Model uncertainty of SPT-based method for evaluation of seismic soil liquefaction potential using multi-gene genetic programming

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Abstract

In this paper, the model uncertainty of the developed standard penetration test (SPT)-based model for evaluation of liquefaction potential of soil is estimated within the framework of the first-order reliability method (FORM). First, an empirical model to determine the cyclic resistance ratio (CRR) of the soil is developed, based on the post-liquefaction SPT data using an evolutionary artificial intelligence technique, multi-gene genetic programming (MGGP). This developed resistance model along with an existing cyclic stress ratio (CSR) model forms a limit state function for reliability-based approach for liquefaction triggering analysis. The uncertainty of the developed limit state model is represented by a lognormal random variable, in terms of its mean and the coefficient of variation, estimated through an extensive reliability analysis following a trial and error approach using Bayesian mapping functions calibrated with a high quality post-liquefaction case history database. A deterministic model with a mapping function relating the probability of liquefaction ($P_L$) and the factor of safety against liquefaction ($F_s$) is also developed for use in absence of parameter uncertainties. Two examples are presented to compare the present MGGP-based reliability method with the available regression-based reliability method.

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Keywords: Standard penetration test; Liquefaction index; Multi-gene genetic programming; Probability of liquefaction; Bayesian mapping function; Reliability index; Notional probability

1. Introduction

The first and perhaps the most important step toward mitigating liquefaction-induced damage is the evaluation of the liquefaction potential of a soil subjected to seismic loading. Though, different approaches like cyclic strain-based, energy-based and cyclic stress-based approaches are in use, the stress-based approach is the most widely used method for the evaluation of the liquefaction potential of soil (Kramer, 1996). Seed and Idriss (1971) pioneered the stress-based simplified method and the procedure has been modified and improved by Seed et al. (1983, 1985) using standard penetration test (SPT)-based field performance data. The National Center for Earthquake Engineering Research (NCEER) workshop, 1998, published the reviews of in-situ test-based simplified method with recommendations for the evaluation of liquefaction potential of soil (Youd et al., 2001). Deterministic methods were discussed, which allow the liquefaction potential of soil to be evaluated in terms of the factor of safety against liquefaction ($F_s$), defined as the ratio of cyclic resistance ratio (CRR) to the cyclic stress ratio (CSR). However, due to parameter and model uncertainties, $F_s > 1$ may not always indicate non-liquefaction cases, and similarly, $F_s \leq 1$ may not always correspond to liquefaction (Juang et al., 2000). The boundary curve that separates liquefaction and non-liquefaction

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cases in the deterministic methods is considered as a performance function or “limit state function” and is generally biased toward the conservative side by encompassing most of the liquefied cases. The degree of conservatism, however, is not quantified (Juang et al., 2000). In order to overcome the above mentioned difficulties in the deterministic approach, a probabilistic evaluation of liquefaction potential has been performed where liquefaction potential is expressed in terms of the probability of liquefaction (\(P_L\)). Few attempts have been made by researchers to quantify the unknown degree of conservatism associated with the limit state function and to assess liquefaction potential in terms of the probability of liquefaction using statistical or probabilistic approaches. Haldar and Tang (1979) carried out second moment statistical analyses of the SPT-based test data using the limit state function introduced by Seed and Idriss (1971) to estimate the \(P_L\). Lio et al. (1988), Youd and Nobble (1997) and Toprak et al. (1999) used logistic regression analyses of post-liquefaction field performance data to develop empirical equations for assessing \(P_L\). These models are all data-driven as they are based on statistical analyses of the databases of post-liquefaction case histories. The calculation of \(P_L\) using these empirical models requires only the mean values of the input variables, whereas the uncertainty in the parameters and the model is excluded from the analysis. Thus, resulting \(P_L\) is subject to error if the effect of the parameter or the model uncertainty is significant. These difficulties can be overcome by adopting a reliability-based probabilistic analysis of liquefaction, which considers both model and parameter uncertainties. Juang et al. (1999) used the advanced first-order second moment (AFOSM) method to determine the reliability index (\(\beta\)) for liquefaction and non-liquefaction cases and developed a relationship between \(\beta\) and \(P_L\) using a Bayesian mapping function based on post-liquefaction CPT data. They used the ellipsoid method (Low and Tang, 1997) to determine the reliability index. Juang et al. (2000) developed a simplified method based on a post-liquefaction SPT database using the Bayesian mapping function approach to relate \(F_L\) with \(P_L\). Juang et al. (2002) found that the Bayesian mapping function approach is better than the logistic regression approach for the site-specific probability of

### Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
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<tbody>
<tr>
<td>AAE</td>
<td>average absolute error</td>
</tr>
<tr>
<td>CDF</td>
<td>cumulative distribution function</td>
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<tr>
<td>COV</td>
<td>coefficient of variation</td>
</tr>
<tr>
<td>CRR</td>
<td>cyclic resistance ratio</td>
</tr>
<tr>
<td>CSR(_{7.5})</td>
<td>cyclic stress ratio adjusted to a benchmark earthquake magnitude of 7.5</td>
</tr>
<tr>
<td>FORM</td>
<td>first order reliability method</td>
</tr>
<tr>
<td>FOSM</td>
<td>first order second moment method</td>
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<tr>
<td>GP</td>
<td>genetic programming</td>
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<td>LI</td>
<td>liquefaction index</td>
</tr>
<tr>
<td>MAE</td>
<td>maximum absolute error</td>
</tr>
<tr>
<td>MGGP</td>
<td>multi-gene genetic programming</td>
</tr>
<tr>
<td>MSF</td>
<td>magnitude scaling factor</td>
</tr>
<tr>
<td>PDF</td>
<td>probability density function</td>
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<td>RMSE</td>
<td>root mean square error</td>
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### Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>(C_B)</td>
<td>correction for borehole diameter</td>
</tr>
<tr>
<td>(C_E)</td>
<td>correction for hammer energy efficiency</td>
</tr>
<tr>
<td>(C_N)</td>
<td>factor to normalize (N_m) to a common reference effective overburden stress</td>
</tr>
<tr>
<td>(C_R)</td>
<td>correction for “short” rod length</td>
</tr>
<tr>
<td>(C_S)</td>
<td>correction for non-standardized sampler configuration</td>
</tr>
<tr>
<td>(E)</td>
<td>Nash–Sutcliffe coefficient of efficiency</td>
</tr>
<tr>
<td>(E_f)</td>
<td>error function</td>
</tr>
<tr>
<td>(f)</td>
<td>MGGP functions defined by the user</td>
</tr>
<tr>
<td>(F)</td>
<td>liquefaction index function</td>
</tr>
<tr>
<td>(FC)</td>
<td>fines content in percentage</td>
</tr>
<tr>
<td>(F_s)</td>
<td>factor of safety against occurrence of liquefaction</td>
</tr>
<tr>
<td>(g)</td>
<td>acceleration due to gravity</td>
</tr>
<tr>
<td>(G_{\text{max}})</td>
<td>maximum number of genes</td>
</tr>
<tr>
<td>(K_{\sigma})</td>
<td>overburden correction factor</td>
</tr>
<tr>
<td>(L)</td>
<td>liquefied cases</td>
</tr>
<tr>
<td>(LI)</td>
<td>liquefaction index</td>
</tr>
<tr>
<td>(M_w)</td>
<td>earthquake magnitude on moment magnitude scale</td>
</tr>
<tr>
<td>(N_{\text{gen}})</td>
<td>number of generations</td>
</tr>
<tr>
<td>(NL)</td>
<td>non-liquefied cases</td>
</tr>
<tr>
<td>(n)</td>
<td>number of terms of target expression</td>
</tr>
<tr>
<td>(N_m)</td>
<td>measured SPT blow count</td>
</tr>
<tr>
<td>(N_{1,60})</td>
<td>corrected SPT blow count (i.e., corresponds to the (N_m) value after correction for overburden, energy, equipment and procedural effects in SPT method)</td>
</tr>
<tr>
<td>(N_{1,60,\text{cor}})</td>
<td>the equivalent clean-sand overburden stress corrected SPT blow count</td>
</tr>
<tr>
<td>(P_L)</td>
<td>probability of liquefaction</td>
</tr>
<tr>
<td>(R)</td>
<td>correlation coefficient</td>
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<tr>
<td>(Z)</td>
<td>performance function</td>
</tr>
<tr>
<td>(\sigma'_v)</td>
<td>effective vertical stress at the depth under consideration</td>
</tr>
<tr>
<td>(\sigma_v)</td>
<td>total vertical stress at the depth under consideration</td>
</tr>
<tr>
<td>(a_{\text{max}})</td>
<td>peak horizontal ground surface acceleration</td>
</tr>
<tr>
<td>(d_{\text{max}})</td>
<td>maximum depth of gene</td>
</tr>
<tr>
<td>(c_0)</td>
<td>bias</td>
</tr>
<tr>
<td>(\mu_z)</td>
<td>mean of performance function</td>
</tr>
<tr>
<td>(\sigma_z)</td>
<td>standard deviation of performance function</td>
</tr>
<tr>
<td>(\beta)</td>
<td>reliability index</td>
</tr>
<tr>
<td>(p_f)</td>
<td>probability of failure</td>
</tr>
<tr>
<td>(\Phi(\cdot))</td>
<td>CDF of standard normal variable</td>
</tr>
<tr>
<td>(\epsilon_{\text{cmf}})</td>
<td>model factor</td>
</tr>
<tr>
<td>(\mu_{\epsilon_{\text{cmf}}})</td>
<td>mean of (\epsilon_{\text{cmf}})</td>
</tr>
<tr>
<td>(\beta_1)</td>
<td>reliability index without considering model uncertainty</td>
</tr>
<tr>
<td>(\beta_2)</td>
<td>reliability index considering model uncertainty</td>
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liquefaction evaluation. Cetin (2000) and Cetin et al. (2004) developed SPT based on probabilistic models for the evaluation of liquefaction potential using the first-order reliability method (FORM) and a Bayesian updating technique. Hwang et al. (2004) used first order and second moment (FOSM) reliability analyses to evaluate the liquefaction potential of soil in terms of $P_L$ on the basis of post-liquefaction SPT data obtained after the 1999, Chi-Chi earthquake. Juang et al. (2008) used the first-order reliability method (FORM) along with the Bayesian mapping function approach for the probabilistic assessment of soil liquefaction potential based on SPT data and carried out a sensitivity analysis to characterize uncertainties associated with the CRR model as presented by Youd et al. (2001). Recently, Bagheripour et al. (2012) used an advanced first order second moment method (AFOSM) to evaluate the liquefaction potential of soil in terms of the probability of liquefaction using SPT data. However, there are certain limitations in the above reliability-based liquefaction potential evaluation. Bagheripour et al. (2012) used the existing regression-based CRR and CSR models as per Youd et al. (2001) to formulate the performance function in their reliability analysis. They did not develop any new model for CRR. It is pertinent to mention here that the CSR model as presented in Youd et al. (2001) was revised by Idriss and Boulanger (2006) by incorporating the modified stress reduction factor ($\gamma_r$), earthquake magnitude scaling factor (MSF) and overburden correction factor ($K_o$) as per the behavior identified by analytical and experimental studies. They have used normal distribution for variables instead of more relevant lognormal distribution (Jefferyes et al., 1988).

