\mathrm{O}(h^{n+2}) \tag{3.1}$$

where $\omega^*(p)=\sum_{l=0}^{n}(p-l)$.

When i is not a multiple of n, say $i=nk+\nu$, then end rules of degree $(n+1),(n+2),\ldots,(2n-1)$ will be used over the last few intervals. The error using the end rule will be

$$E_{i,n+\nu}=\int_{t_{i-n-\nu}}^{t_i}\frac{\tilde{\omega}(s)\phi^{(n+\nu+1)}(\theta_j)\,\mathrm{d}s}{(n+\nu+1)!(t_i-s)^{\alpha}},\quad \theta_j\in(t_{i-n-\nu},t_i),\quad \nu=1,2,\ldots,n-1,$$

where $\tilde{\omega}(s)=\prod_{l=0}^{n+\nu}(s-t_{i-n-\nu+l})$. Hence

$$E_{i,n+\nu}=h^{n+\nu+2-\alpha}\int_0^{n+\nu}\frac{\tilde{\omega}(p)\phi^{(n+\nu+1)}(\theta_j)\,\mathrm{d}p}{(n+\nu+1)!(n+\nu-p)^{\alpha}}=\mathrm{O}(h^{n+\nu+2-\alpha})$$

on putting $s=t_{i-n-\nu}+ph$ where $\tilde{\omega}(p)=\prod_{l=0}^{n+\nu}(p-l)$. This clearly will not affect the global error of the method e.g. when $n=2$ and $\nu=1$ the product three-eighths rule gives an error of $\mathrm{O}(h^{5-\alpha})$. Hence when i is not a multiple of n the error will be determined by (3.1) with nk replaced by $nk+\nu$.

To determine the correct order of consistency requires the following lemma.

Lemma 3.1. *If $F(x)$ is continuously differentiable and $0<p<1$, then*

$$nh\sum_{j=0}^{k-1}\frac{F(t_{nj}+nph)}{(t_{i+u}-t_{nj}-nph)^{\alpha}}=\int_0^{t_i}\frac{F(s)\,\mathrm{d}s}{(t_i-s)^{\alpha}}+\mathrm{O}(h^{1-\alpha}F(t_i))+\mathrm{O}\left(\frac{h}{t_i^{\alpha}}\right),$$

where $i=nk$, and $u=0,1,2,\ldots,n-1$.

This Lemma is an extension of that used by De Hoog and Weiss [6] and results from the work done by Lyness and Ninham [15] on Euler–Maclaurin summation formulae for functions with algebraic singularities.

When $i=nk$, then we can use the lemma with $u=0$ in (3.1) to give a constant times

$$h^{n+1}\int_0^1 \omega^*(np)\left\{\int_0^{t_i}\frac{\phi^{(n+1)}(s)\,\mathrm{d}s}{(t_i-s)^{\alpha}}+\mathrm{O}(h^{1-\alpha})\right\}\mathrm{d}p+\mathrm{O}(h^{n+2})$$

so that the global error will be $\mathrm{O}(h^{n+2-\alpha})$ whenever $\int_0^1\omega^*(np)\,\mathrm{d}p=0$. This will be true when $n=2,4,6,\ldots$. When $i=nk+\nu$, we use the lemma again with $u=1,2,\ldots,n-1$ and again the error will be $\mathrm{O}(h^{n+2-\alpha})$ whenever n is even. Hence the three- and five-point formulae are consistent of $\mathrm{O}(h^{4-\alpha})$ and $\mathrm{O}(h^{6-\alpha})$ while the four-point formula is consistent of $\mathrm{O}(h^4)$ only.

The definition of consistency can be stated formally as follows.

Definition 3.2. A method will be said to be consistent of order p at a point $x(t) \in C^{2p-2}[0, T]$ if for $h \in (0, h_0]$, $h_0 > 0$ and $i \geqslant r$, there exists a constant C_1, independent of h, such that

$$\left| h^{1-\alpha} \sum_{j=0}^{i} w_{ij} k(t_i, t_j, x(t_j)) - \int_0^{t_i} \frac{k(t_i, s, x(s))\, \mathrm{d}s}{(t_i - s)^\alpha} \right| \leqslant C_1 h^p.$$

Convergence of the starting values is also required.

Definition 3.3 The starting values are said to be convergent of order p if for $i = 0, 1, \ldots, r-1$ there exists a constant C_2, independent of h, such that

$$|y(t_i) - \tilde{y}_i| \leqslant C_2 h^p.$$

4. Convergence

Firstly it is necessary to obtain a bound, independent of h, on the quadrature weights. From (2.3) the weights will be sums of integrals of the form

$$I_1 = \int_0^n \frac{\rho(p)\, \mathrm{d}p}{(q-p)^\alpha}$$

where $\rho(p)$ is a polynomial. Hence

$$|I_1| \leqslant \max_{p \in [0, n]} |\rho(p)| \int_0^n (q-p)^{-\alpha}\, \mathrm{d}p$$

and so

$$|I_1| \leqslant M/q^\alpha \tag{4.1}$$

for some M independent of h. First we require the generalised Gronwall lemma:

