Vertex-arboricity of planar graphs without intersecting triangles

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ABSTRACT

The vertex-arboricity \( a(G) \) of a graph \( G \) is the minimum number of subsets into which vertex set \( V(G) \) can be partitioned so that each subset induces an acyclic graph. In this paper, we prove one of the conjectures proposed by Raspaud and Wang (2008) [15] which says that \( a(G) = 2 \) for any planar graph without intersecting triangles.

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1. Introduction

We consider only simple graphs in this paper unless otherwise stated. A plane graph is a particular drawing of a planar graph on the Euclidean plane. For a plane graph \( G \), we use \( V(G) \), \( E(G) \), \( F(G) \), \( |G| \), \( \Delta(G) \), and \( \delta(G) \) to denote its vertex set, edge set, face set, order, maximum degree, and minimum degree, respectively. A triangle is synonymous with a 3-cycle. We say that two cycles (or faces) are adjacent if they share at least one common (boundary) edge. Two cycles (or faces) are intersecting if they share at least one common (boundary) vertex. The distance, denoted by \( \text{dist}(x, y) \), between two vertices \( x \) and \( y \) is the length of a shortest path connecting them in \( G \). The distance between two triangles \( T \) and \( T' \) is defined as the value \( \text{min}\{\text{dist}(x, y) | x \in V(T) \text{ and } y \in V(T')\} \).

The vertex-arboricity \( a(G) \) of a graph \( G \) is the minimum number of subsets into which vertex set \( V(G) \) can be partitioned so that each subset induces an acyclic graph; such a partition is called an acyclic partition of \( V(G) \). Clearly, \( a(G) \geq 1 \) for every nonempty graph \( G \) and \( a(G) = 1 \) if and only if \( G \) itself is acyclic. There is an equivalent definition to the vertex-arboricity in terms of the coloring version. An acyclic \( k \)-coloring of a graph \( G \) is a mapping \( \pi \) from \( V(G) \) to the set \( \{1, \ldots, k\} \) such that each color class induces an acyclic subgraph, i.e., a forest. The vertex-arboricity \( a(G) \) of \( G \) is the smallest integer \( k \) such that \( G \) has an acyclic \( k \)-coloring.

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This vertex version of arboricity was first introduced by Chartrand, Kronk et al. [5] in 1968, who named it point-arboricity. They proved that $a(G) \leq \left\lfloor \frac{1+\Delta(G)}{2} \right\rfloor$ for any graph $G$ and $a(G) \leq 3$ for any planar graph. Chartrand and Kronk [6] showed this bound is sharp, by giving a planar graph which has vertex-arboricity 3. In fact, this graph was discovered by Professor Tutte, which was used to disprove the conjecture of Tait that the graph of every cubic convex polyhedron is Hamiltonian (see [18]).

The upper bound 3 for $a(G)$ on planar graphs has also been studied by Chartrand and Kronk [6], Goddard [10], Grünbaum [11] and Poh [14]. Among them, Goddard [10] and Poh [14], independently, proved a stronger result that the vertex set of any planar graph can be partitioned into three sets such that each set induces a linear forest. The path version of vertex-arboricity, called linear vertex-arboricity, has also been studied extensively in [14,1,2,13].

It was known [9] that determining the vertex-arboricity of a graph is NP-hard. Hakimi and Schmeichel [12] showed that determining whether $a(G) \leq 2$ is NP-complete for maximal planar graphs Stein [17] characterizes completely maximal planar graph $G$ with at least 4 vertices by proving that $a(G) = 2$ if and only if its dual graph $G^*$ is Hamiltonian. This result was further strengthened by Hakimi and Schmeichel [12] by showing that a plane graph $G$ has $a(G) = 2$ if and only if its dual graph $G^*$ contains a connected Eulerian spanning subgraph. The reader is referred to [3,4,7,8,16,19] for other results about the vertex-arboricity of graphs.

Recently, Raspaud and Wang [15] gave some sufficient conditions on a planar graph to have vertex-arboricity 2. More precisely, they proved the following theorem.

**Theorem 1.** Let $G$ be a planar graph.

1. If $G$ contains no $k$-cycles for some fixed $k \in \{3, 4, 5, 6\}$, then $a(G) \leq 2$.
2. If $G$ contains no triangles at distance less than 2, then $a(G) \leq 2$.

Our main purpose in this paper is to give a positive answer to the conjecture proposed by Raspaud and Wang in [15]. More precisely, we prove the following

**Theorem 2.** Every planar graph $G$ without intersecting triangles has vertex-arboricity at most 2.

**Some notation:** The degree of a face is the length of its boundary walk. We will write $d(x)$ for $d_G(x)$ the degree of the vertex $x$ in $G$ when no confusion can arise. A $k$-vertex, $k^+$-vertex, or $k^-$-vertex is a vertex of degree $k$, at least $k$, or at most $k$. Similarly, we can define $k$-face, $k^+$-face, $k^-$-face, etc. Suppose that $f$ and $f'$ are two adjacent faces by sharing a common edge $e$. We say that $f$ and $f'$ are normally adjacent if $|V(f) \cap V(f')| = 2$. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of $f$ and write $f = \{u_1u_2\cdots u_n\}$ if $u_1$, $u_2$, ..., $u_n$ are the vertices of $b(f)$ appearing in a boundary walk of $f$. Sometimes, we write simply $V(f) = V(b(f))$. An $m$-face $f = \{v_1v_2\cdots v_m\}$ is called an $(a_1, a_2, \ldots, a_m)$-face if the degree of the vertex $v_i$ is $a_i$ for $i = 1, 2, \ldots, m$. For $x \in V(G) \cup F(G)$ and integer $i \geq 1$, let $m_i(x)$ denote the number of $i$-faces incident or adjacent to $x$. Let $N(v)$ denote the set of neighbors of a vertex $v$. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$. In particular, we write $G - S = G[V(G) \setminus S]$.

For all figures in this paper, a vertex is represented by a solid point when all of its incident edges are drawn; otherwise it is represented by a hollow point.

2. **Proof of Theorem 2**

Suppose to the contrary that the theorem is not true. Let $G$ be a counterexample with the least number of vertices. Thus, $G$ is connected. Since $G$ contains no intersecting triangles, every subgraph of $G$ also contains no intersecting triangles. This straightforward fact is tacitly used in the following proofs. In the following, let $C = \{a, b\}$ denote the color set. We first investigate the structural properties of $G$, then use Euler’s formula and the technique to derive a contradiction.

**Claim 1.** The minimum degree $\delta(G) \geq 4$.

**Proof.** Assume to the contrary that $G$ contains a $3^-$-vertex $v$. By the minimality of $G$, $G - \{v\}$ is acyclically 2-colorable. It is easy to show that any acyclic 2-coloring of $G - \{v\}$ can be extended to an acyclic 2-coloring of $G$. This completes the proof of **Claim 1.** $\square$
Lemma 1. Let \( f = [v_1 v_2 \cdots v_5] \) be a light 4-face and \( H = G - V(f) \). If an acyclic 2-coloring \( \pi \) of \( G - V(f) \) cannot be extended to \( G \), then the following conditions hold.

(1) All vertices in \( \bigcup_{i=1}^{j=4} N_{G}(v_i) \) must be assigned with the same color, say \( a \), see Fig. 1.

(2) \( f \) is adjacent to at least one 5-face.

Proof. For \( i \in \{1, 2, 3, 4\} \), let \( x_i, y_i \) be the other two neighbors of \( v_i \) not on \( f \). Suppose \( \pi \) is an acyclic 2-coloring of \( G - V(f) \) which cannot be extended to \( G \). Let \( f_i \) be the face adjacent to \( f \) by the common edge \( v_i v_{i+1} \), where \( i \) is taken modulo 4. Let \( S(a) \) denote the subset of \( \{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_4, y_4\}\} \) which satisfies that all vertices in \( S(a) \) get the same color \( a \) in the coloring \( \pi \). Thus \( 0 \leq |S(a)| \leq 4 \). We will make contradiction to show (1),(2).

(1) Suppose to the contrary that \( |S(a)| \neq 4 \). It implies that \( 0 \leq |S(a)| \leq 3 \). It is easy to see that \( v_1 v_3 \notin E(G) \) and \( v_2 v_4 \notin E(G) \) since \( G \) contains no adjacent triangles. We have to consider the following four cases, depending on the value of \( |S(a)| \).

- \( |S(a)| = 3 \). Without loss of generality, assume that \( \pi(x_i) = \pi(y_i) = a \) for all \( i = 1, 2, 3 \) and one of \( x_4 \) and \( y_4 \) is colored with \( b \). We can color \( v_1, v_2, v_3 \) with \( b \) and \( v_4 \) with \( a \).

- \( |S(a)| = 2 \). First assume, without loss of generality, that \( \pi(x_1) = \pi(y_1) = \pi(x_2) = \pi(y_2) = a \) and \( \pi(x_3) = \pi(x_4) = b \). If both \( y_3 \) and \( y_4 \) are colored with \( b \), we color \( v_1, v_2 \) with \( b \) and \( v_3, v_4 \) with \( a \). Otherwise, w.l.o.g., assume that \( \pi(y_3) = a \). We color \( v_1, v_2, v_3 \) with \( b \) and \( v_4 \) with \( a \). Now assume, w.l.o.g., that \( \pi(x_1) = \pi(y_1) = \pi(x_3) = \pi(y_3) = a \) and \( \pi(x_2) = \pi(x_4) = b \). If \( \pi(y_2) = \pi(y_4) = b \), then color \( v_1, v_3 \) with \( b \) and \( v_2, v_4 \) with \( a \). Otherwise, at least one of \( y_2 \) and \( y_4 \) is colored with \( b \), say \( y_2 \). Thus color \( v_1, v_2, v_3 \) with \( b \) and \( v_4 \) with \( a \).

- \( |S(a)| = 1 \). Without loss of generality, assume that \( \pi(x_1) = \pi(y_1) = a \) and \( \pi(x_2) = \pi(x_3) = \pi(x_4) = b \). If at least two of \( y_2, y_3, y_4 \) are colored with \( b \), then reduce the proof to the former case. If none of \( y_2, y_3, y_4 \) is colored with \( b \), i.e., \( \pi(y_2) = \pi(y_3) = \pi(y_4) = a \), then we color \( v_1, v_3 \) with \( b \) and \( v_2, v_4 \) with \( a \). Now, suppose that exactly one of \( y_2, y_3, y_4 \) is colored with \( b \). If \( \pi(y_2) = b \), then \( \pi(y_3) = \pi(y_4) = a \) and thus we may color \( v_1, v_3 \) with \( b \) and \( v_2, v_4 \) with \( a \). If \( \pi(y_3) = b \), then \( \pi(y_2) = \pi(y_4) = a \) and therefore we color \( v_1, v_4 \) with \( b \) and \( v_2, v_3 \) with \( a \).

