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Signalizers and balance in groups of finite Morley rank ☆

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ABSTRACT

There is a longstanding conjecture, due to Gregory Cherlin and Boris Zilber, that all simple groups of finite Morley rank are simple algebraic groups. The most successful approach to this conjecture has been Borovik's program analyzing a minimal counterexample, or simple K^*-group. We show that a simple K^*-group of finite Morley rank and odd type is either algebraic of else has Prüfer rank at most two. This result signifies a switch from the general methods used to handle large groups, to the specilized methods which must be used to identify PSL_2, PSL_3, PSp_4, and G_2.

The algebraicity conjecture for simple groups of finite Morley rank, also known as the Cherlin–Zilber conjecture, states that simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields. In the last 15 years, the main line of attack on this problem has been the Borovik program of transferring methods from finite group theory. This program has led to considerable progress; however, the conjecture itself remains decidedly open. We divide groups of finite Morley rank into four types, odd, even, mixed, and degenerate, according to the structure of their Sylow 2-subgroups. For even and mixed type the algebraicity conjecture has been proven, and connected degenerate type groups are now known to have trivial Sylow 2-subgroups [BBC07]. The present paper is part of the program to analyze a minimal counterexample to the conjecture in *odd type*, where the Sylow 2-subgroup is divisible-abelian-by-finite. It is the final paper in a sequence proving that such a minimal counterexample, or simple nonalgebraic K^*-group, has Prüfer 2-rank at most two.

High Prüfer Rank Theorem. *A simple K^*-group of finite Morley rank with Prüfer 2-rank at least three is algebraic.*

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This will be a consequence of the following so-called trichotomy, which is proved in the present paper. Here the traditional term "trichotomy" refers to the fact that there is also the Prüfer 2-rank $\leqslant 2$ case, which is largely unexplored at present.

Generic Trichotomy Theorem. *Let G be a simple K^*-group of finite Morley rank and odd type with Prüfer 2-rank $\geqslant 3$. Then either*

1. *G has a proper 2-generated core, or*
2. *G is an algebraic group over an algebraically closed field of characteristic not 2.*

High Prüfer Rank Theorem then follows by applying the next two results.

Strong Embedding Theorem. *(See [BBN08].) Let G be a simple K^*-group of finite Morley rank and odd type, with normal 2-rank $\geqslant 3$ and Prüfer 2-rank $\geqslant 2$. Suppose that G has a proper 2-generated core M. Then G is a minimal connected simple group, and M is strongly embedded.*

Minimal Simple Theorem. *(See [BCJ07].) Let G be a minimal connected simple group of finite Morley rank and of odd type. Suppose that G contains a proper definable strongly embedded subgroup M. Then G has Prüfer 2-rank one.*

It may seem odd that the first of these results is appearing last. In fact, an earlier version of the trichotomy theorem began this sequence of developments. Namely, Borovik first proved the trichotomy theorem under a *tameness* assumption in [Bor95], and the present author had explored eliminating tameness in [Bur04b]. In [Bor95], Borovik produces the proper 2-generated core with a tame nilpotent signalizer functor theorem [BN94, Theorem B.30] (see also [Bur04b, Theorem 6.2]), an approach mirrored in the present paper. In [Bur04b], we show that the "most unipotent part" of a solvable signalizer functor is a nilpotent signalizer functor. This was believed to quickly eliminate tameness from [Bor95]. However, more careful investigations revealed that obtaining a signalizer functor remained problematic.

In Section 2.1 of the present paper, we resolve this difficulty by constructing signalizer functors of a "sufficiently unipotent" reduced rank. The most serious obstacle is explained in Example 2.1. Our approach forces subsequent analysis to restrict itself to components of the centralizers of involutions which involve sufficiently large fields, a worrying but ultimately harmless restriction. Indeed, all complexities introduced by this approach are dispensed with in Section 2.1. This seems to be a different approach from that used by finite group theorists, who work with so-called *weakly balanced* signalizer functors [GLS94, §29]; a similar method might work here as well. We call our approach partial balance.

The first section of this article covers necessary background material, including the definitions of a signalizer functor and the 2-generated core.

The second section contains the delicate definitions of partial balance, and of the associated family $\tilde{\mathcal{E}}_X$ of components from the centralizers of involutions. This section also contains a version of Asar's theorem (Theorem 2.12) which states that $\tilde{\mathcal{E}}_X \neq \emptyset$, as well as a criterion for $\langle \tilde{\mathcal{E}}_X \rangle = G$ (Theorem 2.18). Borovik's earlier unpublished work on the analysis of Lie rank two components [Bor03] has heavily influenced this final result, although partial balance has given these results a more technical flavor.

The third section provides a suitable version of Berkman and Borovik's Generic Identification Theorem [BB04]. It is the role of section two to verify the two hypotheses of this argument: reductivity for, and generation by, the centralizers of involutions. Our partial balance approach provides only a weak form of the reductivity hypothesis, which necessitates some alterations in the proof of the Generic Identification Theorem. All such critical changes are confined to Section 3.1, but there are important modifications throughout Section 3. The reader unfamiliar with the Generic Identification Theorem should consider exploring Section 3 before Section 1 or Section 2.

This is by no means the end of the story. The Prüfer 2-rank $\geqslant 3$ hypothesis used here is weaker than the normal 2-rank $\geqslant 3$ hypothesis originally used by Borovik [Bor95]. As part of the ongoing program in odd type, [BB08] will show that Borovik's original trichotomy holds, without the tameness hypothesis. We view [BB08] as a bridge between the "generic case" which is treated here, and the "quasi-thin" case (the identification of PSp_4, G_2, and PSL_3).

1. Preliminaries

This first section recalls various definitions and facts which are used throughout Section 2, and less pervasively in Section 3.

1.1. K-groups

We proceed, in this paper, by analyzing a so-called simple K^*-group of finite Morley rank. A K^*-group is a group whose *proper* definable simple sections are all algebraic. Similarly, a K-group is a group whose definable simple sections are all algebraic. So the proper subgroups of our K^*-group are clearly K-groups. One major K-group fact used throughout this article is the following generation principle.

Fact 1.1. *(See [Bor95, Theorem 5.14]; see also [Bur04a, Theorem 3.25].) Let G be a connected K-group of finite Morley rank and odd type. Let V be a four-subgroup acting definably on G. Then*

$$G = \langle C_G^\circ(v) \mid v \in V^\# \rangle$$

Another major K-group fact worth recalling at the outset is the following "reductivity" criterion, which requires two definitions.

Definition 1.2. A quasisimple subnormal subgroup of a group G is called a *component* of G (see [BN94, p. 118, (2)]). We define $E(G)$ to be the connected part of the product of components of G, or equivalently the product of the components of G° (see [BN94, Lemma 7.10iv]). Such components are normal in G° by [BN94, Lemma 7.1iii], and indeed $E(G) \lhd G$.

Definition 1.3. The *odd part* $O(G)$ of a group G of finite Morley rank is the maximal definable connected normal $2^\perp$-subgroup of G.

Clearly $O(G)$ is solvable if G is a K-group.

Fact 1.4. *(See [Bor95, Theorem 5.12].) Let H be a connected K-group of finite Morley rank and odd type with $O(H) = 1$. Then $H = F^\circ(H) * E(H)$ is isomorphic to a central product of quasisimple algebraic groups over algebraically closed fields of characteristic not 2 and of a definable normal divisible abelian group $F^\circ(H)$.*

This fact motivates signalizer functor theory, whose goal is to show that $O(H) = 1$, or something similar, when H is the centralizer of an involution. The version we aim at here is Corollary 2.11 below.

1.2. Algebraic groups

A key tool in our program is the fact that a group of finite Morley rank acting faithfully as a group of automorphisms of an algebraic group must itself be algebraic.

Definition 1.5. Given an algebraic group G, a maximal torus T of G, and a Borel subgroup B of G which contains T, we define the group Γ of *graph automorphisms* associated to T and B, to be the group of algebraic automorphisms of G which normalize both T and B.

Fact 1.6. *(See [BN94, Theorem 8.4].) Let $G \rtimes H$ be a group of finite Morley rank where G and H are definable, G an infinite quasisimple algebraic group over an algebraically closed field, and $C_H(G)$ is trivial. Then, viewing H as a subgroup of* $\mathrm{Aut}(G)$*, we have $H \leqslant \mathrm{Inn}(G)\Gamma$, where* $\mathrm{Inn}(G)$ *is the group of inner automorphisms of G and Γ is the group of graph automorphisms of G, relative to a fixed choice of Borel subgroup B and maximal torus T contained in B.*

An algebraic group is said to be *reductive* if it has no unipotent radical. Such a group is a central product of semisimple algebraic groups and algebraic tori. The centralizer of an involution in a reductive algebraic group over a field of characteristic $\neq 2$ is itself reductive.

Fact 1.7. *(See [Ste68, Theorem 8.1].) Let G be a quasisimple algebraic group over an algebraically closed field. Let ϕ be an algebraic automorphism of G whose order is finite and relatively prime to the characteristic of the field. Then $C_G^\circ(\phi)$ is nontrivial and reductive.*

Proof. We know both that $C_H(\phi) \leqslant C_H(\phi \bmod Q)$ as well as $[\phi, C_H(\phi \bmod Q)] \leqslant Q$. There is a homomorphism $C_H(\phi \bmod Q) \to Q$ given by $x \mapsto [\phi, x]$. As $Z(G)$ is finite, t centralizes $C_H^\circ(\phi \bmod Q)$, as desired. So $C_{G/Z(G)}^\circ(\phi) = C_G^\circ(T)/Z(G)$ We may therefore assume that G is its own universal central extension.

Since ϕ is algebraic and has finite order, $G \rtimes \langle\phi\rangle$ is an algebraic group which contains ϕ as an inner automorphism. Since the order of ϕ is finite and relatively prime to the characteristic, ϕ is a semisimple automorphism of G. So the result follows from Theorem 8.1 of [Ste68]. $\square$

More specialized facts about algebraic groups will appear in Section 3.

1.3. Unipotent groups

While there is no intrinsic definition of unipotence in a group of finite Morley rank, there are various analogs of the "unipotent radical": the Fitting subgroup, the p-unipotent operators U_p, for p prime, and their "characteristic zero" analogs $U_{0,r}$ from [Bur04b,Bur04a]. We recall their definitions below.

Definition 1.8. The *Fitting subgroup* $F(G)$ of a group G of finite Morley rank is the subgroup generated by all its nilpotent normal subgroups.

The Fitting subgroup is itself nilpotent and definable [Bel87,Nes91,BN94, Theorem 7.3], and serves as a rough notion of unipotence in some contexts. However, the Fitting subgroup of a solvable group H may not be contained in the Fitting subgroup of a solvable group containing H.

Definition 1.9. A connected definable p-subgroup of bounded exponent in a group H of finite Morley rank is said to be *p-unipotent*. We write $U_p(H)$ for the subgroup generated by all p-unipotent subgroups of H.

Clearly $U_p(H)$ need not be solvable when H is a nonsolvable algebraic group in characteristic p; however, a p-unipotent K-group is solvable, and hence nilpotent by the following.

Fact 1.10. *(See [CJ04, Corollary 2.16], [ABC97, Fact 2.36].) Let H be a connected solvable group of finite Morley rank. Then $U_p(H) \leqslant F^\circ(H)$ is itself p-unipotent, and hence nilpotent.*

Thus the p-unipotent radical U_p will automatically behave well, inside a solvable group. Its only weakness is that it may be trivial.

Fact 1.11. *(See [BN94, Theorem 9.29 and §6.4].) Let G be a connected solvable group of finite Morley rank. Then a Sylow p-subgroup P of G is connected, and $P = U_p(G) * T$ for a divisible abelian p-group T.*

The present paper relies on the theory of "characteristic zero" unipotence introduced in [Bur04b]. We now turn our attention to this definition, as well as some facts from [Bur04b,Bur06,Bur04a].

