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Ternary derivations of finite-dimensional real division algebras

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Abstract

We present for each $n \in \{1, 2, 4, 8\}$ a group acting on the set of all division algebra structures on $\mathbb{R}^n$, and an invariant, the Lie algebra of ternary derivations, for this action. An exploration of these structures is conducted in terms of this new invariant obtaining simple descriptions of the division algebras involved. In the course of the investigation another family of algebras is considered, among them the algebra $sl(4, F)$ of 4×4 traceless matrices with the symmetric product $xy + yx - \frac{1}{2}t(xy)I$ shows an exceptional behavior.

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1. Introduction

It is commonly accepted that the symmetry of an algebraic structure is measured by its group of automorphisms. In the case of algebras, this group is approximated by the Lie algebra of derivations – the Lie algebra of the group of automorphisms in the real case. For

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instance, in [3,4,10] an investigation of finite-dimensional real division algebras was conducted under this premise. Recall that an (not necessarily associative) algebra is said to be a division algebra if for every $x \neq 0$ the left and right multiplication operators L_x, R_x are invertible. Milnor and Bott [21] and Kervaire [18] proved that finite-dimensional real division algebras only appear in dimensions 1, 2, 4 and 8. The classification of these algebras has been solved in dimensions 1 and 2 [1,6,7,8,17,26] but only partial results are known in dimensions 4 and 8.

Isomorphisms preserve all algebraic information, so when studying a family of algebras defined by some algebraic conditions, an isomorphic copy of a member in the family will remain in the family. This makes automorphisms and isomorphisms valuable tools for dealing with arbitrary families of algebras. However, sometimes automorphisms fail to detect certain hidden symmetries of algebras. The following two-parametric family will illustrate this point: given scalars α, β in a field F of characteristic 0 with $\alpha \neq 0$ or $\beta \neq 0$, define a new product on $sl(n+1, F)$ $n \geqslant 1$ by

$$x * y = \alpha xy + \beta yx - \frac{\alpha+\beta}{n+1} t(xy) I \tag{1}$$

with xy the usual product of matrices, $t(x)$ the trace of x and I the identity matrix. For any fixed $n \geqslant 2$, independently of α and β, the derivation algebra is $\{ad_a : x \mapsto ax - xa \mid a \in sl(n+1, F)\}$. Up to where automorphisms are concerned, we should declare all these algebras 'of similar interest'. However in the literature we find that the case $n = 2$ and $\alpha = -w^2\beta$, with $1 \neq w$ a cubic root of unit, corresponds to an Okubo algebra, an algebra that has many interesting properties [12,13,14,15]. This algebra admits a quadratic form permitting composition although it lacks a unit element. Having neglected this exceptional case, the study of this family using automorphisms seems misleading.

When dealing with a particular family of algebras, a larger group than the group of automorphisms might also be more natural (see [25] and references therein). The finite-dimensional real division algebras constitute probably such a family. Since many products on the same vector space will appear in this example, sometimes the notation in [17] will be convenient. An algebra will be a pair (A, P) with $P: A \times A \to A$ a bilinear map on a vector space A. The product of $x, y \in A$ is denoted by xPy. The left and right multiplication operators by x are denoted by $L_P(x)$ and $R_P(x)$ respectively. Fixed a finite-dimensional real division algebra (A, P), the following products also provide division algebras:

(i) $xP^{\mathrm{op}}y = yPx$, the opposite algebra of (A, P).
(ii) $xP^{\underline{\varphi}}y = \varphi_1^{-1}(\varphi_2(x)P\varphi_3(y))$ with $\underline{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, $\varphi_i \in \mathrm{GL}(A)$, $i = 1, 2, 3$. In the literature $P^{\underline{\varphi}}$ is called an isotope of P. We will also say that $(A, P^{\underline{\varphi}})$ is obtained by isotopy from (A, P) or that it is isotopic to (A, P).
(iii) $xP^*y = L_P(x)^*(y)$ where $L_P(x)^*$ denotes the adjoint of $L_P(x)$ with respect to some nondegenerate symmetric bilinear form on A. We will say that (A, P^*) is an adjoint of (A, P).