- \( |S(a)| = 0 \). It implies that \( \{\pi(x_i), \pi(y_i)\} = \{a, b\} \) for all \( i = 1, 2, 3, 4 \). Hence, it suffices to color \( v_1, v_3 \) with \( a \) and \( v_2, v_4 \) with \( b \).

It is easy to verify that in each possible case the extended coloring is an acyclic 2-coloring of \( G \), driving a contradiction.
(2) Assume to the contrary that $3 \leq d(f_i) \leq 4$ for all $i = 1, 2, 3, 4$. It means that either $y_i = x_{i+1}$ or $y_i x_{i+1} \in E(G)$ for each $i \in \{1, 2, 3, 4\}$ and $i$ is taken modulo 4. Since $\pi$ cannot be extended to $V(f)$, we may assume that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1, 2, 3, 4$ by (1). If there exists a vertex $v$ which can be given the color $a$ without arising any monochromatic cycle, then we color the remaining vertices with $b$ to obtain an acyclic 2-coloring of $G$, a contradiction. Otherwise, suppose that for each $i \in \{1, 2, 3, 4\}$ there exists a path $P_i$ connecting $x_i$ and $y_i$ in $H$ such that all vertices in $P_i$ are colored with $a$. Therefore, a monochromatic cycle $C$ formed by $\bigcup_{i=1}^{4} P_i$ and some edges $y_1 x_2, y_2 x_3, y_3 x_4$ and $y_4 x_1$ (if exist) is established in $H$. This contradicts the choice of $H$.

Therefore, we complete the proof of Lemma 1.

\textbf{Claim 2.} There are no adjacent light 4-faces in $G$.

\textbf{Proof.} Suppose to the contrary that there are 4-faces $f_1 = [v_1 v_2 v_3 v_4]$ and $f_2 = [v_2 v_3 v_4 v_5]$ adjacent by sharing one common edge $v_2 v_5$ such that $d(v_i) = 4$ for all $i = 1, 2, 3, 4, 5, 6$, see Fig. 2. One can easily check that $v_1, \ldots, v_6$ are mutually distinct by the absence of adjacent triangles in $G$. Let

$$H = G - V(f_1).$$

Then $H$ admits an acyclic 2-coloring $\pi$ by the minimality of $G$. If $\pi$ can be extended to $G$, then we are done. Otherwise, by Lemma 1, we suppose that $x_1, y_1, x_2, v_3, v_4, x_5, x_6, y_6$ are all colored with the same color $a$. If at least one vertex in $\{x_3, y_3, x_4, y_4\}$ is colored with $a$, i.e., $\pi(x_3) = a$, then recolor $v_3$ with $b$, color $v_1, v_5, v_6$ with $b$ and $v_2$ with $a$. Otherwise, it suffices to color $v_1, v_5, v_6$ with $b$ and $v_2$ with $a$. It is easy to see that $\pi$ is extended to the whole graph $G$ in each possible case. This complete the proof of Claim 2.

The following claim is proved by Raspaud and Wang in [15].

\textbf{Claim 3.} $G$ contains no a 5-cycle $C = v_1 v_2 \cdots v_5 v_1$ with a chord $v_2 v_5$ such that $d(v_i) = 4$ for all $i = 1, 2, 4, 5$.

\textbf{Claim 4.} A light 4-face cannot be adjacent to a light 5-face.

\textbf{Proof.} Suppose to the contrary that $f = [v_1 v_2 v_3 v_4]$ is a (4, 4, 4, 4)-face adjacent to a (4, 4, 4, 4, 4)-face $f' = [v_2 v_3 u_1 u_2 u_3]$ by sharing a common edge $v_2 v_3$, see Fig. 3. By the definition, it is easy to know that $d(v_i) = 4$ for all $i = 1, \ldots, 4$ and $d(u_j) = 4$ for all $j = 1, 2, 3$. Moreover, $u_1, u_3 \not\in V(f)$ by the absence of adjacent triangles in $G$. If $u_2 = v_1$, then $C = u_2 v_1 v_3 v_4 v_5 u_3$ is a 5-cycle with a chord $v_2 v_4$ such that all vertices in $C$ are of degree 4. This contradicts Claim 3. Thus, $V(f) \cap V(f') = \{v_2, v_3\}$. By the minimality of $G$, $G - V(f)$ admits an acyclic 2-coloring $\pi$. If $\pi$ can be extended to $G$, then $H$ is done. Otherwise, by Lemma 1, we suppose that $y_1, y_3, x_2, u_1, x_3, x_4, y_4$ are all assigned with the same color $a$. The following discussion is divided into two cases, according to the color of $u_2$.

- $\pi(u_2) = a$. If at most one of $s_1$ and $t_1$ is colored with $b$, we recolor $u_1$ with $b$ and then color $v_1, v_2, v_4$ with $b$ and $v_3$ with $a$. So assume $\pi(s_1) = \pi(t_1) = b$. By symmetry, we also assume
Claim 5. If a $(5, 4, 4, 4, 4)$-face is adjacent to a light 5-face, then they are normally adjacent.

Proof. Suppose that $f^* = [v_1v_2 \cdots v_5]$ is a $(5, 4, 4, 4, 4)$-face adjacent to a $(4, 4, 4, 4)$-face $f$. Obviously, $|V(f^*) \cap V(f)| \neq 4$. If $|V(f^*) \cap V(f)| = 2$, then we are done. So, in what follows, we assume that $|V(f^*) \cap V(f)| = 3$. By symmetry, we only need to consider the following two cases.

Case 1 $V(f^*) \cap V(f) = \{v_2, v_3, v_4\}$.

We first assume that $f = [v_2v_3wv_4]$. Clearly, $w \notin \{v_4, v_5\}$. Then two adjacent triangles $v_2v_3wv_2$ and $v_2v_3wv_3$ are formed, a contradiction. Now assume that $f = [v_4v_3wv_2]$. Similarly, $w \notin \{v_1, v_3\}$. It is easy to observe that a 3-cycle $v_2v_3wv_2$ is adjacent to a 3-cycle $v_2v_3wv_2$, a contradiction.

Case 2 $V(f^*) \cap V(f) = \{v_2, v_3, v_5\}$.

We first assume that $f = [v_2v_3v_5w]$. Clearly, $w \notin \{v_1, v_4\}$. It is easy to see that $C = v_4v_3v_2wv_5v_4$ is a 5-cycle with a chord $v_3v_5$ such that all vertices in $C$ are of degree 4. This contradicts Claim 3.

Now, assume that $f = [v_2v_3wv_5]$. Notice that $w \notin \{v_1, v_4\}$. Let $w_1, w_2$ be the neighbors of $w$ different from $v_3$ to $v_5$. Let $x_4, y_4$ be the neighbors of $v_4$ different from $v_3$ to $v_5$. Let $x_2$ be the neighbor of $v_2$ different from $v_1, v_3$ to $v_5$. Let $x_3$ be the neighbor of $v_2$ different from $v_2, v_4$ and $w$. By the minimality of $G$, $G - V(f)$ admits an acyclic 2-coloring $\pi$. If $\pi$ can be extended to $G$, then it contradicts the choice of $G$. Otherwise, by Lemma 1, we suppose that $v_1, x_2, x_3, v_4, w_1, w_2$ are all colored with the same color $a$. If neither $x_4$ nor $y_4$ is colored with $a$, then color $v_3$ with $a$ and $v_2, v_5, w$ with $b$. Otherwise, we first recolor $v_4$ with $b$, and then color $v_5$ with $a$ and $v_2, v_3, w$ with $b$. In each case, we extend $\pi$ to $G$ successfully, a contradiction.

Therefore, we complete the proof of Claim 5. \hfill \Box

Claim 6. Suppose that $f_1 = [v_1v_2v_3]$ and $f_2 = [v_4v_5v_6]$ are two light 4-faces which intersect at the unique vertex $v$. Then $G$ does not contain the configuration (B1) and (B2) as shown in Fig. 4.

Proof. In each case, let $H = G - \{v, v_1, v_2, v_3\}$. By the minimality of $G$, $H$ admits an acyclic 2-coloring $\pi$. Next, we will show that $\pi$ can be extended to $G$ and thus arrive at a contradiction.
(1) Assume $G$ contains (B1). If $\pi$ cannot be extended to $\{v, v_1, v_2, v_3\}$, by Lemma 1, we suppose that $x_1, y_1, x_2, v_4, x_3, y_3, v_6$ are all colored with $a$. In this case, we color $v$ with $a$ and $v_1, v_2, v_3$ with $b$. If the resulting coloring is not acyclic, one of $x_4$ and $v_5$ must be colored with $a$. Then, we further recolor $v_4$ with $b$.

(2) Assume $G$ contains (B2). Similarly, if $\pi$ cannot be extended to $\{v, v_1, v_2, v_3\}$, by Lemma 1, we suppose that $x_1, y_1, x_2, v_5, x_3, y_3, v_4, v_6$ are all colored with $a$. In this case, we first recolor $v_5$ with $b$ and then extend $\pi$ to the remaining uncolored vertices easily by (1) of Lemma 1.

Thus, we complete the proof of Lemma 6. □

Lemma 2. Suppose that $f^* = [vu_1u_2v_1v_2]$ is a $(5, 4, 4, 4, 4)$-face adjacent to two light 4-faces $f_1 = [v_1v_2v_3v_4]$ and $f_2 = [u_1u_2u_3u_4]$ by the common edge $v_1v_2$ and $u_1u_2$, respectively, see Fig. 5. Let $H = G - V(f_1)$. If an acyclic 2-coloring $\pi$ of $G - V(f_1)$ cannot be extended to $G$, then either $f_1$ or $f_2$ is adjacent to at least two $5^+$-faces.