The coefficient of variation (COV) of variables and coefficients of correlation among the variables were assumed from the available literature rather than using their actual values: this may be due to the non-availability of the required data in the database, which might have caused the over- or under-estimation of the probability of liquefaction. Furthermore, the model uncertainty of the performance function was not considered, though it is well known that semi-empirical models are associated with some degree of uncertainty (Juang et al., 2006). Hence, since the limit state boundary curves are generally biased toward the conservative side (i.e., PL < 50%) in the Bagheripour et al. (2012) reliability method, it is difficult to make an unbiased design decision from the obtained probability of liquefaction.

Artificial intelligence (AI) techniques, such as the artificial neural network (ANN) (Goh, 1994; Juang et al., 2000; Hanna et al., 2007; Samui and Sitharam, 2011), the support vector machine (SVM) (Pal, 2006; Goh and Goh, 2007; Samui and Sitharam, 2011) and the relevance vector machine (RVM) (Samui, 2007), have been used to develop various liquefaction prediction models based on an in-situ test database. While these models are found to be more efficient than statistical methods, there are problems associated with them. The poor generalization of the ANN approach is attributed to attainment of local minima during training and needs iterative learning steps to obtain better learning performances. Even thought the SVM has better generalization than ANN, the parameters ‘C’ and insensitive loss function (ε) need to be fine-tuned by the user. Moreover, these techniques will not produce a comprehensive relationship between the inputs and output, and, as such, are referred to as ‘black box’ systems.

In the recent past, genetic programming (GP) based on Darwinian theory of natural selection is being used as an alternate AI technique. The GP is defined as the next generation AI technique and is referred to as a ‘grey box’ model (Giustolisi et al., 2007). It allows the mathematical structure of the model to be derived, and provides further information of the system behavior. The models based on GP and its variants have been applied to some difficult geotechnical engineering problems (Yang et al., 2004; Javadí et al., 2006; Rezania and Javadi, 2007; Alavi et al., 2011; Gandomi and Alavi, 2012b; Muduli et al., 2013; Alkroosh and Nikraz, 2014) with success. However, its use in liquefaction susceptibility assessment is very limited (Gandomi and Alavi, 2012b; Muduli and Das, 2013). The main advantage of GP and its variants over traditional statistical methods and other artificial intelligence techniques is its ability to develop a compact and explicit prediction equation in terms of different model variables, which can be used for further analysis.

In the present study, an attempt has been made to evaluate the reliability-based liquefaction potential of soil in terms of the probability of liquefaction using FORM based on the post-liquefaction SPT database (Cetin, 2000). Muduli et al. (2013) observed that the efficacy of the GP-based predictive model in determining the uplift capacity of suction caisson exceeded that of other soft computing techniques (ANN, SVM, RVM) based on prediction models in terms of different statistical performance criteria. Thus, in this study the multi-gene GP (MGGP), a variant of GP, is used to develop the CRR model of soil. The developed CRR model along with the updated CSR model (Idriss and Boulanger, 2006) forms the limit state model of liquefaction for the reliability analysis. The uncertainties of the input parameters, which are considered as random variables, are obtained from the database (Cetin, 2000). As Jefferyes et al. (1988) observed that the measured geotechnical parameters are well fitted to logarithmic normal distribution. In this study, the parameters are assumed to follow logarithmic distribution in reliability analysis. Here, in the reliability analysis coefficient of variation (COV) of the parameters and coefficients of correlation among parameters have been considered as per the actual as determined and presented in Cetin (2000) and Juang et al. (2008), respectively. Also, a rigorous reliability analysis associated with the Bayesian mapping function approach is carried out to estimate the model uncertainty of the limit state, which is represented by a lognormal random variable, and is characterized in terms of its two statistics, namely, the mean and the coefficient of variation. The probability of liquefaction, obtained from the above reliability analysis using the estimated model uncertainty of the limit state function for a future seismic event, will help in making an unbiased design decision. A mapping function is also developed on the basis of Bayesian theory to relate $F_s$ with $P_L$, which can be used in absence of parameter uncertainties.

2. Methodology

In the present study, first, the MGGP is used to develop a liquefaction field performance observation function termed as
liquefaction index \((LI)\) (Juang et al., 2000). The \(LI\) is a function of various soil and seismic parameters and is used to separate liquefaction and non-liquefaction cases. In the second step, artificial data points are generated for the unknown boundary curve separating liquefied cases from non-liquefied cases using a search technique (Juang et al., 2000). The boundary curve referred as a “limit state function” representing the CRR of the soil is approximated with the generated data points using MGGP. The developed CRR model along with the CSR model (Idriss and Boulanger, 2006) forms the performance functions or limit state model of liquefaction for reliability analysis. Here, FORM (Hasofer and Lind, 1974) is used to evaluate the liquefaction potential of soil in terms of \(P_L\), which requires the knowledge of both parameter and model uncertainties. The uncertainty associated with proposed limit state model is determined following the extensive sensitive analysis adopted by Juang et al. (2006) through a rigorous reliability analysis associated with Baysian mapping function approach. Bayesian theory of conditional probability is used to create a mapping function to relate \(F_1\) with \(P_L\).