Definition 1.12. We say that a connected abelian group of finite Morley rank is *indecomposable* if it has a unique maximal proper definable connected subgroup, denoted $J(A)$ (see [Bur04b, Lemma 2.4]). We define the *reduced rank* $\bar{r}(A)$ of a definable indecomposable abelian group A to be the Morley rank of the quotient $A/J(A)$, i.e. $\bar{r}(A) = \mathrm{rk}(A/J(A))$. For a group G of finite Morley rank, and any integer r, we define

$$U_{0,r}(G) = \left\langle A \leqslant G \;\middle|\; \begin{array}{l} A \text{ is a definable indecomposable group,} \\ \bar{r}(A) = r, \text{ and } A/J(A) \text{ is torsion-free} \end{array} \right\rangle.$$

We say that G is a $U_{0,r}$*-group* (alternatively $(0,r)$*-unipotent* group) if $U_{0,r}(G) = G$. We also set $\bar{r}_0(G) = \max\{r \mid U_{0,r}(G) \neq 1\}$.

We view the reduced rank parameter r as a *scale of unipotence*, with larger values being more unipotent. By the following fact, analogous to Fact 1.10, the "most unipotent" groups, in this scale, are nilpotent.

Fact 1.13. *(See [Bur04a, Theorem 2.21], [Bur04b, Theorem 2.16].) Let H be a connected solvable group of finite Morley rank. Then $U_{0,\bar{r}_0(H)}(H) \leqslant F(H)$.*

Fact 1.14. *(See [Bur06, Corollary 4.6].) Let $G = HT$ be a group of finite Morley rank, with H and T definable and nilpotent, and $H \lhd G$. Suppose that T is a $U_{0,r}$-group for some $r \geqslant \bar{r}_0(H)$. Then G is nilpotent.*

A *good torus* is a divisible abelian group of finite Morley rank whose definable connected subgroups are the definable hulls of their torsion. We arrive at a good torus when all our various notions of unipotence are trivial.

Fact 1.15. *(See [Bur04a, Theorem 2.19], [Bur04b, Theorem 2.15].) Let H be a connected solvable group of finite Morley rank. Suppose $U_p(H) = 1$ for all p prime, and $U_{0,\bar{r}_0(H)}(H) = 1$. Then H is a good torus.*

In a similar vein, the notion of $(0,r)$-unipotence provides a useful decomposition of a nilpotent group.

Fact 1.16. *(See [Bur06, Corollary 3.6], [Bur04a, Theorem 2.31].) Let G be a connected nilpotent group of finite Morley rank. Then $G = D * B$ is a central product of definable characteristic subgroups $D, B \leqslant G$ where D is divisible and B is connected of bounded exponent. Let T be the torsion part of D. Then we have decompositions of D and B as follows.*

$$D = d(T) * U_{0,1}(G) * U_{0,2}(G) * \cdots,$$

$$B = U_2(G) \times U_3(G) \times U_5(G) \times \cdots.$$

The next fact tells us when q-unipotence is preserved by taking centralizers, a fact used to produce a signalizer functor in Lemma 2.5 below.

Fact 1.17. *(See [Bur04b, Fact 3.4], [ABCC01].) Let G be a connected solvable $p^\perp$-group of finite Morley rank, and let P be a finite p-group of definable automorphisms of G. Then $C_G(P)$ is connected.*

There is also a "characteristic zero" analog of the foregoing.

Fact 1.18. *(See [Bur04b, Lemma 3.6].) Let G be a nilpotent $(0,r)$-unipotent $p^{\perp}$-group of finite Morley rank, and let P be finite p-group of definable automorphisms of G. Then $C_G(P)$ is $(0,r)$-unipotent.*

In a similar vein, commutator subgroups of connected or $(0,r)$-unipotent groups tend to retain these properties.

Fact 1.19. *(See [BN94, Corollary 5.29].) Let H be a definable connected subgroup of a group G of finite Morley rank and let $X \subset G$ be any subset of G. Then the group $[H,X]$ is definable and connected.*

Fact 1.20. *(See [Bur06, Corollary 3.6].) Let G be a solvable group of finite Morley rank, let $S \subset G$ be any subset, and let H be a nilpotent $U_{0,r}$-group which is normal in G. Then $[H,S] \leqslant H$ is a $U_{0,r}$-group.*

1.4. 2-Local structure

As the goal of our project is to constrain the 2-local structure, we need a few parameters to measure the complexity of a Sylow 2-subgroup. We define the *2-rank* $m(G)$ of a group G to be the maximum rank of its elementary abelian 2-subgroups. The *Prüfer 2-rank* $\mathrm{pr}(G)$ is the maximum k such that there is a Prüfer 2-subgroup $\mathbb{Z}(2^\infty)^k$ inside G, and the *normal 2-rank* $n(G)$ is the maximum 2-rank of a normal elementary abelian 2-subgroup of G. In an odd type group of finite Morley rank, these various ranks are all finite, and we have

$$m(G) \geqslant n(G) \geqslant \mathrm{pr}(G).$$

These notions are well-defined because the Sylow 2-subgroups of a group of finite Morley rank are conjugate [BP90,BN94, Theorem 10.11].

We use $\mathcal{E}_k(H)$ to denote the set of elementary abelian 2-subgroups $U \leqslant H$ with $m(U) \geqslant k$. We give $\mathcal{E}_2(H)$ a *graph structure* by placing an edge between $U, V \in \mathcal{E}_2(H)$ whenever $[U,V]=1$. We say H is *2-connected* if the graph $\mathcal{E}_2(H)$ is connected, and we refer to the components of $\mathcal{E}_2(H)$ as *2-connected components* otherwise.

Fact 1.21. *(Compare [Asc93, 46.2].) Let S be a locally finite 2-group. Then*

1. *If $m(S) > 2$ then the graph $\mathcal{E}_2(S)$ has a unique nonsingleton 2-connected component given by*

$$\mathcal{E}_2^0(S) := \{X \in \mathcal{E}_2(S)\colon\ m(C_S(X)) > 2\},$$

   and $\mathcal{E}_2^0(S)$ contains any $X \in \mathcal{E}_2(S)$ with $X \lhd S$.
2. *If $n(S) > 2$ then S is 2-connected.*

Proof. Since S is locally finite, this reduces to the finite case, found in [Asc93, 46.2]. □

1.5. Proper 2-generated core

Definition 1.22. Consider a group G of finite Morley rank and a 2-subgroup S of G with $m(S) \geqslant 3$. We define the *2-generated core* $\Gamma_{S,2}(G)$ of G (associated to S) to be the definable hull of the group generated by all normalizers of groups in $\mathcal{E}_2(S)$:

$$\Gamma_{S,2}(G) = d(\langle N_G(U)\colon\ U \in \mathcal{E}_2(S)\rangle).$$

We also define the *weak 2-generated core* $\Gamma^0_{S,2}(G)$ of G (associated to S) to be the definable hull of all normalizers of groups in the nonsingleton 2-connected component $\mathcal{E}_2^0(S)$.

$$\Gamma^0_{S,2}(G) = d(\langle N_G(U)\colon\ U \in \mathcal{E}_2(S),\ m(C_S(U)) > 2\rangle).$$

We say that G has a *proper* 2-*generated core*, or a *proper weak* 2-*generated core*, when, for a Sylow 2-subgroup S, $\Gamma_{S,2}(G) < G$ or $\Gamma^0_{S,2}(G) < G$, respectively.

Both notions of 2-generated core are well-defined, by the conjugacy of Sylow 2-subgroups. By Fact 1.21.2, the 2-generated core and the weak 2-generated core coincide when $n(G) \geqslant 3$, as is the case for much of the rest of this paper. When they differ, the weak 2-generated core is the more useful notion.

For an elementary abelian 2-group V acting definably on G, we define $\Gamma_V(G)$ to be the group generated by the connected centralizers of involutions in V.

$$\Gamma_V(H) = \langle C^\circ_H(v):\ v \in V^\# \rangle.$$

Proposition 1.23. *Let G be a simple K^*-group of finite Morley rank and odd type, with $m(G) \geqslant 3$, and let S be a Sylow 2-subgroup of G. Suppose that $\Gamma_E(G) < G$ for some $E \in \mathcal{E}^0_2(S)$. Then G has a proper weak 2-generated core.*

This depends on a lemma.

Lemma 1.24. *Let G be a simple K^*-group of finite Morley rank and odd type. Then $\Gamma_U(G) = \Gamma_V(G)$ for any U, V in the same connected component of the graph $\mathcal{E}_2(G)$.*

Proof. It is enough to prove the result for U, V with $[U, V] = 1$. For any $v \in V^\#$, simplicity implies that $C^\circ_G(v)$ is a proper subgroup of G, and hence a K-group. Since U normalizes $C^\circ_G(v)$, $C^\circ_G(v) = \Gamma_U(C^\circ_G(v))$ by Fact 1.1. So $\Gamma_V(G) \leqslant \Gamma_U(G)$, and the result follows by symmetry. $\square$

Proof of Proposition 1.23. We may assume $E \leqslant S$ by conjugacy of Sylow 2-subgroups. Since involutions of G have infinite centralizers by [BN94, Ex. 13 & 15, p. 79], the result will follow from the following claim, and simplicity.

$$\Gamma^0_{S,2}(G) \leqslant N_G(\Gamma_E(G)) \quad \text{for any } E \in \mathcal{E}^0_2(S).$$

By Lemma 1.24 and Fact 1.21.1, $\Gamma_E(G) = \Gamma_U(G)$ for any $U \in \mathcal{E}^0_2(S)$. For any $U \in \mathcal{E}^0_2(S)$,

$$N_G(U) \leqslant N_G(\Gamma_U(G)) = N_G(\Gamma_E(G)).$$

Thus $\Gamma^0_{S,2}(G) \leqslant N_G(\Gamma_E(G))$, as desired. $\square$

We will encounter a variation of the preceding in the next section (see Lemma 2.15 and Proposition 2.17).

The following black hole principle for proper 2-generated cores reverses the roles of the subgroups $\Gamma^0_{S,2}(G)$ and $\Gamma_E(G)$ in Proposition 1.23.

Lemma 1.25. *Let G be an infinite simple K^*-group of finite Morley rank and odd type, and let S be a 2-subgroup of G satisfying $m(S) \geqslant 3$. Then $C^\circ_G(x) \leqslant \Gamma^0_{S,2}(G)$ for every $x \in I(S)$ with $m(C_S(x)) > 2$.*

Proof. There is an $E \in \mathcal{E}^0_2(S)$ with $x \in E$ and $m(E) \geqslant 3$ by Fact 1.21.1. So there is an $E_1 \in \mathcal{E}^0_2(S)$ with $E_1 \leqslant E$ and $E_1 \cap \langle x \rangle = 1$. For any $y \in E_1^\#$, we have $C_{C_G(x)}(y) \leqslant C_G(y, x)$ and $\langle y, x \rangle \in \mathcal{E}^0_2(S)$. By simplicity, Fact 1.1 yields

$$C^\circ_G(x) = \Gamma_{E_1}(C^\circ_G(x)) \leqslant \Gamma^0_{S,2}(G). \qquad \square$$

In particular, given a simple K^*-group G, Lemma 1.25 says

$$\Gamma_E(G) \leqslant \Gamma_{E,2}(G) \quad \text{for any } E \in \mathcal{E}_3(G).$$

Now Proposition 1.23 and Lemma 1.25 yield the following.

Proposition 1.26. *Let G be a simple K^*-group of finite Morley rank and odd type, with $m(G) \geqslant 3$, and let S be a Sylow 2-subgroup of G. If $\Gamma_{E,2}(G) < G$ for some $E \in \mathcal{E}_3(G)$, then G has a proper weak 2-generated core, i.e. $\Gamma^0_{S,2}(G) < G$.*

1.6. Signalizer functors

Signalizer functors are used in both the finite case and in the finite Morley rank case to produce a dichotomy between a proper 2-generated core, and a reductivity condition for centralizers of involutions.

Definition 1.27. Consider a group G of finite Morley rank, and an elementary abelian 2-subgroup $E \in \mathcal{E}_3(G)$. An *E-signalizer functor* on G is a family $\{\theta(s)\}_{s\in E^\#}$ of definable E-invariant $2^\perp$-subgroups of G satisfying:

(a) $\theta(s) \lhd C_G(s)$ for each $s \in E^\#$;
(b) $\theta(s) \cap C_G(t) \leqslant \theta(t)$ for any $s, t \in E^\#$.