Proof. By Claim 5, we see that $\{v_3, v_4\} \cap \{v, u_1, u_2\} = \emptyset$ and $\{u_3, u_4\} \cap \{v, v_1, v_2\} = \emptyset$. If $u_3 = v_4$, then $C = u_2v_3v_4v_1u_2v_3$ is a 5-cycle with a chord $v_1v_2$ such that all vertices in $C$ have degree 4. This contradicts Claim 3. If $u_3 = v_3$, then $f_1$ intersects $f_2$ at $v_3$ such that $v_1$ is adjacent to $u_2$, contradicting to (B1). So, suppose that $u_3 \notin \{v_3, v_4\}$. If $u_4 = v_4$, then $f_1$ intersects $f_2$ at $v_4$ such that $v_2u_2 \in E(G)$, contradicting to (B1). If $u_4 = v_3$, then $f_1$ and $f_2$ intersect at $v_3$ such that $v_1u_2 \in E(G)$, which is a contradiction to (B2). Thus, in the following argument, we suppose that $\{u_3, u_4\} \cap \{v_3, v_4\} = \emptyset$. Let $g_{i-1}$ denote the face adjacent to $f_1$ by the common edge $v_iv_{i+1}$, where $i \in \{2, 3, 4\}$ and $i$ is taken
modulo 4. Let $h_{j-1}$ denote the face adjacent to $f_j$ by the common edge $u_ju_{j+1}$, where $j \in \{2, 3, 4\}$ and $j$ is taken modulo 4, see Fig. 5.

Assume to the contrary that $3 \leq d(g_i) \leq 4$ and $3 \leq d(h_i) \leq 4$ for all $i, j = 1, 2, 3$. Denote $H = G - V(f_j)$. By the minimality of $G$, $H$ has an acyclic 2-coloring $\pi$. If $\pi$ can be extended to $G$, then we arrive at a contradiction to the assumption on $G$. Otherwise, assume w.l.o.g., that $u_2, x_1, x_2, v, x_3, y_3, x_4, y_4$ are all colored with $a$ by Lemma 1. We have to deal with the following five cases.

**Case 1** Assume that at most one of $u_1, u_3, s_2$ is colored with $b$.
Then recolor $u_2$ with $b$, color $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$.

**Case 2** Assume that all $u_1, u_3, s_2$ are colored with $b$.
Then color $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$.

**Case 3** Assume that $\pi(u_1) = a$ and $\pi(u_2) = \pi(s_2) = b$.
If there is no monochromatic cycle arising after recoloring $u_1$ with $b$, then recolor $u_1$ with $b$ firstly and then go back to the previous Case 2. Otherwise, suppose that $\pi(s_1) = \pi(u_4) = b$. If one of $s_3$ and $t_3$ is colored with $b$, then recolor $v_3$ with $a, u_2$ with $b$ and then color $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$. So assume that neither $s_3$ nor $t_3$ is colored with $b$. If at least one of $s_4$ and $t_4$ is colored with $b$, then recolor $u_4$ with $a$, $u_1$ with $b$ and then reduce the proof to the former Case 2. Now, assume that $b \notin \{s_4, \pi(t_4)\}$. Therefore, we first recolor $u_2$ with $b$, and then extend $\pi$ to $G$ by coloring $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$.

**Case 4** Assume that $\pi(u_1) = a$ and $\pi(u_2) = \pi(s_2) = b$.
If the color $b$ did not appear on $s_1$ and $u_4$, then recolor $u_2$ with $b$, and color $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$. If the color $a$ did not appear on $s_1$ and $u_4$, then switch the colors of $u_1$ and $u_2$, then color $v_1$ with $a$ and finally recolor $v_2, v_3, v_4$ with $b$. Otherwise, suppose that $\{\pi(s_1), \pi(u_4)\} = \{a, b\}$. We have two possibilities below.

• $\pi(s_1) = b$ and $\pi(u_4) = a$. If at most one of $s_4$ and $t_4$ is colored with $b$, then recolor $u_2, u_4$ with $b$, $u_1$ with $a$, and color $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$. Hence, assume $\pi(s_4) = \pi(t_4) = b$. If at most one of $s_3$ and $t_3$ is colored with $b$, then recolor $u_3$ with $b$ and then go back to the previous Case 2. Otherwise, set $\pi(s_3) = \pi(t_3) = b$. In this case, we may first switch the colors of $u_1$ and $u_2$ and then color $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$ successfully.

• $\pi(s_1) = a$ and $\pi(u_4) = b$. If $b \notin \{\pi(s_4), \pi(t_4)\}$, then recolor $u_2$ with $b$ and color $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$ successfully. If $a \notin \{\pi(s_3), \pi(t_3)\}$, then color $v_1$ with $a$ and finally recolor $v_2, v_3, v_4$ with $b$. So, w.l.o.g., assume that $\pi(s_3) = a$ and $\pi(s_4) = b$. In this case, we can first switch the colors of $u_3$ and $u_4$ and then reduce the proof to the former Case 2.

**Case 5** Assume that $\pi(s_2) = a$ and $\pi(u_1) = \pi(u_3) = b$.
First we consider the case that $\pi(u_4) = a$. If either $\pi(s_1) \neq b$ or $b \notin \{\pi(s_3), \pi(t_3)\}$, then recolor $u_2$ with $b$, color $v_1$ with $a$ and $v_2, v_3, v_4$ with $b$. So, w.l.o.g., assume that $\pi(s_1) = b$ and $\pi(s_2) = b$. We first switch the colors of $u_1$ and $u_2$, then color $v_1$ with $a$ and finally recolor $v_2, v_3, v_4$ with $b$. If the resulting coloring is not acyclic, at least one of $s_4$ and $t_4$ is colored with $a$. Thus, we further recolor $v_3$ with $a$ and $u_4$ with $b$.

Now we consider the case that $\pi(u_4) = b$. If at most one of $s_3, t_3$ is colored with $a$, then first switch the colors of $u_2$ and $u_3$, then color $v_1$ with $a$ and finally recolor $v_2, v_3, v_4$ with $b$. So assume that $\pi(s_3) = \pi(t_3) = a$. If at most one of $s_4, t_4$ is colored with $a$, then recolor $u_4$ with $a$ and then go back to the previous above case. Hence, $\pi(s_4) = \pi(t_4) = a$. If $\pi(s_2) \neq a$, then switch the colors of $u_1$ and $u_2$, and assign color $a$ to $v_1$ and $b$ to $v_2, v_3, v_4$, respectively. So now assume $\pi(s_2) = a$. Notice that each of $g_i$ and $h_i$ is of degree at most 4 with $i = 1, 2, 3$. Moreover, for $i \in \{1, 2, 3, 4\}$, in $H$, there exists a path denoted by $P_i$ connecting two vertices of $N_H(v_i)$ such that all vertices in $P_i$ are colored with $a$. Similarly, for $j \in \{1, 2, 3, 4\}$, in $H$, there exists a path denoted by $P_j$ connecting two vertices of $N_H(u_j)$ such that all vertices in $P_j$ are colored with $a$. However, a monochromatic cycle $C$ is formed in $H$ by $\bigcup_{i=1}^{4} P_i, \bigcup_{j=1}^{4} P_j$ and some edges $x_1x_4, y_4x_3, y_3x_2, s_1s_4, t_4s_3$ and $t_3s_2$ (if exist). This contradicts the choice of $H$. Therefore, we complete the proof of Lemma 2. □

**Claim 7.** $G$ does not contain two $(4, 4, 4, 5)$-faces $f_1 = [v_2v_1v_6v_5]$ and $f_2 = [v_2v_3v_4v_5]$ sharing a unique common edge $v_2v_5$ and $d(v_5) = 5$. 

Proof. Suppose on the contrary that $G$ contains such adjacent $(4, 4, 4, 5)$-faces $f_1$ and $f_2$, see Fig. 6. Since there is no adjacent triangles, $v_1v_5 \notin E(G)$ and $v_2v_6 \notin E(G)$. It implies that $v_1v_2 \cdots v_6v_1$ is a 6-cycle. Let $H = G - \{v_1, \ldots , v_6\}$. Then $H$ admits an acyclic 2-coloring $\pi$ by the minimality of $G$. Let $S(a)$ denote the subset of $\{x_1, y_1\}, \{x_3, y_3\}, \{x_4, y_4\}, \{x_6, y_6\}$ which satisfies that all vertices in $S(a)$ get the same color $a$ in the coloring $\pi$. Thus $0 \leq |S(a)| \leq 4$. The following proof is divided into five cases as follows, depending on the value of $|S(a)|$.

Case 1 $|S(a)| = 4$.

It implies that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1, 3, 4, 6$. If at most one of $x_5, y_5$ is colored with $b$, color $v_1, v_3, v_4, v_5, v_6$ with $b$ and $v_2$ with $a$. Otherwise, color $v_1, v_3, v_4, v_6$ with $b$ and $v_2, v_5$ with $a$.

Case 2 $|S(a)| = 3$.

By symmetry, we have two possible cases below.

- Assume that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1, 3, 4$. W.l.o.g., assume that $\pi(x_6) = b$. If $\pi(x_5) = \pi(y_5) = b$, then color $v_1, v_2, v_3, v_4$ with $b$ and $v_5, v_6$ with $a$. Otherwise, color $v_1, v_3, v_4, v_5$ with $b$ and $v_2, v_6$ with $a$.

- Assume that $\pi(x_i) = \pi(y_i) = a$ for all $i = 3, 4, 6$. W.l.o.g., assume that $\pi(x_1) = b$. We first color $v_3, v_4, v_6$ with $b$ and $v_1$ with $a$. If the color $a$ appears at most once on the set $x_5, y_5$, then further color $v_2$ with $b$ and $v_5$ with $a$. Otherwise, we assign $v_2$ and $v_5$ with $b$ to extend $\pi$ to $G$ successfully.

Case 3 $|S(a)| = 2$.

By symmetry, we have four possible cases below.

- Assume that $\pi(x_1) = \pi(y_1) = \pi(x_3) = \pi(y_3) = a$. W.l.o.g., suppose that $\pi(x_4) = \pi(x_6) = b$. We first color $v_1, v_3$ with $b$ and $v_4, v_6$ with $a$. If at least one of $x_5, y_5$ is colored with $a$, then further color $v_2$ with $a$ and $v_5$ with $b$. Otherwise, suppose that $\pi(x_5) = \pi(y_5) = b$. In this case, we color $v_2, v_5$ with $a$. If the resulting coloring is not acyclic, we assert that at least one of $y_4$ and $y_6$ is colored with $a$, say $y_4$. And thus we can reassign color $b$ to $v_6$ to derive an acyclic 2-coloring of $G$, a contradiction.