The GP and its variant, the MGGP, have been used in limited geotechnical engineering problems and are not very common to geotechnical engineering professionals; hence, a brief description is presented in the following sections. A brief description about the determination of reliability index and its corresponding \(P_L\) using GA as optimization tool is also presented.

2.1. Genetic programming

Genetic programming is a pattern recognition technique where the model is developed on the basis of adaptive learning over a number of cases of provided data, developed by Koza (1992). It mimics biological evolution of living organisms and makes use of the principles of genetic algorithms (GA). It provides a solution in the form of a tree structure or in the form of a compact equation using the given dataset. A brief description about the GP is presented for the completeness, but the details can be found in Koza (1992).

The GP model is composed of nodes, which resembles a tree structure and thus, it is also known as the GP tree. Nodes are the elements either from a functional set or terminal set. A functional set may include arithmetic operators (+, -, *, /), mathematical functions (\(\sin()\), \(\cos()\), \(\tanh()\) or \(\ln()\)), Boolean operators (AND, OR, NOT, etc.), logical expressions (IF, or THEN) or any other suitable functions defined by the user. The terminal set includes variables (like \(x_1\), \(x_2\), \(x_3\), \(x_4\)) or constants (like 3, 5, 6, 9, etc.) or both. The functions and terminals are randomly chosen to form a GP tree with a root node and the branches extending from each function nodes to end in terminal nodes as shown in Fig. 1.

Initially a set of GP trees, as per user defined population size, is randomly generated using various functions and terminals assigned by the user. The fitness criterion is calculated by the objective function and it determines the quality of each individual in the population competing with the rest. At each generation a new population is created by selecting individuals as per the merit of their fitness from the initial population and then, implementing various evolutionary mechanisms like reproduction, crossover and mutation (Muduli and Das, 2013). This process is iterated until the termination criterion, which can be either a threshold fitness value or maximum number of generations, is satisfied. The best GP model, based on its fitness value that appeared in any generation, is selected as the result of genetic programming.

2.2. Multi-genre genetic programming

MGGP is a variant of GP and is designed to develop an empirical mathematical model, which is a weighted linear combination of a number of GP trees. It is also referred to as symbolic regression. Each tree represents lower order non-linear transformations of input variables and is called a ‘gene’. “Multi-genre” refers to the linear combination of these genes. Fig. 2 shows a typical flow diagram of MGGP procedure in which \(N_{gen}\) is the number of generations, \(P_s\), \(P_c\), and \(P_m\) are the probability of reproduction, crossover and mutation, respectively.

Fig. 3 shows an example of MGGP model where the output is represented as a linear combination of two genes (Gene-1 and Gene-2) that are developed using four input variables \((x_1, x_2, x_3, x_4)\). Each gene is a nonlinear model as it contains nonlinear terms (\(\sin()\), \(\log()\)). In the MGGP model development, it is important to make a tradeoff between accuracy and complexity in terms of maximum allowable number of genes \((G_{max})\) and maximum depth of GP tree \((d_{max})\). The user specifies the values of \(G_{max}\) and \(d_{max}\) to have a control over the complexity of MGGP model. Thus, there are optimum values of \(G_{max}\) and \(d_{max}\), which produce a relatively compact model (Searson et al., 2010). The linear coefficients (weights) of Gene-1 and Gene-2 \((c_1\) and \(c_2\)) and the bias \((c_0)\) of the model are obtained from the training data using statistical regression analysis (ordinary least squares method).

In the MGGP procedure, the initial population is generated by creating individuals that contain randomly evolved genes from the user-defined functions and variables. In addition to the standard GP evolution mechanisms as discussed earlier there are some special MGGP crossover and mutation mechanisms.
Searson et al., 2010; Gandomi and Alavi, 2012a), which allow the exchange of genes between individuals.

The probabilities of the various evolutionary mechanisms can be set by the user to achieve the best MGGP model. These mechanisms are grouped into categories referred to as events. Therefore, the probability of crossover, mutation and direct reproduction events is to be specified by the user in such a way that the sum of these probabilities is 1.0.

The general form of MGGP-based model of the present study can be presented as

\[ LI_p = \sum_{i=1}^{n} F(X,f(X),c_i) + c_0 \]  

where \( LI_p \) = predicted value of \( LI \), \( F=\)the function created by the MGGP referred herein as liquefaction index function, \( X=\)vector of input variables={\( N_{1,60}, FC, \sigma_v, CSR_{7.5} \)}, \( N_{1,60}=\)normalized standard penetration resistance (Idriss and Boulanger, 2006), \( FC=\)fines content (Idriss and Boulanger, 2006), \( \sigma_v=\)vertical effective stress of soil at the depth studied, and \( CSR_{7.5}=\)cyclic stress ratio adjusted to a benchmark earthquake of moment magnitude (\( M_w \)) of 7.5 and to an equivalent \( \sigma_v' \) of 101 kPa (Idriss and Boulanger, 2006):

\[ CSR_{7.5} = 0.65 \left( \frac{\sigma_v}{\sigma_{v'}} \right) \left( \frac{a_{\text{max}}}{g} \right) (r_d) / \text{MSF} / K_a \]  

where \( \sigma_v=\)vertical total stress of soil at the depth studied, \( a_{\text{max}}=\)peak horizontal ground surface acceleration, \( g=\)acceleration due to gravity, \( r_{d}=\)shear stress reduction coefficient, \( \text{MSF} \) is the
magnitude scaling factor, \( K_o \) = overburden correction factor, \( c_i \) = constant, \( f(X) \) = function defined by the user from the functional set of MGGP, \( n \) is the number of terms of target expression and \( c_0 \) = bias. The MGGP as per Searson et al. (2010) is used and the present model is developed and implemented using Matlab (Math Works Inc., 2005).

2.3. Reliability analysis

In order to overcome the limitations of the conventional factor of safety approach in the liquefaction potential evaluation as discussed in the earlier section, reliability analyses have been performed in the present study using the first-order reliability method, FORM (Hasofer and Lind, 1974). The Hasofer-Lind approach is one of the most widely used reliability methods (Haldar and Mahadevan, 2000; Baecher and Christian, 2003). It is the improvement over the first order second moment (FOSM) reliability method developed by Cornell (1969) and avoids its lack of invariance problem. A brief description of the formulation of the present problem as per FORM is discussed below.

In the liquefaction potential evaluation, the CSR (loading) and the CRR (resistance) are denoted by \( Q \) and \( R \), respectively. The margin of safety, \( Z \) (Baecher and Christian, 2003), is defined as the difference between the resistance and the load, which is also the performance function for liquefaction potential assessment and is presented by

\[
Z = R - Q
\]  

(3)

If \( Z < 0 \), it indicates the occurrence of liquefaction. If \( Z > 0 \), it suggests that there will be no liquefaction. If \( Z = 0 \), the performance state is designated as a limit state, which is the boundary between liquefaction and non-liquefaction. It is to be noted that both \( R \) and \( Q \) are uncertain, and thus can be treated as random variables and reliability index (\( \beta \)) can be presented by Eq. (4) following Baecher and Christian (2003).

\[
\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_Q}{\sqrt{(\sigma_R^2 + \sigma_Q^2 - 2\rho_{RQ}\sigma_R\sigma_Q)}}
\]  

(4)

If the load and resistance are uncorrelated (i.e., correlation coefficient is zero), Eq. (4) reduces to

\[
\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_Q}{\sqrt{\sigma_R^2 + \sigma_Q^2}}
\]  

(5)

In Eqs. (4) and (5), \( \mu_R, \mu_Q \) are the mean values of \( R \) and \( Q \), respectively; \( \sigma_R, \sigma_Q \) are the standard deviations of \( R \) and \( Q \), respectively; \( \sigma_R^2, \sigma_Q^2 \) are the variances of \( R \) and \( Q \), respectively, and \( \rho_{RQ} \) is the correlation coefficient between \( R \) and \( Q \). The reliability index \( \beta \) defined by Eq. (5) is the same as the first order second moment (FOSM) reliability method developed by Cornell (1969) using the first-order Taylor series expansion approximation.