We observe that the second condition is equivalent to the "balance" condition

$$\theta(s) \cap C_G(t) = \theta(t) \cap C_G(s) \quad \text{for any } s, t \in E^\#.$$

In practice, we will only be interested in signalizer functors satisfying the following stronger invariance condition, which is used to produce a proper 2-generated core.

$$\theta(s)^g = \theta\big(s^g\big) \quad \text{for all } s \in E^\# \text{ and all } g \in G \text{ for which } s^g \in E. \tag{†}$$

As one would expect, we say θ is a *connected* or *nilpotent* signalizer functor if the groups $\theta(s)$ are connected or nilpotent, respectively, for *all* $s \in E^\#$.

We now show that signalizer functors yield a proper weak 2-generated core.

Theorem 1.28. *Let G be a simple K^*-group of finite Morley rank and odd type, and let S be a Sylow 2-subgroup of G. Suppose that, for some $E \in \mathcal{E}_3(S)$, G admits a nontrivial connected nilpotent E-signalizer functor θ satisfying (†) Then G has a proper weak 2-generated core.*

The key fact underlying this result is the Nilpotent Signalizer Functor Theorem.

Definition 1.29. We say that an E-signalizer functor on a group G of finite Morley rank is *complete* if

(a) $\theta(E) = \langle\theta(s)\colon\ s \in E^\#\rangle$ is a solvable $p^\perp$-group, and
(b) $\theta(s) = C_{\theta(E)}(s)$ for any $s \in E^\#$.

Nilpotent Signalizer Functor Theorem. *(See [Bor95,Bur04b], [BN94, Theorem B.30].) Let G be a group of finite Morley rank, and let $E \leqslant G$ be a finite elementary abelian 2-group of rank at least 3. Let θ be a connected nilpotent E-signalizer functor. Then θ is complete and $\theta(E)$ is nilpotent.*

We shall work with the proper group $\theta(E)$ in the same manner as we did with $\Gamma_E(G)$ in Proposition 1.23.

We also recall a variation on Fact 1.1.

Fact 1.30. *(See [Bur04b, Fact 3.7].) Let H be a solvable $p^\perp$-group of finite Morley rank. Let E be a finite elementary abelian p-group acting definably on H. Then*

$$H = \langle C_H(E_0) \mid E_0 \leqslant E,\ [E : E_0] = p \rangle.$$

Proof of Theorem 1.28. It suffices to show that $\Gamma_{E,2}(G) < G$ by Proposition 1.26. Nilpotent Signalizer Functor Theorem says that θ is complete and $\theta(E)$ is nilpotent. Since G is simple, our result will follow from

$$\Gamma_{E,2}(G) \leqslant N_G\big(\theta(E)\big).$$

For any $U, V \in \mathcal{E}_2(E)$, we have that $\theta(U) = \theta(V)$ because

$$\theta(u) \leqslant \langle C_{\theta(u)}(v) \colon\ v \in V^\# \rangle \leqslant \theta(V) \quad \text{for any } u \in U.$$

by Fact 1.30 and the signalizer functor property. Thus $\theta(U) = \theta(E)$. For any $U \in \mathcal{E}_2(E)$ and any $g \in N_G(U)$, our hypothesis (†) yields

$$\theta(E)^g = \theta(U)^g = \theta\big(U^g\big) = \theta(U) = \theta(E).$$

Thus $\Gamma_{E,2}(G) \leqslant N_G(\theta(E)) < G$, as desired. □

2. Balance and components

In the tame setting of [Bor95], Borovik states that $O(C_G(i))$ is a nilpotent signalizer functor. In view of Theorem 1.28, it then follows that either G has a proper 2-generated core, if $O(C_G(i)) \neq 1$, or else $C_G(i)$ is "reductive" in the sense of Fact 1.4. It also follows, from Proposition 1.23, that either G has a proper 2-generated core, or else $\Gamma_{\Omega_1(S^\circ)}(G) = G$. These two facts constitute the reductivity and generation conditions of the Generic Trichotomy Theorem [BB04], so a tame version of the Generic Trichotomy Theorem then follows. In fact, [BB04] uses these two conditions merely to establish that G is generated by the quasisimple components of the centralizers of toral involutions, and hence by their root SL_2-subgroups. The remainder of the argument focuses on these root SL_2-subgroups, treating them as an abstract family of root SL_2-subgroups for G, and eventually applying the Curtis–Tits theorem.

In this section, we turn our attention towards "unbalanced groups" where the group $O(C_G(i))$ is not necessarily a signalizer functor, in order to eliminate the hypothesis of tameness. Instead, we use the "most unipotent" parts of $O(C_G(i))$ as signalizer functors. In Theorem 2.9, these signalizer functors are used to prove a dichotomy between a proper 2-generated core, and our $\tilde{B}$-property (see Section 2.1 below). Corollary 2.11 then provides a limited form of the reductivity proved in Fact 1.4. However, this weaker form of reductivity does not admit such a quick proof of generation by components. So our version of this result, Theorem 2.18 below, requires a considerably more delicate argument.

2.1. Partial balance

We require an example to explain the failure of balance.

Example 2.1. Consider a field $(k, T, +, \cdot)$ of finite Morley rank, with $T < k^*$ torsion-free, and $G := \mathrm{SO}_8(k)$ (D_4). By Table 4.3.1 on p. 145 of [GLS98], there are involutions i, j in G, lying in a common torus, such that

$$C_G(i) \cong \mathrm{SL}_4(k) * k^* \quad \text{and} \quad C_G(j) \cong \mathrm{SL}_3(k) * \mathrm{SL}_3(k).$$

So $O(C_G(i)) = O(k^*) = T \neq 1$ and $O(C_G(j)) = 1$. However, every inner involutive automorphism of SL_n is a central product with one copy of k^*. So $O(C_G(\cdot))$ is not a signalizer functor. In fact, the reductivity hypothesis of Fact 1.4 fails too, although its conclusion still holds since the centralizer is still reductive.

Our solution to this is to choose a reduced rank $\bar{r}^*(\cdot)$ which is the largest possible problematic reduced rank in k^*, and work above it by using the fact that $\mathrm{rk}(k^*) > \bar{r}_0(k^*)$.

Definition 2.2. Consider a simple K^*-group G of finite Morley rank and let X be a subgroup of G with $m(X) \geqslant 3$. We write

$$I^0(X) := \{i \in I(X)\colon\ m(C_X(i)) \geqslant 3\}$$

for the set of involutions from eight-groups in $\mathcal{E}_3(X)$. We define

$$\bar{r}^O(X) := \sup\{\bar{r}_0(O(C_G(i)))\colon\ i \in I^0(X)\}$$

as the supremum of the reduced ranks of the odd parts of the centralizers in G of the involutions in $I^0(X)$. We also define $\bar{r}^*(X)$ to be the supremum of $\bar{r}_0(k^*)$ as k ranges over the base fields of the quasisimple components of the quotients $C_G^\circ(i)/O(C_G(i))$ associated to involutions $i \in I^0(X)$.

One can easily check that

$$\bar{r}^O(G) = \max_{E \in \mathcal{E}_3(G)} \bar{r}^O(E) \quad \text{and} \quad \bar{r}^*(G) = \max_{E \in \mathcal{E}_3(G)} \bar{r}^*(E).$$

We recall that, for a nonsolvable group L of finite Morley rank, $U_{0,r}(L)$ and $U_p(L)$ need not be solvable, as quasisimple algebraic groups are generated by the unipotent radicals of their Borel subgroups. Proposition 2.13 below will shed further light on the definition of $\bar{r}^*(\cdot)$ by providing a converse to this observation.

Definition 2.3. We continue in the notation of Definition 2.2. For a definable subgroup H of G, we define $\tilde{U}_X(H)$ to be the subgroup of H generated by $U_p(H)$ for p prime as well as by $U_{0,r}(H)$ for $r > \bar{r}^*(X)$. As an abbreviation, we use $\tilde{F}_X(H)$ to denote $F^\circ(\tilde{U}_X(H))$, and $\tilde{E}_X(H)$ to denote $E(\tilde{U}_X(H))$. We use $\tilde{\mathcal{E}}_Y^X$ to denote the set of components of $\tilde{E}_X(C_G(i)) = E(\tilde{U}_X(C_G(i)))$ for $i \in I^0(Y)$ with $Y \leqslant X$, and we set $\tilde{\mathcal{E}}_X = \tilde{\mathcal{E}}_X^X$.

$\tilde{U}_X(H)$ is the subgroup of H which is generated by its unmistakably unipotent subgroups. These definitions are all sensitive to the choice of X, which is usually a fixed eight-group.

Definition 2.4. We say that a simple K^*-group G with $m(G) \geqslant 3$ satisfies the *$\tilde{B}$-property* if, for every 2-subgroup $X \leqslant G$ with $m(X) \geqslant 3$ and every $t \in I^0(X)$, the group $\tilde{U}_X(O(C_G^\circ(t)))$ is trivial. This is equivalent to

($\tilde{B}$-1) $U_p(O(C_G(t))) = 1$ for all $t \in I^0(G)$ and every prime p.
($\tilde{B}$-2) $\bar{r}^O(X) \leqslant \bar{r}^*(X)$ for every 2-subgroup $X \leqslant G$ with $m(X) \geqslant 3$.

The $\tilde{B}$-property is an unbalanced alternative to Borovik's B-conjecture: that $O(C_C(i)) = 1$ for all $i \in I(G)$. Although the $\tilde{B}$-property is significantly more delicate than the strong B-property, the next two subsections will establish results about the components in $\tilde{\mathcal{E}}_X$ which are similar to Borovik's.

Our goal in this subsection is to verify that the failure of the $\tilde{B}$-property leads to a proper weak 2-generated core. For this, we need two appropriate signalizer functors.

Lemma 2.5. *Let G be a simple K^*-group of finite Morley rank and odd type with $m(G) \geqslant 3$, and let $E \in \mathcal{E}_3(G)$. Then $\{U_p(O(C_G(t))) \mid t \in E^\#\}$ is a connected nilpotent E-signalizer functor satisfying*

$$\theta(s)^g = \theta\big(s^g\big) \quad \textit{for all } s \in E^\# \textit{ and all } g \in G \textit{ for which } s^g \in E. \tag{†}$$

We need the following two facts.

Fact 2.6. *(See [BN94, Ex. 11, p. 93, Ex. 13c, p. 72].) Let G be a group of finite Morley rank and let $H \lhd G$ be a definable subgroup. If $x \in G$ is an element such that $\bar{x} \in G/H$ is a p-element, then xH contains a p-element. Furthermore, if H and G/H are $p^\perp$-groups, then G is a $p^\perp$-group.*

Fact 2.7. *(See [ABCC01], [Bur04b, Fact 3.2].) Let the group $G = H \rtimes T$ be a semidirect product of finite Morley rank. Suppose T is a solvable π-group of bounded exponent and $Q \lhd H$ is a definable solvable T-invariant $\pi^\perp$-subgroup. Then*

$$C_H(T)Q/Q = C_{H/Q}(T).$$

Proof of Lemma 2.5. Let $\theta(t) := U_p(O(C_G(t)))$. We observe that $\theta(s)^g = \theta(s^g)$ for every involution $s \in I(G)$ and every $g \in G$. By Fact 1.10, $\theta(s)$ is nilpotent. $\theta(s)$ is connected by definition.