Lemma 4.1. [17]. *If x_j, $j = 0, 1, \ldots, N$ is a sequence of real numbers with $|x_0| < \delta$ and*

$$|x_i| \leqslant h^{1-\alpha} M \sum_{j=0}^{i-1} \frac{|x_j|}{(i-j)^\alpha} + \delta, \quad i = 1, 2, \ldots, N,$$

where $M > 0$, $\delta > 0$ and $0 \leqslant \alpha < 1$, then

$$\|x\|_\infty \leqslant (\delta' + hM'T^{n-1-n\alpha}\delta) \exp(M'T^{n-n\alpha}),$$

where

$$M' = \begin{cases} M, & \text{if } n = 1, \\ M^n \prod_{k=1}^{n-1} B(k(1-\alpha), 1-\alpha), & \text{if } n \geqslant 2, \end{cases}$$

and

$$\delta' = \delta \left\{ (1 + h^{1-\alpha} M) \sum_{j=0}^{n-2} \gamma^j + \gamma^{n-1} \right\}, \quad n \geqslant 1,$$

where $\gamma = MT^{1-\alpha}/(1-\alpha)$ and where n is the smallest positive integer such that $n\alpha \leqslant n-1$.

Theorem 4.2. *If the discretisation is consistent of order p and the starting values are convergent of order p, then it is convergent of order p.*

Proof. Let $e_i = y(t_i) - y_i$ so that

$$e_i = h^{1-\alpha} \sum_{j=0}^{i} w_{ij}\{k(t_i, t_j, y_i) - k(t_i, t_j, y(t_j))\}$$
$$- \int_0^{t_i} \frac{k(t_i, s, y(s))\,\mathrm{d}s}{(t_i - s)^\alpha} + h^{1-\alpha} \sum_{j=0}^{i} w_{ij} k(t_i, t_j, y(t_j)),$$
$$i = r, r+1, \ldots.$$

and so

$$|e_i| \leqslant h^{1-\alpha} \sum_{j=0}^{i} |w_{ij}||k(t_i, t_j, y(t_j)) - k(t_i, t_j, y_j)| + C_1 h^p,$$
$$i = r, r+1, \ldots.$$

For the starting values

$$|e_i| \leqslant C_2 h^p, \quad i = 0, 1, \ldots, r-1.$$

Hence

$$|e_i| \leqslant h^{1-\alpha} L \sum_{j=0}^{i} |w_{ij}||e_j| + Ch^p, \quad i = 1, 2, \ldots, N,$$

where $C = \max\{C_1, C_2\}$ and L is the Lipschitz constant with respect to the third variable of the kernel. Further from (4.1) we observe that there exists some $\bar{w} > 0$, independent of i and j, such that

$$|w_{ij}| \leqslant \frac{\bar{w}}{(i-j)^\alpha}$$

and so

$$|e_i| \leqslant h^{1-\alpha} \bar{w} L \sum_{j=0}^{i} \frac{|e_j|}{(i-j)^\alpha} + Ch^p, \quad i = 1, 2, \ldots, N.$$

Thus for h sufficiently small Lemma 4.1 can be applied and the result is immediate. □

5. Backward difference product integration methods

Backward difference methods for equation (2.1) involve the approximation of the kernel $k(t_i, s, y(s))$ by some $(n+1)$-point formula over $[t_j, t_{j+1}]$ using back-values of t, i.e. the points $t_{j+1}, t_j, \ldots, t_{j+1-n}$ for a method of order $n+1$.

For the first few intervals a forward interpolation polynomial of degree n is used and n initial values are required. The second-order method is simply the product trapezoidal rule. Thus for

$i \geqslant n$, equation (2.1) becomes

$$y(t_i) + h^{1-\alpha} \sum_{j=0}^{n-2} \int_0^1 \frac{R_n^{(j)}(p)\,\mathrm{d}p}{(i-j-p)^\alpha} + \sum_{j=n-1}^{i-1} \int_{-1}^0 \frac{P_n^{(j)}(p)\,\mathrm{d}p}{(i-j-1-p)^\alpha} = f(t_i)$$

$$i = n, n+1, \ldots, N, \tag{5.1}$$

on making the substitution $s = t_{j+\nu} + ph$, $\nu = 0$ and 1 respectively. The term $R_n^{(j)}(p)$ denotes the forward interpolation formula of degree n over $[t_0, t_n]$ and $P_n^{(j)}(p)$ the Newton–Gregory interpolating polynomial of degree n given by

$$P_n^{(j)}(p) = \phi_{j+1} + p\nabla\phi_{j+1} + \tfrac{1}{2}p(p+1)\nabla^2\phi_{j+1} + \cdots + \frac{1}{n!}\prod_{k=0}^{n-1}(p+k)\nabla^n\phi_{j+1}$$

with $\phi_{j+1} = \phi(t_{j+1}) = k(t_i, t_{j+1}, y(t_{j+1}))$.