If \( R \) and \( Q \) are the random variables with normal distribution, then the performance function, \( Z = R - Q \), is also normally distributed. Fig. 4 shows the resulting probability density function (PDF) of \( Z \). The probability of liquefaction is defined as the probability that \( Z \leq 0 \). The dark area of the PDF of \( Z \) as shown in Fig. 4 indicates the probability of liquefaction. The probability of liquefaction is indicated by the dark region: the larger the dark region, the higher the probability of liquefaction. Then, the probability of liquefaction, \( p_f (P_L) \), can be calculated from Eq. (6) as presented below:

\[
p_f = P_L = P[Z \leq 0] = \Phi \left( \frac{-\mu_Z}{\sigma_Z} \right) = \Phi(-\beta) = 1 - \Phi(\beta)
\]  

(6)

where \( \Phi(\cdot) \) is the cumulative distribution function (CDF) for a standard normal variable.

As per the Hasofer-Lind approach, all the normal random variables are transformed to their reduced form in the standard normal space with zero mean and unit standard deviation. Thus, \( R \) and \( Q \) in the liquefaction analysis can be expressed as standard normal variables as given below.

\[
R' = \frac{R - \mu_R}{\sigma_R}, \quad Q' = \frac{Q - \mu_Q}{\sigma_Q}
\]  

(7)

If \( R \) and \( Q \) are uncorrelated, Eq. (3) for the performance function becomes

\[
Z = R - Q = R'\sigma_R - Q'\sigma_Q + \mu_R - \mu_Q
\]  

(8)

Fig. 5 shows a plot of the liquefaction limit state criterion using the standard normal variables as the axes. The origin is the point at which both \( R \) and \( Q \) equal to their mean values. The distance, \( d \), between the origin and the limit state line, \( Z=0 \), is

\[
d = \frac{\mu_R - \mu_Q}{\sqrt{\sigma_R^2 + \sigma_Q^2}}
\]  

(9)

which is identical to the definition of the reliability index \( \beta \) given by Eq. (5). This result suggests that the reliability index can be interpreted geometrically as the shortest distance from the point defined by the mean values of the variables to performance function surface defining the limit state. The liquefaction performance function, \( Z \), however, depends on \( R \) and \( Q \), which are also the functions of multiple basic variables such as

![Fig. 4. Probability density function of liquefaction performance function, Z (modified from Baecher and Christian (2003)).](image-url)
measured standard penetration resistance \((N_m)\), \(FC\), \(\sigma_v\), \(\sigma_v'\), \(a_{max}\) and \(M_w\). Thus, the performance function can be presented as \(Z=R-Q=g(z)\), where \(z\) is a vector of uncorrelated random variables, i.e., \(z=[N_m, FC, \sigma_v, \sigma_v', a_{max}, M_w]\). Each variable, \(z_i\), is defined in terms of its mean \(\mu_z\) and its standard deviation \(\sigma_z\).

The above interpretation of reliability index (Eq. (9)) can be generalized for \(n\) (6) number of random variables, which are first converted to standard normal variables \((z'_{i})\) as per Eq. (7). In the multi-dimensional standard normal space, the distance from the origin to a point on the liquefaction limit state is

\[
d = \sqrt{z_{1}'^2 + z_{2}'^2 + \ldots + z_{n}'^2} = \sqrt{z'^T z'}
\]

(10)

where the superscript \(T\) denotes the transpose of the vector \(z'\). Thus, determination of \(\beta\) using Hasofer-Lind first order reliability formulation for liquefaction potential assessment can be stated as follows:

\[
\text{Minimize} \quad \beta = \min (z'^T z')^{1/2}
\]

(11a)

Subjected to

\[
g(z) = 0
\]

(11b)

This is a constrained optimization problem that can be solved using various tools, including Lagrange multipliers and the Taylor series. The Hasofer-Lind reliability approach described above with regard to its applicability to a reliability-based liquefaction triggering analysis can be extended to the non-linear limit state function and/or correlated to non-normal random variables by suitable transformation algorithms.

In the present study, the Cholesky approach (Baecher and Christian, 2003) is used to convert uncorrelated standard normal variables to correlated standard normal variable using the correlation matrix of variables. Furthermore, the random variables, \(z=[N_m, FC, \sigma_v, \sigma_v', a_{max}, M_w]\), are assumed to follow lognormal distribution in this study; thus, the mean and standard deviation of equivalent normal variables are calculated following Der Kiureghian et al. (1987).

While the Rackwitz and Fiessler (1978) iterative algorithm is widely used in the reliability problem to determine the minimum value of \(\beta\) from various algorithms available in the literature (Lin and Der Kiureghian, 1991), because of the tendency to attain local minima by most of the algorithms in complex and non-linear limit state functions, this algorithm fails to produce the true \(\beta\) value. The problem can be overcome by using full population-based iterative procedures such as the Monte Carlo or heuristic optimization algorithm, GA (Xue and Gavin, 2007; Gavin and Xue, 2008, 2009). In the present study, GA is used as the optimization tool for the reliability analysis.

The GA is a random search algorithm based on the concept of natural selection inherent in natural genetics, which presents a robust method to search for the optimum solution to the complex problems. In the present study, the GA was implemented using a pseudo code (toolbox) available in Matlab (Math Works Inc., 2005).

3. Results and discussion

In the present study, the models are developed based on the post-liquefaction SPT database compiled and reassessed by Cetin (2000). The total database consists of 198 cases from different earthquakes around the world. A total of 163 cases are initially considered, and 35 cases of proprietary data from the 1995 Hyogoken-Nambu (“Kobe”) earthquake could not be considered because details are not available. The database contains the following information about soil and seismic parameters: \(N_m\), correction for “short” rod length (\(C_p\)), correction for non-standardized sampler configuration (\(C_s\)), correction for the borehole diameter (\(C_b\)), correction for hammer energy efficiency (\(C_e\)), \(FC\), \(\beta\), \(\sigma_v\), \(\sigma_v'\), \(a_{max}\), \(M_w\) and liquefaction field performance observation, \(LI\). The soil parameters are \(N_m, FC, \sigma_v\) and \(\sigma_v'\), and \(a_{max}\) and \(M_w\) are the seismic parameters considered in the present study for the development of the CRR model. The soil in these cases ranges from sand to silt mixtures (sandy and clayey silt). As per Cetin (2000), the case histories in the database are classified into three groups, Class A, Class B and Class C, in decreasing order according to the quality of informational content. A total 160 cases, including 43 cases out of 44 Class A data, 111 cases from 113 Class B data and all the 6 Class C data were considered with three marginally liquefied cases not considered. Of these, 160 cases considered for the development of the model, 92 cases are liquefied and 68 cases are non-liquefied. The database contains the mean and standard deviation of the above parameters calculated from the SPT bore log for each in the critical depth range where soil is susceptible to the liquefaction (Cetin, 2000). The coefficient of variation (COV), which is the ratio of standard deviation and mean, for the above variables was calculated and the summarized extract of the database used for model development is provided in Table 1 in terms of the
range of mean and COV of the various variables considered in the present investigation as inputs and output.

In the MGGP procedure, a number of potential models are evolved at random and each model is trained and tested using the training and testing cases. The fitness of each model is determined by minimizing the root mean square error (RMSE) between the predicted and actual value of the output variable (LI) as the objective function or error function (E):

\[
RMSE = E_j = \sqrt{\frac{\sum_{i=1}^{N}(LI - LI_p)^2}{N}}
\]

where \(LI = 1\) (liquefied case), 0 (non-liquefied case), \(N=\) number of cases in the fitness group. If the errors calculated by using Eq. (12) for all the models in the existing population do not satisfy the termination criteria, the generation of new population continues until the best model is developed as indicated in the earlier discussion.