Let $s, t \in E^\#$; in particular $[s, t] = 1$. Also let $K_s = O(C_G(s))$. Since $C_G(s)$ is a K-group, Fact 1.4 says $C^\circ_G(s)/K_s = G_1 * \cdots * G_n * F$ is the central product of finitely many quasisimple algebraic groups $G_1, \ldots, G_n$ and of a definable divisible abelian group F. Since F is abelian, $O(C^\circ_F(t)) = 1$ by Fact 2.6. We now consider the action of t on the components. For any component G_k, either t normalizes G_k, or else t swaps G_k with another component $G_l = G_k^t$. In the second case, the centralizer of t is some diagonal subgroup of $G_k^t * G_k$, i.e. $C^\circ_G(t) \cong \{(g, \sigma(g)) \mid g \in G_k\} \cong G_k$ for some automorphism σ of G_k. So we may assume that t normalizes each G_k with $k \leqslant m$, and $C^\circ_{G_{m+1} * \cdots * G_n}(t) \cong G_{\frac{n+m}{2}} * \cdots * G_n$. By Facts 1.6 and 1.7, $C^\circ_{G_k}(t)$ is reductive for $k \leqslant m$. So $O(C^\circ_{G_k}(t))$ is a subgroup of an algebraic torus, and hence divisible abelian. Hence $U_p(O(C^\circ_{G_1 * \cdots * G_n}(t))) = 1$. So $U_p(O(C^\circ_{C^\circ_G(s)/K_s}(t))) = 1$. Since $C_{C^\circ_G(s)}(t)K_s/K_s = C_{C^\circ_G(s)/K_s}(t)$ by Fact 2.7, $U_p(O(C_{C^\circ_G(t)}(s))) = U_p(O(C_{C^\circ_G(s)}(t))) \leqslant K_s$.

For any $t \in E^\#$, the group $\theta(t)$ is p-unipotent, and so $2^\perp$ and nilpotent. By Fact 1.17, $C_{\theta(t)}(s)$ is a connected. So $C_{\theta(t)}(s) = U_p(O(C_{\theta(t)}(s))) \leqslant U_p(O(C_{C^\circ_G(t)}(s))) \leqslant U_p(K_s) = \theta(s)$. □

Lemma 2.8. *Let G be a simple K^*-group of finite Morley rank and odd type with $m(G) \geqslant 3$. Let $E \in \mathcal{E}_3(G)$ and set $r := \bar{r}^0(E)$. If $r > \bar{r}^*(E)$ then $U_{0,r}(O(C_G(t)))$ is a connected nilpotent E-signalizer functor again satisfying (†).*

Proof. Let $\theta(t) := U_{0,r}(O(C_G(t)))$. We observe that $\theta(s)^g = \theta(s^g)$ for every involution $s \in I(G)$ and every $g \in G$. $\theta(s)$ is clearly connected and solvable. So $\theta(t)$ is nilpotent by Theorem 1.13.

Let $s, t \in E^\#$; in particular $[s, t] = 1$. Also let $K_s = O(C_G(s))$. Since $C_G(s)$ is a K-group, Fact 1.4 says $C^\circ_G(s)/K_s = G_1 * \cdots * G_n * F$ is the central product of finitely many quasisimple algebraic groups $G_1, \ldots, G_n$ and of a definable divisible abelian group F. Since F is abelian, $O(C^\circ_F(t)) = 1$. We next show that $U_{0,r}(O(C^\circ_{G_1 * \cdots * G_n}(t))) = 1$. For any component G_k, either t normalizes G_k, or else t swaps G_k with another component $G_l = G_k^t$. In the second case, the centralizer of t is some diagonal subgroup of $G_k^t * G_k$, i.e. $C^\circ_G(t) \cong \{(g, \sigma(g)) \mid g \in G_k\} \cong G_k$ for some automorphism σ of G_k. So we may assume that t normalizes each G_k with $k \leqslant m$, and $C^\circ_{G_{m+1} * \cdots * G_n}(t) \cong G_{\frac{n+m}{2}} * \cdots * G_n$. Consider a connected definable indecomposable abelian subgroup A of $O(C^\circ_{G_1 * \cdots * G_m}(t))$ with $A/J(A)$ torsion-free.

Then there is a $k \leqslant m$ with a nontrivial projection map $\pi : A \to O(C^\circ_{G_k/Z(G_k)}(t))$. By [Bur04b, Lemma 2.9], the image $\pi(A)$ is also indecomposable abelian, and $\bar{r}_0(A) = \bar{r}_0(\pi(A))$. By Facts 1.6 and 1.7, $C^\circ_{G_k/Z(G_k)}(t)$ is reductive, and hence $O(C^\circ_{G_k/Z(G_k)}(t))$ is an algebraic torus. It follows that

$$\bar{r}_0(A) = \bar{r}_0\bigl(\pi(A)\bigr) \leqslant \bar{r}_0\bigl(O\bigl(C^\circ_{G_k/Z(G_k)}(t)\bigr)\bigr) \leqslant \bar{r}^*(E).$$

Since $r > \bar{r}^*(E)$, we have $U_{0,r}(O(C^\circ_{G_1 * \cdots * G_n}(t))) = 1$. Since $C_{C^\circ_G(s)}(t)K_s/K_s = C_{C^\circ_G(s)/K_s}(t)$ by Fact 2.7, $U_{0,r}(O(C_{C^\circ_G(t)}(s))) = U_{0,r}(O(C_{C^\circ_G(s)}(t))) \leqslant K_s$.

For any $t \in E^\#$, the centralizer $C_{\theta(t)}(s)$ is a connected $(0,r)$-unipotent $2^\perp$-group by Fact 1.18. Thus

$$C_{\theta(t)}(s) \leqslant U_{0,r}\bigl(O\bigl(C_{C^\circ_G(t)}(s)\bigr)\bigr) \leqslant U_{0,r}(K_s) = \theta(s),$$

and θ is a signalizer functor. □

We can now verify the $\tilde{B}$-property, in the absence of a proper 2-generated core.

Theorem 2.9. *Let G be a simple K^*-group of finite Morley rank and odd type with $m(G) \geqslant 3$. Then either*

1. *G has a proper weak 2-generated core, or else*
2. *G satisfies the $\tilde{B}$-property, i.e. $\tilde{U}_X(O(C_G(t)) = 1$ for every 2-subgroup $X \leqslant G$ with $m(X) \geqslant 3$ and every $t \in I^0(X)$.*

Proof. We first suppose that ($\tilde{B}$-1) fails, i.e. $U_q(O(C_G(i))) \neq 1$ for some involution $i \in I^0(G)$. There is an $E \in \mathcal{E}_3(G)$ containing i. By Lemma 2.5, $\theta(t) := U_p(O(C_G(t)))$ is a connected nilpotent E-signalizer functor satisfying (†). So G has a proper weak 2-generated core by Theorem 1.28.

We next suppose that ($\tilde{B}$-2) fails, i.e. $\bar{r}^O(X) > \bar{r}^*(X)$ for some 2-subgroup $X \leqslant G$ with $m(X) \geqslant 3$. There is an $E \in \mathcal{E}_3(X)$ such that $\bar{r}^O(E) = \bar{r}^O(X)$. Let $\theta(t) := U_{0,r}(O(C_G(t)))$ where $r := \bar{r}^O(E)$ is the largest reduced rank appearing inside $O(C(i))$ for involutions $i \in E^\#$. Now $\theta(t) := U_{0,r}(O(C_G(t)))$ is a connected nilpotent E-signalizer functor satisfying (†) by Lemma 2.8. By the choice of r, $\theta(i)$ is nontrivial for some involution $i \in E^\#$. So G again has a weak proper 2-generated core by Theorem 1.28. □

2.2. Existence of components in $\tilde{\mathcal{E}}_X$

In this subsection, we will use the $\tilde{B}$-property to show that G is a group "of component type" in the sense that

$$\tilde{\mathcal{E}}_X \neq \emptyset \quad \text{for every 2-subgroup } X \leqslant G \text{ with } m(X) \geqslant 3.$$

We employ the p-unipotent and 0-unipotent signalizer functors found in Section 2.1, via Theorem 2.9. Our major tool will be the following analog of Fact 1.4 which allows us to exploit the $\tilde{B}$-property.

Lemma 2.10. *Let H be a K-group of finite Morley rank and odd type, and let $\tilde{H}$ be the subgroup of H generated by*

(a) *$U_{0,r}(H)$ for $r > \bar{r}_0(O(H))$, as well as*
(b) *$U_p(H)$ for any prime p satisfying $U_p(O(H)) = 1$.*

Then $\tilde{H} = E(\tilde{H}) * F^\circ(\tilde{H})$ and $F^\circ(\tilde{H})$ is abelian.

Proof. We first show that $[F(O(H)), \tilde{H}] = 1$. Let A be a definable connected nilpotent subgroup of H with either

(a) $U_{0,r}(A) = A$ for some $r > \bar{r}_0(O(H))$, or
(b) $U_p(A) = A$ for some prime p for which $U_p(O(H)) = 1$.

Then $A \cdot F(O(H))$ is nilpotent by Fact 1.14. Since $r > \bar{r}_0(O(H))$, we have $[A, F(O(H))] = 1$ by Fact 1.16. So $[F(O(H)), \tilde{H}] = 1$.

Since $F(O(\tilde{H}))$ is definably characteristic in $\tilde{H}$, we have $F(O(\tilde{H})) \leqslant F(O(H))$, and so $F(O(\tilde{H})) \leqslant Z(\tilde{H})$. Hence $O(\tilde{H})$ is nilpotent, and

$$O(\tilde{H}) = F\big(O(\tilde{H})\big) \leqslant Z(\tilde{H}). \tag{$\star$}$$

Consider $K := \tilde{H}/O(\tilde{H})$. By Fact 1.4, $K = E(K) * F(K)$ and $F(K)$ is abelian. By ($\star$), the inverse image of $F(K)$ in $\tilde{H}$ is nilpotent, and thus equals $F^\circ(\tilde{H})$. As any quasisimple component L of $E(K)$ is perfect, such a component L admits only finite central extensions by [AC99]. By ($\star$), the inverse image $\hat{L}$ of L in $\tilde{H}$ is isomorphic to a central product $L * O(\tilde{H})$, which thus contains a component of $\tilde{H}$. Now $\tilde{H} = E(\tilde{H}) * F^\circ(\tilde{H})$, as desired.

Therefore $F^\circ(\tilde{H})$ satisfies our generation hypotheses. Clearly $U_p(F^\circ(\tilde{H})) \leqslant U_p(O(H)) = 1$. By Fact 1.16, $F^\circ(\tilde{H})$ is a central product of the $U_{0,r}(F^\circ(\tilde{H}))$ with $r > \bar{r}_0(O(H)) \geqslant \bar{r}_0(O(\tilde{H})$. It follows that $F^\circ(\tilde{H})$ is abelian since $F(K)$ was abelian. □

The $\tilde{B}$-property states that the centralizers of appropriate involutions satisfy the hypotheses of Lemma 2.10.

Corollary 2.11. *Let G be a simple K^*-group of finite Morley rank and odd type with $m(G) \geqslant 3$ which satisfies the $\tilde{B}$-property. Then, for every 2-subgroup $X \leqslant G$ with $m(X) \geqslant 3$ and every $i \in I^0(X)$, we have $\tilde{U}_X(C_G(i)) = \tilde{E}_X(C_G(i)) * \tilde{F}_X(C_G(i))$ and $\tilde{F}_X(C_G(i))$ is abelian.*

We can now verify that $\tilde{\mathcal{E}}_X$ is nonempty.

Theorem 2.12. *Let G be a simple K^*-group of finite Morley rank and odd type with $m(G) \geqslant 3$. Then either*

1. *G has a proper weak 2-generated core, or else*
2. *$\tilde{\mathcal{E}}_X \neq \emptyset$ for every 2-subgroup $X \leqslant G$ with $m(X) \geqslant 3$.*

Proof. By Theorem 2.9, we may assume that G satisfies the $\tilde{B}$-property. Consider a 2-subgroup $X \leqslant G$ with $m(X) \geqslant 3$. There is an $E \in \mathcal{E}_3(X)$ with $\bar{r}^*(E)$ maximal. So $\bar{r}^*(E) = \bar{r}^*(X)$ and $\tilde{\mathcal{E}}_E \subset \tilde{\mathcal{E}}_X$.

