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Journal of Combinatorial Theory, Series A 106 (2004) 165–191

Journal of Combinatorial Theory Series A

http://www.elsevier.com/locate/jcta

Bose–Mesner algebras attached to invertible Jones pairs

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Received 7 May 2003

Communicated by Bruce Rothschild

Abstract

In 1989, Vaughan Jones introduced spin models and showed that they could be used to form link invariants in two different ways—by constructing representations of the braid group, or by constructing partition functions. These spin models were subsequently generalized to the so-called four-weight spin models by Bannai and Bannai; these could be used to construct partition functions, but did not lead to braid group representations in any obvious way. Jaeger showed that spin models were intimately related to certain association schemes. Yamada gave a construction of a symmetric spin model on $4n$ vertices from each four-weight spin model on n vertices.

In this paper, we build on recent work with Munemasa to give a different proof to Yamada's result, and we analyze the structure of the association scheme attached to this spin model.

Keywords: Spin model; Jones pair; Association scheme; Type II matrix

1. Introduction

Spin models are a special class of matrices introduced by Jones [8] as a tool for creating link invariants. There are two strands to their subsequent development that are of interest to us. First, Jaeger and Nomura showed that all spin models could be

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0097-3165/$ - see front matter
doi:10.1016/j.jcta.2004.01.007

realized as matrices in association schemes (see [10]). Hence spin models have a combinatorial aspect and, perhaps more importantly, the search for new spin models was reduced to the search for certain special classes of association schemes. (This means that the search space is discrete rather than continuous.)

The second strand was the development of more general classes of models, culminating in the four-weight spin models of Bannai and Bannai [2]. These models are formed from a pair of matrices; they still provided link invariants, but apparently lacked the intimate connection to association schemes.

In [4], Munemasa and the present authors developed a new approach to spin models, based on what we called *Jones pairs*. We showed that these included the four-weight spin models as a special case. As a result, we were able to show that each four-weight spin model determines a pair of association schemes.

In [11], Yamada showed that each four-weight spin model of order n embeds in a very natural way in a spin model of order $4n$. We give a complete and different proof to Yamada's result. In addition, the tools we develop in Sections 2–5 allow us to analyze the structure of $\mathcal{N}_V$, which was not investigated in [11].

2. Invertible Jones pairs

Given two matrices A and B of the same order, we use $A \circ B$ to denote their Schur product, which has

$$(A \circ B)_{i,j} = A_{i,j} B_{i,j}.$$

If all entries of A are non-zero, then we say A is *Schur invertible* and define its *Schur-inverse*, $A^{(-)}$, by

$$A^{(-)}_{i,j} = \frac{1}{A_{i,j}}.$$

Equivalently, we have $A^{(-)} \circ A = J$, where J is the matrix of all ones.

For any $n \times n$ matrix C, we define two linear operators X_C and Δ_C as follows:

$$X_C(M) := CM, \quad \Delta_C(M) := C \circ M \quad \text{for all } M \in \mathbf{M}_n(\mathbb{C}).$$

Given a linear operator Y on $\mathbf{M}_n(\mathbb{C})$, we use Y^{T} to denote its adjoint relative to the non-degenerate bilinear form $tr(M^{\mathrm{T}} N)$ on $\mathbf{M}_n(\mathbb{C})$, and call it the *transpose* of Y. It is easy to see that

$$X_C^{\mathrm{T}} = X_{C^{\mathrm{T}}}, \quad \Delta_C^{\mathrm{T}} = \Delta_C.$$

A *Jones pair* is a pair of $n \times n$ complex matrices (A, B) such that X_A and Δ_B are invertible and

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B, \tag{2.1}$$

$$X_A \Delta_{B^{\mathrm{T}}} X_A = \Delta_{B^{\mathrm{T}}} X_A \Delta_{B^{\mathrm{T}}}. \tag{2.2}$$

Note that X_A and Δ_B are invertible only if A is invertible and B is Schur invertible. It is also easy to observe that (A, B) is a Jones pair if and only if (A, B^{T}) is a Jones pair. Jones pairs are designed to give representation of braid groups using Jones' construction. Please see Section 2 of [4] for a description of the construction.

An $n \times n$ matrix W is a *type-II matrix* if

$$WW^{(-)\mathrm{T}} = nI.$$

Note that a type-II matrix is invertible with respect to both matrix multiplication and the Schur product. We say that a Jones pair (A, B) is *invertible* if A is Schur invertible and B is invertible. Theorems 7.1 and 7.2 of [4] imply that a Jones pair (A, B) is invertible if and only if A and B are type-II matrices.

Let W_1, W_2, W_3 and W_4 be $n \times n$ complex matrices and let d be such that $d^2 = n$. A *four-weight spin model* is a 5-tuple $(W_1, W_2, W_3, W_4; d)$ that satisfies

$$W_3 = W_1^{(-)\mathrm{T}}, \quad W_2 = W_4^{(-)\mathrm{T}}, \tag{2.3}$$

$$W_1 W_3 = nI, \quad W_2 W_4 = nI, \tag{2.4}$$

$$\sum_{h=1}^{n} (W_1)_{k,h}(W_1)_{h,i}(W_4)_{h,j} = d(W_4)_{i,j}(W_1)_{k,i}(W_4)_{k,j}, \tag{2.5}$$

$$\sum_{h=1}^{n} (W_1)_{h,k}(W_1)_{i,h}(W_4)_{j,h} = d(W_4)_{j,i}(W_1)_{i,k}(W_4)_{j,k}. \tag{2.6}$$

From (2.3) and (2.4), we see that both W_1 and W_4 are type-II matrices and they determine W_3 and W_2, respectively. Furthermore, it is straightforward to verify that Eqs. (2.5) and (2.6) are equivalent to Eqs. (2.1) and (2.2) when $W_1 = dA$ and $W_4 = B$.

Jaeger showed in [6] that (A, B) and (C, B) are invertible Jones pairs if and only if $C = DAD^{-1}$ for some invertible diagonal matrix D. We say that these two invertible Jones pairs are *odd-gauge equivalent*. Proposition 7 of [6] states that for every invertible Jones pair (A, B), there exists an invertible diagonal matrix D such that DAD^{-1} is symmetric. Since odd-gauge equivalent invertible Jones pairs give the same link invariants, we suffer no loss by considering only invertible Jones pairs whose first matrix is symmetric.

3. Nomura algebras

We start this section by defining the *Nomura algebras* $\mathcal{N}_{A,B}$ and $\mathcal{N}'_{A,B}$ of a pair of $n \times n$ matrices. When A is a type-II matrix and $B = A^{(-)}$, our construction gives the Nomura algebras discussed in [7,10]. The definitions here are taken from [4].

Let A and B be $n \times n$ matrices, let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbb{C}^n$ and form the n^2 column vectors

$$Ae_i \circ Be_j \quad \text{for } i,j = 1, \ldots, n.$$

We define $\mathcal{N}_{A,B}$ to be the set of matrices of which $Ae_i \circ Be_j$ is an eigenvector, for all $i, j = 1, \dots, n$. This set of matrices is closed under matrix multiplication and contains the identity matrix I_n.

For each matrix $M \in \mathcal{N}_{A,B}$, we use $\Theta_{A,B}(M)$ to denote the $n \times n$ matrix that satisfies

$$M(Ae_i \circ Be_j) = \Theta_{A,B}(M)_{i,j}(Ae_i \circ Be_j).$$

We view $\Theta_{A,B}$ as a linear map from $\mathcal{N}_{A,B}$ to $\mathbf{M}_n(\mathbb{C})$ and we use $\mathcal{N}'_{A,B}$ to denote the image of $\mathcal{N}_{A,B}$. By the definition of $\Theta_{A,B}$, we have

$$\Theta_{A,B}(MN) = \Theta_{A,B}(M) \circ \Theta_{A,B}(N).$$

Consequently, the space $\mathcal{N}'_{A,B}$ is closed under the Schur product. Since $I_n \in \mathcal{N}_{A,B}$, the matrix $\Theta_{A,B}(I_n) = J_n$ belongs to $\mathcal{N}'_{A,B}$. We conclude that $\mathcal{N}'_{A,B}$ is a commutative algebra with respect to the Schur product.

If A is invertible, then the columns of A are linearly independent. Further if B is Schur invertible, then for any j

$$\{Ae_1 \circ Be_j, Ae_2 \circ Be_j, \dots, Ae_n \circ Be_j\}$$

is a basis of $\mathbb{C}^n$. In this case, the map $\Theta_{A,B}$ is an isomorphism from $\mathcal{N}_{A,B}$, as an algebra with respect to the matrix multiplication, to $\mathcal{N}'_{A,B}$, as an algebra with respect to the Schur product. We conclude from the commutativity of $\mathcal{N}'_{A,B}$ that $\mathcal{N}_{A,B}$ is commutative with respect to matrix multiplication.

The following result is called the *Exchange Lemma*. It will serve as a powerful tool in Sections 6 and 7. The proof of Theorem 3.2 also demonstrates the usefulness of this lemma.

Lemma 3.1 (Chan et al. [4, Lemma 5.1] [Exchange]). *If* $A, B, C, Q, R, S \in \mathbf{M}_n(\mathbb{C})$ *then*

$$X_A \Delta_B X_C = \Delta_Q X_R \Delta_S$$

if and only if

$$X_A \Delta_C X_B = \Delta_R X_Q \Delta_{S^\mathrm{T}}.$$

Theorem 3.2. *If A and B are* $n \times n$ *type-II matrices, then the following are equivalent*:

(a) $R \in \mathcal{N}_{A,B}$ *and* $S = \Theta_{A,B}(R)$.
(b) $X_R \Delta_B X_A = \Delta_B X_A \Delta_S$.
(c) $X_R \Delta_A X_B = \Delta_A X_B \Delta_{S^\mathrm{T}}$.
(d) $\Delta_{B^\mathrm{T}} X_{B^{(-)\mathrm{T}}} \Delta_{nR} = X_{S^\mathrm{T}} \Delta_{A^{(-)\mathrm{T}}} X_{A^\mathrm{T}}$.
(e) $\Delta_{A^{(-)\mathrm{T}}} X_{A^\mathrm{T}} \Delta_{nR^\mathrm{T}} = X_S \Delta_{B^\mathrm{T}} X_{B^{(-)\mathrm{T}}}$.

Proof. The equivalence of (a) and (b) follows from Theorem 6.2 of [4].

Applying the Exchange Lemma to (b) gives (c), which is equivalent to

$$\Delta_{A^{(-)}} X_R \Delta_A = X_B \Delta_{S^\mathrm{T}} X_{B^{-1}}. \tag{3.1}$$

Applying the Exchange Lemma to Eq. (3.1) again, we get

$$\Delta_R X_{A^{(-)}} \Delta_{A^{\mathrm{T}}} = X_B \Delta_{B^{-1}} X_{S^{\mathrm{T}}}.$$

Now we have $B^{-1} = n^{-1}B^{(-)\mathrm{T}}$ and $A^{(-)} = n(A^{-1})^{\mathrm{T}}$ because A and B are type-II matrices. The above equation becomes

$$\Delta_R X_{n(A^{-1})^{\mathrm{T}}} \Delta_{A^{\mathrm{T}}} = X_B \Delta_{n^{-1}B^{(-)\mathrm{T}}} X_{S^{\mathrm{T}}}$$

which leads to

$$\Delta_{B^{\mathrm{T}}} X_{B^{-1}} \Delta_{nR} = X_{S^{\mathrm{T}}} \Delta_{A^{(-)\mathrm{T}}} X_{n^{-1}A^{\mathrm{T}}}. \tag{3.2}$$

We get (d) after multiplying both sides of Eq. (3.2) by n and replacing nB^{-1} by $B^{(-)\mathrm{T}}$.