- Assume that $\pi(x_1) = \pi(y_1) = \pi(x_4) = \pi(y_4) = a$. W.l.o.g., assume that $\pi(x_3) = \pi(x_6) = b$. We first color $v_1, v_4$ with $b$ and $v_3, v_6$ with $a$. If $\pi(x_5) = \pi(y_5) = b$, then further color $v_2$ with $b$ and $v_5$ with $a$. Otherwise, W.l.o.g., suppose that $\pi(x_5) = a$. We further color $v_2, v_5$ with $b$. Similarly, if the resulting coloring is not acyclic, we assert that $\pi(x_2) = \pi(y_5) = b$ and thus reassign $v_2$ with $a$ to obtain an acyclic 2-coloring of $G$. This contradicts the choice of $G$.

- Assume that $\pi(x_1) = \pi(y_1) = \pi(x_6) = \pi(y_6) = a$. W.l.o.g., assume that $\pi(x_3) = \pi(x_4) = b$. First assume that $\pi(x_3) = \pi(y_4) = b$. If at least one of $x_5, y_5$ is colored with $a$, then color $v_1, v_5, v_6$ with $b$ and $v_2, v_3, v_4$ with $a$. Otherwise, assume that $\pi(x_5) = \pi(y_5) = b$ and thus color $v_1, v_2, v_5, v_6$ with $b$ and $v_3, v_4, v_5$ with $a$. Next assume that $\pi(y_2) = b$ and $\pi(y_4) = a$. If at least one of $x_5, y_5$ is colored with $b$, then color $v_1, v_2, v_4, v_6$ with $b$ and $v_3, v_5$ with $a$. Otherwise, assume that $\pi(x_5) = \pi(y_5) = a$ and hence we may color $v_1, v_4, v_5, v_6$ with $b$ and $v_2, v_3$ with $a$. Finally assume that $\pi(x_5) = \pi(y_5) = a$ and hence we may color $v_1, v_3, v_5, v_6$ with $b$ and $v_2, v_4$ with $a$. 

\[ \text{Fig. 6.} \] $f_1$ and $f_2$ are adjacent $(4,4,5)$-faces.
Claim 8. $G$ contains no a 5-cycle $C = v_1 v_2 \cdots v_5 v_1$ with a chord $v_2 v_5$ such that $d(v_i) = 4$ for all $i = 1, 3, 4, 5$ and $d(v_2) = 5$.

**Proof.** Suppose to the contrary that $G$ contains a 5-cycle $C = v_1 v_2 \cdots v_5 v_1$ with a chord $v_2 v_5$ such that $d(v_i) = 4$ for all $i = 1, 3, 4, 5$ and $d(v_2) = 5$, see Fig. 7. Let $H = G - \{v_1, \ldots, v_5\}$. Then $H$ admits an acyclic 2-coloring $\pi$ by the minimality of $G$. For $a \in C$, let $S(a)$ denote the subset of $\{\{x_1, y_1\}, \{x_3, y_3\}, \{x_4, y_4\}\}$ which satisfies that all vertices in $S(a)$ get the same color $a$ in the coloring $\pi$. Thus $0 \leq |S(a)| \leq 3$. The following proof is divided into four cases as follows, according to the value of $|S(a)|$.

Case 1 $|S(a)| = 3$.

It implies that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1, 3, 4$. If at most one of $x_2, y_2$ is colored with $b$, then color $v_1, v_2, v_3, v_4$ with $b$ and $v_5$ with $a$. Otherwise, color $v_1, v_3, v_4, v_5$ with $b$ and $v_2$ with $a$.

Case 2 $|S(a)| = 2$.

Thus, we complete the proof of Claim 7. $\square$

Claim 8. The configuration in Claim 8.
Claim 9. \( G \) does not contain the configuration (F1), as shown in Fig. 8 where \( f_1, f_2, f_3 \) are all faces.

**Proof.** Assume \( G \) contains (F1). By Claim 8, \( d(v_9) \geq 5 \). By Claim 5, we deduce that \( f_1 \) and \( f_2 \) are normally adjacent. In other words, \( V(f_1) \cap V(f_2) = \{v_1, v_4\} \). First we claim that \( V(f_1) \cap V(f_2) = \{v_1\} \). It suffices to show that \( v_9 \notin \{v_2, v_3, v_4\} \). It is easy to see that \( v_9 \neq v_4 \). If \( v_9 = v_2 \), a 3-cycle \( v_1v_{10}v_2v_1 \) is adjacent to a 3-cycle \( v_2v_3v_1v_1 \), a contradiction. If \( v_9 = v_3 \), then \( v_8 = v_4 \), a contradiction since \( d(v_4) = 4 \). Next we claim that \( V(f_1) \cap V(f_3) = \{v_1, v_7\} \). To see that, we only need to show that \( v_9 \neq v_5 \) and \( v_{10} \notin \{v_5, v_6\} \).

If \( v_5 = v_9 \), then \( v_8 = v_4 \), a contradiction. If \( v_{10} = v_5 \), then a 5-cycle \( v_1v_2v_3v_4v_5v_1 \) with a chord \( v_1v_4 \).
such that \(d(v_i) = 4\) for all \(i = 1, \ldots, 5\) exists in \(G\), contradicting to Claim 3. If \(v_{10} = v_6\), then a 3-cycle \(v_1v_7v_6v_1\) is adjacent to a 3-cycle \(v_9v_7v_6v_9\), a contradiction. Thus, in what follows, we assume that all vertices in the set \(\{v_1, v_2, \ldots, v_{10}\}\) are mutually distinct.

Let \(H = G - V(f_1)\). By the minimality of \(G\), \(H\) admits an acyclic 2-coloring \(\pi\). If \(\pi\) cannot be extended to \(G\), by (1) of Lemma 1, we deduce that all vertices in \(\bigcup_{i=1}^{h} N_H(v_i)\) get the same color in the coloring \(\pi\). Without loss of generality, suppose that \(v_7, v_{10}, x_2, y_2, x_3, y_3, x_4, v_5\) are all colored with \(a\). We have to consider two cases below by the color of \(v_6\).

**Case 1** Assume \(\pi(v_6) = a\).

If at most one of \(x_5\) and \(y_5\) is colored with \(b\), we recolor \(v_5\) with \(b\), color \(v_1, v_2, v_3\) with \(b\) and \(v_4\) with \(a\). So suppose that \(\pi(x_5) = \pi(y_5) = b\). If at most one of \(x_6\) and \(y_6\) is colored with \(b\), we recolor \(v_6\) with \(b\), color \(v_1, v_2, v_3, v_7, v_8, v_4\) with \(a\). Now suppose that \(\pi(x_6) = \pi(y_6) = b\). If at most one of \(x_7, y_8, v_9\) is colored with \(b\), then recolor \(v_7\) with \(b\) and thus we can color \(v_1\) with \(a\) and finally color \(v_2, v_3, v_4\) with \(b\). If the color \(a\) did not appear on the set \(\{x_7, v_8, v_9\}\), we can extend \(\pi\) to \(G\) by coloring \(v_1\) with \(a\) and \(v_2, v_3, v_4\) with \(b\). Thus, in what follows, assume that exactly two of \(x_7, v_8, v_9\) are colored with \(a\) and one is colored with \(b\). We need to discuss three possibilities below.

- **\(\pi(x_7) = a\) and \(\pi(v_8) = \pi(v_9) = b\)**. It is easy to derive that one of \(x_{10}\) and \(y_{10}\) is colored with \(a\). Otherwise, we may give the color \(a\) to \(v_1\) and the color \(b\) to other three remaining uncolored vertices. Therefore, we can first recolor \(v_7, v_{10}\) with \(b, v_9\) with \(a\) and then extend \(\pi\) to \(G\) by coloring \(v_1\) with \(a\) and \(v_2, v_3, v_4\) with \(b\).
- **\(\pi(v_8) = a\) and \(\pi(x_7) = \pi(v_9) = b\)**. Similarly, we deduce that one of \(x_{10}\) and \(y_{10}\) is colored with \(a\). Otherwise, we can color \(v_1\) with \(a\) and \(v_2, v_3, v_4\) with \(b\) to derive an acyclic 2-coloring of \(G\), a contradiction. Thus, we recolor \(v_{10}\) with \(b\), color \(v_1, a\) and \(v_2, v_3, v_4\) with \(b\). If the resulting coloring is not acyclic, \(x_9\) must be colored with \(b\). Then we further switch the colors of \(v_7\) and \(v_9\).
- **\(\pi(x_7) = a\) and \(\pi(v_8) = \pi(v_9) = b\)**. If at most one of \(x_{10}\) and \(y_{10}\) is colored with \(b\), we recolor \(v_{10}\) with \(b\), color \(v_1\) with \(a\) and \(v_2, v_3, v_4\) with \(b\). Now suppose that \(\pi(x_{10}) = \pi(y_{10}) = b\). If \(\pi(x_9) = b\), we color \(v_1, v_2, v_3\) with \(b\) and \(v_4\) with \(a\). Otherwise, recolor \(v_9\) with \(b\) and then color \(v_1\) with \(a\) and \(v_2, v_3, v_4\) with \(b\).

**Case 2** Assume \(\pi(v_5) = b\).

One can easily observe that one of \(x_5, y_5\) is assigned with \(a\). Otherwise, we may color \(v_4\) with \(a\) and \(v_1, v_2, v_3\) with \(b\). If the color \(b\) did not appear on the set \(\{x_5, y_5\}\), we first recolor \(v_5\) with \(b\) and color \(v_4\) with \(a\) and \(v_1, v_2, v_3\) with \(b\). So, w.l.o.g., assume that \(\pi(x_5) = a\) and \(\pi(y_5) = b\). By a similar argument, we can deduce that \(\{\pi(x_6), \pi(y_6)\} = \{a, b\}\). If at most one of \(x_7, v_8, v_9\) is colored with \(b\), then recolor \(v_7, v_5\) with \(b, v_6\) with \(a\), and thus color \(v_1\) with \(a\) and finally color \(v_2, v_3, v_4\) with \(b\). If the color \(a\) did not appear on the set \(\{x_7, v_8, v_9\}\), we can extend \(\pi\) to \(G\) by coloring \(v_1\) with \(a\) and \(v_2, v_3, v_4\) with \(b\).
Thus, in what follows, assume that exactly two of $x_7$, $v_8$, $v_9$ are colored with $b$ and one is colored with $a$. The following proof is similar to the previous Case 1.