We first consider the case where $C^\circ_G(i)$ is solvable for all $i \in E^\#$. In particular, $\bar{r}^*(E) = 0$. For all $i \in E^\#$, $U_p(C_G(i)) = U_p(O(C_G(i)) = 1$ and $\bar{r}^O(E) \leqslant \bar{r}^*(E) = 0$, since G satisfies the $\tilde{B}$-property. By Fact 1.15, $O(C^\circ_G(i))$ is a good torus, and hence central in $C^\circ_G(i)$ by [BN94, Theorem 6.16]. Since $C^\circ_G(i)/O(C^\circ_G(i))$ is abelian by Fact 1.4. $C^\circ_G(i)$ is nilpotent, and divisible. Now $C^\circ_G(i)'$ is torsion-free by [BN94, Theorem 6.9]. As $\bar{r}^O(E) = 0$, $C^\circ_G(i)$ is in fact abelian. By Fact 1.30, $C^\circ_G(i) = \langle C^\circ_{C^\circ_G(i)}(E_0)\colon\ E_0 \leqslant E,\ [E : E_0] = 2\rangle$. As $C^\circ_G(i) \neq 1$ by [BN94, Ex. 13 & 15, p. 79], there is some four-group $E_1 \leqslant E$ with $H := C^\circ_G(E_1) \neq 1$. Since each $C^\circ_G(i)$ is abelian, $H \lhd C^\circ_G(i)$ for all $i \in E_1^\#$, and thus $H \lhd \Gamma_{E_1}(G)$. Since G is simple, $\Gamma_{E_1}(G) \leqslant N_G(H) < G$, and G has a proper weak 2-generated core $\Gamma^0_{S,2}(G) < G$ by Proposition 1.23. So we may assume that $C^\circ_G(i)$ is nonsolvable for some $i \in E^\#$.

We now fix an $i \in E^\#$ and a component L of $C^\circ_G(i)/O(C^\circ_G(i))$ so that $\bar{r}_0(k^*) = \bar{r}^*(E)$ where k is the base field of L. Since k is algebraically closed, k^* contains torsion, and hence

$$\mathrm{rk}(k_+) = \mathrm{rk}\big(k^*\big) > \bar{r}_0\big(k^*\big) = \bar{r}^*(E).$$

Suppose toward a contradiction that $\tilde{E}_E(C_G(i)) = 1$. By Corollary 2.11, we have $\tilde{U}_E(C_G(i)) = \tilde{E}_E(C_G(i)) * \tilde{F}_E(C_G(i))$ and $\tilde{F}_E(C_G(i))$ is abelian for every $i \in E^\#$. So $\tilde{U}_E(C_G(i)) = \tilde{F}_E(C_G(i))$ is abelian. If

$\mathrm{char}(k) > 0$ then $U_{\mathrm{char}(k)}(C_G^\circ(i)) \leqslant \tilde{F}_E(C_G^\circ(i))$ is abelian. If $\mathrm{char}(k) = 0$ then $U_{0,\mathrm{rk}(k_+)}(C_G^\circ(i)) \leqslant \tilde{F}_E(C_G(i))$ is abelian. Either case contradicts the existence of L. □

We also observe that the definition of $\tilde{\mathcal{E}}_X$ restricts the fields involved as follows.

Proposition 2.13. *Let H be a group of finite Morley rank which is isomorphic to a linear algebraic group over an algebraically closed field k. Then*

1. *If $U_{0,r}(H) \neq 1$ for some $r > \bar{r}_0(k^*)$ then* $\mathrm{char}(k) = 0$ *and* $\mathrm{rk}(k) = r$.
2. *If $U_p(H) \neq 1$ then* $\mathrm{char}(k) = p$.

If H is quasisimple, these conditions imply $U_p(H) = H$ and $U_{0,r}(H) = H$, respectively.

Proof. If $\mathrm{char}(k) \neq p$, then H has bounded p-rank, so the second point follows.

We now consider the first point. Let A be a nontrivial $(0,r)$-unipotent abelian group, and let $\hat{A}$ be the Zariski closure of A. Then $\hat{A} = S \times U$ where S is semisimple, and U is the unipotent radical of A. If A has nonunipotent elements, then $\bar{A} := AU/U$ is a nontrivial subgroup of the semisimple group $\hat{A}/U$. As $\hat{A}/U$ is linear, $\hat{A}/U \hookrightarrow (k^*)^n$ for some n. But $U_{0,r}(\bar{A}) = \bar{A}$ by [Bur04b, Lemma 2.11], contradicting $r > \bar{r}_0(k^*)$. So A consists of unipotent elements, i.e. $A \leqslant U$ $(= \hat{A})$. Hence $\mathrm{char}(k) = 0$. As U is linear, $U \hookrightarrow (k_+)^n$ for some n. By [Poi87, Corollary 3.3], there are no definable subgroups of k^+. So $U_{0,r}(U) = 1$ unless $r = \mathrm{rk}(k)$.

The last remark follows from the fact that quasisimple algebraic groups are generated by the unipotent radicals of their Borel subgroups, or indeed by any conjugacy class of elements. □

2.3. Generation by components in $\tilde{\mathcal{E}}_X$

We next show that components in $\tilde{\mathcal{E}}_X$ generate G, i.e.

$$\langle \tilde{\mathcal{E}}_{\Omega_1(S^\circ)} \rangle = G \quad \text{when } \mathrm{pr}(S) \geqslant 3.$$

However, these results will be proven in a form usable also when $\mathrm{pr}(S) < 3$.

For any group H of finite Morley rank, any 2-subgroup X acting definably on H, and any $V \in \mathcal{E}_2^0(X)$, we define

$$\tilde{\Gamma}_{X,V}(H) = \langle \tilde{U}_X\big(C_H^\circ(v)\big):\ v \in V^\# \rangle.$$

Lemma 2.14. *Let G be a K^*-group of finite Morley rank and odd type. Let X be a 2-subgroup of G with $m(X) \geqslant 3$. Suppose that G satisfies the $\tilde{B}$-property and that there is a four-group $E \in \mathcal{E}_2^0(X)$ which centralizes a Sylow° 2-subgroup T of G. Let $H := \langle \tilde{E}_X(C_G(z)):\ z \in E^\# \rangle$. Then the following hold.*

1. *For any $x, y \in E^\#$, we have $[\tilde{F}_X(C_G(x)), \tilde{F}_X(C_G(y))] = 1$ and the group $\tilde{F}_X(C_G(x))$ normalizes $\tilde{E}_X(C_G(y))$.*
2. *$\tilde{U}_X(O(H)) = 1$.*
3. *$\tilde{\Gamma}_{X,E}(G) \leqslant N_G^\circ(\tilde{E}_X(H))$.*

Proof. As a notational convenience, let $F_x := \tilde{F}_X(C_G(x))$ and $H_x := \tilde{E}_X(C_G(x))$ for $x \in E^\#$. Since G satisfies the $\tilde{B}$-property, Corollary 2.11 says $\tilde{U}_X(C_G(x)) = H_x * F_x$ and F_x is abelian. Since $y \in E^\#$ normalizes F_x for any $x \in E^\#$, there is a homomorphism $f_y : F_x \to F_x$ given by $u \mapsto [y, u]$. Since E centralizes T and $F_x \cap T$ is the Sylow 2-subgroup of F_x, there is no 2-torsion in $F_x \setminus \ker(f_y)$. So $[F_x, E] \leqslant O(F_x) \leqslant O(C_G^\circ(x))$ by Fact 2.6 (and Fact 1.19). By Fact 1.20, $[U_{0,r}(F_x), E]$ is a $U_{0,r}$-group. Clearly $[U_p(F_x), E]$ is p-unipotent. Since $\tilde{U}_X(C_G^\circ(x)) = H_x * F_x$, we have $F_x = \tilde{U}_X(F_x)$. So $[F_x, E] \leqslant \tilde{U}_X(O(C_G^\circ(x))) = 1$ by the $\tilde{B}$-property. Now $F_x \leqslant \tilde{U}_X(C_G(y))$ for every $x, y \in E^\#$. Since F_y is central in

$\tilde{U}_X(C_G(y))$, we have $[F_x, F_y] = 1$. Also F_x normalizes H_y, as $H_y \lhd \tilde{U}_X(C_G(y))$. Thus $F := \langle F_z\colon z \in E^\# \rangle$ is an abelian group of automorphisms of $H = \langle H_z\colon z \in E^\# \rangle$.

Since H_x is characteristic in $C_G(x)$ for all $x \in E^\#$, E normalizes H. Suppose towards a contradiction that $\tilde{U}_X(O(H)) \neq 1$. So either

(1) $K := U_{0,r}(O(H)) \neq 1$ for some $r > \bar{r}^*(X)$, or else
(2) $K := U_p(O(H)) \neq 1$ for some prime p.

In either case, $K = \Gamma_E(K)$ by Fact 1.1. So a $K_x := C_K(x)$ is nontrivial. In case (1), we may choose r maximal, so K is nilpotent by Theorem 1.13. Hence K_x is $(0,r)$-unipotent by Lemma 1.18. In case (2), K is nilpotent by Fact 1.10. Hence K_x is p-unipotent by Fact 1.17. In either case, K_x is nilpotent and normalized by H_x. Since $K_x \leqslant \tilde{U}_X(C_G(x))$, and H_x is semisimple, we have $K_x \leqslant F_x$, in contradiction to the $\tilde{B}$-property. Thus $\tilde{U}_X(O(H)) = 1$, as desired.

For the last part, we may assume $H < G$ is a K-group. So $H = \tilde{E}_X(H) * \tilde{F}_X(H)$ by Corollary 2.11. Since $\tilde{E}_X(H)$ is characteristic in H, F normalizes $\tilde{E}_X(H)$. So $\tilde{\Gamma}_{X,E}(G) = F \cdot H \leqslant N_G^\circ(\tilde{E}_X(H))$. □

We now provide the promised variant of Lemma 1.24 above.

Lemma 2.15. *Let G be a K^*-group of finite Morley rank and odd type with $m(G) \geqslant 3$, and which satisfies the $\tilde{B}$-property. Let T be a Sylow° 2-subgroup of G, and let $X \leqslant C_G(T)$ be a 2-subgroup which centralizes T and has $m(X) \geqslant 3$. Then $\tilde{\Gamma}_{X,U}(G) = \tilde{\Gamma}_{X,V}(G)$ for any two four-groups $U, V \in \mathcal{E}_2^0(X)$.*

We need the following algebraic fact.

Fact 2.16. *Let G be a quasisimple algebraic group over an algebraically closed field of characteristic not 2 which is not of type $(P)\mathrm{SL}_2$, and let V be a four-group of algebraic automorphisms of G which centralizes a maximal 2-torus of G. Then*

$$G = \langle E(C_G(x))\colon x \in V^\# \rangle.$$

Proof. Let T be a maximal 2-torus of G centralized by V. For $i \in V^\#$, Fact 1.7 says $C_G^\circ(i)$ is reductive. Hence $C_G^\circ(i) = F_i * H_i$ where $H_i := E(C_G^\circ(i))$ and $F_i := F^\circ(C_G^\circ(i))$ is an algebraic torus. So F_i is the Zariski closure of $F_i \cap T$, and hence is centralized by V. Now $F := \langle F_x \mid x \in V^\# \rangle$ is an algebraic torus normalizing $H = \langle H_x \mid x \in V^\# \rangle$.

We now show that H is reductive. Suppose that H has a nontrivial unipotent radical U. By Fact 1.1, $U_j := C_U(j) \neq 1$ for some $j \in V^\#$. But $U_j \lhd C_G^\circ(j)$, contradicting the reductivity of $C_G^\circ(j)$. So H must be reductive.