Taking the transpose of both sides of Eq. (3.2) gives

$$\Delta_{nR} X_{(B^{-1})^{\mathrm{T}}} \Delta_{B^{\mathrm{T}}} = X_{n^{-1}A} \Delta_{A^{(-)\mathrm{T}}} X_S$$

and

$$\Delta_{A^{\mathrm{T}}} X_{A^{(-)\mathrm{T}}} \Delta_{nR} = X_S \Delta_{B^{(-)\mathrm{T}}} X_{B^{\mathrm{T}}}.$$

We get (e) after applying the Exchange Lemma to the above equation. □

Now we state an easy consequence of Theorem 3.2(b).

Corollary 3.3 (Chan et al. [4, Lemma 10.2]). *Let A and B be $n \times n$ type-II matrices. If $R \in \mathcal{N}_{A,B}$ then*

$$R^{\mathrm{T}} \in \mathcal{N}_{A^{(-)},B^{(-)}} \quad \textit{and} \quad \Theta_{A^{(-)},B^{(-)}}(R^{\mathrm{T}}) = \Theta_{A,B}(R).$$

4. Nomura algebras of a type-II matrix

When A is a type-II matrix and $B = A^{(-)}$, existing papers such as [7] use $\mathcal{N}_A$, $\mathcal{N}'_A$ and Θ_A to denote $\mathcal{N}_{A,B}$, $\mathcal{N}'_{A,B}$ and $\Theta_{A,B}$, respectively. The algebra $\mathcal{N}_A$ is called the Nomura algebra of A. We now present some results on $\mathcal{N}_A$ due to Jaeger et al. [7] which we will use later.

When $B = A^{(-)}$, Condition 3.2(e) becomes

$$\Delta_{A^{(-)\mathrm{T}}} X_{A^{\mathrm{T}}} \Delta_{nR^{\mathrm{T}}} = X_S \Delta_{A^{(-)\mathrm{T}}} X_{A^{\mathrm{T}}}$$

and it implies

$$\Theta_{A^{\mathrm{T}}}(S) = \Theta_{A^{\mathrm{T}}}(\Theta_A(R)) = nR^{\mathrm{T}}. \tag{4.1}$$

We conclude that if $R \in \mathcal{N}_A$ then $\Theta_A(R) \in \mathcal{N}_{A^{\mathrm{T}}}$ and $R^{\mathrm{T}} \in \mathcal{N}'_{A^{\mathrm{T}}}$. Hence

$$\mathcal{N}'_A \subseteq \mathcal{N}_{A^{\mathrm{T}}} \quad \text{and} \quad \dim(\mathcal{N}_A) = \dim(\mathcal{N}_A^{\mathrm{T}}) \leqslant \dim(\mathcal{N}'_{A^{\mathrm{T}}}).$$

Similarly A^{T} is also a type-II matrix, so

$$\mathcal{N}'_{A^{\mathrm{T}}} \subseteq \mathcal{N}_A \quad \text{and} \quad \dim(\mathcal{N}_{A^{\mathrm{T}}}) \leqslant \dim(\mathcal{N}'_A).$$

Therefore $\mathcal{N}'_A = \mathcal{N}_{A^T}$ and $\mathcal{N}'_{A^T} = \mathcal{N}_A$, which implies that $\mathcal{N}_A$ and $\mathcal{N}_{A^T}$ are closed under both matrix multiplication and the Schur product. It also implies that $\mathcal{N}_A = \mathcal{N}'_{A^T}$ is closed under the transpose. Since A is invertible and $A^{(-)}$ is Schur invertible, the map Θ_A is an isomorphism from $\mathcal{N}_A$ to $\mathcal{N}'_A$. Hence $\mathcal{N}_A$ is commutative with respect to matrix multiplication. In summary, the algebra $\mathcal{N}_A$ is commutative with respect to matrix multiplication, is also closed under the transpose and the Schur product, and contains I and J. In other words, $\mathcal{N}_A$ is a *Bose–Mesner algebra*.

We now investigate the properties of the map Θ_A. Let M and N be matrices in $\mathcal{N}_A$. Since $\Theta_{A^T} : \mathcal{N}_{A^T} \to \mathcal{N}_A$ is an isomorphism, there exist M' and N' in $\mathcal{N}_{A^T}$ such that $\Theta_{A^T}(M') = M$ and $\Theta_{A^T}(N') = N$. Hence

$$\begin{aligned}\Theta_A(M \circ N) &= \Theta_A(\Theta_{A^T}(M') \circ \Theta_{A^T}(N'))\\ &= \Theta_A(\Theta_{A^T}(M'N'))\end{aligned}$$

which equals $n(M'N')^T$ by Eq. (4.1). Since

$$\Theta_A(M) = \Theta_A(\Theta_{A^T}(M')) = nM'^T$$

and $\Theta_A(N) = nN'^T$, we have

$$\begin{aligned}\Theta_A(M \circ N) &= \frac{1}{n}(nN'^T)(nM'^T)\\ &= \frac{1}{n}\Theta_A(N)\Theta_A(M)\\ &= \frac{1}{n}\Theta_A(M)\Theta_A(N),\end{aligned}$$

the last equality results from the commutativity of $\mathcal{N}'_A$. Now we conclude that Θ_A swaps matrix multiplication with the Schur product.

Furthermore, applying $\frac{1}{n}\Theta_A$ to the two rightmost terms of Eq. (4.1) gives

$$\frac{1}{n}\Theta_A(\Theta_{A^T}(\Theta_A(R))) = \Theta_A(R^T).$$

It follows from Eq. (4.1) that the left-hand side equals $\Theta_A(R)^T$. Thus Θ_A and the transpose commute. From Corollary 3.3, we see that

$$\Theta_{A^{(-)}}(R) = \Theta_A(R)^T.$$

Also note that by Eq. (4.1), we have

$$\Theta_A(J) = \Theta_A(\Theta_{A^T}(I)) = nI.$$

We call Θ_A a *duality map* from $\mathcal{N}_A$ to $\mathcal{N}_{A^T}$ and say that these two Bose–Mesner algebras form a *formally dual pair*. If $\mathcal{N}_A = \mathcal{N}_{A^T}$ and $\Theta_A = \Theta_{A^T}$, we say that it is *formally self-dual*.

A *spin model* is an $n \times n$ matrix W such that $(W, W, W^{(-)}, W^{(-)}; d)$ is a four-weight spin model, for $d^2 = n$. It follows from Section 9 of [4] that W is a spin model if and only if $(d^{-1}W, W^{(-)})$ is an invertible Jones pair. In [7], Jaeger et al. gave the following characterization of a spin model W using its Nomura algebra $\mathcal{N}_W$.

Theorem 4.1 (Jaeger et al. [7, Theorem 11]). *Suppose W is a type-II matrix. Then $W \in \mathcal{N}_W$ if and only if cW is a spin model for some non-zero scalar c. In this case*,

$$\mathcal{N}_W = \mathcal{N}_{W^T}$$

is a formally self-dual Bose–Mesner algebra with duality map $\Theta_W = \Theta_{W^T}$.

5. Nomura algebras of an invertible Jones pair

We study the relation among the different Nomura algebras of an invertible Jones pair.

Theorem 5.1 (Bannai [1, Theorem 3]). *If (A, B) is an invertible Jones pair, then*

$$\mathcal{N}_A = \mathcal{N}_{A^T} = \mathcal{N}_B = \mathcal{N}_{B^T},$$

the duality maps satisfy $\Theta_A = \Theta_{A^T}$ and $\Theta_B = \Theta_{B^T}$.

Bannai et al. [1] proved this result for four-weight spin models, which are equivalent to invertible Jones pairs. For an alternate proof using the Nomura algebras of A and B, see Section 10 of [4].

Let A and B be type-II matrices. We see from Theorem 3.2(a) and (b) that (A, B) is an invertible Jones pair if and only if $A \in \mathcal{N}_{A,B} \cap \mathcal{N}_{A,B^T}$, $\Theta_{A,B}(A) = B$ and $\Theta_{A,B^T}(A) = B^T$. The next two results provide some insights to the relations among $\mathcal{N}_{A,B}$, $\mathcal{N}'_{A,B}$ and $\mathcal{N}_A$.

Theorem 5.2 (Chan et al. [4, Theorem 10.3]). *Let A and B be $n \times n$ type-II matrices. If $F \in \mathcal{N}_A$, $G \in \mathcal{N}_{A,B}$ and $H \in \mathcal{N}_B$, then $F \circ G$, and $G \circ H$ belong to $\mathcal{N}_{A,B}$ and*

$$\Theta_{A,B}(F \circ G) = n^{-1} \Theta_A(F) \Theta_{A,B}(G),$$
$$\Theta_{A,B}(G \circ H) = n^{-1} \Theta_{A,B}(G) \Theta_B(H)^T.$$

Theorem 5.3 (Chan et al. [4, Theorem 10.4]). *Let A and B be $n \times n$ type-II matrices. If $F, G \in \mathcal{N}_{A,B}$, then $F \circ G^T \in \mathcal{N}_A \cap \mathcal{N}_B$ and*

$$\Theta_A(F \circ G^T) = n^{-1} \Theta_{A,B}(F) \Theta_{A,B}(G)^T,$$
$$\Theta_B(F \circ G^T) = n^{-1} \Theta_{A,B}(F)^T \Theta_{A,B}(G). \tag{5.1}$$

We list two consequences of Theorems 5.2 and 5.3.

Theorem 5.4 (Chan et al. [4, Theorem 10.6]). *Let A and B be $n \times n$ type-II matrices. If $\mathcal{N}_{A,B}$ contains a Schur invertible matrix G and $H = \Theta_{A,B}(G)$, then*

$$\mathcal{N}_{A,B} = G \circ \mathcal{N}_A, \quad \mathcal{N}'_{A,B} H^{\mathrm{T}} = \mathcal{N}_{A^{\mathrm{T}}}.$$

Corollary 5.5 (Chan et al. [4, Corollary 10.9]). *If (A, B) is an invertible Jones pair, then*

$$\Theta_B(M)^{\mathrm{T}} = B^{-1} \Theta_A(M) B$$

for all $M \in \mathcal{N}_A$.

Now we present an important application of Theorems 5.2 and 5.3, which implies that the Nomura algebras $\mathcal{N}_A$, $\mathcal{N}_{A,B}$ and $\mathcal{N}'_{A,B}$ have the same dimension.

Theorem 5.6. *Let (A, B) be an invertible Jones pair. Then*

$$\mathcal{N}_{A,B} = A \circ \mathcal{N}_A, \quad \mathcal{N}'_{A,B} B^{\mathrm{T}} = \mathcal{N}_A$$

and

$$\mathcal{N}'_{A,B} = (\mathcal{N}'_{A,B^{\mathrm{T}}})^{\mathrm{T}}.$$

Proof. We get the first equality by letting $G = A$ in Theorem 5.4. Since $B = \Theta_{A,B}(A)$, we have

$$\mathcal{N}'_{A,B} B^{\mathrm{T}} = \mathcal{N}_{A^{\mathrm{T}}}.$$

By Theorem 5.1, we have $\mathcal{N}_{A^{\mathrm{T}}} = \mathcal{N}_A$ and hence the second equality holds.