Therefore, we complete the proof of Claim 9.  

Claim 10. $G$ does not contain the configuration (F2), as shown in Fig. 9.

Proof. Assume $G$ contains (F2). Clearly, $\{v_3, v_4\} \cap \{v_6, v_7\} = \emptyset$, since $G$ contains no adjacent triangles. It follows that $C = v_1v_2 \cdots v_7v_1$ is a 7-cycle. Moreover, it is easy to see that $x_2 \notin C$. By the minimality of $G$, $G - \{v_2\}$ admits an acyclic 2-coloring $\pi$. It is easy to observe that if there exists a color $c$ appearing at most once on the set $\{x_2, v_1, v_3, v_5\}$, we can color $v_2$ with $c$ to obtain an acyclic 2-coloring of $G$. So, in the following, we always assume that the colors $a$ and $b$ appear exactly twice on the set $\{x_2, v_1, v_3, v_5\}$, respectively. We need to handle the following cases.

Case 1 $\pi(x_2) = \pi(v_3) = a$ and $\pi(v_1) = \pi(v_5) = b$.

First consider the case that $\pi(v_4) = a$. If $a \in \{\pi(x_3), \pi(y_3)\}$, recolor $v_3$ with $b$ and color $v_2$ with $a$. So assume $\pi(x_3) = \pi(y_3) = b$. If neither $x_4$ nor $y_4$ is colored with $a$, we color $v_2$ with $a$. If neither $x_4$ nor $y_4$ is colored with $b$, recolor $v_4$ with $b$ and color $v_2$ with $a$. Thus, in what follows, w.l.o.g., assume that $\pi(x_4) = a$ and $\pi(y_4) = b$. If at most one of $x_5$, $v_6$ is colored with $a$, then recolor $v_5$ with $a$, $v_4$ with $b$, and color $v_2$ with $b$. Otherwise, suppose that $\pi(x_5) = \pi(v_6) = a$. If $x_1, y_1, v_7$ are all colored with $a$, then recolor $v_4$ with $b$ and thus we can color $v_2$ with $a$. If at least two of $x_1, y_1, v_7$ are colored with $b$, then recolor $v_4$ with $a$ and then color $v_2$ with $b$. Otherwise, assume that exactly two of $x_1, y_1, v_7$ are colored with $a$ and one is colored with $b$. By symmetry, we need to consider two subcases as follows.

- $\pi(v_7) = b$ and $\pi(x_1) = \pi(y_1) = a$. If neither $x_6$ nor $y_6$ is colored with $a$, switch the colors of $v_4$ and $v_5$ and color $v_2$ with $b$ and afterward color $v_2$ with $b$. If neither $x_7$ nor $y_7$ is colored with $b$, recolor $v_4$ with $b$ and color $v_2$ with $a$. So, w.l.o.g., assume that $\pi(x_6) = a$ and $\pi(x_7) = b$. In this case, we may first switch the colors of $v_5$ and $v_7$ and then go back to the previous case.

- $\pi(x_1) = b$ and $\pi(y_1) = \pi(v_7) = a$. If one of $x_6, y_6$ is colored with $a$, recolor $v_4, v_6$ with $b$, $v_5$ with $a$ and color $v_2$ with $b$. So assume that $\pi(x_6) = \pi(v_6) = b$. Similarly, if one of $x_7, y_7$ is colored with $b$, recolor $v_7$ with $b$, $v_1$ with $a$ and color $v_2$ with $b$. So assume that $\pi(x_7) = \pi(v_7) = b$. Now, we can recolor $v_5$ with $a$, $v_4$ with $b$, and color $v_2$ with $b$ to extend $\pi$ to $G$ successfully.

Now consider the case that $\pi(v_4) = b$. If $\pi(x_3) = \pi(y_3) = a$, recolor $v_3$ with $b$ and color $v_2$ with $a$. If $\pi(x_3) = \pi(y_3) = b$, color $v_2$ with $a$. So, w.l.o.g., assume that $\pi(x_3) = a$ and $\pi(y_3) = b$. If $\pi(x_4) = \pi(y_4) = b$, recolor $v_4$ with $a$ and then go back to the previous case. If $\{\pi(x_4), \pi(y_4)\} = \{a, b\}$, switch the colors of $v_3$ and $v_4$ and color $v_2$ with $a$. Now, suppose that $\pi(x_4) = \pi(y_4) = a$. If at most one of $x_5, v_6$ is colored with $a$, then recolor $v_3$ with $a$, and color $v_2$ with $b$. Otherwise, suppose that $\pi(x_5) = \pi(v_6) = a$. The following proof is similar to the first case.

Case 2 $\pi(x_2) = \pi(v_5) = a$ and $\pi(v_1) = \pi(v_3) = b$.

We first consider the case that $\pi(v_4) = a$. If $\pi(x_3) = \pi(y_3) = a$, color $v_2$ with $b$. If $\pi(x_3) = \pi(y_3) = b$, recolor $v_3$ with $a$ and color $v_2$ with $b$. So, assume that $\pi(x_3) = a$ and $\pi(y_3) = b$. If $a \in \{\pi(x_4), \pi(y_4)\}$, recolor $v_4$ with $b$, $v_3$ with $a$ and color $v_2$ with $b$. Now, suppose that $\pi(x_4) = \pi(y_4) = b$. If neither $v_6$ nor $x_5$ is colored with $a$, then color $v_2$ with $a$. If neither $v_6$ nor $x_5$ is colored with $b$, then recolor $v_5$ with $b$ and color $v_2$ with $a$. So, assume that $\{\pi(x_5), \pi(v_6)\} = \{a, b\}$. We have two cases below.
• \( \pi(v_6) = a \) and \( \pi(x_5) = b \). If \( x_1, y_1, v_7 \) are all colored with \( a \), then recolor \( v_5 \) with \( b \) and color \( v_2 \) with \( a \). If at least two of \( x_1, y_1, v_7 \) are colored with \( b \), then recolor \( v_1, v_3 \) with \( a, v_5 \) with \( b \), and \( v_2 \) with \( b \). Otherwise, assume that exactly two of \( x_1, y_1, v_7 \) are colored with \( a \) and one is colored with \( b \). By symmetry, we need to handle the following two possibilities.

- \( \pi(x_1) = b \) and \( \pi(y_1) = \pi(v_7) = a \). If \( a \in \{ \pi(x_6), \pi(y_6) \} \), recolor \( v_6 \) with \( b \) and then reduce the proof to the former case. Otherwise, set \( \pi(x_6) = \pi(y_6) = b \). If \( a \in \{ \pi(x_7), \pi(y_7) \} \), recolor \( v_5, v_7 \) with \( b, v_1, v_3 \) with \( a \), and color \( v_2 \) with \( b \). Now we assert that \( \pi(x_6) = \pi(y_6) = b \). In this case, we can color \( v_2 \) with \( a \). It is easy to verify that the resulting coloring of \( G \) is an acyclic 2-coloring, a contradiction.

- \( \pi(v_7) = b \) and \( \pi(x_1) = \pi(y_1) = a \). If \( a \notin \{ \pi(x_6), \pi(y_6) \} \), recolor \( v_3 \) with \( a \) and color \( v_2 \) with \( b \). If \( b \notin \{ \pi(x_7), \pi(y_7) \} \), color \( v_2 \) with \( b \). Otherwise, w.l.o.g., assume that \( \pi(x_6) = a \) and \( \pi(x_7) = b \). We may first switch the colors of \( v_6 \) and \( v_7 \), and then color \( v_2 \) with \( a \).

• \( \pi(v_6) = b \) and \( \pi(x_5) = a \). Similarly, we deduce that exactly two of \( x_1, y_1, v_7 \) are colored with \( a \) and one is colored with \( b \). By symmetry, we need to handle the following two possibilities.

- \( \pi(x_7) = b \) and \( \pi(y_1) = \pi(v_7) = a \). If \( a \notin \{ \pi(x_5), \pi(y_5) \} \), recolor \( v_1, v_3 \) with \( a, v_5 \) with \( b \) and color \( v_2 \) with \( a \). If \( b \notin \{ \pi(x_6), \pi(y_6) \} \), recolor \( v_5 \) with \( b \) and color \( v_2 \) with \( a \). Otherwise, recolor \( v_1, v_5 \) with \( a, v_3, v_7 \) with \( b \) and color \( v_2 \) with \( b \).

- \( \pi(v_7) = b \) and \( \pi(x_1) = \pi(y_1) = a \). If \( b \in \{ \pi(x_7), \pi(y_7) \} \), recolor \( v_7 \) with \( a \) and then go back to the previous case. Now, assume \( \pi(x_7) = \pi(y_7) = a \). Similarly, if \( b \notin \{ \pi(x_6), \pi(y_6) \} \), then color \( v_6 \) with \( a, v_5 \) with \( b \) and color \( v_2 \) with \( a \). So, assume \( \pi(x_6) = \pi(y_6) = a \). Therefore, we may color \( v_2 \) with \( b \) successfully.

Now we consider the case that \( \pi(v_4) = b \). If \( b \in \{ \pi(x_1), \pi(y_1) \} \), recolor \( v_3 \) with \( a \) and color \( v_2 \) with \( b \). So assume \( \pi(x_3) = \pi(y_3) = a \). If \( \pi(x_4) = \pi(y_4) = a \), color \( v_2 \) with \( b \). If \( \pi(x_4) = \pi(y_4) = b \), recolor \( v_4 \) with \( a \) and color \( v_2 \) with \( b \). Now, suppose that \( \{ \pi(x_4), \pi(y_4) \} = \{a, b\} \). If neither \( v_5 \) nor \( x_5 \) is colored with \( a \), then color \( v_2 \) with \( a \). If neither \( v_5 \) nor \( x_5 \) is colored with \( b \), then recolor \( v_5 \) with \( b, v_4 \) with \( a \), and color \( v_2 \) with \( a \). So, assume that both colors \( a \) and \( b \) appear exactly once on the set \( \{x_5, v_6\} \). The following discussion is similar to the previous case.