It is pertinent to mention here that to make a substantive claim that the developed MGGP-based \(LI_p\) is the “best”, some sort of repeated validation scheme, such as \(K\)-fold cross-validation or repeated random sub-sampling needs to be followed. Thus, in the present study, a \(K\)-fold cross validation (Oommen and Baise, 2010), which is a reliable and robust approach, was used to obtain the most “efficient” \(LI_p\)-based predictive model by the MGGP. Here, the original data (160 cases) of the present database is split into \(K\) (4) equal folds. For each \(K\) split, \((K-1)\)-folds are used for training and the remaining one fold is used for testing the developed model, as shown in Fig. 6. Therein, the filled rectangles represent testing data, whereas the open rectangles represent the training data for each split. Hence, in each split, out of the mentioned 160 data, 120 data are selected for training and remaining 40 data are used for testing the developed model. The advantage of \(K\)-fold cross validation is that all the cases in the database are ultimately used for both training and testing. For each split, several \(LI_p\) models were obtained by varying the MGGP parameters like population from 1000 to 5000 individuals, generations from 100 to 500, maximum number of genes \((G_{max})\) from 2 to 5, maximum depth of GP tree \(d_{max}\) from 2 to 4, reproduction probability in the range of \([0.01, 0.07]\), with a crossover probability in the range of \([0.75, 0.9]\) and a mutation probability in the range of \([0.05, 0.15]\). Then, the developed models were analyzed with respect to the physical interpretation of \(LI\) of soil, and after careful consideration of various alternatives, four models, the “best” of each split, are selected. It is important to note that the efficiency of different models should be compared in terms of testing data rather than training data (Das and Basudhar, 2008). Hence, the efficiency of each of the four developed \(LI_p\) models are evaluated by calculating the rate of successful prediction in percentage on test data of each of the \(K\) (4) splits. The “best” MGGP-based \(LI_p\) model was obtained with population size of 4000 individuals at 150 generations with reproduction probability of 0.05, crossover probability of 0.85 and mutation probability of 0.1. It is relevant to mention here that the RMSE was decreased with increase in \(G_{max}\) and \(d_{max}\) but, at the same time, the model complexity was increased. In this study, the optimum result was obtained with

\[
\text{Table 1 Summary of the database used for development of different models in the present study.}
\]

<table>
<thead>
<tr>
<th>Model variables</th>
<th>Type</th>
<th>Range of mean value</th>
<th>Range of COV value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D) (m)</td>
<td>Input</td>
<td>1.100–20.400</td>
<td>–</td>
</tr>
<tr>
<td>(N_{IS})</td>
<td>Input</td>
<td>1.500–37.000</td>
<td>0.007–0.815</td>
</tr>
<tr>
<td>FC (%)</td>
<td>Input</td>
<td>0.000–92.000</td>
<td>0.000–2.000</td>
</tr>
<tr>
<td>(\sigma_c) (kPa)</td>
<td>Input</td>
<td>15.470–383.930</td>
<td>0.031–0.280</td>
</tr>
<tr>
<td>(\sigma_v) (kPa)</td>
<td>Input</td>
<td>8.140–198.660</td>
<td>0.044–0.378</td>
</tr>
<tr>
<td>(d_{max}) (G)</td>
<td>Input</td>
<td>0.090–0.693</td>
<td>0.011–0.300</td>
</tr>
<tr>
<td>(M_w)</td>
<td>Input</td>
<td>5.900–8.000</td>
<td>0.000–0.025</td>
</tr>
<tr>
<td>(LI)</td>
<td>Output</td>
<td>0.000–1.000</td>
<td>–</td>
</tr>
</tbody>
</table>

![Fig. 6. An example of \(K\)-fold cross validation approach where the data are split into \(K\) (4) equal folds (modified from Oommen and Baise (2010)).](image)

\[
G_{max} \text{ as 4 and } d_{max} \text{ as 3. These optimum values of } G_{max} \text{ and } d_{max} \text{ were obtained based on some guidelines suggested by Searson et al. (2010) and after using a trial and error approach. The developed “best” } LI_p \text{ model can be described as}
\]

\[
LI_p = 1.823 \tan [6.024CSR_{7.5} - 0.0368 \{N_{1,60} + CSR_{7.5} - \cos N_{1,60}\} + \frac{26.16CSR_{7.5} \sin (FC)}{\sigma_v^2}] - 0.3728
\]

The statistical performances of both training and testing data for the developed \(LI_p\) model (Eq. (13)) in terms of correlation coefficient \((R)\), Nash–Sutcliffe coefficient of efficiency \((E)\) (Das and Basudhar, 2008), average absolute error (AAE), maximum absolute error (MAE) and RMSE were found to be comparable showing good generalization of the developed model, which also ensures that the model is not over-fitting to training data. A prediction in terms of \(LI_p\) is said to be successful if it agrees with field manifestation of the database. As per Eq. (13), the rate of successful prediction of liquefied and non-liquefied cases is 85% for training and 88% for testing data.

The Eq. (13) can be used by geotechnical engineering professionals with the help of a spreadsheet to predict the occurrence of liquefaction based on soil properties in a future seismic event without going into the complexities of the model development. In the present study, the developed \(LI_p\) model is further used for the development of a proposed CRR model.
3.1. Generation of artificial points on the limit state curve

To approximate a limit state function that will separate liquefied cases from the non-liquefied ones, artificial data points on the boundary curve are generated using Eq. (13) and following a simple but robust search technique developed by Juang et al. (2000). The technique is explained conceptually with the help of Fig. 7. Let a liquefied case, ‘L’ (target output $L=1$) of the database, as shown in Fig. 7, be brought on to the boundary or limit state curve (i.e., when the case becomes just non-liquefied according to the evaluation by the developed $L_I$ model equation) if $CSR_{7.5}$ is allowed to decrease (path $P$) or $N_{1,60}$ is allowed to increase (path $Q$). Furthermore, for a non-liquefied case, ‘NL’ (target output $L=0$) of the database, the search for a point on the boundary curve involves an increase in $CSR_{7.5}$ (path $S$) or a decrease in $N_{1,60}$ (path $R$) and the desired point is obtained when the case just becomes liquefied as adjudged by Eq. (13). Fig. 8 shows the detailed flow chart of this search technique for paths ‘$P$’ and ‘$S$’. A multi-dimensional ($N_{1,60}$, $FC$, $\sigma_v'$, $CSR_{7.5}$) data point on the unknown boundary curve is obtained from each successful search. In this study, the limit state is defined as the ‘limiting’ $CSR_{7.5}$, which a soil can resist without the occurrence of liquefaction and beyond which the soil will liquefy. Thus, for a particular soil at its in-situ condition, this limit state specifies its $CRR$ value. A total of 240 multi-dimensional artificial data points ($N_{1,60}$, $FC$, $\sigma_v'$, $CSR_{7.5}$) located on the boundary curve are generated using the developed MGGP-based model (Eq. (13)) and the technique explained in Figs. 7 and 8. These data points are used to approximate the limit state function in the form of $CRR=\frac{P_{LI}}{N_{1,60}}$. (Eq. (14)) using MGGP and this is explained in the next section.

3.2. MGGP model for CRR

The $CRR$ model is developed using 240 artificially generated data points using the MGGP. The K-fold (4) cross validation procedure is also adopted to find the “best” MGGP-based $CRR$ model. Here, out of 240 generated data points in each of $K$ (4) split, 180 data points are selected for training and the remaining 60 for testing the developed model. For each split, several CRR models were obtained by varying the MGGP parameters as mentioned earlier for the development of the $L_I$ model. Similarly, four models, the “best” one in each split are selected and their statistical performances in terms of $R$, $E$, $AAE$, $MAE$ and $RMSE$ on the basis of testing data are evaluated and presented in Table 2. The model obtained from split-1 is found to be the “best” among these four models on the basis of above statistical performances and is described below as

$$CRR = 1.235 \times 10^{-5} \left( N_{1,60} \right)^2 \left[ N_{1,60} + 8.706 \right] - 0.0001253 \left( N_{1,60} \right)^2 \sin \left( FC \right) - \frac{6.371 \sin \left( FC \right)}{\sigma_v' - 3.302} + \frac{8.398 \sin \left( FC \right)}{N_{1,60} + \sigma_v'} + 0.1129$$ (14)

The statistical performances of the “best” MGGP-based $CRR$ model in terms of $R$, $E$, $RMSE$, $AAE$ and $MAE$ as presented in

![Fig. 7. Conceptual model of the search technique for artificial data points on the boundary curve that separates liquefied and non-liquefied cases (modified from Juang et al. (2000)).](image-url)
The calculated $F_s$ values for different cases of the present database are grouped according to the field performance observation of liquefaction ($L$) and non-liquefaction ($NL$). After considering several different probability density functions, it is found that both the liquefied and non-liquefied groups best fitted by lognormal distribution with parameters $(\mu, \sigma)$ are $(0.441, 0.437)$ and $(0.419, 0.506)$, respectively, as shown in Fig. 9(a) and (b). According to Juang et al. (1999), the probability of liquefaction occurrence of a case in the database for which the $F_s$ has been calculated, can be determined using Bayes’ theorem of search algorithm for generation of artificial data points on the boundary curve that separates liquefied and non-liquefied cases.