If we replace B by B^{T} in the above equality, then we get

$$\mathcal{N}'_{A,B^{\mathrm{T}}} B = \mathcal{N}_{A^{\mathrm{T}}}.$$

Since multiplication by B is injective, the dimensions of $\mathcal{N}_A = \mathcal{N}_{A^{\mathrm{T}}}$ and $\mathcal{N}'_{A,B^{\mathrm{T}}}$ are equal. Now we let G equal A and replace B by B^{T} in Eq. (5.1). We get

$$\mathcal{N}'_{B^{\mathrm{T}}} \subseteq (\mathcal{N}'_{A,B^{\mathrm{T}}})^{\mathrm{T}} B^{\mathrm{T}}.$$

By Theorem 5.1, $\mathcal{N}_A = \mathcal{N}_B = \mathcal{N}'_{B^{\mathrm{T}}}$. Since $\mathcal{N}_A$ and $\mathcal{N}'_{A,B^{\mathrm{T}}}$ have the same dimension, we have

$$\mathcal{N}_A = (\mathcal{N}'_{A,B^{\mathrm{T}}})^{\mathrm{T}} B^{\mathrm{T}}.$$

Thus $\mathcal{N}'_{A,B} B^{\mathrm{T}} = (\mathcal{N}'_{A,B^{\mathrm{T}}})^{\mathrm{T}} B^{\mathrm{T}}$, which leads to the last equality of the theorem. □

Corollary 5.7. *Let (A, B) be an invertible Jones pair. Then*

$$\mathcal{N}_{A,B} = \mathcal{N}_{A,B^{\mathrm{T}}}.$$

Moreover, if A is symmetric, then

$$\mathcal{N}_{A,B} = (\mathcal{N}_{A,B})^{\mathrm{T}}.$$

Proof. Applying Theorem 5.6 to the invertible Jones pairs (A, B) and (A, B^{T}) gives

$$\mathcal{N}_{A,B} = A \circ \mathcal{N}_A = \mathcal{N}_{A,B^{\mathrm{T}}}.$$

Using the same equation, we have $\mathcal{N}^{\mathrm{T}}_{A,B} = A^{\mathrm{T}} \circ \mathcal{N}^{\mathrm{T}}_A$. Since $\mathcal{N}_A$ is closed under the transpose and A is symmetric, we conclude that $\mathcal{N}_{A,B} = \mathcal{N}^{\mathrm{T}}_{A,B}$. □

6. A Bose–Mesner algebra of order $4n$

From now on, we assume that (A, B) is an invertible Jones pair and A is symmetric.

Lemma 6.1. *For each H in $\mathcal{N}_{A,B}$, there exists a unique matrix K in $(\mathcal{N}_{A,B^{\mathrm{T}}})^{\mathrm{T}}$ such that*

$$\Theta_{A,B}(H) = \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}}. \tag{6.1}$$

Proof. Existence follows directly from the last equality in Theorem 5.6, while uniqueness holds because $\Theta_{A,B^{\mathrm{T}}}$ is an isomorphism. □

Given any matrix H in $\mathcal{N}_{A,B}$, we say that the unique K in $\mathcal{N}'_{A,B}$ satisfying Eq. (6.1) is *paired* with H.

Lemma 6.2. *For each H in $\mathcal{N}_{A,B}$, K in $\mathcal{N}'_{A,B}$ is paired with H if and only if K^{T} is paired with H^{T}. Moreover we have*

$$\Theta_A(H \circ A) = \Theta_{B^{\mathrm{T}}}(K^{\mathrm{T}} \circ A). \tag{6.2}$$

Proof. Multiplying each side of Eq. (6.1) by $n^{-1}\Theta_{A,B}(A)^{\mathrm{T}} = n^{-1}\Theta_{A,B^{\mathrm{T}}}(A)$ gives

$$n^{-1}\Theta_{A,B}(H)\Theta_{A,B}(A)^{\mathrm{T}} = n^{-1}\Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}}\Theta_{A,B^{\mathrm{T}}}(A).$$

We apply Theorem 5.3 to both sides of the above equation to get

$$\Theta_A(H \circ A^{\mathrm{T}}) = \Theta_{B^{\mathrm{T}}}(K^{\mathrm{T}} \circ A^{\mathrm{T}}).$$

Since A is symmetric, we see that Eq. (6.1) is equivalent to Eq. (6.2).

In addition, taking the transpose of both sides gives

$$\Theta_A(H^{\mathrm{T}} \circ A) = \Theta_{B^{\mathrm{T}}}(K \circ A).$$

Therefore H and K satisfy Eq. (6.1) if and only if H^{T} and K^{T} satisfy Eq. (6.1). □

For any $F \in \mathcal{N}_A$ and $H, G \in \mathcal{N}_{A,B}$, we define the $4n \times 4n$ matrix $\mathcal{M}(F, G, H)$ to be

$$\begin{pmatrix} \Theta_A(F)+H & \Theta_A(F)-H & \Theta_{A,B}(G) & \Theta_{A,B}(G) \\ \Theta_A(F)-H & \Theta_A(F)+H & \Theta_{A,B}(G) & \Theta_{A,B}(G) \\ \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}} & \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}} & \Theta_{B^{(-)}}(F)+K & \Theta_{B^{(-)}}(F)-K \\ \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}} & \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}} & \Theta_{B^{(-)}}(F)-K & \Theta_{B^{(-)}}(F)+K \end{pmatrix},$$

where K is paired with H. We consider the space

$$\mathcal{B} := \{\mathcal{M}(F, G, H) : F \in \mathcal{N}_A \text{ and } H, G \in \mathcal{N}_{A,B}\}. \tag{6.3}$$

Now we show that $\mathcal{B}$ is a Bose–Mesner algebra. It turns out that $\mathcal{B}$ contains the $4n \times 4n$ type-II matrix V defined at the beginning of Section 7 and it is a subscheme of $\mathcal{N}_V$. This leads to the main result of this paper which says that V is a spin model if and only if (A, B) is an invertible Jones pair.

To convince ourselves that $\mathcal{B}$ is a Bose–Mesner algebra, we need to check that $\mathcal{B}$ contains the identity matrix I_{4n} and the matrix of all ones J_{4n}; it is closed under the transpose; it is a commutative algebra with respect to matrix multiplication; it is closed under the Schur product.

Lemma 6.3. *The vector space $\mathcal{B}$ contains I_{4n} and J_{4n}.*

Proof. The matrix K that is paired with $\frac{1}{2}I_n$ satisfies

$$\Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}} = \Theta_{A,B}\left(\frac{1}{2}I_n\right) = \frac{1}{2}J_n.$$

Since $\Theta_{A,B^{\mathrm{T}}}$ is an isomorphism, we conclude that $K = \frac{1}{2}I_n$. Note that $\Theta_A(\frac{1}{2n}J_n) = \frac{1}{2}I_n$. Thus $\mathcal{M}(\frac{1}{2n}J_n, \mathbf{0}, \frac{1}{2}I_n) = I_{4n}$ belongs to $\mathcal{B}$.

Since $\Theta_A(I_n) = \Theta_{A,B}(I_n) = J_n$, the matrix $\mathcal{M}(I_n, I_n, \mathbf{0}) = J_{4n}$ belongs to $\mathcal{B}$. □

Lemma 6.4. *The vector space $\mathcal{B}$ is closed under transpose.*

Proof. Let $\mathcal{M}(F, G, H) \in \mathcal{B}$. Now $\mathcal{M}(F, G, H)^{\mathrm{T}}$ equals

$$\begin{pmatrix} \Theta_A(F)^{\mathrm{T}}+H^{\mathrm{T}} & \Theta_A(F)^{\mathrm{T}}-H^{\mathrm{T}} & \Theta_{A,B}(G^{\mathrm{T}}) & \Theta_{A,B}(G^{\mathrm{T}}) \\ \Theta_A(F)^{\mathrm{T}}-H^{\mathrm{T}} & \Theta_A(F)^{\mathrm{T}}+H^{\mathrm{T}} & \Theta_{A,B}(G^{\mathrm{T}}) & \Theta_{A,B}(G^{\mathrm{T}}) \\ \Theta_{A,B}(G)^{\mathrm{T}} & \Theta_{A,B}(G)^{\mathrm{T}} & \Theta_{B^{(-)}}(F)^{\mathrm{T}}+K^{\mathrm{T}} & \Theta_{B^{(-)}}(F)^{\mathrm{T}}-K^{\mathrm{T}} \\ \Theta_{A,B}(G)^{\mathrm{T}} & \Theta_{A,B}(G)^{\mathrm{T}} & \Theta_{B^{(-)}}(F)^{\mathrm{T}}-K^{\mathrm{T}} & \Theta_{B^{(-)}}(F)^{\mathrm{T}}+K^{\mathrm{T}} \end{pmatrix}.$$

Since $\mathcal{N}_{A,B}$ is closed under the transpose, the matrices G^{T} and H^{T} belong to $\mathcal{N}_{A,B}$. It follows from Lemma 6.2 that K^{T} is paired with H^{T}. Moreover, $\Theta_A(F)^{\mathrm{T}} = \Theta_A(F^{\mathrm{T}})$. As a result we conclude that

$$\mathcal{M}(F, G, H)^{\mathrm{T}} = \mathcal{M}(F^{\mathrm{T}}, G^{\mathrm{T}}, H^{\mathrm{T}})$$

and the vector space $\mathcal{B}$ is closed under the transpose. □

Lemma 6.5. *The vector space $\mathscr{B}$ is a commutative algebra under matrix multiplication.*

Proof. Let $M = \mathscr{M}(F, G, H)$ and $M_1 = \mathscr{M}(F_1, G_1, H_1)$ be any matrices in $\mathscr{B}$.
By Theorem 5.3, we have

$$\Theta_{A,B}(G)\Theta_{A,B}(G_1^{\mathrm{T}})^{\mathrm{T}} = n\Theta_A(G \circ G_1).$$

Hence the top left $2n \times 2n$ block of MM_1 equals

$$\begin{pmatrix} 2n\Theta_A(F \circ F_1 + G \circ G_1) + 2HH_1 & 2n\Theta_A(F \circ F_1 + G \circ G_1) - 2HH_1 \\ 2n\Theta_A(F \circ F_1 + G \circ G_1) - 2HH_1 & 2n\Theta_A(F \circ F_1 + G \circ G_1) + 2HH_1 \end{pmatrix}.$$

Similarly, by Theorem 5.3

$$\begin{aligned} \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}\Theta_{A,B}(G_1) &= n\Theta_B(G^{\mathrm{T}} \circ G_1^{\mathrm{T}}) \\ &= n\Theta_B(G \circ G_1)^{\mathrm{T}} \\ &= n\Theta_{B^{(-)}}(G \circ G_1). \end{aligned}$$

Consequently the bottom right $2n \times 2n$ block of MM_1 equals

$$\begin{pmatrix} 2n\Theta_{B^{(-)}}(F \circ F_1 + G \circ G_1) + 2KK_1 & 2n\Theta_{B^{(-)}}(F \circ F_1 + G \circ G_1) - 2KK_1 \\ 2n\Theta_{B^{(-)}}(F \circ F_1 + G \circ G_1) - 2KK_1 & 2n\Theta_{B^{(-)}}(F \circ F_1 + G \circ G_1) + 2KK_1 \end{pmatrix},$$

where K and K_1 are paired with H and H_1, respectively. Now we need to show that KK_1 is paired with HH_1. From Eq. (6.1), we have

$$\Theta_{A,B}(H) = \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}} \quad \text{and} \quad \Theta_{A,B}(H_1) = \Theta_{A,B^{\mathrm{T}}}(K_1^{\mathrm{T}})^{\mathrm{T}}.$$

Therefore

$$\begin{aligned} \Theta_{A,B}(HH_1) &= \Theta_{A,B}(H) \circ \Theta_{A,B}(H_1) \\ &= \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}} \circ \Theta_{A,B^{\mathrm{T}}}(K_1^{\mathrm{T}})^{\mathrm{T}} \\ &= \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}}K_1^{\mathrm{T}})^{\mathrm{T}}. \end{aligned}$$