Now $\Gamma_V(G) = FH \leqslant N_G^\circ(E(H))$. We observe that $H_j \neq 1$ for $j \in V^\#$, since $G \not\cong (P)\mathrm{SL}_2$. So H is not solvable, and $E(H) \neq 1$. Since $\Gamma_V(G) = G$ by Fact 1.1, it follows that $E(H) = G$ because G is quasisimple, and hence $H = G$. □

Proof of Lemma 2.15. We may assume that $[U, V] = 1$ since U and V lie in the same 2-connected component. It is enough to show that $\tilde{U}_X(C_G(u)) \leqslant \tilde{\Gamma}_{X,V}(\tilde{U}_X(C_G(u))$ for any $u \in U^\#$. Since G satisfies the $\tilde{B}$-property, $\tilde{U}_X(C_G(u)) = \tilde{E}_X(C_G(u)) * \tilde{F}_X(C_G(u))$ by Corollary 2.11. By Fact 1.16, the abelian group $F_u := \tilde{F}_X(C_G(u))$ may be written as a product of various $U_p(F_u)$, with p prime, and $U_{0,r}(F_u)$, with $r > \bar{r}^*(X)$. By Fact 1.18, $C_{U_{0,r}(F_u)}(v)$ is $(0,r)$-unipotent. By Fact 1.17, $C_{U_p(F_u)}(v)$ is p-unipotent. So $C_{F_u}(v) \leqslant \tilde{U}_X(C_G(u))$. Fact 1.1 yields

$$\tilde{F}_X(C_G(u)) \leqslant \tilde{\Gamma}_{X,V}(\tilde{U}_X(C_G(u)).$$

Now consider a component $L \lhd \tilde{E}_X(C_G(u))$. It suffices to show that $L = \tilde{\Gamma}_{X,V}(L)$.

In the case that $L \ncong (P)\mathrm{SL}_2$, Fact 2.16 yields

$$L = \langle E(C_L(x)):\ x \in V^{\#}\rangle.$$

By Proposition 2.13, we have $E(C_L(x)) = \tilde{E}_X(C_L(x))$, and hence $L = \tilde{\Gamma}_{X,V}(L)$.

In the case that $L \cong (P)\mathrm{SL}_2$, $(P)\mathrm{SL}_2$ has no graph automorphisms, so every $x \in V^{\#}$ acts by some inner automorphism by Fact 1.6. We observe that $T \leqslant C_G(u)$, and thus T is a Sylow° 2-subgroup of $C_G(u)$. So $L \cap T$ is a Sylow° 2-subgroup of L. Since $(P)\mathrm{SL}_2$ contains no four-group centralizing a torus, there is now some $x \in V^{\#}$ which centralizes L, and the claim follows. □

We now prove a version of Proposition 1.23.

Proposition 2.17. *Let G be an infinite simple K^*-group of finite Morley rank and odd type with $m(G) \geqslant 3$. Let T be a Sylow° 2-subgroup of G, and let $X \leqslant C_G(T)$ be a 2-subgroup which centralizes T with $m(X) \geqslant 3$. Suppose that $\tilde{\Gamma}_{X,E}(G) < G$ for some $E \in \mathcal{E}_2^0(X)$. Then G has a proper weak 2-generated core.*

Proof. By Proposition 1.23, it is enough to show that $\Gamma_E(G) < G$. Let $A \in \mathcal{E}_3(X)$ be an eight-subgroup of X containing E. By Lemma 1.25, $\Gamma_E(G) \leqslant \Gamma_{A,2}(G)$. So the result will follow from the following claim and simplicity.

$$\Gamma_{A,2}(G) \leqslant N_G(\tilde{\Gamma}_{X,E}(G))$$

By Lemma 2.15 and Fact 1.21.1, $\tilde{\Gamma}_{X,E}(G) = \tilde{\Gamma}_{X,U}(G)$ for any $U \leqslant A$. For any four-group $U \leqslant A$,

$$N_G(U) \leqslant N_G(\tilde{\Gamma}_{X,U}(G)) = N_G(\tilde{\Gamma}_{X,E}(G)).$$

Thus $\Gamma_{A,2}(G) \leqslant N_G(\tilde{\Gamma}_{X,E}(G))$, as desired. □

We now prove that our components generate G.

Theorem 2.18. *Let G be a K^*-group of finite Morley rank and odd type with $m(G) \geqslant 3$. Suppose that there is a four-group $E \in \mathcal{E}_2^0(G)$ which centralizes a Sylow° 2-subgroup T of G, and that there is an eight-group $X \in \mathcal{E}_3(C_G(T))$ containing E. Then either*

1. *G has a proper weak 2-generated core, or*
2. *$\langle \tilde{\mathcal{E}}_E^X \rangle = \langle \tilde{E}_X(C_G(x)):\ x \in E^{\#}\rangle = G$.*

We need the following fact about involutive automorphisms of algebraic groups, which follows immediately from Table 4.3.1 on p. 145 of [GLS98] and Fact 1.6.

Fact 2.19. *Let G be a quasisimple algebraic group over an algebraically closed field of characteristic not 2 and let α be a definable involutive automorphism of G. If $G \ncong (P)\mathrm{SL}_2$, then $E(C_G(\alpha)) \neq 1$.*

Proof of Theorem 2.18. We may assume that G satisfies the $\tilde{B}$-property by Theorem 2.9.

We first show that $\tilde{E}_X(C_G^\circ(z)) \neq 1$ for some $z \in E^{\#}$ (note $m(E) \geqslant 2$). We may assume that $\tilde{\mathcal{E}}_X \neq 1$ by Theorem 2.12, so there is a component $L \leqslant \tilde{E}_X(C_G(x)) \neq 1$ for some $x \in X^{\#}$. We observe that E normalizes L since X is an eight-group containing E. So any $z \in E^{\#}$ acts on L via an algebraic automorphism, by Fact 1.6, and hence $C_L^\circ(z)$ is reductive by Fact 1.7. By Proposition 2.13, $\tilde{U}_X(U) = U$ for any unipotent group in L. As quasisimple groups are generated by their unipotent subgroups, $\tilde{U}_X(C_G(z)) \geqslant \tilde{E}_X(C_L(z)) = E(C_L^\circ(z))$. If $L \ncong (P)\mathrm{SL}_2$, then $E(C_L^\circ(z)) \neq 1$ by Fact 2.19, and hence $\tilde{E}_X(C_G(z)) \neq 1$ by Corollary 2.11. So we may assume $L \cong (P)\mathrm{SL}_2$. Since $(P)\mathrm{SL}_2$ has no graph automorphisms, any $z \in E^{\#}$ acts via inner automorphism, by Fact 1.6. Since $(P)\mathrm{SL}_2$ contains no four-group centralizing a torus, there is now some $z \in E^{\#}$ which centralizes L, and $\tilde{U}_X(C_G(z))$ is nonabelian. By Corollary 2.11, $\tilde{U}_X(C_G(z)) = \tilde{E}_X(C_G(z)) * \tilde{F}_X(C_G(z))$ and $\tilde{F}_X(C_G(z))$ is abelian. So $\tilde{E}_X(C_G^\circ(z) \neq 1$ for some $z \in E^{\#}$.

Let $H := \langle \tilde{E}_X(C_G(x))\colon x \in E^{\#}\rangle$. Since $\tilde{U}_X(O(H)) = 1$ by Lemma 2.14, $H = \tilde{E}_X(H) * \tilde{F}_X(H)$ and $\tilde{F}_X(H)$ is abelian by Corollary 2.11. Since $\tilde{E}_X(C_G^\circ(x)) \neq 1$, we have $\tilde{E}_X(H) \neq 1$.

Since E centralizes T, Lemma 2.14 says that $\tilde{\Gamma}_{X,E}(G) \leqslant N_G^\circ(\tilde{E}_X(H))$. Therefore $\tilde{\Gamma}_{X,E}(G) \leqslant N_G^\circ(\tilde{E}_X(H)) < G$, and the theorem follows from Proposition 2.17. □

3. Generic Trichotomy Theorem

We now turn our attention toward proving the following, our main result.

Generic Trichotomy Theorem. *Let G be a simple K^*-group of finite Morley rank and odd type with* $\mathrm{pr}(G) \geqslant 3$. *Then either*

1. *G has a proper 2-generated core, i.e. $\Gamma_{S,2}(G) < G$, or else*
2. *G is an algebraic group over an algebraically closed field of characteristic not 2.*

Our strategy is to replicate the proof by Berkman and Borovik of the Generic Identification Theorem [BB04], being careful to use only "safe" components, under the assumption that (1) does not occur. So we adopt the following standing hypotheses and notation.

Hypothesis 3.1. We consider a simple K^*-group G of finite Morley rank and odd type with $\mathrm{pr}(G) \geqslant 3$, and fix a Sylow 2-subgroup S of G. We also suppose that the 2-generated core of G is not proper, i.e. $\Gamma_{S,2}(G) = G$.

As $n(2) \geqslant \mathrm{pr}(G) \geqslant 3$, we have $\Gamma_{S,2}^0(G) = \Gamma_{S,2}(G) = G$ by Fact 1.21.2. So, by Theorem 2.9,

$$G \text{ satisfies the } \tilde{B}\text{-property.} \tag{1}$$

3.1. Root SL_2-subgroups

The first stage in our analysis is to select, and establish the properties of a family of abstract "root SL_2-subgroups" of G. The root SL_2-subgroups of an *algebraic group* associated to a maximal torus T may be defined as those *Zariski closed* subgroups of G which are normalized by T and are isomorphic to $(P)\mathrm{SL}_2$, or alternatively in terms of groups generated by opposite root groups. We employ several facts about root SL_2-subgroups of algebraic groups.

Fact 3.2. *(Cf. [BB04, Fact 2.1].) Let G be a quasisimple algebraic group over an algebraically closed field. Let T be a maximal torus in G and let K, L be* Zariski closed *subgroups of G that are isomorphic to* SL_2 *or* PSL_2 *and are normalized by T. Then*

1. *Either K and L commute or $\langle K, L\rangle$ is a quasisimple algebraic group of type A_2, C_2, or G_2.*
2. *The subgroups K and L are root* SL_2*-subgroups of $\langle K, L\rangle$.*
3. *If $\langle K, L\rangle$ is of type G_2, then $G = \langle K, L\rangle$.*

More generally, a semisimple subgroup of a simple algebraic group G which is normalized by a maximal torus T is called *subsystem* subgroup of G, associated to T. Berkman and Borovik refer to the full classification of semisimple subsystem subgroups [Sei83, 2.5] (see also §3.1 of [Sei95]) for the proof of this fact. The elementary argument here is based on the following fact.

Fact 3.3. *(See [Sei95, Proposition 3.1], [Sei83, 2.5].) Let G be a simple algebraic group, let T be a maximal torus of G, and let X be a closed connected subgroup of G which contains T. Then $X = DZU$ where D is a subsystem subgroup normalized by T, Z is a torus, and U is the unipotent radical of X.*

Proof of Fact 3.2. By Fact 3.3, $\langle K, L\rangle = DZU$ where D is a subsystem subgroup, Z is a torus, and U is the unipotent radical of $\langle K, L\rangle$. There is an automorphism ϕ of the root system for G which sends any root $\alpha \in I$ to its negative $-\alpha$, and ϕ translates to an automorphism ϕ of the group G such that ϕ normalizes T and $\phi(X_\alpha) = X_{-\alpha}$ [Car89]. Since K and L each contain one positive and one negative root from I, we find that K, L, and $\langle K, L\rangle$ are all normalized by ϕ. If U is nontrivial, it contains a root group X_α. Since U is characteristic in $\langle K, L\rangle$, the nilpotent group U must also contain $X_{-\alpha}$, and hence contains a copy of $(P)\mathrm{SL}_2$ [Car89]. So $U = 1$, and $\langle K, L\rangle = D$. Since D is semisimple, $D \cong A_1 * A_1$, A_2, C_2, or G_2, as desired. □

Fact 3.4. *(See [Car93, p. 19].) Let G be a semisimple algebraic group over an algebraically closed field, and fix a maximal algebraic torus T of G. Then the following hold.*

1. *G is generated by its root SL_2-subgroups associated with T.*
2. *The intersection $T \cap K$ of T and a root SL_2-subgroup K associated to T is a maximal algebraic torus of K.*

Proof. The fact that G is generated by those root SL_2-subgroups which are normalized by T can be found on p. 19 of [Car93]. For the second part, we observe that the maximal algebraic tori of $N_G(K)$, one of which is T and one of which extends a maximal algebraic torus of K, are conjugate in $N_G(K)$. □

We also need to know that a root SL_2-subgroup is "cut out" by the centralizer of a 2-torus in the associated maximal torus. We remark that this is an essential point if one hopes to apply Fact 3.2, but it remains somewhat obscure in [BB04].