### Table 2

<table>
<thead>
<tr>
<th>Data (numbers)</th>
<th>$R$</th>
<th>$E$</th>
<th>AAE</th>
<th>MAE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Split-1</td>
<td>0.98</td>
<td>0.95</td>
<td>0.01</td>
<td>0.05</td>
<td>0.02</td>
</tr>
<tr>
<td>Split-2</td>
<td>0.89</td>
<td>0.79</td>
<td>0.02</td>
<td>0.18</td>
<td>0.04</td>
</tr>
<tr>
<td>Split-3</td>
<td>0.93</td>
<td>0.86</td>
<td>0.02</td>
<td>0.16</td>
<td>0.03</td>
</tr>
<tr>
<td>Split-4</td>
<td>0.97</td>
<td>0.94</td>
<td>0.01</td>
<td>0.08</td>
<td>0.02</td>
</tr>
<tr>
<td>Summary</td>
<td>0.95</td>
<td>0.89</td>
<td>0.02</td>
<td>0.12</td>
<td>0.03</td>
</tr>
</tbody>
</table>

### Table 3

<table>
<thead>
<tr>
<th>Data (numbers)</th>
<th>$R$</th>
<th>$E$</th>
<th>AAE</th>
<th>MAE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Training (180)</td>
<td>0.96</td>
<td>0.92</td>
<td>0.01</td>
<td>0.13</td>
<td>0.02</td>
</tr>
<tr>
<td>Testing (60)</td>
<td>0.98</td>
<td>0.95</td>
<td>0.01</td>
<td>0.05</td>
<td>0.02</td>
</tr>
</tbody>
</table>

3.3. $P_L - F_s$ mapping function

The calculated $F_s$ values for different cases of the present database are grouped according to the field performance...
probability:
\[
P(L|F_i) = \frac{P(F_i|L)P(L)}{P(F_i|L)P(L) + P(F_i|NL)P(NL)}
\]
where \(P(L|F_i)\) is the probability of liquefaction for a given \(F_i\); \(P(F_i|L)\) is the distribution function of \(F_i\) assumed that liquefaction did occur; \(P(F_i|NL)\) is the distribution function of \(F_i\) assuming that liquefaction did not occur; \(P(L)\) is the prior probability of liquefaction and \(P(NL)\) is the prior probability of non-liquefaction. \(P(F_i|L)\) and \(P(F_i|NL)\) can be obtained using Eqs. (16a) and (16b), respectively.

\[
P(F_i|L) = \int_{F_i}^{F_i+\Delta F_i} f_L(x)dx
\]

\[
P(F_i|NL) = \int_{F_i}^{F_i+\Delta F_i} f_{NL}(x)dx
\]

where \(f_L(x)\) and \(f_{NL}(x)\) are PDFs of \(F_i\) for liquefied cases and non-liquefied cases of the database, respectively. As \(\Delta F_i \to 0\), Eq. (15) can be expressed as

\[
P(L|F_i) = \frac{f_L(F_i)P(L)}{f_L(F_i)P(L) + f_{NL}(F_i)P(NL)}
\]

If the information of prior probabilities \(P(L)\) and \(P(NL)\) is available, Eq. (17) can be used to determine the probability of liquefaction for a given \(F_i\). In absence of \(P(L)\) and \(P(NL)\) values, it can be assumed that \(P(L) = P(NL)\) on the basis of the maximum entropy principle (Juang et al., 1999). Thus, under the assumption that \(P(L) = P(NL)\), Eq. (17) can be presented as

\[
P_L = \frac{f_L(F_i)}{f_L(F_i) + f_{NL}(F_i)}
\]

where \(f_L(F_i)\) and \(f_{NL}(F_i)\) are the lognormal PDFs of \(F_i\) for liquefied cases and non-liquefied cases, respectively. Based on the obtained probability density functions, \(P_L\) is calculated using Eq. (18) for each case in the database. The \(F_i\) and the corresponding \(P_L\) of the total 144 cases of database are plotted and the mapping function is approximated through curve fitting as shown in Fig. 10. The mapping function is presented as Eq. (19) with a high correlation coefficient value \((R)\) of 0.99.

\[
P_L = \frac{1}{1 + (F_i/\alpha)^7}
\]

where \(\alpha (1.003)\) and \(b (4)\) are the parameters of the fitted logistic curve. Once the \(F_i\) is calculated using the proposed MGGP-based deterministic method (Eqs. (2) and (14)), the corresponding \(P_L\) can be determined using the developed mapping function. The proposed CRR model is also characterized with a probability of 50.3% \((P_L=0.503)\) according to Eq. (19) when \(F_i=1.0\), indicating that the CRR model is unbiased relative to the CSR model (Juang et al., 2000). Ideally, a boundary surface (i.e., CRR model) is said to be unbiased if its probability of occurrence of liquefaction, \(P_L\), is 50% corresponding to \(F_i=1\). Thus, the degree of conservatism of the developed limit state boundary surface should be close to the ideal condition, quantified in terms of \(P_L\) as 50.3%. In comparison, Juang et al. (2008) developed a SPT-based \(P_L-F_i\) mapping function, which characterizes their adopted CRR model (Youd et al., 2001) by a \(P_L\) of 55, corresponding to \(F_i=1\).

3.4. Estimation of model uncertainty from reliability analysis

The most unbiased evaluation of liquefaction potential of soil is possible with a boundary surface with the liquefied and non-liquefied cases separated, and a \(P_L\) of 50%. Such a limit state model is considered to have no model uncertainty. Alternatively, unbiased evaluation of liquefaction potential is possible by quantifying the model uncertainty of the limit state and incorporating the correct model uncertainty into the reliability analysis when evaluating the liquefaction potential. The model uncertainty of the liquefaction limit state (Eq. (3)) may be represented with a random variable ‘\(c_{\text{unc}}\)’ and is referred herein as the model factor (Juang et al., 2006). Thus, the liquefaction limit state model can be presented as follows:

\[
g(z) = c_{\text{mf}}R - Q = c_{\text{mf}}CRR - CSR
\]

where \(g(z)\) is the limit state function considering model uncertainty and \(z\) is the vector of input parameters. Here, only the uncertainty in the CRR model (Eq. (14)) is considered, and the effect of the unrealized uncertainty associated with the CSR model is realized in the CRR model as the CRR model is developed using CSR model (Eq. (2)) as a reference (Juang et al., 2006).

In the present study, the model factor ‘\(c_{\text{unc}}\)’ is treated as a random variable and then, combining it with the basic input parameters from the CRR (Eq. (14)) and CSR (Eq. (2)) models, the limit state function for the reliability analysis can be presented as

\[
g(z) = c_{\text{mf}}CRR - CSR = g(c_{\text{mf}}, N_m, FC, \sigma, \sigma', a_{\text{max}}, M_{\text{so}})
\]

Each of the six basic input parameters in Eq.(21), \(N_m, FC, \sigma, \sigma', a_{\text{max}}, M_{\text{so}}\), is considered as a random variable and is assumed to follow a lognormal distribution, which has been shown to provide a good fit to the measured geotechnical parameters (Jefferies et al. 1988). As discussed earlier, the mean and coefficient of variation (COV) of each of the input parameters for the 160 cases considered in the present study for reliability analysis are obtained from Cetin (2000), no site specific COV of the parameter \(M_{\text{so}}\) is contained within this. Hence, only 94 cases (59 liquefied and 35 non-liquefied cases) of the total 160 databases with a site-specific COV of the parameter \(M_{\text{so}}\) as presented in Moss (2003), are considered for the reliability analysis. It should be noted that Juang et al. (2008) assumed a single value of COV of \(M_{\text{so}}\) as 0.1 for each case of database instead of site-specific COV for the SPT-based reliability analysis. The model factor ‘\(c_{\text{unc}}\)’ is also assumed to follow lognormal distribution, which is very well accepted in the reliability analysis (Juang et al., 2006). The model factor ‘\(c_{\text{unc}}\)’ is also characterized with a mean (\(\mu_{\text{unc}}\)) and a COV. Thus, the estimation of the model uncertainty includes the determination of these two statistical parameters, ‘\(c_{\text{unc}}\)’.