Since $\mathscr{N}_{A,B^{\mathrm{T}}}$ is commutative with respect to matrix multiplication,

$$\Theta_{A,B}(HH_1) = \Theta_{A,B^{\mathrm{T}}}((KK_1)^{\mathrm{T}})^{\mathrm{T}}.$$

We now consider the top right $2n \times 2n$ block of MM_1. Note that

$$\begin{aligned} &2\Theta_A(F)\Theta_{A,B}(G_1) + 2\Theta_{A,B}(G)\Theta_{B^{(-)}}(F_1) \\ &\quad = 2\Theta_A(F)\Theta_{A,B}(G_1) + 2\Theta_{A,B}(G)\Theta_B(F_1)^{\mathrm{T}}. \end{aligned}$$

Applying Theorem 5.2 to each term, we get

$$2n\Theta_{A,B}(F \circ G_1 + G \circ F_1).$$

Thus the top right $2n \times 2n$ block of MM_1 is

$$\begin{pmatrix} 2n\Theta_{A,B}(F \circ G_1 + G \circ F_1) & 2n\Theta_{A,B}(F \circ G_1 + G \circ F_1) \\ 2n\Theta_{A,B}(F \circ G_1 + G \circ F_1) & 2n\Theta_{A,B}(F \circ G_1 + G \circ F_1) \end{pmatrix}.$$

Consider the bottom left $2n \times 2n$ block of MM_1, we have

$$2\Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}\Theta_A(F_1) + 2\Theta_{B^{(-)}}(F)\Theta_{A,B}(G_1^{\mathrm{T}})^{\mathrm{T}}$$
$$= 2\Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}\Theta_A(F_1) + 2\Theta_B(F)^{\mathrm{T}}\Theta_{A,B}(G_1^{\mathrm{T}})^{\mathrm{T}}.$$

Since each of Θ_A and Θ_B commutes with the transpose, the above expression becomes

$$2\Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}\Theta_A(F_1^{\mathrm{T}})^{\mathrm{T}} + 2\Theta_B(F^{\mathrm{T}})\Theta_{A,B}(G_1^{\mathrm{T}})^{\mathrm{T}}$$

which equals

$$2n\Theta_{A,B}(F_1^{\mathrm{T}}\circ G^{\mathrm{T}} + G_1^{\mathrm{T}}\circ F^{\mathrm{T}})^{\mathrm{T}}$$

by Theorem 5.2. Hence the bottom left $2n \times 2n$ block of MM_1 is

$$\begin{pmatrix} 2n\Theta_{A,B}(F_1^{\mathrm{T}}\circ G^{\mathrm{T}} + G_1^{\mathrm{T}}\circ F^{\mathrm{T}})^{\mathrm{T}} & 2n\Theta_{A,B}(F_1^{\mathrm{T}}\circ G^{\mathrm{T}} + G_1^{\mathrm{T}}\circ F^{\mathrm{T}})^{\mathrm{T}} \\ 2n\Theta_{A,B}(F_1^{\mathrm{T}}\circ G^{\mathrm{T}} + G_1^{\mathrm{T}}\circ F^{\mathrm{T}})^{\mathrm{T}} & 2n\Theta_{A,B}(F_1^{\mathrm{T}}\circ G^{\mathrm{T}} + G_1^{\mathrm{T}}\circ F^{\mathrm{T}})^{\mathrm{T}} \end{pmatrix}.$$

Now we conclude that

$$MM_1 = \mathscr{M}(2nF\circ F_1 + 2nG\circ G_1, 2nF\circ G_1 + 2nG\circ F_1, 2HH_1)$$

belongs to $\mathscr{B}$.

It follows from the commutativity of $\mathscr{N}_{A,B} = \mathscr{N}_{A,B^{\mathrm{T}}}$ that $HH_1 = H_1H$ and $KK_1 = K_1K$. Therefore all four $2n \times 2n$ blocks of MM_1 remain unchanged after swapping F with F_1, G with G_1, H with H_1 and K with K_1. Consequently, the matrices M and M_1 commute. □

Lemma 6.6. *The algebra $\mathscr{B}$ is closed under the Schur product.*

Proof. Let $M = \mathscr{M}(F, G, H)$ and $M_1 = \mathscr{M}(F_1, G_1, H_1)$ be two matrices in $\mathscr{B}$. We want to write $M\circ M_1$ as $\mathscr{M}(F', G', H')$, for some F' in $\mathscr{N}_A$ and G' and H' in $\mathscr{N}_{A,B}$. If we divide $M\circ M_1$ into sixteen $n \times n$ blocks naturally, then the $(1,1)$- and $(2,2)$-blocks of $M\circ M_1$ are equal to

$$\Theta_A(F)\circ\Theta_A(F_1) + H\circ H_1 + \Theta_A(F)\circ H_1 + \Theta_A(F_1)\circ H$$
$$= (\Theta_A(FF_1) + H\circ H_1) + (\Theta_A(F)\circ H_1 + \Theta_A(F_1)\circ H).$$

The $(1,2)$- and $(2,1)$-blocks of $M\circ M_1$ are equal to

$$(\Theta_A(FF_1) + H\circ H_1) - (\Theta_A(F)\circ H_1 + \Theta_A(F_1)\circ H).$$

The $(3,3)$- and $(4,4)$-blocks of $M\circ M_1$ are equal to

$$(\Theta_{B^{(-)}}(FF_1) + K\circ K_1) + (\Theta_{B^{(-)}}(F)\circ K_1 + \Theta_{B^{(-)}}(F_1)\circ K).$$

The $(3,4)$- and $(4,3)$-blocks of $M\circ M_1$ are equal to

$$(\Theta_{B^{(-)}}(FF_1) + K\circ K_1) - (\Theta_{B^{(-)}}(F)\circ K_1 + \Theta_{B^{(-)}}(F_1)\circ K).$$

To determine F', we need to show that there exists $\hat{F}\in\mathscr{N}_A$ such that

$$H\circ H_1 = \Theta_A(\hat{F}) \quad \text{and} \quad K\circ K_1 = \Theta_{B^{(-)}}(\hat{F})$$

and $F' = FF_1 + \hat{F}$. Now the matrix K is paired with H. Right-multiplying both sides of Eq. (6.1) by $B^{(-)\mathrm{T}}$ yields

$$\Theta_{A,B}(H)B^{(-)\mathrm{T}} = \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}}B^{(-)\mathrm{T}}$$

which is rewritten as

$$\Theta_{A,B}(H)\Theta_{A,B}(A^{-1})^{\mathrm{T}} = \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}}\Theta_{A,B^{\mathrm{T}}}(A^{-1}).$$

Since $A^{-1} = \frac{1}{n}A^{(-)\mathrm{T}}$, the above equation is equivalent to

$$\frac{1}{n}\Theta_{A,B}(H)\Theta_{A,B}(A^{(-)\mathrm{T}})^{\mathrm{T}} = \frac{1}{n}\Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}}\Theta_{A,B^{\mathrm{T}}}(A^{(-)\mathrm{T}}).$$

Applying Theorem 5.3 to each side, we get

$$\Theta_A(H\circ A^{(-)}) = \Theta_{B^{\mathrm{T}}}(K^{\mathrm{T}}\circ A^{(-)}).$$

Applying Corollary 3.3 to the right-hand side, we get

$$\begin{aligned}\Theta_A(H\circ A^{(-)}) &= \Theta_{B^{(-)\mathrm{T}}}(K\circ A^{(-)\mathrm{T}})\\ &= \Theta_{B^{(-)\mathrm{T}}}(K\circ A^{(-)}).\end{aligned}$$

Similarly, $M_1 \in \mathscr{B}$. By Lemma 6.2, the matrices H_1 and K_1 satisfy Eq. (6.2)

$$\Theta_A(H_1\circ A) = \Theta_{B^{\mathrm{T}}}(K_1^{\mathrm{T}}\circ A) = \Theta_{B^{(-)\mathrm{T}}}(K_1\circ A^{\mathrm{T}}).$$

Since A is symmetric,

$$\Theta_A(H\circ A^{(-)})\Theta_A(H_1\circ A) = \Theta_{B^{(-)\mathrm{T}}}(K\circ A^{(-)})\Theta_{B^{(-)\mathrm{T}}}(K_1\circ A)$$

and

$$\Theta_A(H\circ A^{(-)}\circ H_1\circ A) = \Theta_{B^{(-)\mathrm{T}}}(K\circ A^{(-)}\circ K_1\circ A)$$

which simplifies to

$$\Theta_A(H\circ H_1) = \Theta_{B^{(-)\mathrm{T}}}(K\circ K_1).$$

If we let $\hat{F} = \frac{1}{n}\Theta_A(H\circ H_1)^{\mathrm{T}}$, then

$$\begin{aligned}\Theta_A(\hat{F}) &= \frac{1}{n}\Theta_A(\Theta_A(H\circ H_1)^{\mathrm{T}})\\ &= \frac{1}{n}\Theta_A(\Theta_A(H\circ H_1))^{\mathrm{T}}.\end{aligned}$$

Since A is symmetric, it follows from Eq. (4.1) that $\Theta_A(\hat{F}) = H\circ H_1$ and

$$\begin{aligned}\Theta_{B^{(-)}}(\hat{F}) &= \frac{1}{n}\Theta_{B^{(-)}}(\Theta_{B^{(-)\mathrm{T}}}(K\circ K_1)^{\mathrm{T}})\\ &= K\circ K_1.\end{aligned}$$

As a result we have $F' = FF_1 + \frac{1}{n}\Theta_A(H\circ H_1)^{\mathrm{T}}$.

We see from the $(1,1)$- and $(1,2)$-blocks of $M\circ M_1$ that H' should be equal to $\Theta_A(F)\circ H_1 + \Theta_A(F_1)\circ H$. We now need to verify that $\Theta_{B^{(-)}}(F)\circ K_1 + \Theta_{B^{(-)}}(F_1)\circ K$

is paired with H'. That is,

$$\Theta_{A,B}(\Theta_A(F)\circ H_1 + \Theta_A(F_1)\circ H) = \Theta_{A,B^{\mathrm{T}}}((\Theta_{B^{(-)}}(F)\circ K_1 + \Theta_{B^{(-)}}(F_1)\circ K)^{\mathrm{T}})^{\mathrm{T}}. \tag{6.4}$$

Applying Theorem 5.2 gives

$$\begin{aligned}&\Theta_{A,B}(\Theta_A(F)\circ H_1 + \Theta_A(F_1)\circ H)\\ &\quad = \frac{1}{n}\Theta_A(\Theta_A(F))\Theta_{A,B}(H_1) + \frac{1}{n}\Theta_A(\Theta_A(F_1))\Theta_{A,B}(H)\\ &\quad = F^{\mathrm{T}}\Theta_{A,B}(H_1) + F_1^{\mathrm{T}}\Theta_{A,B}(H).\end{aligned}$$

By Eq. (6.1), the above expression equals

$$F^{\mathrm{T}}\Theta_{A,B^{\mathrm{T}}}(K_1^{\mathrm{T}})^{\mathrm{T}} + F_1^{\mathrm{T}}\Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}} = (\Theta_{A,B^{\mathrm{T}}}(K_1^{\mathrm{T}})F + \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})F_1)^{\mathrm{T}}.$$

By Eq. (4.1), we see that $F = n^{-1}\Theta_{B^{\mathrm{T}}}(\Theta_B(F))^{\mathrm{T}}$ and consequently the above expression is equal to

$$\left(\frac{1}{n}\Theta_{A,B^{\mathrm{T}}}(K_1^{\mathrm{T}})\Theta_{B^{\mathrm{T}}}(\Theta_B(F))^{\mathrm{T}} + \frac{1}{n}\Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})\Theta_{B^{\mathrm{T}}}(\Theta_B(F_1))^{\mathrm{T}}\right)^{\mathrm{T}}.$$

Applying Theorem 5.2 yields

$$\begin{aligned}&\Theta_{A,B^{\mathrm{T}}}(K_1^{\mathrm{T}}\circ\Theta_B(F))^{\mathrm{T}} + \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}}\circ\Theta_B(F_1))^{\mathrm{T}}\\ &\quad = \Theta_{A,B^{\mathrm{T}}}(K_1^{\mathrm{T}}\circ\Theta_{B^{(-)}}(F)^{\mathrm{T}} + K^{\mathrm{T}}\circ\Theta_{B^{(-)}}(F_1)^{\mathrm{T}})^{\mathrm{T}}\\ &\quad = \Theta_{A,B^{\mathrm{T}}}((K_1\circ\Theta_{B^{(-)}}(F) + K\circ\Theta_{B^{(-)}}(F_1))^{\mathrm{T}})^{\mathrm{T}}.\end{aligned}$$

Hence, Eq. (6.4) is satisfied and $H' = \Theta_A(F)\circ H_1 + \Theta_A(F_1)\circ H$.