Case 3 \( \pi(x_2) = \pi(v_1) = a \) and \( \pi(v_3) = \pi(v_5) = b \).

First consider the case that \( \pi(v_4) = a \). If \( \pi(x_3) = \pi(y_3) = a \), color \( v_2 \) with \( b \) and color \( v_2 \) with \( b \). If \( \pi(x_3) = \pi(y_3) = b \), recolor \( v_3 \) with \( a \) and color \( v_2 \) with \( b \). So, assume that \( \{ \pi(x_3), \pi(y_3) \} = \{a, b\} \). If \( \pi(x_4) = \pi(y_4) = a \), recolor \( v_4 \) with \( b, v_5 \) with \( a \) and color \( v_2 \) with \( b \). If \( \pi(x_4) = \pi(y_4) = b \), recolor \( v_3 \) with \( a \), and color \( v_2 \) with \( b \). So, now assume that \( \{ \pi(x_4), \pi(y_4) \} = \{a, b\} \). If neither \( v_6 \) nor \( x_5 \) is colored with \( a \), then recolor \( v_3, v_5 \) with \( a, v_4 \) with \( b \), and color \( v_2 \) with \( b \). If neither \( v_6 \) nor \( x_5 \) is colored with \( b \), then color \( v_2 \) with \( b \). So, assume that both colors \( a \) and \( b \) appear on the set \( \{x_5, v_6\} \). We have two cases below.

• \( \pi(v_6) = a \) and \( \pi(x_5) = b \). If \( x_1, y_1, v_7 \) are all colored with \( b \), then color \( v_2 \) with \( a \). If at least two of \( x_1, y_1, v_7 \) are colored with \( a \), then recolor \( v_1, v_4 \) with \( b, v_3, v_5 \) with \( a \), and \( v_2 \) with \( b \). Otherwise, assume that exactly two of \( x_1, y_1, v_7 \) are colored with \( b \) and one is colored with \( a \). By symmetry, we need to deal with the following two possibilities.

- \( \pi(v_7) = a \) and \( \pi(x_1) = \pi(y_1) = b \). If at least one of \( x_7, y_7 \) is colored with \( a \), then recolor \( v_7 \) with \( b \) and color \( v_2 \) with \( a \). Otherwise, assume \( \pi(x_6) = \pi(y_6) = b \). If \( a \notin \{ \pi(x_6), \pi(y_6) \} \), color \( v_2 \) with \( a \). Otherwise, recolor \( v_4, v_6 \) with \( b \) and \( v_3, v_5 \) with \( a \) and color \( v_2 \) with \( b \).

- \( \pi(x_1) = a \) and \( \pi(v_7) = \pi(y_1) = b \). If \( b \notin \{ \pi(x_7), \pi(y_7) \} \), recolor \( v_1, v_4 \) with \( b, v_3, v_5 \) with \( a \), and color \( v_2 \) with \( b \). If \( a \notin \{ \pi(x_6), \pi(y_6) \} \), recolor \( v_3, v_5 \) with \( a, v_4 \) with \( b \), and color \( v_2 \) with \( b \). Otherwise, we can first recolor \( v_3, v_5, v_7 \) with \( a \) and \( v_1, v_4, v_6 \) with \( b \).

• \( \pi(v_6) = b \) and \( \pi(x_5) = a \). By a similar argument as above, we may suppose that exactly two of \( x_1, y_1, v_7 \) are colored with \( b \) and one is colored with \( a \). By symmetry, we need to deal with the following two possibilities.

- \( \pi(v_7) = a \) and \( \pi(x_1) = \pi(y_1) = b \). If either \( a \notin \{ \pi(x_7), \pi(y_7) \} \) or \( b \notin \{ \pi(x_6), \pi(y_6) \} \), then color \( v_2 \) with \( a \) or \( b \). Otherwise, set \( \pi(x_7) = a \) and \( \pi(x_6) = b \). Then, switch the colors of \( v_6 \) and \( v_7 \) and then color \( v_2 \) with \( a \) successfully.
\[\pi(x_1) = a \quad \text{and} \quad \pi(x_7) = \pi(y_1) = b.\] If \(b \in \{\pi(x_6), \pi(y_6)\}\), recolor \(v_6\) with \(a\) and color \(v_2\) with \(b\).

Hence \(\pi(x_6) = \pi(y_6) = a\). If \(b \in \{\pi(x_7), \pi(y_7)\}\), recolor \(v_7\) with \(a\), \(v_1\) with \(b\), and color \(v_2\) with \(a\). Otherwise, color \(v_2\) with \(b\) easily.

Now consider the case that \(\pi(v_4) = b\). If \(b \in \{\pi(x_3), \pi(y_3)\}\), recolor \(v_3\) with \(a\) and color \(v_2\) with \(b\). Otherwise, assume that \(\pi(x_3) = \pi(y_3) = a\). If \(b \in \{\pi(x_4), \pi(y_4)\}\), recolor \(v_4\) with \(a\) and then go back to the previous case. So we may assume that \(\pi(x_4) = \pi(y_4) = a\). If \(v_5\) can be given a new color \(a\) without arising any monochromatic cycle, we can further color \(v_2\) with \(b\) successfully. Otherwise, we have the following two cases.

First assume that \(\pi(v_6) = \pi(x_5) = a\). If \(x_1, y_1, x_7\) are all colored with \(b\), then color \(v_2\) with \(a\). If at least two of \(x_1, y_1, v_7\) are colored with \(a\), then recolor \(v_1\) with \(a\) and color \(v_2\) with \(b\). Otherwise, assume that exactly two of \(x_1, y_1, x_7\) are colored with \(b\) and one is colored with \(a\). By symmetry, we need to deal with two possibilities below.

- \(\pi(x_7) = a\) and \(\pi(x_1) = \pi(y_1) = b\). If \(a \in \{\pi(x_7), \pi(y_7)\}\), recolor \(v_7\) with \(b\) and color \(v_2\) with \(a\). So assume that \(\pi(x_7) = \pi(y_7) = b\). If \(a \in \{\pi(x_6), \pi(y_6)\}\), recolor \(v_6\) with \(b\), \(v_5\) with \(a\) and color \(v_2\) with \(b\). Thus, \(\pi(x_6) = \pi(y_6) = b\). In this case, we can color \(v_2\) with \(a\) to derive an acyclic 2-coloring of \(G\), a contradiction.
- \(\pi(x_1) = a\) and \(\pi(x_7) = \pi(y_1) = b\). If \(b \notin \{\pi(x_7), \pi(y_7)\}\), recolor \(v_1\) with \(b\) and color \(v_2\) with \(a\). So, w.l.o.g., assume \(\pi(x_1) = b\). If \(a \notin \{\pi(x_6), \pi(y_6)\}\), recolor \(v_7\) with \(a\), \(v_1\) with \(b\) and finally color \(v_2\) with \(a\). Otherwise, recolor \(v_1, v_6\) with \(b\), \(v_7\) with \(a\), and color \(v_2\) with \(a\).

Now assume that \(\{\pi(v_6), \pi(x_5)\} = \{a, b\}\). The proof is similar to the previous case.

Therefore, we complete the proof of Claim 10. \(\square\)

### 3. Proof of Theorem 2

We define a weight function \(\omega\) on the vertices and faces of \(G\) by letting \(\omega(v) = 2d(v) - 6\) if \(v \in V(G)\) and \(\omega(f) = d(f) - 6\) if \(f \in F(G)\). It follows from Euler's formula \(|V(G)| - |E(G)| + |F(G)| = 2\) and the relation \(\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|\) that the total sum of weights of the vertices and faces is equal to

\[
\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12. \tag{1}
\]

We shall design appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function \(\omega^*\) is produced. The total sum of weights is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, the new weight function satisfies \(\omega^*(x) \geq 0\) for all \(x \in V(G) \cup F(G)\). This leads to the following obvious contradiction,

\[-12 = \sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} \omega^*(x) \geq 0\]

and hence demonstrates that no such counterexample exists.

Suppose \(v\) is a 5-vertex. Let \(v_1, v_2, \ldots, v_5\) be the neighbors of \(v\) in a cyclic order. Let \(f_i\) be the face with \(vv_i\) and \(vv_{i+1}\) as two boundary edges for \(i = 1, 2, \ldots, 5\), where indices are taken modulo 5. We call \(v\) a special 5-vertex of \(f_i\) if the following conditions hold:

1. \(d(f_1) = 3\);
2. \(d(f_i) = 4\) for all \(i = 2, 3, 4, 5\);
3. \(f_2\) and \(f_4\) are both \((5, 4, 4, 4)\)-faces.

Moreover, we call \(f_2\) a special 4-face with respect to \(v\). Fig. 10 shows a special 5-vertex \(v\). By Claim 8 and Claim 10, we have to notice that such special 4-face is either a \((5, 4, 4, 6^+)\)-face or a \((5, 4, 5^+, 5^+)\)-face. These two observations will be used directly in the following proof.

For \(x, y \in V(G) \cup F(G)\), let \(\tau(x \to y)\) denote the amount of weights transferred from \(x\) to \(y\).

Our discharging rules are as follows:

\(\text{(R1)}\) Every \(6^+\)-vertex sends 1 to each incident \(3^+\)-face.

\(\text{(R2)}\) Let \(v\) be a 5-vertex incident to a face \(f\). Then

\(\text{(R2.1)}\) \(\tau(v \to f) = 1\), if \(f\) is either a 3-face or \((5, 4, 4, 4)\)-face;
Every 4-vertex sends at least $\frac{1}{3}$ to each incident 4-face.

(R2.2) $\tau(v \to f) = \frac{2}{3}$, if $f$ is either a non-special 4-face or a bad 5-face.

(R2.3) $\tau(v \to f) = \frac{3}{4}$, if $f$ is either a special 4-face or a good 5-face.

(R3) Let $v$ be a 4-vertex and $f_1, f_2, f_3, f_4$ denote the faces of $G$ incident to $v$ in a cyclic order. Assume $m_3(v) = 0$. Then

(R3.1) If $l(v) = 0$, then $\tau(v \to f_i) = \frac{1}{2}$ for each $i = 1, 2, 3, 4$.

(R3.2) If $l(v) = 1$, say $f_1$, then $\tau(v \to f_1) = \frac{3}{4}$, $\tau(v \to f_3) = \frac{3}{4}$, and $\tau(v \to f_i) = \frac{1}{2}$ for each $i = 2, 4$.