The coefficients of the quadrature expression are calculated analytically.

Example 5.1. The third-order method, with $n = 2$, requires two starting values y_0 and y_1. Hence for $i = 2, 3, \ldots$ we have

$$y(t_i) + h^{1-\alpha}\Bigg\{\beta_0(i)\phi_0 + \beta_1(i)\phi_1 + \beta_2(i)\phi_2 + \sum_{j=1}^{i-1}\left(\alpha_0(i-j-1)\phi_{j-1} + \alpha_1(i-j-1)\phi_j + \alpha_2(i-j-1)\phi_{j+1}\right)\Bigg\} = f(t_i),$$

where the coefficients of the forward rule are

$$\beta_0(i) = \tfrac{1}{2}\int_0^1 \frac{(p-1)(p-2)\,\mathrm{d}p}{(i-p)^\alpha}, \qquad \beta_1(i) = -\int_0^1 \frac{p(p-2)\,\mathrm{d}p}{(i-p)^\alpha},$$

$$\beta_2(i) = \tfrac{1}{2}\int_0^1 \frac{p(p-1)\,\mathrm{d}p}{(i-p)^\alpha},$$

and of the backward rule are

$$\alpha_0(\nu) = \tfrac{1}{2}\int_{-1}^0 \frac{p(p+1)\,\mathrm{d}p}{(\nu-p)^\alpha}, \qquad \alpha_1(\nu) = -\int_{-1}^0 \frac{p(p+2)\,\mathrm{d}p}{(\nu-p)^\alpha}$$

and

$$\alpha_2(\nu) = \tfrac{1}{2}\int_{-1}^0 \frac{(p+1)(p+2)\,\mathrm{d}p}{(\nu-p)^\alpha}.$$

In general the discretisation is described by (2.4) and when the kernel is linear by (2.5).

The convergence arguments of the previous section are equally applicable to those methods, but, in this case, the convergence will be of order h^n and not $h^{n+\alpha}$. The extra h^α convergence arose, essentially, because of the ability to choose an odd number of equally spaced points over the integration interval thus making

$$\int_0^1 \omega^*(np)\,\mathrm{d}p = 0.$$

This is not the case for these backward difference methods.

Table 1
Order three

h	t				
	0.2	0.4	0.6	0.8	1.0
0.01	1.08×10^{-9}	6.07×10^{-10}	3.56×10^{-10}	2.16×10^{-10}	1.34×10^{-10}
0.005	9.64×10^{-11}	5.35×10^{-11}	3.12×10^{-11}	1.89×10^{-11}	1.16×10^{-11}
Ratio	1:11.2	1:11.3	1:11.4	1:11.4	1:11.5

Order $3\frac{1}{2}$ would give a ratio of 1:11.3.

Table 2
Order four

h	t				
	0.2	0.4	0.6	0.8	1.0
0.01	1.26×10^{-10}	2.11×10^{-10}	4.31×10^{-10}	5.03×10^{-10}	6.12×10^{-10}
0.005	7.76×10^{-12}	1.60×10^{-11}	2.62×10^{-11}	3.08×10^{-11}	4.37×10^{-11}
Ratio	1:16	1:13	1:16	1:16	1:14

Convergence of order $4\frac{1}{2}$ would give a ratio of 1:23.

Table 3
Order five

h	t				
	0.2	0.4	0.6	0.8	1.0
0.01	7.1×10^{-13}	3.0×10^{-13}	1.3×10^{-13}	6.2×10^{-14}	2.8×10^{-14}
0.005	1.5×10^{-14}	6.4×10^{-15}	2.8×10^{-15}	1.3×10^{-15}	5.6×10^{-16}
Ratio	1:46	1:47	1:47	1:48	1:49

Convergence of order $5\frac{1}{2}$ would give a ratio of 1:45.

6. Numerical results

The first example is that used by Linz [12] and also by Phillips [18] for their third-order method. It is solved here with methods of orders 3 (Table 1), 4 (Table 2) and 5 (Table 3) and ratios are computed to illustrate the orders of convergence.

Example 6.1.

$$y(t)+\tfrac{1}{4}\int_0^t (t-s)^{-1/2}y(s)\,\mathrm{d}s=(1+t)^{-1/2}+\tfrac{1}{8}\pi-\tfrac{1}{4}\sin^{-1}\left(\frac{1-t}{1+t}\right)$$

with true solution $y(t)=(1+t)^{-1/2}$.

7. Concluding remarks

This paper has been concerned with a generalisation of some results of De Hoog and Weiss [6]. It was shown that interpolatory product integration methods based on odd-order interpolation, say n, converged like $h^{n+1-\alpha}$ while those based on even-order interpolation, say n, converged like h^n. Backward difference methods were introduced briefly.

Since these methods with odd order do not have a $1-\alpha$ increase in their convergence rate they may seem to have little to recommend them, and this is generally true except if a high-accuracy method (order 5 and above) were required. It was found that multiple precision was necessary for the interpolatory methods of order greater than three whereas ordinary precision was sufficient for the high-accuracy backward difference methods.

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