Since

$$\Theta_{A,B}(G)\circ\Theta_{A,B}(G_1) = \Theta_{A,B}(GG_1)$$

and

$$\begin{aligned}\Theta_{A,B}(G_1^{\mathrm{T}})^{\mathrm{T}}\circ\Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}} &= \Theta_{A,B}(G_1^{\mathrm{T}}G^{\mathrm{T}})^{\mathrm{T}}\\ &= \Theta_{A,B}((GG_1)^{\mathrm{T}})^{\mathrm{T}},\end{aligned}$$

the top right $2n\times 2n$ and the bottom left $2n\times 2n$ blocks of $M\circ M_1$ are

$$\begin{pmatrix}\Theta_{A,B}(GG_1) & \Theta_{A,B}(GG_1)\\ \Theta_{A,B}(GG_1) & \Theta_{A,B}(GG_1)\end{pmatrix}$$

and

$$\begin{pmatrix}\Theta_{A,B}((GG_1)^{\mathrm{T}})^{\mathrm{T}} & \Theta_{A,B}((GG_1)^{\mathrm{T}})^{\mathrm{T}}\\ \Theta_{A,B}((GG_1)^{\mathrm{T}})^{\mathrm{T}} & \Theta_{A,B}((GG_1)^{\mathrm{T}})^{\mathrm{T}}\end{pmatrix},$$

respectively. We conclude that $G' = GG_1$ and that

$$M \circ M_1 = \mathscr{M}\left(FF_1 + \frac{1}{n}\Theta_A(H \circ H_1)^{\mathrm{T}}, GG_1, \Theta_A(F) \circ H_1 + \Theta_A(F_1) \circ H\right)$$

belongs to $\mathscr{B}$. □

Theorem 6.7. *The algebra $\mathscr{B}$ is a Bose–Mesner algebra whose dimension is three times the dimension of $\mathscr{N}_A$.*

Proof. It follows from Lemmas 6.3–6.6 that $\mathscr{B}$ is a Bose–Mesner algebra. By the definition of the matrices in $\mathscr{B}$, the algebra $\mathscr{B}$ is the direct sum of three vector spaces. The first one consists of matrices $\mathscr{M}(F, \mathbf{0}, \mathbf{0})$ for all $F \in \mathscr{N}_A$. This space is isomorphic to $\mathscr{N}_A$. The second vector space consists of matrices $\mathscr{M}(\mathbf{0}, G, \mathbf{0})$ for all $G \in \mathscr{N}_{A,B}$. The third one consists of matrices $\mathscr{M}(\mathbf{0}, \mathbf{0}, H)$ for all $H \in \mathscr{N}_{A,B}$. Both the second and the third vector spaces are isomorphic to $\mathscr{N}_{A,B}$. By Theorem 5.4, $\mathscr{N}_A$ and $\mathscr{N}_{A,B}$ have the same dimension. Therefore, the dimension of $\mathscr{B}$ is three times the dimension of $\mathscr{N}_A$. □

7. A $4n \times 4n$ symmetric spin model

Let A and B be $n \times n$ type-II matrices, and assume A is symmetric. Let d be such that $d^2 = n$. In [11], Yamada defined a symmetric $4n \times 4n$ matrix

$$V := \begin{pmatrix} dA & -dA & B^{(-)} & B^{(-)} \\ -dA & dA & B^{(-)} & B^{(-)} \\ B^{(-)\mathrm{T}} & B^{(-)\mathrm{T}} & dA & -dA \\ B^{(-)\mathrm{T}} & B^{(-)\mathrm{T}} & -dA & dA \end{pmatrix}$$

and showed that V is a spin model if and only if (A, B) is an invertible Jones pair. This extends Nomura's result in [9] which covers only the invertible Jones pairs (A, B) where both A and B are symmetric. We give below a different proof for Yamada's result.

First, it is straightforward to check that V is also a type-II matrix. Let $\mathscr{B}$ be the Bose–Mesner algebra of order $4n$ defined in the previous section.

Theorem 7.1. *If (A, B) is an invertible Jones pair and A is symmetric, then V belongs to $\mathscr{B}$.*

Proof. Let $H = dA$. By Eq. (6.1), the matrix K paired with H satisfies

$$\Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}} = \Theta_{A,B}(dA) = dB.$$

Since $\Theta_{A,B^{\mathrm{T}}}$ is an isomorphism and $\Theta_{A,B^{\mathrm{T}}}(dA)^{\mathrm{T}} = dB$, we conclude that $K = dA^{\mathrm{T}} = dA$. Hence V is equal to $\mathscr{M}(\mathbf{0}, A^{-1}, dA)$ and it belongs to $\mathscr{B}$. □

Assume (A, B) is an invertible Jones pair and A is an $n \times n$ symmetric matrix. We use the next four lemmas to show that $\mathscr{B} \subseteq \mathscr{N}_V$. If $M = \mathscr{M}(F, G, H)$ in $\mathscr{B}$, we want to show that $Ve_r \circ V^{(-)}e_s$ is an eigenvector of M for all $r, s = 1, \ldots, 4n$.

In the following, we divide V into sixteen $n \times n$ blocks. We use $\mathbf{Y}_{i,j}^{\alpha,\beta}$ to denote $Ve_r \circ V^{(-)}e_s$ when Ve_r is the ith column of the αth block and $V^{(-)}e_s$ is the jth column of the βth block. We display the vectors $\mathbf{Y}_{i,j}^{\alpha,\beta}$ to make checking the computation easier.

$$\mathbf{Y}_{i,j}^{1,1} = \mathbf{Y}_{i,j}^{2,2} = \begin{pmatrix} Ae_i \circ A^{(-)}e_j \\ Ae_i \circ A^{(-)}e_j \\ B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j \\ B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j \end{pmatrix}, \quad \mathbf{Y}_{i,j}^{1,2} = \mathbf{Y}_{i,j}^{2,1} = \begin{pmatrix} -Ae_i \circ A^{(-)}e_j \\ -Ae_i \circ A^{(-)}e_j \\ B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j \\ B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j \end{pmatrix},$$

$$\mathbf{Y}_{i,j}^{3,3} = \mathbf{Y}_{i,j}^{4,4} = \begin{pmatrix} B^{(-)}e_i \circ Be_j \\ B^{(-)}e_i \circ Be_j \\ Ae_i \circ A^{(-)}e_j \\ Ae_i \circ A^{(-)}e_j \end{pmatrix}, \quad \mathbf{Y}_{i,j}^{3,4} = \mathbf{Y}_{i,j}^{4,3} = \begin{pmatrix} B^{(-)}e_i \circ Be_j \\ B^{(-)}e_i \circ Be_j \\ -Ae_i \circ A^{(-)}e_j \\ -Ae_i \circ A^{(-)}e_j \end{pmatrix},$$

$$\mathbf{Y}_{i,j}^{1,3} = -\mathbf{Y}_{i,j}^{2,4} = \begin{pmatrix} dAe_i \circ Be_j \\ -dAe_i \circ Be_j \\ d^{-1}B^{(-)\mathrm{T}}e_i \circ A^{(-)}e_j \\ -d^{-1}B^{(-)\mathrm{T}}e_i \circ A^{(-)}e_j \end{pmatrix},$$

$$\mathbf{Y}_{i,j}^{1,4} = -\mathbf{Y}_{i,j}^{2,3} = \begin{pmatrix} dAe_i \circ Be_j \\ -dAe_i \circ Be_j \\ -d^{-1}B^{(-)\mathrm{T}}e_i \circ A^{(-)}e_j \\ d^{-1}B^{(-)\mathrm{T}}e_i \circ A^{(-)}e_j \end{pmatrix},$$

$$\mathbf{Y}_{i,j}^{3,1} = -\mathbf{Y}_{i,j}^{4,2} = \begin{pmatrix} d^{-1}B^{(-)}e_i \circ A^{(-)}e_j \\ -d^{-1}B^{(-)}e_i \circ A^{(-)}e_j \\ dAe_i \circ B^{\mathrm{T}}e_j \\ -dAe_i \circ B^{\mathrm{T}}e_j \end{pmatrix}$$

and

$$\mathbf{Y}_{i,j}^{4,1} = -\mathbf{Y}_{i,j}^{3,2} = \begin{pmatrix} d^{-1}B^{(-)}e_i \circ A^{(-)}e_j \\ -d^{-1}B^{(-)}e_i \circ A^{(-)}e_j \\ -dAe_i \circ B^{\mathrm{T}}e_j \\ dAe_i \circ B^{\mathrm{T}}e_j \end{pmatrix}.$$

Lemma 7.2. *Let* $M = \mathscr{M}(F, G, H)$ *be in* $\mathscr{B}$. *Then for* $i, j = 1, \ldots, n$, $\mathbf{Y}_{i,j}^{1,1}$, $\mathbf{Y}_{i,j}^{1,2}$, $\mathbf{Y}_{i,j}^{2,1}$ *and* $\mathbf{Y}_{i,j}^{2,2}$ *are eigenvectors of* M.

Proof. Note that $M\mathbf{Y}_{i,j}^{1,1}$ equals

$$2\begin{pmatrix} \Theta_A(F)(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \\ \Theta_A(F)(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \\ \Theta_{B^{(-)}}(F)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) + \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(Ae_i \circ A^{(-)}e_j) \\ \Theta_{B^{(-)}}(F)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) + \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(Ae_i \circ A^{(-)}e_j) \end{pmatrix}$$

which in turn equals

$$2\begin{pmatrix} \Theta_A(\Theta_A(F))_{i,j}(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \\ \Theta_A(\Theta_A(F))_{i,j}(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \\ \Theta_{B^{(-)\mathrm{T}}}(\Theta_{B^{(-)}}(F))_{i,j}(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) + \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(Ae_i \circ A^{(-)}e_j) \\ \Theta_{B^{(-)\mathrm{T}}}(\Theta_{B^{(-)}}(F))_{i,j}(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) + \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(Ae_i \circ A^{(-)}e_j) \end{pmatrix}.$$

Now, we show that $\mathbf{Y}_{i,j}^{1,1}$ is an eigenvector of M and compute the corresponding eigenvalue, which is the ijth entry in the $(1,1)$-block of $\Theta_V(M)$. Since A is symmetric, it follows from Eq. (4.1) that

$$\begin{aligned} \Theta_A(\Theta_A(F)) &= \Theta_{A^{\mathrm{T}}}(\Theta_A(F)) \\ &= nF^{\mathrm{T}} \\ &= \Theta_{B^{(-)\mathrm{T}}}(\Theta_{B^{(-)}}(F)). \end{aligned} \tag{7.1}$$