(R3.3) Assume $f_2$ is a light 4-face. Then

(a1) If $f_3$ is a 4-face, then $\tau(v \to f_i) = \frac{1}{2}$ for each $i = 2, 3, 4$.

(a2) If $f_3$ is a 6+ face, then $\tau(v \to f_2) = \frac{2}{3}$ and $\tau(v \to f_4) = \frac{1}{3}$.

(a3) Assume $f_3$ is a 5-face. Then

(a1.1) If either $f$ is a good 5-face or $m_3^+(f_2) = 1$, then $\tau(v \to f_2) = \frac{2}{3}$ and $\tau(v \to f_4) = \frac{1}{3}$.

(a3.2) Assume $f_3$ is a bad 5-face and $f_2$ is adjacent to another 5+ face $f^*$ different from $f_3$.

(a3.2.1) If $f^*$ is a bad 5-face, then $\tau(v \to f_2) = \frac{1}{2}$, $\tau(v \to f_3) = \frac{1}{6}$, and $\tau(v \to f_4) = \frac{1}{3}$.

(a3.2.2) Otherwise, $\tau(v \to f_3) = \frac{1}{3}$ for each $i \in \{2, 3, 4\}$.

(R3.3) Assume $f_3$ is a light 4-face. Then

(b1) If one of $f_2$ and $f_4$ is of degree at least 6, say $f_2$, then $\tau(v \to f_3) = \frac{2}{3}$ and $\tau(v \to f_4) = \frac{1}{3}$.

(b2) If $m_5(v) = 0$, then $\tau(v \to f_1) = \frac{1}{2}$ for each $i = 2, 3, 4$.

(b3) Assume $m_5(v) = 2$ such that $f_2$ and $f_4$ are both 5-faces.

(b3.1) If one of $f_2, f_4$ is a good 5-face, say $f_2$, then $\tau(v \to f_1) = \frac{2}{3}$ and $\tau(v \to f_4) = \frac{1}{3}$.

(b3.2) Otherwise, $\tau(v \to f_2) = \tau(v \to f_4) = \frac{1}{6}$ and $\tau(v \to f_3) = \frac{2}{3}$.

(b4) Assume $m_5(v) = 1$ such that $f_2$ is a 4-face and $f_4$ is a 5-face.

(b4.1) If $f_4$ is a good 5-face, then $\tau(v \to f_2) = \frac{1}{3}$ and $\tau(v \to f_3) = \frac{2}{3}$.

(b4.2) Assume $f_4$ is a bad 5-face.

(b4.2.1) If $m_5^+(f_3) = 1$, then $\tau(v \to f_2) = \frac{1}{2}$ and $\tau(v \to f_3) = \frac{2}{3}$.

(b4.2.2) Assume $f_3$ is adjacent to another 5+ face $f^*$ different from $f_4$. Then

(b4.2.2.1) If $f^*$ is a bad 5-face, then $\tau(v \to f_2) = \frac{1}{3}$, $\tau(v \to f_3) = \frac{1}{2}$, and $\tau(v \to f_4) = \frac{1}{6}$.

(b4.2.2.2) Otherwise, $\tau(v \to f_i) = \frac{1}{3}$ for each $i \in \{2, 3, 4\}$.

For simplicity, in Fig. 11, we use the notation "$L^*$" to denote a light 4-face. By a careful observation, (R3) includes all possible incident cases for any vertex of degree 4. Thus, combining (R1) and (R2), the following statement holds.

**Observation 1.** Every 4+ vertex sends at least $\frac{1}{3}$ to each incident 4-face.
Let us check that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

Let $v \in V(G)$. Since $\delta(G) \geq 4$, $d(v) \geq 4$. In what follows, let $v_1, v_2, \ldots, v_{d(v)}$ denote the neighbors of $v$ in a cyclic order, and let $f_i$ denote the incident face of $v$ with $vv_i$ and $vv_{i+1}$ as two boundary edges for $i = 1, 2, \ldots, d(v)$, where indices are taken modulo $d(v)$. We have to handle the following cases, depending on the size of $d(v)$.

Case 1. If $d(v) \geq 6$, then it is trivial that $\omega^*(v) = 2d(v) - 6 - d(v) = d(v) - 6 \geq 0$ by (R1).
Claim 7, cases, depending on the situation of the incident light 4-face. τ = ω are both (5, 4, 4, 4)-faces. Note that following, we assume that m(v) at most one light 4-face. If m(v) = 1 or m(v) = 2, by Claim 5 and 7, there is only one possible case that f2 and f4 are both (5, 4, 4, 4)-faces. Note that d(v1) ≥ 5 by Claim 8. This fact implies that f3 cannot be a (5, 4, 4, 4, 4)-face.

Case 2. If d(v) = 5, then ω(v) = 4. Let m2(v) be the number of incident (5, 4, 4, 4)-faces. By Claim 7, m2(v) ≤ 2. Moreover, m2(v) ≤ 1 by the absence of intersecting triangles. If m2(v) = 0, then ω(v) ≥ 4 − m2(v) − 2(5 − m2(v)) = 2 − 4m2(v) ≥ 0 by (R2).

Now, without loss of generality, assume that f1 = {v1v2} is a 3-face. By (R2.1), τ(v → f1) = 1. If m2(v) ≤ 1, then ω(v) ≥ 4 − 1 − m2(v) − 2(4 − m2(v)) = 1 − 1 − m2(v) ≥ 0 by (R2). So, in the following, we assume that m2(v) = 2. By Claims 5 and 7, there is only one possible case that f2 and f4 are both (5, 4, 4, 4)-faces. Note that d(v1) ≥ 5 by Claim 8. This fact implies that f3 cannot be a (5, 4, 4, 4, 4)-face.

• If d(f2) ≥ 6, then v sends nothing to f3 by (R2) and hence ω(v) ≥ 4 − 1 − 1 × 2 − 2 = 1.
• If d(f2) = 5, then f3 cannot be adjacent to any light 4-face by Claim 9. It follows immediately from the definition that f3 is not a bad 5-face. So, by (R2.3), τ(v → f3) = 1/2. Therefore, we derive that ω(v) = 4 − 1 − 1 × 2 − 2 − 1 = 0.

Now, suppose that f3 = [uvw3w4v] is a 4-face. Moreover, f2 is a (5, 4, 5+, 4)-face and thus it gets at most 2 from v by (R2.2). If we can show that f3 gets at most 1 from v and thus we obtain that ω0(v) = 4 − 1 − 1 × 2 − 2 = 0. To see that, we have two cases. If f3 is not a 4-face, then v sends at most 1 to f3 since f3 cannot be a (5, 4, 4, 4, 4)-face. Now we assume that f3 is a 4-face. It implies that f3 is a special face with respect to v and therefore v sends 1 to f3 by (R2.3).

Case 3. If d(v) = 4, then ω(v) = 2. Clearly, m2(v) ≤ 1. First assume that m2(v) = 0. By Claim 2, if v is incident to at most two light 4-faces. It is easy to derive that ω0(v) ≥ 2 − 2 − 2 = 0 by (R3.1.1), or ω0(v) ≥ 2 − 2 − 2 = 2 × 2 = 0 by (R3.1), or ω0(v) ≥ 2 − 2 − 2 = 2 × 2 = 0 by (R3.1.3).

Now assume that m2(v) = 1 and f1 is a 3-face. By (R3.2), τ(v → f1) = 1. By (R2), we notice that v only sends charge to incident face. So, in the following each case, it remains to show that ∑i=24 τ(v → f1) ≤ 1 and therefore we have that ω0(v) ≥ 2 − 1 − 1 = 0. For simplicity, we write τ for ∑i=24 τ(v → f1). By Claims 2 and 3, we obtain that l(v) ≤ 1. In other words, v is incident to at most one light 4-face. If l(v) = 0, then τ(v → f1) = 1/3 for each i = 2, 3, 4 by (R3.2.1) and thus τ = 1/3 × 3 = 1. Now assume that l(v) = 1. By symmetry, the following proof is divided into two cases, depending on the situation of the incident light 4-face.

• Assume that f2 is a light 4-face. If f2 is a 4-face, by (a1), we have τ = 1/3 × 3 = 1. If f2 is a 6+-face, by (a2), we have τ = 1/3 + 1/3 = 1. Now assume d(f2) = 5. If either m5+(f2) = 1 or f3 is a good 5-face, then τ = 2/3 + 1/3 = 1 by (a3.1). Otherwise, assume that f3 is a bad 5-face and f2 is adjacent to another 5+-face f* different from f3. We also obtain that τ = 1/3 × 3 = 1 by (a3.2) or τ = 2/3 + 1/3 = 1 by (a3.2.1).
• Assume that f2 is a light 4-face. If at least one of f2 and f4 is a 6+-face, say f2, then by (b1), we have that τ = 2/3 + 1/3 = 1. So, in the following, suppose that f3 is either a 4-face or a 5-face for each i ∈ {2, 4}. If d(f2) = d(f4) = 4, then τ = 1/3 × 3 = 1 by (b2). Assume that d(f2) = d(f4) = 5. If at least one of f2, f4 is a 5-face, then τ = 2/3 + 1/3 = 1 by (b3.2). Otherwise, τ = 1/6 × 2 + 2/3 = 1 by (b3.1). Now, by symmetry, assume that d(f2) = 4 and d(f4) = 5. If f4 is a good 5-face, then τ = 1/3 + 2/3 = 1 by (b4.1). Now assume f3 is a bad 5-face. If m5+(f3) = 1, then τ = 1/3 + 2/3 = 1 by (b4.2.1). Otherwise, it is clear that f3 is adjacent to another 5+-face f* different from f4. If f* is bad, then τ = 1/3 + 2/3 = 1 by (b4.2.2). Otherwise, we deduce that τ = 1/3 × 3 = 1 by (b4.2.2).

It remains to show that ω0(f) ≥ 0 for f ∈ F(G). Let f = [v1v2 . . . vn] be an m-face. Denote fi be the adjacent face to f by sharing a common edge vi and where indices are taken modulo m. The proof is divided into four cases below according to the value of d(f).

Case 4. If d(f) ≥ 6, then ω0(f) = d(f) − 6 ≥ 0 by (R1)–(R3).

Case 5. If d(f) = 3, then ω(f) = −3. By Claim 1, f is incident to three 4+-vertices and thus ω(f) = −3 + 1 × 3 = 0 by (R1)–(R3).