In the present study, the correlations among the input variables are incorporated in the reliability analysis. The correlation coefficients between each pair of parameters used in the proposed limit state are provided in Table 4 as estimated by Juang et al. (2008)
from the original database of Cetin (2000). As per Phoon and Kulhawy (2005), the model factor, \( c_{mf} \), is very weakly correlated to input variables. Thus, in the present study no correlation is assumed between \( c_{mf} \) and the other six input parameters considered herein.

In the present FORM analysis, the population-based optimization algorithm, GA, is used as the optimization tool to obtain the reliability index, \( \beta \), and avoid the local minima problem. The limit state function, \( g(z) = 0 \) (Eq. (21)) is used as constrained function and Eq. (11a) is the objective function, which is to be minimized. Some of the GA parameters such as the initial population size (\( N_{pop} \)), the probability of cross over (\( P_c \)), the probability of mutation (\( P_m \)) and the maximum number of generation (\( MaxGen \)) affect the convergence rate. Thus, through a sensitivity analysis the following appropriate values are found and applied to a GA analysis: \( N_{pop} = 200 \), \( P_c = 0.75 \), \( P_m = 0.05 \) and \( MaxGen = 100 \). Following the flow
chart as shown in Fig. 11, a code is developed in MATLAB (Math Works Inc., 2005) to estimate $\beta$. Then, the notional probability of liquefaction, $P_L$, is obtained using Eq. (6).

Since the model uncertainty is not known initially, calculating the reliability index without considering model uncertainty or without considering any assumed value will result in an incorrect calculation of $\beta$ and corresponding notional probability $P_L$. Hereafter, the reliability index, calculated without taking the model uncertainty into account is designated as $\beta_1$, whereas the reliability index calculated considering any value of model uncertainty is denoted as $\beta_2$.

In the first step of model uncertainty determination procedure, the reliability index, $\beta_1$, is calculated for each of the 94 cases of the database considered. Following the Bayesian mapping function approach and calibrating with field manifestations of the database as explained earlier, the following $P_L - \beta$ relationship is obtained:

$$P_L = \frac{f_L(\beta)}{f_L(\beta) + f_{NL}(\beta)}$$  \hspace{1cm} (22)

where $f_L(\beta)$ and $f_{NL}(\beta)$ are the PDFs of the calculated $\beta$ of group $L$ and $NL$ cases, respectively. $P_L$ for each case of the database is calculated using Eq. (22). Fig. 12 shows a plot of $P_L - \beta_1$ relationship obtained from the reliability analyses of 94 cases of database without considering the model uncertainty. The notional probability of each case of the database using Eq. (6) is calculated and plotted in the same figure. A difference is observed between the national concept-based $P_L$-curve and the Bayesian mapping function-based $P_L$-curve. The latter curve is calibrated empirically with the field manifestations of case history database considered in the present study, and thus, it is assumed to be the most probable evaluation of the “true” probability of liquefaction. It can also be observed from the Bayesian mapping function-based $P_L$-curve in Fig. 12 that $P_L$ is 0.52 when $\beta_1$ is 0. This result is consistent with the 50.3% probability of the developed CRR model as suggested by $P_L-F$, mapping function (Eq. (19)), which was discussed in an earlier section. This indicates the robustness of the proposed methodology. The accuracy of calculated probability on the basis of notional concept depends on the accuracy with which $\beta$ is calculated. Because $\beta$ is calculated without considering the limit state model uncertainty in a reliability analysis, it is subjected to some error and thus, the resulting notional probability may not be completely accurate. Still, the notional probability concept always yields: $P_L = 0.5$ at $\beta = 0$. If the “true” model uncertainty can be incorporated with the limit state model, the resulting reliability index ($\beta_2$) at 0 will produce a $P_L$ value of 0.5 from the calibrated Bayesian mapping function approach.

The methodology for estimating the uncertainty of the adopted limit state model (Eq. (20)) is based on the proposition that a calibrated Bayesian mapping function produces a most accurate estimate of the “true probability of liquefaction” for any given case. With the above idea, a simple but trial-and-error procedure is adopted to estimate model uncertainty. The “true” model uncertainty is the one that yields the reliability indices and the corresponding notional probabilities matching best with those probabilities calculated from the calibrated $P_L$-mapping function (Juang et al., 2006). In addition, the plot of $\beta_2$ versus $P_L$ as obtained from the calibrated Bayesian mapping function produces a $P_L$ value of 0.5 at $\beta_2 = 0$.

In the first phase of the model uncertainty estimation procedure, a series of reliability analyses for the 94 cases, which make up the database, are performed to study the effect of the COV component of the model factor, $c_{cmf}$. Four cases of model uncertainty, each with the mean of the model factor kept equal to 1 ($\mu_{cmf} = 1.0$) and a different COV of 0.0, 0.1, 0.2, 0.3 are studied. For each case of model uncertainty, $\beta_2$ values are calculated for each of the 94 cases in the database. A Bayesian mapping function is obtained for each model uncertainty scenario. From the developed mapping function for each of the above-mentioned model uncertainty scenario, liquefaction probabilities are obtained from the corresponding $\beta_2$ values. Fig. 13 shows the plot between $\beta_2$ versus $P_L$ for each of the above-mentioned model uncertainty. It is clear that within the considered range of COV value, [0–0.3], the COV component of the model uncertainty has a significant effect on the calculated probability. It may be mentioned here that, Juang et al. (2008) did not consider the effect of the COV component of model uncertainty on $P_L$ in their reliability analysis, citing (Juang et al., 2004) it as insignificant.

In a second phase of investigation COV component of model uncertainty is kept constant at 0.1, whereas the mean value of $c_{cmf}$ is varied from 0.9 to 1.1 at an interval of 0.05 (i.e., $\mu_{cmf} = 0.9, 0.95, 1.0, 1.05$ and 1.1). For each of the above scenarios of model uncertainty, a Bayesian mapping function is developed using all 94 cases of liquefaction and non-liquefaction for the reliability analysis. The $P_L$ values calculated from the mapping functions are then plotted against the corresponding reliability index ($\beta_2$) for different cases of model uncertainty as mentioned above and presented in Fig. 14. It can also be observed from the above figures that the mapping function shifts from left to right as $\mu_{cmf}$ increases, and also the probability corresponding to $\beta_2 = 0$ increases. As per Fig. 14, at $\mu_{cmf} = 1$ and COV = 0.1, the $P_L$ at $\beta = 0$, whereas $\mu_{cmf} = 0.95$ produces a lower value of $P_L$ (0.48) at $\beta = 0$. Thus, an intermediate value of $\mu_{cmf} = 0.98$ is considered for further reliability analysis. Since COV was shown to have a significant effect on calculated $P_L$, $\mu_{cmf}$ is kept at 0.98 and the COV component was changed from 0 to 0.30, a series of reliability analysis for all 94 cases is performed and similarly, the Bayesian mapping functions are obtained. The $P_L$ versus $\beta_2$ plot for the above cases is shown in Fig. 15. It is clear from the mapping function curves for the two uncertainty scenarios that when $\mu_{cmf}$ is 0.98, COV is 0, and when $\mu_{cmf}$ is 0.98, COV = 0.1,
and the $P_L$ was determined to be 0.5 at $\beta = 0$. The latter scenario of model uncertainty is considered the “true” model factor. This is explained in more detail below.