Moreover, applying Theorem 3.2(e) with R equal to G, we have

$$X_{\Theta_{A,B}(G)}\Delta_{B^{\mathrm{T}}}X_{B^{(-)\mathrm{T}}} = \Delta_{A^{(-)\mathrm{T}}}X_{A^{\mathrm{T}}}\Delta_{nG^{\mathrm{T}}}.$$

Since A is symmetric, the above equation is equivalent to

$$\Theta_{A,B}(G)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) = nG_{i,j}^{\mathrm{T}}(Ae_i \circ A^{(-)}e_j) \tag{7.2}$$

for $i, j = 1, \ldots, n$. Similarly, applying Theorem 3.2(d) with R equals to G^{T} gives

$$X_{\Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}}\Delta_{A^{(-)}}X_A = \Delta_{B^{\mathrm{T}}}X_{B^{(-)\mathrm{T}}}\Delta_{nG^{\mathrm{T}}}$$

which implies

$$\Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(Ae_i \circ A^{(-)}e_j) = nG_{i,j}^{\mathrm{T}}(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \tag{7.3}$$

for $i,j = 1, \ldots, n$. From Eqs. (7.1)–(7.3), we see that

$$M\mathbf{Y}_{i,j}^{1,1} = \begin{pmatrix} 2nF_{j,i}(Ae_i \circ A^{(-)}e_j) + 2nG_{j,i}(Ae_i \circ A^{(-)}e_j) \\ 2nF_{j,i}(Ae_i \circ A^{(-)}e_j) + 2nG_{j,i}(Ae_i \circ A^{(-)}e_j) \\ 2nF_{j,i}(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) + 2nG_{j,i}(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \\ 2nF_{j,i}(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) + 2nG_{j,i}(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \end{pmatrix} = 2n(F_{j,i} + G_{j,i})\mathbf{Y}_{i,j}^{1,1}$$

and the $(1,1)$-block of $\Theta_V(M)$ is equal to $2n(F^{\mathrm{T}} + G^{\mathrm{T}})$. Since $\mathbf{Y}_{i,j}^{1,1} = \mathbf{Y}_{i,j}^{2,2}$, the $(2,2)$-block of $\Theta_V(M)$ is also $2n(F^{\mathrm{T}} + G^{\mathrm{T}})$. For $(\alpha, \beta) \in \{(1,2),(2,1)\}$,

$$M\mathbf{Y}_{i,j}^{\alpha,\beta} = 2\begin{pmatrix} -\Theta_A(F)(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \\ -\Theta_A(F)(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) \\ \Theta_{B^{(-)}}(F)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) - \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(Ae_i \circ A^{(-)}e_j) \\ \Theta_{B^{(-)}}(F)(B^{(-)\mathrm{T}}e_i \circ B^{\mathrm{T}}e_j) - \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(Ae_i \circ A^{(-)}e_j) \end{pmatrix}.$$

Using the above argument, the $(1,2)$- and $(2,1)$-blocks of $\Theta_V(M)$ are equal to $2n(F^{\mathrm{T}} - G^{\mathrm{T}})$. □

Lemma 7.3. *Let $M = \mathcal{M}(F,G,H)$ be in $\mathcal{B}$. Then for $i,j = 1, \ldots, n$, $\mathbf{Y}_{i,j}^{3,3}$, $\mathbf{Y}_{i,j}^{3,4}$, $\mathbf{Y}_{i,j}^{4,3}$ and $\mathbf{Y}_{i,j}^{4,4}$ are eigenvectors of M.*

Proof. We have $M\mathbf{Y}_{i,j}^{3,3}$ equals

$$2\begin{pmatrix} \Theta_A(F)(B^{(-)}e_i \circ Be_j) + \Theta_{A,B}(G)(Ae_i \circ A^{(-)}e_j) \\ \Theta_A(F)(B^{(-)}e_i \circ Be_j) + \Theta_{A,B}(G)(Ae_i \circ A^{(-)}e_j) \\ \Theta_{B^{(-)}}(F)(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(B^{(-)}e_i \circ Be_j) \\ \Theta_{B^{(-)}}(F)(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(B^{(-)}e_i \circ Be_j) \end{pmatrix}$$

which is equal to

$$2\begin{pmatrix} \Theta_{B^{(-)}}(\Theta_A(F))_{i,j}(B^{(-)}e_i \circ Be_j) + \Theta_{A,B}(G)(Ae_i \circ A^{(-)}e_j) \\ \Theta_{B^{(-)}}(\Theta_A(F))_{i,j}(B^{(-)}e_i \circ Be_j) + \Theta_{A,B}(G)(Ae_i \circ A^{(-)}e_j) \\ \Theta_A(\Theta_{B^{(-)}}(F))_{i,j}(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(B^{(-)}e_i \circ Be_j) \\ \Theta_A(\Theta_{B^{(-)}}(F))_{i,j}(Ae_i \circ A^{(-)}e_j) + \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(B^{(-)}e_i \circ Be_j) \end{pmatrix}.$$

We now show that $\mathbf{Y}_{i,j}^{3,3}$ is an eigenvector of M, and compute the corresponding eigenvalue which is the ijth entry in the $(3,3)$-block of $\Theta_V(M)$. By Corollary 5.5,

$$\begin{aligned} \Theta_{B^{(-)}}(\Theta_A(F)) &= \Theta_B(\Theta_A(F))^{\mathrm{T}} \\ &= B^{-1}\Theta_A(\Theta_A(F))B \\ &= nB^{-1}F^{\mathrm{T}}B. \end{aligned} \tag{7.4}$$

Applying Corollary 5.5 to the Jones pair (A, B^{T}),

$$\begin{aligned}\Theta_A(\Theta_{B^{(-)}}(F)) &= B^{\mathrm{T}}\Theta_{B^{\mathrm{T}}}(\Theta_{B^{(-)}}(F))^{\mathrm{T}}(B^{-1})^{\mathrm{T}}\\ &= B^{\mathrm{T}}\Theta_{B^{(-)\mathrm{T}}}(\Theta_{B^{(-)}}(F))(B^{-1})^{\mathrm{T}}\\ &= nB^{\mathrm{T}}F^{\mathrm{T}}(B^{-1})^{\mathrm{T}}\\ &= nB^{-1}(BB^{\mathrm{T}})F^{\mathrm{T}}(B^{-1})^{\mathrm{T}}.\end{aligned}$$

Since $B \in \mathcal{N}'_{A,B}$, it follows from Theorem 5.6 that $BB^{\mathrm{T}} \in \mathcal{N}_A$. Now F^{T} belongs to $\mathcal{N}_A$, the commutativity of $\mathcal{N}_A$ implies

$$\begin{aligned}\Theta_A(\Theta_{B^{(-)}}(F)) &= nB^{-1}F^{\mathrm{T}}(BB^{\mathrm{T}})(B^{-1})^{\mathrm{T}}\\ &= nB^{-1}F^{\mathrm{T}}B.\end{aligned} \tag{7.5}$$

From Theorem 5.6, there exists $G_1^{\mathrm{T}} \in \mathcal{N}_{A,B^{\mathrm{T}}}$ such that

$$\Theta_{A,B}(G) = \Theta_{A,B^{\mathrm{T}}}(G_1^{\mathrm{T}})^{\mathrm{T}}.$$

Hence G_1 is paired with G. Applying Theorem 3.2(d) with R equals to G_1^{T} in $\mathcal{N}_{A,B^{\mathrm{T}}}$ yields

$$X_{\Theta_{A,B^{\mathrm{T}}}(G_1^{\mathrm{T}})^{\mathrm{T}}}\Delta_{A^{(-)\mathrm{T}}}X_{A^{\mathrm{T}}} = \Delta_B X_{B^{(-)}}\Delta_{nG_1^{\mathrm{T}}}$$

which is equivalent to

$$X_{\Theta_{A,B}(G)}\Delta_{A^{(-)}}X_A = \Delta_B X_{B^{(-)}}\Delta_{nG_1^{\mathrm{T}}}.$$

Consequently

$$\Theta_{A,B}(G)(Ae_i \circ A^{(-)}e_j) = n(G_1^{\mathrm{T}})_{i,j}(B^{(-)}e_i \circ Be_j). \tag{7.6}$$

By Lemma 6.2, G_1^{T} is also paired with G^{T}. Applying Theorem 3.2(e) to $R = G_1$ in $\mathcal{N}_{A,B^{\mathrm{T}}}$ gives

$$X_{\Theta_{A,B^{\mathrm{T}}}(G_1)}\Delta_B X_{B^{(-)}} = \Delta_{A^{(-)\mathrm{T}}}X_{A^{\mathrm{T}}}\Delta_{nG_1^{\mathrm{T}}}$$

which is equivalent to

$$X_{\Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}}\Delta_B X_{B^{(-)}} = \Delta_{A^{(-)}}X_A\Delta_{nG_1^{\mathrm{T}}}$$

and

$$\Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}(B^{(-)}e_i \circ Be_j) = n(G_1^{\mathrm{T}})_{i,j}(Ae_i \circ A^{(-)}e_j). \tag{7.7}$$

It follows from Eqs. (7.4)–(7.7) that

$$\begin{aligned}M\mathbf{Y}_{i,j}^{3,3} &= 2\begin{pmatrix} n(B^{-1}F^{\mathrm{T}}B)_{i,j}(B^{(-)}e_i \circ Be_j) + n(G_1^{\mathrm{T}})_{i,j}(B^{(-)}e_i \circ Be_j)\\ n(B^{-1}F^{\mathrm{T}}B)_{i,j}(B^{(-)}e_i \circ Be_j) + n(G_1^{\mathrm{T}})_{i,j}(B^{(-)}e_i \circ Be_j)\\ n(B^{-1}F^{\mathrm{T}}B)_{i,j}(Ae_i \circ A^{(-)}e_j) + n(G_1^{\mathrm{T}})_{i,j}(Ae_i \circ A^{(-)}e_j)\\ n(B^{-1}F^{\mathrm{T}}B)_{i,j}(Ae_i \circ A^{(-)}e_j) + n(G_1^{\mathrm{T}})_{i,j}(Ae_i \circ A^{(-)}e_j)\end{pmatrix}\\ &= 2n(B^{-1}F^{\mathrm{T}}B + G_1^{\mathrm{T}})_{i,j}\mathbf{Y}_{i,j}^{3,3}.\end{aligned}$$

Note that $\mathbf{Y}_{i,j}^{3,3} = \mathbf{Y}_{i,j}^{4,4}$. Hence the $(3,3)$- and $(4,4)$-blocks of $\Theta_V(M)$ are equal to $2n(B^{-1}F^{\mathrm{T}}B + G_1^{\mathrm{T}})$. It is easy to see from the block structure of $\mathbf{Y}_{i,j}^{3,4}$ and $\mathbf{Y}_{i,j}^{4,3}$ that the $(3,4)$- and $(4,3)$-blocks of $\Theta_V(M)$ are equal to $2n(B^{-1}F^{\mathrm{T}}B - G_1^{\mathrm{T}})$. □

Lemma 7.4. *Let $M = \mathscr{M}(F,G,H)$ be in $\mathscr{B}$. Then for $i,j = 1,\dots,n$, $\mathbf{Y}_{i,j}^{1,3}$, $\mathbf{Y}_{i,j}^{1,4}$, $\mathbf{Y}_{i,j}^{2,3}$ and $\mathbf{Y}_{i,j}^{2,4}$ are eigenvectors of M.*