Case 6. If d(f) = 4, then ω(f) = −2. By Claim 1, we see that d(vi) ≥ 4 for all i = 1, 2, 3, 4. Moreover, for i ∈ {1, 2, 3, 4}, vi sends at least 1/3 to f by Observation 1. This observation will be used frequently.
without further notice. If \( f \) is incident to at least one \( 6^+ \)-vertex, say \( v_1 \), then \( \tau(v_1 \to f) = 1 \) by (R1) and thus \( \omega^*(f) \geq -2 + 1 + \frac{2}{3} \times 3 = 0 \). Now, in the following, we assume that \( 4 \leq d(v_i) \leq 5 \) for all \( i = 1, 2, 3, 4 \). By symmetry, we only need to consider six subcases below.

First assume that \( d(v_i) = 4 \) for all \( i = 1, 2, 3, 4 \). Namely, \( f \) is a light 4-face. By (2) of Lemma 1, \( f \) is adjacent to at least one \( 5^- \)-face. Without loss of generality, assume that \( f_1 \) is a \( 5^- \)-face. If \( d(f_1) \geq 6 \), then \( \tau(v_1 \to f) = \tau(v_2 \to f) = \frac{2}{3} \) by (R3.1), (a2) and (b1). Therefore, \( \omega^*(f) \geq -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0 \). So assume that \( f_1 = [v_1u_1u_2u_3v_2] \) is a \( 5^- \)-face. If \( f_1 \) is a good 5-face, by (R3.1), (a3.1), (b3.2) and (b4.1), we see that each of \( v_1 \) and \( v_2 \) sends \( \frac{2}{3} \) to \( f \), respectively. Thus \( \omega^*(f) \geq -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0 \). Now assume \( f_1 \) is a bad 5-face. If \( f_2, f_3, f_4 \) are all \( 4^- \)-faces, then similarly we obtain that \( \omega^*(f) \geq -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0 \) by (R3.1), (a3.1), (b3.1) and (b4.2.1). So, in the following, we may suppose that \( f_1 \) is a \( 5^+ \)-face for some fixed \( i \in \{2, 3, 4\} \). Moreover, we may suppose that \( f_i \) is a bad 5-face. If not, we can reduce the argument to the previous cases. By symmetry, we have two cases below.

- Assume \( f_3 \) is a bad 5-face. It follows from (R3.1), (a3.2.1), (b1), (b3.1), (b3.2) and (b4.2.2) that \( \tau(v_1 \to f) \geq \frac{1}{3} \) for each \( i = 1, 2, 3, 4 \). Thus, \( \omega^*(f) \geq -2 + \frac{1}{3} \times 3 = 0 \).

- Assume \( f_4 \) is a bad 5-face. It implies that \( v_1 \) is a 4-vertex which incident to two opposite bad 5-faces. By (b3.1), \( \tau(v_1 \to f) = \frac{2}{3} \). Similarly, by (R3.1), (a3.2.1), (b1), (b3.1), (b3.2) and (b4.2.2)

Next assume that \( d(v_1) \geq 5 \) and \( d(v_i) = 4 \) for all \( i = 2, 3, 4 \). By (R1) and (R2), \( v_1 \) sends 1 to \( f \). Hence, \( \omega^*(f) \geq -2 + \frac{1}{3} \times 3 = 0 \).

Next assume that \( d(v_1) = d(v_2) = 5 \) and \( d(v_3) = d(v_4) = 4 \). Since each special 4-face is either a \((5, 4, 4, 4, 6^-)\)-face or a \((5, 4, 5^+ , 5^-)\)-face, neither \( v_1 \) nor \( v_2 \) can be a special 5-vertex of \( f \). Thus \( \omega^*(f) \geq -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0 \).

Next assume that \( d(v_1) = d(v_2) = 5 \) and \( d(v_3) = d(v_4) = 4 \). The discussion is similar to the above case.

Now assume that \( d(v_1) = d(v_2) = d(v_3) = 5 \) and \( d(v_4) = 4 \). We first notice that \( v_2 \) cannot be a special vertex since neither \( f_1 \) nor \( f_2 \) is a \((5, 4, 4, 4, 4^-)\)-face. If at most one of \( v_1, v_3 \) is a special vertex, then it is easy to derive that \( \omega^*(f) \geq -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0 \) by (R2.2) and (R2.3). Otherwise, suppose that \( v_1 \) and \( v_3 \) are both special 5-vertices. By the definition, we obtain immediately that \( f_1 \) and \( f_2 \) are both 3-faces while \( f_3, f_4 \) are both \((5, 4, 4, 4^-)\)-faces. This contradicts the assumption on \( G \).

Finally assume that \( d(v_1) = 5 \) for all \( i = 1, 2, 3, 4 \). Notice again that none of \( v_1, v_2, v_3, v_4 \) is a special 5-vertex. Consequently, \( \omega^*(f) \geq -2 + \frac{2}{3} \times 4 + \frac{2}{3} = 0 \) by (R2.2).

Claim 11. Suppose \( v \) is a 4-vertex. Let \( f_1, f_2, f_3, f_4 \) denote the faces of \( G \) incident to \( v \) in a cyclic order such that \( f_1 \) is a 5-face. If neither \( f_2 \) nor \( f_4 \) is a light 4-face, then \( \tau(v \to f_1) \geq \frac{1}{3} \).

Proof. First assume that \( l(v) = 0 \). It follows immediately from (R3.1.1) and (R3.2.1) that \( \tau(v \to f_1) \geq \frac{1}{3} \) and thus we are done. Otherwise, assume that \( f_2 \) is a light 4-face. By (a1), (a2) and (a3), it is easy to deduce that \( \tau(v \to f_1) \geq \frac{1}{3} \). Thus, we complete the proof of Claim 11. \( \square \)

Case 4. If \( d(f) = 5 \), then \( \omega^*(f) = -1 \). Notice that \( d(v_i) \geq 4 \) by Claim 1. If \( f \) is incident to at least one \( 6^- \)-vertex, then \( \omega^*(f) \geq -1 + 1 = 0 \) by (R1). So, in the following, assume that \( 4 \leq d(v_i) \leq 5 \) for all \( i = 1, \ldots, 5 \). In what follows, let \( n_5(f) \) denote the number of 5-vertices incident to \( f \). First assume that \( n_5(f) \geq 3 \). It is trivial that \( \omega^*(f) \geq -1 + \frac{1}{3} \times 3 = 0 \) by (R2).

Next assume that \( n_5(f) = 2 \). By (R2), each 5-vertex sends at least \( \frac{1}{3} \) to \( f \). It suffices to show that \( f \) gets at least \( \frac{1}{3} \) from the remaining 4-vertices in total. By symmetry, we have two possibilities:

- Assume \( d(v_1) = d(v_2) = 5 \). It implies that \( d(v_3) = d(v_4) = d(v_5) = 4 \). So there are at most two light 4-faces adjacent to \( f \). If \( l(f) = 2 \), i.e., \( f_3, f_4 \), then \( \tau(v_4 \to f) = \frac{1}{3} \) by (R3.1.3). Suppose \( l(f) = 1 \). By symmetry, suppose that \( f_3 \) is a light 4-face and \( f_4 \) is not. By Claim 11, it is easy to deduce that \( \tau(v_5 \to f) \geq \frac{1}{3} \) since \( f_5 \) is not a light 4-face. Finally suppose that \( l(f) = 0 \). We obtain immediately that \( v_4 \) sends at least \( \frac{1}{3} \) to \( f \) by Claim 11.

- Assume \( d(v_1) = d(v_2) = 5 \). Then \( d(v_3) = d(v_4) = d(v_5) = 4 \). Obviously, neither \( f_1 \) nor \( f_2 \) is a light 4-face. Thus, \( v_2 \) sends at least \( \frac{1}{3} \) to \( f \) by Claim 11.
Now assume \( n_5(f) = 1 \), say \( v_1 \). Then \( d(v_i) = 4 \) for all \( i = 2, 3, 4, 5 \) and \( l(f) \leq 3 \). If \( l(f) = 3 \), then \( \tau(v_3 \rightarrow f) = \tau(v_4 \rightarrow f) = \frac{1}{3} \) by (R3.1.3) and \( \tau(v_1 \rightarrow f) \geq \frac{1}{3} \) by (R2.2) and (R2.3). Thus, \( \omega^*(f) \geq -1 + \frac{1}{3} + \frac{1}{3} \times 2 = 0 \). If \( l(f) \leq 1 \), then there exist \( v_i \) and \( v_j \) whose incident light 4-face must be opposite to \( f \). By Claim 11, each of them sends \( \frac{1}{3} \) to \( f \) and hence \( \omega^*(f) \geq -1 + \frac{1}{3} \times 3 = 0 \).

Now assume that \( l(f) = 2 \). If \( f_2, f_3 \) are light 4-faces and \( f_4 \) is not, then \( \tau(v_3 \rightarrow f) = \tau(v_5 \rightarrow f) \geq \frac{1}{3} \) by (R3.1.3) and (R2.2). Thus, \( \omega^*(f) \geq -1 + \frac{1}{3} + \frac{1}{3} \times 3 = 0 \). If \( f_2, f_4 \) are light 4-faces and \( f_3 \) is not, then \( f \) is a light 5-face. By Lemma 2, at least one of \( f_2 \) and \( f_4 \) is adjacent to a \( 5^+ \)-face different from \( f \), say \( f_2 \). By (R3.1), (a.3.2.1), (a.3.2.2), (b1), (b3.1), (b3.2), (b4.2.2), we assert that each of \( v_2, v_3 \) sends at least \( \frac{1}{6} \) to \( f \). Therefore, \( \omega^*(f) \geq -1 + \frac{2}{3} + \frac{1}{6} \times 3 = 0 \) by (R2.2).

Finally assume that \( n_5(f) = 0 \). Namely, \( d(v_i) = 4 \) for all \( i = 1, \ldots, 5 \). In other words, \( f \) is a light 5-face. By Claim 4, none of \( f_1, \ldots, f_5 \) is a light 4-face. It follows directly from Claim 11 that each \( v_i \) sends at least \( \frac{1}{6} \) to \( f \). Therefore, we conclude that \( \omega^*(f) \geq -1 + \frac{1}{3} \times 5 = \frac{2}{3} \). \( \square \)

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**References**