Fig. 16 shows a comparison of the probability of liquefaction for each of the 94 case-histories obtained from two mapping functions, one considering the “true” model uncertainty (i.e., $\mu_{cmf} = 0.98$ and $\text{COV} = 0.1$) and the other not considering the model uncertainty (i.e., $\mu_{cmf} = 1.00$ and $\text{COV} = 0$). In the earlier case, the reliability index, $\beta_2$, for each case is calculated and then, the corresponding mapping function is established using Eq. (22). In the latter case, reliability index $\beta_1$ for each case is determined and then, the corresponding mapping function is developed in a similar manner using Eq. (22). The two sets of probabilities obtained for all 94 cases based on the two sets of mapping functions agree well with each other, which is evident from the statistical parameters ($R = 0.99$, $E = 0.99$ and $RMSE = 0.02$) as mentioned in Fig. 16. Fig. 17 shows the comparison of the notional probabilities obtained for all 94 cases in the present database using the reliability index $\beta_3$ calculated by taking the “true” model factor ($\mu_{cmf} = 0.98$ and $\text{COV} = 0.1$) into account with the probabilities obtained from the $P_L - \beta_1$ mapping function, which were obtained without considering model uncertainty in the reliability analyses. Fig. 17 also shows very good agreement ($R = 0.99$, $E = 0.94$ and $RMSE = 0.10$) between the probabilities obtained from two different concepts, which indicate that the probability of liquefaction can be correctly calculated from the notional concept if the right model uncertainty is incorporated in the reliability analysis. Similar analyses made with the model uncertainty scenario $\mu_{cmf} = 0.98$ and $\text{COV} = 0$ also yielded better results. Thus the “true” model uncertainty of the developed limit
state, considering the present database, is characterized by $\mu_{cmf}=0.98$ and $\text{COV}=0.1$. In comparison, using the same SPT database characterized by the limit state model formed by $\text{CSR}$ and $\text{CRR}$ models as presented in Youd et al. (2001), Juang et al. (2008) observed model uncertainty of $\mu_{cmf}=0.96$ and $\text{COV}=0.04$ by FORM analysis.

Finally, the $P_L$ can be estimated from the developed $P_L-\beta_1$ mapping function using a reliability index, $\beta_1$, calculated by FORM considering only parameter uncertainties. Alternatively, the reliability index $\beta_2$ can be determined by FORM considering both model and parameter uncertainties, and then, the $P_L$ can be obtained with the notional concept using Eq. (6). The notional concept to estimate the $P_L$ of a future case is preferred as the model uncertainty of the adopted limit state has been determined and also it is a well-accepted approach in the reliability theory (Juang et al., 2006).

To explain the above findings one example of liquefaction case from the 1978 Miyagiken-Oki earthquake at Ishinomakai-2 site as presented in the database of Cetin (2000) has been analyzed to
find the $P_L$. The soil and seismic parameters at critical depth ($D=3.7$ m) are given as follows: $N_m=3.7$; $FC=10\%$; $\sigma_v=58.83$ kPa; $\sigma'_v=36.28$ kPa; $a_{\text{max}}=0.2$ g and $M_o=7.4$. The COV of the parameters $N_m$, $FC$, $\sigma_v$, $\sigma'_v$, $a_{\text{max}}$ are 0.189, 0.2, 0.217, 0.164, and 0.2, respectively, whereas the COV of $M_o$ is taken as 0.1 as given by Juang et al. (2008) since it is not available in Cetin (2000) or Moss (2003). Both Eqs. (2) and (14) are used to form the limit state of liquefaction and to consider the model uncertainty ($\mu_{\text{mod}}=0.98$ and COV=0.1), and a FORM analysis is made using the developed code in MATLAB. The reliability index, $\beta_2$, and corresponding notional probability of liquefaction, $P_L$, using Eq. (6) are determined as $-1.3437$ and 0.91, respectively. Juang et al. (2008) also observed the reliability index-based notional $P_L$ as 0.91 for the above example. The results of both the methods confirm the case as liquefied, and the $P_L$ calculated by the proposed MGGP-based reliability method is also found to be equal to that obtained by the statistical regression-based reliability method developed by Juang et al. (2008). Similarly, the probability of liquefaction can be evaluated using $P_L-F_s$ mapping function using only the mean values of the input variables. For the above liquefied case, using Eqs. (2) and (14), $F_s$ is determined to be 0.575, and thus $P_L=0.90$ according to Eq. (19). The results of obtained considering two different approaches are consistent.

Another example of a non-liquefaction case from 1977, the Argentina earthquake at San Juan B-5, as presented in Cetin (2000), has been analyzed to find $P_L$. The mean values of seismic and soil parameters at the critical depth ($D=2.9$ m) are given as follows: $N_m=15.2$; $FC=3\%$; $\sigma_v=45.61$ kPa; $\sigma'_v=38.14$ kPa; $a_{\text{max}}=0.2$ g and $M_o=7.4$ and the corresponding COV of these parameters are 0.026, 0.333, 0.107, 0.085, and 0.075, respectively, whereas the COV of $M_o$ is taken as 0.1 as given by Juang et al. (2008) as it is not reported either in Cetin (2000) or in Moss (2003). Again, both the $CSR$ and the $CRR$ model equations are used to form the limit state of liquefaction and considering the model uncertainty ($\mu_{\text{mod}}=0.98$ and COV=0.1), and FORM analysis was made using the developed code in MATLAB. The reliability index, $\beta_2$, and corresponding notional probability of liquefaction, $P_L$, using Eq. (6) were determined to be 0.0213 and 0.491, respectively, confirming that the case was a non-liquefaction case. The above example was also presented in Juang et al. (2008) and the corresponding reliability index and notional $P_L$ were 0.533 and 0.297. In that example, however, the COV of $N_{1,60}$ was mistaken by Juang et al. (2008) as 0.23 instead of 0.023, as presented in Cetin (2000). This most likely explains the discrepancy in the results in the above example. In the non-liquefaction example above, when the mean values of the parameters are used with Eqs. (2) and (14), $F_s$ is determined to be 1.044; thus, $P_L=0.460$ as given in Eq. (19). It is pertinent to mention here that the proposed MGGP-based reliability method is developed on basis of the most recent CSR formulation, whereas the available reliability method is based on an older CSR model.

Consistent results are obtained in the above two examples, which suggest the robustness of the present methodology. These two examples also illustrate the procedure for the evaluation of $P_L$ of a site in a future seismic event using the proposed reliability-based analysis if the uncertainties of soil and seismic parameters of the site can be obtained. Eq. (19) can be used for a preliminary estimation of $P_L$ in cases where knowledge of parameter uncertainties is lacking.

4. Conclusion

The following conclusions can be drawn from this study:

(i) The MGGP method is found to very efficient in presenting a compact and comprehensive model equation for reliability analysis of liquefaction potential.

(ii) On the basis rigorous FORM analysis of 94 cases of the database, the developed MGGP-based $CCR$ model is characterized with an uncertainty of mean value 0.98 and COV of 0.1. As the mean value of model uncertainty is very close to 1, the $CCR$ model, which represents the boundary surface separating liquefied cases from non-liquefied cases, can be considered un-biased. This is also evident from the proposed $P_L-F_s$ mapping function, which yields $P_L=0.503$ ($P_L=50.3\%$), when $F_s=1$.

(iii) The probability of liquefaction, $P_L$, can be estimated from the developed $P_L-F_s$ mapping function using a reliability index, $\beta_1$, calculated by FORM considering only parameter uncertainties. Alternatively, the reliability index, $\beta_2$, can be determined by FORM considering both model and parameter uncertainties, and then, $P_L$ can be obtained with the notional probability concept (using Eq. (6)). The notional concept to estimate the $P_L$ of a future case is preferred as the model uncertainty of the adopted limit state has been determined and also it is a well-accepted approach in the reliability theory.

(iv) In the absence of parameter uncertainties, the proposed $P_L-F_s$ mapping function as defined by Eq. (19) can be used to estimate the probability of liquefaction, where $F_s$ is calculated based on the $CSR$ and $CRR$ models as presented by Eqs. (2) and (14), respectively.
(v) Using the developed code for FORM to analyze one liquefaction example and another non-liquefaction example, the corresponding probability of liquefaction on the basis of notional probability concept, using the obtained “true” limit state model, uncertainty was determined to be 0.91 and 0.49, respectively.

References