Proof. We have

$$M\mathbf{Y}_{i,j}^{1,3} = 2\begin{pmatrix} dH(Ae_i \circ Be_j) \\ -dH(Ae_i \circ Be_j) \\ d^{-1}K(B^{(-)\mathrm{T}}e_i \circ A^{(-)}e_j) \\ -d^{-1}K(B^{(-)\mathrm{T}}e_i \circ A^{(-)}e_j) \end{pmatrix} = 2\begin{pmatrix} \Theta_{A,B}(H)_{i,j}(dAe_i \circ Be_j) \\ \Theta_{A,B}(H)_{i,j}(-dAe_i \circ Be_j) \\ \Theta_{B^{(-)\mathrm{T}},A^{(-)}}(K)_{i,j}(d^{-1}B^{(-)\mathrm{T}}e_i \circ A^{(-)}e_j) \\ \Theta_{B^{(-)\mathrm{T}},A^{(-)}}(K)_{i,j}(-d^{-1}B^{(-)\mathrm{T}}e_i \circ A^{(-)}e_j) \end{pmatrix}.$$

By Corollary 3.3,

$$\Theta_{B^{(-)\mathrm{T}},A^{(-)}}(K) = \Theta_{B^{\mathrm{T}},A}(K^{\mathrm{T}}) = \Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}}.$$

Since K is paired with H, by Eq. (6.1), the $(1,3)$-block of $\Theta_M(V)$ is $2\Theta_{A,B}(H)$. Similarly, the $(2,4)$-, $(1,4)$-, $(2,3)$-blocks of $\Theta_M(V)$ are equal to $2\Theta_{A,B}(H)$. □

Lemma 7.5. *Let $M = \mathscr{M}(F,G,H)$ be in $\mathscr{B}$. Then for $i,j = 1,\dots,n$, $\mathbf{Y}_{i,j}^{3,1}$, $\mathbf{Y}_{i,j}^{3,2}$, $\mathbf{Y}_{i,j}^{4,1}$ and $\mathbf{Y}_{i,j}^{4,2}$ are eigenvectors of M.*

Proof. We have

$$M\mathbf{Y}_{i,j}^{3,1} = 2\begin{pmatrix} d^{-1}H(B^{(-)}e_i \circ A^{(-)}e_j) \\ -d^{-1}H(B^{(-)}e_i \circ A^{(-)}e_j) \\ dK(Ae_i \circ B^{\mathrm{T}}e_j) \\ -dK(Ae_i \circ B^{\mathrm{T}}e_j) \end{pmatrix} = 2\begin{pmatrix} \Theta_{B^{(-)},A^{(-)}}(H)_{i,j}(d^{-1}B^{(-)}e_i \circ A^{(-)}e_j) \\ \Theta_{B^{(-)},A^{(-)}}(H)_{i,j}(-d^{-1}B^{(-)}e_i \circ A^{(-)}e_j) \\ \Theta_{A,B^{\mathrm{T}}}(K)_{i,j}(dAe_i \circ B^{\mathrm{T}}e_j) \\ \Theta_{A,B^{\mathrm{T}}}(K)_{i,j}(-dAe_i \circ B^{\mathrm{T}}e_j) \end{pmatrix}.$$

By Corollary 3.3, we have

$$\begin{aligned}\Theta_{B^{(-)},A^{(-)}}(H) &= \Theta_{B,A}(H^{\mathrm{T}})\\ &= \Theta_{A,B}(H^{\mathrm{T}})^{\mathrm{T}}\\ &= \Theta_{A,B^{\mathrm{T}}}(K)\end{aligned}$$

and the last equality follows from the fact that K^{T} is paired with H^{T}. Therefore the $(3,1)$-block of $\Theta_V(M)$ is equal to $2\Theta_{A,B}(H^{\mathrm{T}})^{\mathrm{T}}$. Similarly, the $(4,2)$- , $(4,1)$- and $(3,2)$-blocks are equal to $2\Theta_{A,B}(H^{\mathrm{T}})^{\mathrm{T}}$. $\square$

Theorem 7.6. *If (A,B) is an invertible Jones pair and A is symmetric, then $\mathscr{B}$ is a subscheme of $\mathscr{N}_V$.*

Proof. For any $M\in\mathscr{B}$, we have shown in Lemmas 7.2–7.5 that $\mathbf{Y}_{i,j}^{\alpha,\beta}$ is an eigenvector of M for all $\alpha,\beta\in\{1,2,3,4\}$ and $i,j\in\{1,\dots,n\}$. Thus $M\in\mathscr{N}_V$ and $\mathscr{B}\subseteq\mathscr{N}_V$. $\square$

Corollary 7.7. *The Bose–Mesner algebra $\mathscr{B}$ is formally self-dual with duality map Θ_V.*

Proof. We see from the proof of Lemmas 7.2–7.5 that for $M=\mathscr{M}(F,G,H)$ in $\mathscr{B}$, $\Theta_V(M)$ equals

$$2\begin{pmatrix} nF^{\mathrm{T}}+nG^{\mathrm{T}} & nF^{\mathrm{T}}-nG^{\mathrm{T}} & \Theta_{A,B}(H) & \Theta_{A,B}(H)\\ nF^{\mathrm{T}}-nG^{\mathrm{T}} & nF^{\mathrm{T}}+nG^{\mathrm{T}} & \Theta_{A,B}(H) & \Theta_{A,B}(H)\\ \Theta_{A,B}(H^{\mathrm{T}})^{\mathrm{T}} & \Theta_{A,B}(H^{\mathrm{T}})^{\mathrm{T}} & nB^{-1}F^{\mathrm{T}}B+nG_1^{\mathrm{T}} & nB^{-1}F^{\mathrm{T}}B-nG_1^{\mathrm{T}}\\ \Theta_{A,B}(H^{\mathrm{T}})^{\mathrm{T}} & \Theta_{A,B}(H^{\mathrm{T}})^{\mathrm{T}} & nB^{-1}F^{\mathrm{T}}B-nG_1^{\mathrm{T}} & nB^{-1}F^{\mathrm{T}}B+nG_1^{\mathrm{T}}\end{pmatrix},$$

where G_1 is paired with G, that is

$$\Theta_{A,B}(G)=\Theta_{A,B^{\mathrm{T}}}(G_1^{\mathrm{T}})^{\mathrm{T}}.$$

Since $nF^{\mathrm{T}}\in\mathscr{N}_A$ and $\mathscr{N}_A=\mathscr{N}'_A$, there exists a matrix $\hat{F}\in\mathscr{N}_A$ such that $nF^{\mathrm{T}}=\Theta_A(\hat{F})$. By Corollary 5.5, we have

$$B^{-1}nF^{\mathrm{T}}B=B^{-1}\Theta_A(\hat{F})B=\Theta_B(\hat{F})^{\mathrm{T}}=\Theta_{B^{(-)}}(\hat{F}).$$

By Corollary 5.7, we have $G^{\mathrm{T}}\in\mathscr{N}_{A,B}$. It follows from Lemma 6.2 that G_1^{T} is also paired with G^{T}, whence we have

$$\Theta_V(M)=\mathscr{M}(2\hat{F},2H,2nG^{\mathrm{T}})$$

belongs to $\mathscr{B}$. Moreover, the map Θ_V restricted to $\mathscr{B}$ is a duality map of $\mathscr{B}$. $\square$

We are ready to prove Yamada's result.

Theorem 7.8 (Yamada [11, Theorem 1]). *Let A be a symmetric $n\times n$ matrix. Then (A,B) is an invertible Jones pair if and only if V is a spin model.*

Proof. Suppose (A, B) is an invertible Jones pair. By Theorems 7.1 and 7.6, the matrix V is equal to $\mathscr{M}(\mathbf{0}, A^{-1}, dA)$ and hence it belongs to $\mathscr{N}_V$. By Corollary 7.7,

$$\Theta_V(V) = \mathscr{M}(\mathbf{0}, 2dA, 2nA^{-1}).$$

If K is paired with $H = 2nA^{-1}$, then

$$\Theta_{A,B^{\mathrm{T}}}(K^{\mathrm{T}})^{\mathrm{T}} = 2n\Theta_{A,B}(A^{-1}) = 2nB^{(-)},$$

which implies $K = 2nA^{-1} = 2A^{(-)}$. Therefore

$$\Theta_V(V) = 2\begin{pmatrix} A^{(-)} & -A^{(-)} & dB & dB \\ -A^{(-)} & A^{(-)} & dB & dB \\ dB^{\mathrm{T}} & dB^{\mathrm{T}} & A^{(-)} & -A^{(-)} \\ dB^{\mathrm{T}} & dB^{\mathrm{T}} & -A^{(-)} & A^{(-)} \end{pmatrix} = 2\,dV^{(-)}.$$

By Theorem 3.2, we have

$$X_V \Delta_{V^{(-)}} X_V = \Delta_{V^{(-)}} X_V \Delta_{2dV^{(-)}}.$$

Since V is symmetric, we conclude that $(\frac{1}{2d}V, V^{(-)})$ is an invertible Jones pair, which is equivalent to saying V is a spin model.

Conversely, let V be a spin model, or equivalently, let $(\frac{1}{2d}V, V^{(-)})$ be an invertible Jones pair. Since the $(1,3)$-block of $V^{(-)}$ is equal to B, we have

$$V\mathbf{Y}^{1,3}_{i,j} = 2dB_{i,j}\mathbf{Y}^{1,3}_{i,j}.$$

This equation implies that

$$A(Ae_i \circ Be_j) = B_{i,j}(Ae_i \circ Be_j) \quad \text{for all } i,j = 1, \dots, n.$$

By Theorem 3.2, we have

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B.$$

Similarly, the $(3,1)$-block of $V^{(-)}$ is equal to B^{T}, we get

$$V\mathbf{Y}^{3,1}_{i,j} = 2d(B^{\mathrm{T}})_{i,j}\mathbf{Y}^{3,1}_{i,j}$$

which implies

$$A(Ae_i \circ B^{\mathrm{T}}e_j) = B^{\mathrm{T}}_{i,j}(Ae_i \circ B^{\mathrm{T}}e_j) \quad \text{for all } i,j = 1, \dots, n$$

and

$$X_A \Delta_{B^{\mathrm{T}}} X_A = \Delta_{B^{\mathrm{T}}} X_A \Delta_{B^{\mathrm{T}}}.$$

Thus (A, B) is an invertible Jones pair. □

It follows from Theorem 4.1 and the above theorem that the Bose–Mesner algebra $\mathscr{N}_V$ is formally self-dual and Θ_V is a duality map of $\mathscr{N}_V$.

Given any invertible Jones pair (C, B), it is easy find an odd-gauge equivalent invertible Jones pair (A, B) in which A is symmetric, see Section 8 of [4]. By the above theorem, we can always construct a symmetric spin model V from every invertible Jones pair, or equivalently, every four-weight spin model.

8. Subschemes and induced schemes

Suppose A and B are $n \times n$ type-II matrices. It is easy to verify that the $2n \times 2n$ matrix

$$W = \begin{pmatrix} A & B^{(-)} \\ -A & B^{(-)} \end{pmatrix}$$

is also a type-II matrix. Furthermore, if (A, B) is an invertible Jones pair and A is symmetric, then we have

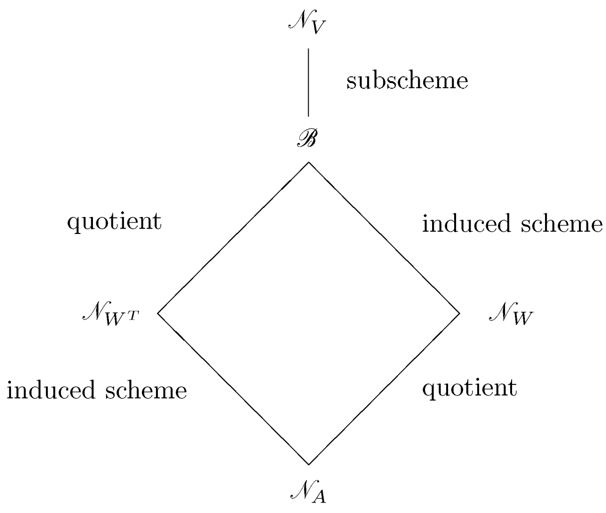
$$\mathcal{N}_W = \left\{ \begin{pmatrix} F+G & F-G \\ F-G & F+G \end{pmatrix} : F \in \mathcal{N}_A, G \in \mathcal{N}_{A,B} \right\} \tag{8.1}$$

and

$$\mathcal{N}_{W^T} = \left\{ \begin{pmatrix} \Theta_A(F) & \Theta_{A,B}(G) \\ \Theta_{B^{(-)},A^{(-)}}(G) & \Theta_{B^{(-)}}(F) \end{pmatrix} : F \in \mathcal{N}_A, G \in \mathcal{N}_{A,B} \right\}. \tag{8.2}$$

Hence the dimensions of $\mathcal{N}_W$ and $\mathcal{N}_{W^T}$ equal twice the dimension of $\mathcal{N}_A$. For details, please see Section 11 of [4].