Fact 3.5. *Let G be a quasisimple algebraic group over an algebraically closed field of characteristic not 2. Let T be a maximal algebraic torus of G, and let L be a root SL_2-subgroup of G normalized by T. Then $L = E(C_G(C_{S^\circ}(L)))$ where S° is the Sylow° 2-subgroup of T.*

Proof. We may assume that G is not isomorphic to $(P)\mathrm{SL}_2$ because otherwise $G = L$. Let S be a Sylow 2-subgroup of G such that $S^\circ \leqslant T$. Any connected definable group of automorphisms of G must be inner by Fact 1.6. Since L is normalized by S°, we have $\mathrm{pr}(C_{S^\circ}(L)) = \mathrm{pr}(G) - 1$. By Fact 1.7 (see also [Car93, Theorem 3.5.4]), $C_G(C_{S^\circ}(L))$ is reductive. So $\mathrm{pr}(K) \leqslant 1$ where $K := E(C_G(C_{S^\circ}(L)))$. Since $L \leqslant K$ and L and K are both algebraic subgroups, we have $L = K$. □

We now proceed with the analysis of groups satisfying Hypothesis 3.1.

Lemma 3.6. *For any $i \in \Omega_1(S^\circ)^\#$ and any definable connected quasisimple algebraic $L \leqslant E(C_G^\circ(i))$ which is normalized by S°, we have*

1. *$S^\circ \cap L$ is a Sylow° 2-subgroup of L,*
2. *$T_L := C_L(S^\circ \cap L)$ is a maximal algebraic torus of L,*
3. *$S^\circ = C^\circ_{S^\circ}(L)(S^\circ \cap L)$, and $\mathrm{pr}(G) = \mathrm{pr}(S^\circ) = \mathrm{pr}(C^\circ_{S^\circ}(L)) + \mathrm{pr}(S^\circ \cap L)$.*

For this, we need the following fact about algebraic groups.

Fact 3.7. *Let G be a quasisimple algebraic group over an algebraically closed field of characteristic not 2, and let T be a Sylow° 2-subgroup of G. Then $C_G(T)$ is a maximal algebraic torus of G.*

Proof. In an algebraic group over an algebraically closed field, the maximal algebraic torus is the Zariski closure of T, and is thus centralized by anything centralizing T. But a maximal algebraic torus is self-centralizing by [Hum75, 24.1]. So the result follows. □

Proof of Lemma 3.6. Since S° is a Sylow° 2-subgroup of $N_G^\circ(L)$, the group $S^\circ \cap L$ is a Sylow° 2-subgroup of L. By Fact 3.7, T_L is a maximal algebraic torus of L. By Fact 1.6, the connected definable group $d(S^\circ)$ acts by inner automorphisms on L, so the third condition follows. □

Lemma 3.8. *For any component $M \in \tilde{\mathcal{E}}_{S^\circ}$, M is normalized by S°.*

Proof. Clearly $M \lhd C_G^\circ(i)$ for some $i \in \Omega_1(S^\circ)$ (see [BN94, Lemma 7.1iii]). Since $S^\circ \leqslant C_G^\circ(i)$, M is normalized by S°. □

Definition 3.9. Let Σ be the set of all root SL_2-subgroups of components $K \in \tilde{\mathcal{E}}_{S^\circ}$ which are associated to T_K, i.e. Σ is the set of all *Zariski closed* subgroups of the components $K \in \tilde{\mathcal{E}}_{S^\circ}$ which are normalized by T_K and are isomorphic to $(P)\mathrm{SL}_2$.

Since the root SL_2-subgroups of K generate K by Fact 3.4.1, Theorem 2.18 yields the following:

$$\langle \Sigma \rangle = \langle \tilde{\mathcal{E}}_{S^\circ} \rangle = G. \tag{($\star$)}$$

We view the subgroups in Σ as abstract root SL_2-subgroups for G.

Lemma 3.10. *For any $L \in \Sigma$, we have*

1. *L is normalized by S°,*
2. *$L = E(C_G(C_{S^\circ}(L)))$, and*
3. *L is a Zariski closed subgroup of any definable quasisimple $K < G$ which contains L and which is normalized by S°.*

Proof. Let $R_L := C_{S^\circ}(L)$ and let $M \in \tilde{\mathcal{E}}_{S^\circ}$ be a component containing L as a root SL_2-subgroup associated with T_M. By Lemma 3.6.3, $S^\circ = C_{S^\circ}(M)(M \cap S^\circ)$. Since L is normalized by $M \cap S^\circ \leqslant T_M$, L is normalized by S°. By Fact 3.5, $L = E(C_M(M \cap R_L))$. Fix $i \in \Omega_1(S^\circ)^\#$ with $M \lhd C_G^\circ(i)$. Clearly $E(C_G(R_L)) \leqslant M$ because any other component of $C_G^\circ(i)$ meets R_L in an infinite 2-torus. As $i \in R_L$, $L = E(C_G(R_L))$.

Now for any definable quasisimple $K < G$ which contains L and which is normalized by S°, the group R_L acts on K by inner automorphisms by Fact 1.6, so $L = E(C_K(R_L))$ is Zariski closed. □

Lemma 3.11. *(Cf. [BB04, Lemma 3.1].) For any distinct $K, L \in \Sigma$,*

1. *$C_{S^\circ}(K) \cap C_{S^\circ}(L) \neq 1$ and $M := \langle K, L \rangle$ is a K-group.*
2. *Either K and L commute or M is an algebraic group of type A_2, $B_2 = C_2$, or G_2.*
3. *$S^\circ \cap M = (S^\circ \cap K) * (S^\circ \cap L)$ is a Sylow° 2-subgroup of M.*
4. *K and L are root SL_2-subgroups of M normalized by T_M.*
5. $[T_K, T_L] = 1$.

Proof. Let $R_L := C_{S^\circ}^\circ(L)$. Since S° normalizes K and L by Lemma 3.10.1, and $K, L \cong \mathrm{SL}_2$, we know that $\mathrm{pr}(R_K), \mathrm{pr}(R_L) = \mathrm{pr}(G) - 1$ and $S^\circ = R_K R_L$ by Lemma 3.6.3. So

$$\mathrm{pr}(G) = \mathrm{pr}(R_K) + \mathrm{pr}(R_L) - \mathrm{pr}(R_K \cap R_L) = 2\,\mathrm{pr}(G) - 2 - \mathrm{pr}(R_K \cap R_L)$$

and

$$\mathrm{pr}(R_K \cap R_L) = \mathrm{pr}(G) - 2 \geqslant 1.$$

Thus $C_{S^\circ}(K) \cap C_{S^\circ}(L) \neq 1$ and M has a nontrivial center. Since G is simple, $M < G$ is a K-group.

Let $i \in I(C_{S^\circ}(K) \cap C_{S^\circ}(L))$. By Corollary 2.11 (and the $\tilde{B}$-property), $K, L \leqslant \tilde{E}_{S^\circ}(C_G(i))$. If the groups belong to different components of $C_G^\circ(i)$, then they commute. If they both belong to the same component $H \in \tilde{\mathcal{E}}_{S^\circ}$, then H is a quasisimple algebraic group normalized by S° by Lemma 3.8. By Lemma 3.10.3, K and L are Zariski closed in H. By Fact 3.2.1, $M = \langle K, L\rangle$ is an algebraic group of type A_2, $B_2 = C_2$, or G_2. In either case, S° normalizes M, so $S^\circ \cap M$ is a Sylow$^\circ$ 2-subgroup of M and T_M is a maximal "algebraic" torus of M by Lemma 3.6.

For (4) and (5), we may assume that $[L, K] \neq 1$ and M is a quasisimple algebraic group. By Fact 3.2.2, K and L are root SL_2-subgroups of M. By Fact 3.4.2, $T_M = T_K * T_L$, so $[T_K, T_L] = 1$. □

We give Σ a graph structure by placing an edge between $L, K \in \Sigma$ when $[L, K] \neq 1$. Since G is simple and $\langle \Sigma\rangle = G$ $(\star)$, the graph Σ is connected. By Lemma 3.11.2, any adjacent $L, K \in \Sigma$ are algebraic groups over the same algebraically closed field. So all the elements of Σ are algebraic groups over a common algebraically closed field $\mathbb{F}$. Since G has odd type, $\mathrm{char}(\mathbb{F}) \neq 2$. In particular, $\mathrm{rk}(K) = \mathrm{rk}(L)$ for all $K, L \in \Sigma$.

From this point on, our argument reduces to that given by Berkman and Borovik in [BB04], following the presentation of [BBBC08]. Indeed, we may now lighten our standing hypotheses to the following.

Hypothesis 3.12. We consider a simple group G of finite Morley rank and odd type with $\mathrm{pr}(G) \geqslant 3$, and fix a Sylow 2-subgroup S of G. Also choose *some* family $\tilde{\mathcal{E}}$ of algebraic components from the centralizers of involutions in S°. Let Σ be the set of root SL_2-subgroups, from components in $\tilde{\mathcal{E}}$, which are associated to S°, in the sense of Definition 3.9. Suppose that Lemmas 3.6, 3.10, and 3.11 are satisfied, and also that

$$\langle \Sigma \rangle = G. \qquad ((\star))$$

We give the analysis in full below.

3.2. Weyl group

We now turn our attention to the Weyl group of G, continuing under Hypothesis 3.12.

Lemma 3.13. *The* natural *torus $T := \langle T_L\colon L \in \Sigma\rangle$ is divisible abelian. So $T_L = T \cap L$.*

Proof. By Lemma 3.11.5, the algebraic tori T_K for $K \in \Sigma$ all commute, so the result follows. □

Definition 3.14. For any $L \in \Sigma$, $W(L) := N_L(T_L)/T_L$ is the Weyl group of L and has order 2. We may identify $W(L) \cong N_L(T)/C_L(T)$ with its image in $W := N_G(T)/C_G(T)$, by Lemma 3.13. Now let r_L denote the single involution inside $W(L)$, and define $W_0 := \langle r_L \in W\colon L \in \Sigma\rangle$.

Lemma 3.15. *(Cf. [BB04, Lemma 3.5].) For any $L, K \in \Sigma$, $[K, L] = 1$ if and only if $[r_K, r_L] = 1$.*

Proof. It suffices to check this in $\langle K, L\rangle$. So the result follows from Fact 3.2.2 and Fact 3.4. □

We will analyze W_0 by examining its action on S° and T.

Lemma 3.16. *(Cf. [BB04, Lemmas 3.6 and 3.7].) S° is the Sylow 2-subgroup of T and $C_G(S^\circ) = C_G(T)$. In particular, W_0 acts faithfully on S°.*

Proof. By Lemma 3.13, T is divisible abelian, so its Sylow 2-subgroup is connected by [BN94, Theorem 9.29].

Let $D := S^\circ \cap T$. Suppose toward a contradiction that $D < S^\circ$. For all $K \in \Sigma$, $[S^\circ, r_K] = S^\circ \cap K$, so r_K acts trivially on S°/D. Let $b \in S^\circ$ satisfy $|b/D| \geqslant 4$ and let $a \in S^\circ$ satisfy $a^{|W_0|} = b$. Then $c = \Pi_{w \in W_0} a^w$ satisfies $b/D = c/D$ and $|c| \geqslant 4$. Since $S^\circ = C_{S^\circ}(K) * (S^\circ \cap K)$ and $|C_{S^\circ}(K) \cap (S^\circ \cap K)| \leqslant |Z(K)| = 2$, we have $[C_{S^\circ}(r_K) : C_{S^\circ}(K)] \leqslant 2$. Since $c \in C_{S^\circ}(r_K)$, $c^2 \in C_{S^\circ}(K)$ for all $K \in \Sigma$, and $c^2 \neq 1$. So $c^2 \in C_G(\langle \Sigma \rangle) = Z(G)$, contradicting the simplicity of G. Thus S° is the Sylow 2-subgroup of T, and $C_G(S^\circ) \geqslant C_G(T)$.