Now we have five Bose–Mesner algebras $\mathcal{N}_V$, $\mathcal{B}$, $\mathcal{N}_W$, $\mathcal{N}_{W^T}$ and $\mathcal{N}_A$ associated to each invertible Jones pair (A, B) with A symmetric. The aim of this section is to show that they satisfy the relations described in the following diagram.



Let **B** be a Bose–Mesner algebra on vertex set $\mathcal{V}$. Let Y be a non-empty subset of $\mathcal{V}$. For any $|\mathcal{V}| \times |\mathcal{V}|$ matrix M, we use M_Y to denote the $|Y| \times |Y|$ matrix obtained from the rows and the columns of M indexed by the elements in Y.

We let the set

$$\mathbf{B}_Y := \{M_Y : M \in \mathbf{B}\}.$$

If $\mathbf{B}_Y$ is also a Bose–Mesner algebra, we say it is an *induced scheme* of $\mathbf{B}$. Suppose the vertex sets of $\mathcal{N}_A$, $\mathcal{N}_W$ and $\mathcal{B}$ are $\{1, \dots, n\}$, $\{1, \dots, 2n\}$ and $\{1, \dots, 4n\}$, respectively. If $Y = \{1, \dots, n\}$, then it is obvious from Eq. (8.2) that the set $(\mathcal{N}_{W^T})_Y$ is equal to $\mathcal{N}'_A$. Therefore $\mathcal{N}_A = \mathcal{N}'_A$ is an induced scheme of $\mathcal{N}_{W^T}$. Similarly, let $Y' = \{1, \dots, 2n\}$. It follows from Eqs. (6.3) and (8.1) that $\mathcal{B}_{Y'} = \mathcal{N}_W$.

Let $\mathbf{B}$ be a Bose–Mesner algebra on vertex set $\mathcal{V}$. Let $\pi = (C_1, \dots, C_r)$ be a partition of $\mathcal{V}$. Define the characteristic matrix S of π to be the $n \times r$ matrix with

$$S_{u,k} = \begin{cases} 1 & \text{if } u \in C_k, \\ 0 & \text{otherwise.} \end{cases}$$

We say π is equitable relative to $\mathbf{B}$ if and only if for each matrix M in $\mathbf{B}$, there is an $r \times r$ matrix Z_M satisfying

$$MS = SZ_M.$$

We call the set $\{Z_M : M \in \mathbf{B}\}$ the *quotient* of $\mathbf{B}$ with respect to π. For $i = 1, \dots, n$, let $C_i = \{i, n+i\}$ and let $\pi = (C_1, \dots, C_n)$. The characteristic matrix of π is

$$S = \begin{pmatrix} I_n \\ I_n \end{pmatrix}.$$

Then a matrix

$$M = \begin{pmatrix} F+R & F-R \\ F-R & F+R \end{pmatrix}$$

in $\mathcal{N}_W$ satisfies

$$MS = S(2F).$$

Thus $Z_M = 2F$. By Eq. (8.1), we see that $F \in \mathcal{N}_A$ and thus the quotient of $\mathcal{N}_W$ with respect to π is equal to $\mathcal{N}_A$. Similarly let $C_i = \{i, n+i\}$, for $i = 1, \dots, n, 2n+1, \dots, 3n$. The characteristic matrix of $\pi' = (C_1, \dots, C_n, C_{2n+1}, \dots, C_{3n})$ is

$$S' = \begin{pmatrix} I_n & \mathbf{0} \\ I_n & \mathbf{0} \\ \mathbf{0} & I_n \\ \mathbf{0} & I_n \end{pmatrix}.$$

Then a matrix $\mathscr{M}(F,G,H)$ in $\mathscr{B}$ satisfies

$$\mathscr{M}(F,G,H)S' = 2\begin{pmatrix} \Theta_A(F) & \Theta_{A,B}(G) \\ \Theta_A(F) & \Theta_{A,B}(G) \\ \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}} & \Theta_{B^{(-)}}(F) \\ \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}} & \Theta_{B^{(-)}}(F) \end{pmatrix} = S'\left(2\begin{pmatrix} \Theta_A(F) & \Theta_{A,B}(G) \\ \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}} & \Theta_{B^{(-)}}(F) \end{pmatrix}\right).$$

By Corollary 3.3, we have

$$\Theta_{B^{(-)},A^{(-)}}(G) = \Theta_{A,B}(G^{\mathrm{T}})^{\mathrm{T}}.$$

As a result, $Z_{\mathscr{M}(F,G,H)} \in \mathscr{N}_{W^{\mathrm{T}}}$ and $\mathscr{N}_{W^{\mathrm{T}}}$ is the quotient of $\mathscr{B}$ with respect to π'.

In addition, it is straightforward to check that the span of the following set

$$\{\mathscr{M}(F,\mathbf{0},H) : F \in \mathscr{N}_A \quad \text{and} \quad H \in \mathscr{N}_{A,B}\} \cup \{\mathscr{M}(\mathbf{0},I_n,\mathbf{0})\}$$

is also a Bose–Mesner algebra. Therefore it is a *subscheme* of $\mathscr{B}$ whose dimension equals $2\dim(\mathscr{N}_A)+1$. Similarly, the span of the set

$$\left\{\begin{pmatrix} \Theta_A(F) & \mathbf{0} \\ \mathbf{0} & \Theta_{B^{(-)}}(F) \end{pmatrix} : F \in \mathscr{N}_A\right\} \cup \left\{\begin{pmatrix} \mathbf{0} & J_n \\ J_n & \mathbf{0} \end{pmatrix}\right\}$$

is a subscheme of $\mathscr{N}_{W^{\mathrm{T}}}$ whose dimension equals $\dim(\mathscr{N}_A)+1$.

9. Comments

We now give an explicit description of $\mathscr{N}_V$. Let $\mathscr{R}$ be the space consisting matrices

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & N & -N \\ \mathbf{0} & \mathbf{0} & -N & N \\ N_1 & -N_1 & \mathbf{0} & \mathbf{0} \\ -N_1 & N_1 & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where N and N_1 satisfy

$$X_{A^{-1}}\Delta_{B^{(-)}}X_N\Delta_{A^{(-)}}X_{B^{-1}} = \Delta_S = X_B\Delta_A X_{N_1}\Delta_B X_A,$$

$$X_{B^{\mathrm{T}}}\Delta_A X_N \Delta_{B^{\mathrm{T}}} X_A = \Delta_{S_1} = X_{A^{-1}}\Delta_{B^{(-)\mathrm{T}}}X_{N_1}\Delta_{A^{(-)}}X_{(B^{-1})^{\mathrm{T}}},$$

for some $n \times n$ matrices S and S_1. Then $\mathscr{N}_V$ is equal to the direct sum of $\mathscr{B}$ and $\mathscr{R}$, see p. 124 of [3]. We see that if $\mathscr{N}_A$ has dimension r, then $\dim(\mathscr{N}_V)$ equals $3r + \dim(\mathscr{R})$. Unfortunately, we do not yet know how to determine the dimension of $\mathscr{R}$. We can only conclude that $3r \leqslant \dim(\mathscr{N}_V) \leqslant 3r+n$. For example, for each of the three 4×4 four-weight spin models given in Section 5 of [2], the algebra $\mathscr{N}_A$ has dimension 4 and $\mathscr{N}_V$ has dimension 16. The natural problem is to determine the dimension of $\mathscr{N}_V$ for any invertible Jones pair (A,B).

We get two link invariants from an invertible Jones pair (A, B): one from (A, B) and the other from the spin model V. It is natural to ask how the two invariants are related. In addition, it would be very useful to have a procedure that decides whether any $4n \times 4n$ spin model is gauge equivalent to a spin model that has the same structure as V. Such procedure may lead us to the extraction of invertible Jones pairs from the spin models of order divisible by four.

Any new examples of invertible Jones pair will be extremely desirable since there is a rich family of Bose–Mesner algebras attached. On the other hand, we are also interested in any Bose–Mesner algebras that fit the diagram in Section 8 because they may lead to the discovery of new invertible Jones pairs, hence possibly new link invariants. In particular, we have examined the formally dual pair of Bose–Mesner algebras, $\mathscr{B}_1$ and $\mathscr{B}_2$, constructed from the *Kasami codes* in [5]. These algebras consist of $2^{4t+2} \times 2^{4t+2}$ matrices and they have dimension six. The Schur-idempotents of, say, $\mathscr{B}_1$ have valencies

$$1, 2^{2t+1} - 1, 2^{2t+1} - 1, 2^{2t+1} - 1, (2^{2t} - 1)(2^{2t+1} - 1) \quad \text{and} \quad (2^{2t} - 1)(2^{2t+1} - 1);$$

while the valencies of the Schur-idempotents of $\mathscr{B}_2$ are

$$1, 2^{2t+1} - 1, 2^{t-1}(2^t - 1)(2^{2t+1} - 1), 2^{t-1}(2^t - 1)(2^{2t+1} - 1),$$
$$2^{t-1}(2^t + 1)(2^{2t+1} - 1) \quad \text{and} \quad 2^{t-1}(2^t + 1)(2^{2t+1} - 1).$$

We are interested in these algebras because they are the only known example of a formally dual pair of Bose–Mesner algebras that are not translation schemes. They are candidates for $\mathscr{N}_W$ and $\mathscr{N}_{W^T}$ in our diagram.

In the following, we use the structure of $\mathscr{N}_{W^T}$ to rule out the possibility that $\mathscr{B}_1$ and $\mathscr{B}_2$ fit into the diagram in Section 8. We see from the previous section that

$$\hat{J} = \begin{pmatrix} \mathbf{0} & J_{2^{4t+1}} \\ J_{2^{4t+1}} & \mathbf{0} \end{pmatrix}$$

belongs to $\mathscr{N}_{W^T}$. Therefore if $\mathscr{N}_{W^T}$ equals to $\mathscr{B}_1$, then a subset of the Schur-idempotents of $\mathscr{B}_1$ would sum to $\hat{J}$. In this case, a subset of the valencies of $\mathscr{B}_1$ would sum to 2^{4t+1}. However, we can use elementary computation to prove that it is impossible to find a subset of the numbers in the first list above to sum to 2^{4t+1}. Consequently the algebra $\mathscr{B}_1$ cannot be $\mathscr{N}_{W^T}$. Similarly, simple computation shows that we cannot find a subset of valencies of $\mathscr{B}_2$ to sum to 2^{4t+1}. We conclude that $\mathscr{B}_2$ cannot be $\mathscr{N}_{W^T}$. As a result there does not exist any invertible Jones pair for which $\{\mathscr{B}_1, \mathscr{B}_2\}$ equals $\{\mathscr{N}_W, \mathscr{N}_{W^T}\}$.

Acknowledgments

We thank the referees for their constructive comments and suggestions. Support from a National Sciences and Engineering Council of Canada operating grant is gratefully acknowledged by the second author.

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