For the reverse direction, consider $x \in C_G(S^\circ)$. Then, for every $L \in \Sigma$, x centralizes $C_{S^\circ}(L)$. So x normalizes $L = E(C_G(C_{S^\circ}(L)))$ by Lemma 3.10.2. Since x centralizes the maximal 2-torus $S^\circ \cap L$, x must act on L as an element of T_L by Fact 1.6. Thus $x \in C_G(T)$ and $C_G(S^\circ) \leqslant C_G(T)$. □

We use the action of W_0 on S° to obtain a complex representation.

Lemma 3.17. *(Cf. [BB04, §3.3].) W_0 has a faithful irreducible representation R over $\mathbb{C}$ of dimension* $\mathrm{pr}(G) \geqslant 3$ *in which the r_L act as reflections for $L \in \Sigma$.*

For this, we employ a Tate module over the 2-adics.

Fact 3.18. *(See [Ber01], [BB04, §3.3].) Let T be a p-torus of Prüfer p-rank n in a group of finite Morley rank. Then* $\mathrm{End}(T)$ *can be faithfully represented as the ring of $n \times n$ matrices over the p-adic integers* $\mathbb{Z}_p \cong \mathrm{End}(\mathbb{Z}(p^\infty))$.

Proof of Lemma 3.17. For every $L \in \Sigma$ and every 2-torus $X \leqslant S^\circ$ disjoint from L, X must act on L as elements of $S^\circ \cap L$ by Fact 1.6, so there is some 2-torus $Y \leqslant C_{S^\circ}(L)$ with $X \leqslant Y(S^\circ \cap L)$. So $S^\circ = C_{S^\circ}(r_L) * (S^\circ \cap L)$. By Lemmas 3.10.1 and 3.6, r_L inverts $S^\circ \cap L = [S^\circ, r_L]$. Thus r_L acts as a “reflection” on S°.

By Lemma 3.16, W_0 acts faithfully on S°. By Fact 3.18, W_0 has a faithful representation over the ring of 2-adic integers $\mathbb{Z}_2$ which has dimension $\mathrm{pr}(S^\circ) \geqslant 3$. By tensoring with $\mathbb{C}$, W_0 has a faithful representation R over $\mathbb{C}$ which has dimension $\mathrm{pr}(S^\circ) \geqslant 3$. The r_L's continue to act as reflections in this representation.

Now suppose towards a contradiction that W_0 acts reducibly on R. Since the representation R is completely reducible, $R = R_1 \oplus R_2$ where R_1 and R_2 are proper W_0-invariant subspaces.

Suppose that W_0 acts trivially on R_i. Then there is a 2-torus $\hat{R}_i$ centralized by all r_L, and $G \leqslant C_G(\hat{R}_i)$, a contradiction. So we may assume that W_0 acts nontrivially on both R_1 and R_2.

For $L \in \Sigma$, the -1-eigenspace $[R, r_L]$ of r_L belongs to one of the two subspaces, either R_1 or R_2. So r_L acts as a reflection on that subspace and centralizes the other. Let $\Sigma_i := \{L \in \Sigma\colon\ [R, r_L] \leqslant R_i\}$ for $i = 1, 2$. For $L \in \Sigma_1$ and $K \in \Sigma_2$, we have $[r_L, r_K] = 1$, and thus $[L, K] = 1$ by Lemma 3.15, in contradiction with the fact that Σ is connected. □

To further constrain W_0, we next obtain representations of W_0 over almost all finite fields.

Lemma 3.19. *(Cf. [BB04, §3.4].) For primes $q >$* $\max(|W_0|, 3)$ *with $q \neq$* $\mathrm{char}(L)$ *for any $L \in \Sigma$, W_0 has a faithful irreducible representation over $\mathbb{Z}/q\mathbb{Z}$, where, for any $L \in \Sigma$, the involution r_L acts by reflection.*

Proof. Consider the elementary abelian q-group E_q generated by all elements of order q in T. W_0 clearly acts on E_q. Let $N = N_G(T)$. Since $C_G(T) \leqslant C_N(E_q)$, we may show that W_0 acts faithfully by showing that $C_N(E_q) \leqslant C_G(T)$.

For any $x \in C_N(E_q) \leqslant N$, x acts on Σ by conjugation. For any $L \in \Sigma$, if $L^x \neq L$ then L and L^x either commute or generate a quasisimple group as root SL_2-subgroups by Lemma 3.11.2. In either case, $|L \cap L^x| \leqslant 2$, in contradiction to the fact that $L \cap E_q = L^x \cap E_q$. So x normalizes L, and the element x acts on $T \cap L$ as an element of $N_L(T \cap L)$ by Fact 1.6. Since the Weyl group of SL_2 inverts the torus, any element of $N_L(T \cap L) \setminus C_L(T \cap L)$ inverts some element of E_q. So x centralizes $T \cap L$ for all $L \in \Sigma$. Now x centralizes $T = \langle T \cap L \mid L \in \Sigma \rangle$, and W_0 acts faithfully on E_q.

We also observe that W_0 acts by reflections on E_q because, for every $L \in \Sigma$, $[E_q, r_L]$ has order q and is inverted by r_L, i.e. $|E_q \cap L| = q$.

Now suppose toward a contradiction that W_0 acts reducibly on E_q. Since $q > |W_0|$, the representation is completely reducible, and $E_q = R_1 \oplus R_2$ where R_1 and R_2 are proper W_0-invariant subspaces of E_q.

Suppose that W_0 acts trivially on R_i. For any $L \in \Sigma$, R_i acts by inner automorphisms on L by Fact 1.6. We recall that nontrivial Weyl group elements in $W(L)$ invert the torus T_L. Since R_i centralizes T_L, we know that R_i acts via conjugation by elements of T_L. Since W_0 centralizes R_i, we find that R_i centralizes L. So $R_i \leqslant Z(\langle \Sigma \rangle) = Z(G)$, a contradiction. So we may assume that W_0 acts nontrivially on both R_1 and R_2.

For $L \in \Sigma$, the eigenspace $[E_q, r_L]$ of r_L belongs to one of the two subspaces, either R_1 or R_2. So r_L acts as a reflection on that subspace and centralizes the other. Let $\Sigma_i := \{L \in \Sigma\colon\ [E_q, r_L] \leqslant R_i\}$ for $i = 1, 2$. Then $\Sigma = \Sigma_1 \cup \Sigma_2$. For $L \in \Sigma_1$ and $K \in \Sigma_2$, we have $[r_L, r_K] = 1$, and thus $[L, K] = 1$ by Lemma 3.15, in contradiction with the fact that Σ is connected. □

The two preceding lemmas provide sufficient information to identify the Weyl group W_0.

Lemma 3.20. *(Cf. [BB04, Lemma 3.11].) There exists an irreducible root system I of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, or F_4 on which W_0 acts as a crystallographic reflection group.*

This lemma follows from the following major fact, which depends on a detailed analysis of the irreducible complex reflection groups [ST54,Coh76].

Fact 3.21. *(See [BBBC08, Theorem 2.3].) Let W be a finite group, $I \subseteq W$ a subset, and n an integer, satisfying the following conditions*:

(1) *The set I generates W, consists of involutions, and is closed under conjugation in W.*
(2) *The graph Δ_I with vertices I and edges (i, j) for noncommuting pairs $i, j \in I$ is connected.*
(3) *For all sufficiently large prime numbers ℓ, W has a faithful representation V_ℓ over the finite field $\mathbb{F}_\ell$ in which the elements of I operate as complex reflections, with no common fixed vectors.*

Then one of the following occurs.

(a) *W is a dihedral group acting in dimension $n = 2$, or cyclic of order two.*
(b) *W is isomorphic to an irreducible crystallographic Coxeter group, that is, A_n, B_n, C_n, D_n ($n \geqslant 3$), E_n ($n =$ 6, 7, or 8), or F_n ($n = 4$).*
(c) *W is a semidirect product of a quaternion group of order* 8 *with the symmetric group* Sym_3, *acting naturally, represented in dimension* 2.

If, in addition, over some field, W has an irreducible representation of dimension at least 3, *in which the elements of I act as reflections, then case* (b) *applies.*

Proof of Lemma 3.20. We observe that $\{r_L\colon\ L \in \Sigma\}$ is a normal subset of W_0 which generates W_0. The noncommuting graph on this set is connected by Lemma 3.15. So Lemmas 3.19 and 3.17 complete the verification of the hypotheses of Fact 3.21. □

We also show that all reflections in W_0 come from our root SL_2-subgroups.

Lemma 3.22. *(Cf. [BB04, Lemma 3.12].) Every $r \in W_0$ which is a reflection in the representation R over $\mathbb{C}$ has the form r_K for some $K \in \Sigma$.*

Recall that the reflections of a Coxeter group correspond to roots in the associated root system (see [Hum90, Lemma 5.7]), and hence there are at most two conjugacy classes of reflections.

Fact 3.23. *(See [Hum78, 10.4, Lemma C].) A finite irreducible reflection group of type A_n, D_n, E_6, E_7, or E_8 has only one conjugacy class of reflections. A finite irreducible reflection group of type B_n, C_n, F_4, and G_2 has two conjugacy classes of reflections, corresponding to the short and long roots.*

Since the roots of only one length are closed under the action of the Coxeter group, they form the root system for a proper subgroup.

Fact 3.24. *The subgroup of B_n, C_n, F_4, or G_2 generated by the reflections associated to roots of only one length is a proper subgroup.*

Proof of Lemma 3.22. By Fact 3.23, there are at most two conjugacy classes of reflections in I, corresponding to the short and long roots. So we may assume that I has more than one root length, i.e. $W_0 \cong B_n$, C_n, or F_4, and that the set $S := \{r_L\colon\ L \in \Sigma\}$ consists of only one of these conjugacy classes. By Fact 3.24, $\langle S\rangle < W_0$, a contradiction. □

3.3. Identification

We continue the analysis of the preceding subsections, loosely following [BB04, §3.6]. We will invoke the Curtis–Tits theorem which may be expressed as follows: a simply connected quasisimple algebraic group is the free amalgam of the system of subgroups and inclusion maps corresponding to all root SL_2 subgroups and subgroups generated by pairs of such subgroups, taken relative to a fixed maximal torus [GLS96]. The Generic Identification Theorem of Berkman and Borovik proceeds by passing from the full system of groups and subgroups to the collection of subsystems corresponding to pairs of roots, which are now known. A flexible form of this result is based on a result of Timmesfeld [Tim04].

Fact 3.25. *(See [BBBC08, Proposition 2.3].) Let Φ be an irreducible root system (of spherical type) and rank at least 3, and let Π be a system of fundamental roots for Φ. Let X a group generated by subgroups X_r for $r \in \Pi$. Set $X_{rs} = \langle X_r, X_s\rangle$. Suppose that X_{rs} is a group of Lie type Φ_{rs} over an infinite field, with X_r and X_s corresponding root SL_2-subgroups with respect to some maximal torus of X_{rs}. Then $X/Z(X)$ is isomorphic to a group of Lie type via a map carrying the subgroups X_r to root SL_2-subgroups.*

We now conclude the proof of the Generic Trichotomy Theorem, working, as usual, under Hypothesis 3.12. By Lemma 3.20, I is the desired irreducible root system of spherical type and rank at least 3. For every vertex $i \in I$, there is an $r_i \in W_0$ which is a reflection in the representation R over $\mathbb{C}$. There is a also reflection r_L for every $L \in \Sigma$. By Lemma 3.22, there is an $L_i \in \Sigma$ such that $r_i = r_{L_i}$, and

$$\langle L_i \mid i \in I\rangle = \langle \Sigma\rangle = G.$$

For $i, j \in I$, the group $M := \langle L_i, L_j\rangle$ is of Lie type by Lemma 3.11.2 when $[L_i, L_j] \neq 1$. If $[L_i, L_j] = 1$, then $M = L_i * L_j$ which has Lie type because L_i and L_j are algebraic over the same field. By Lemma 3.11.4, L_i and L_j are root SL_2-subgroups corresponding to a maximal torus T_M of M. Now G is a Chevalley group by Fact 3.25, as desired.

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