The transformations $P \mapsto P^{\mathrm{op}}, P^{\underline{\varphi}}, P^*$ can be thought of as elements in the group $\mathrm{GL}(\mathrm{Hom}(A \otimes A, A))$. The subgroup $\mathscr{G} = \mathscr{G}(A)$ generated by them acts on $\{P \in \mathrm{Hom}(A \otimes A, A) \mid (A, P) \text{is a division algebra}\}$, so it is probably more natural to study real division algebras under the action of this group than restricting ourselves to the action of the smaller group of automorphisms (observe that different choices of nondegenerate symmetric bilinear forms in the construction of (A, P^*) lead to isotopic algebras, which justifies our notation). Having finite-dimensional real division algebras in mind, one is concerned with the groups $\mathscr{G}_n = \mathscr{G}(\mathbb{R}^n)$ $n = 1, 2, 4, 8$. It is natural then to identify any real vector space A of dimension n with $\mathbb{R}^n$, and $\mathscr{G}(A)$ with $\mathscr{G}_n$, so that for instance we may declare two algebras to be isotopic even if the underlying vector spaces are different. The reader should be aware of this abuse of language all over the present paper.

To perform computations an invariant is needed: the Lie algebra of ternary derivations. Given an arbitrary algebra A over a field F, a ternary derivation of A is a triple $(d_1, d_2, d_3) \in \mathrm{End}_F(A)^3$ which satisfies $d_1(xy) = d_2(x)y + xd_3(y)$ for any $x, y \in A$. Ternary derivations form a Lie algebra, denoted $\mathrm{Tder}(A)$, under $[(d_1, d_2, d_3), (d_1', d_2', d_3')] = ([d_1, d_1'], [d_2, d_2'], [d_3, d_3'])$. The orbit of (A, P) under the action $\mathscr{G}(A)$ is compatible with the Lie algebra of ternary derivations in the following sense: given $(d_1, d_2, d_3) \in \mathrm{Tder}((A, P))$ then

$$
\begin{aligned}
&\text{(i) } (d_1, d_3, d_2) \in \mathrm{Tder}((A, P^{\mathrm{op}}))\\
&\text{(ii) } (\varphi_1^{-1} d_1 \varphi_1, \varphi_2^{-1} d_2 \varphi_2, \varphi_3^{-1} d_3 \varphi_3) \in \mathrm{Tder}((A, P^{\underline{\varphi}}))\\
&\text{(iii) } (-d_3^*, d_2, -d_1^*) \in \mathrm{Tder}((A, P^*)).
\end{aligned}
\qquad (*)
$$

This shows that for any $\sigma \in \mathscr{G}(A)$, the Lie algebra $\mathrm{Tder}((A, P))$ is isomorphic to $\mathrm{Tder}((A, P^\sigma))$, making the Lie algebra of ternary derivations a natural tool to study real division algebras.

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are thought to be up to some extent[3] the more symmetric finite-dimensional real division algebras. There are however other real division algebras sharing the same group of automorphisms that are not isomorphic to them [10]. The maximum dimension of the algebra of ternary derivations of a real division algebra of dimension 1, 2, 4 or 8 is 2, 4, 11 or 30 respectively. In Section 2.4 we will see that these dimensions are only reached by algebras isomorphic to those in the orbits of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively. Thus, ternary derivations succeed in presenting these algebras as the more symmetric real division algebras. In [4] another real division algebra is however isolated. This algebra is constructed from $su(3)$, the 3×3 skew-Hermitian traceless complex matrices by replacing the matrix product xy by $x * y = \alpha(xy - yx) \pm \sqrt{3}\alpha \mathrm{i}\{xy + yx - \frac{2}{3}t(xy)I\}$. It happens to be a real form of the Okubo algebra over $\mathbb{C}$, so its Lie algebra of derivations is isomorphic to $su(3)$, far away from the Lie algebra of derivations of $\mathbb{O}$ which is a compact simple Lie algebra of type G_2. What is the place deserved for this algebra when classifying by ternary derivations? The Lie algebra of ternary derivations of this exceptional algebra is isomorphic to that of $\mathbb{O}$, of maximum dimension, so up to isomorphism they belong to the same orbit under the action of $\mathscr{G}_8$.

Ternary derivations detect some kind of hidden symmetry, for instance that of the octonions and the Okubo algebra, not recognized by usual derivations. But, how can an eight-dimensional algebra as the octonions have a 30-dimensional algebra of ternary derivations? The answer is that ternary derivations are tailored to be compatible with alternative algebras. The generalized

[3] The connected component of the identity of the group of automorphisms of each of these finite-dimensional real division algebras has the maximum possible dimension. However, other algebras with larger group of automorphisms exist. As the referee kindly pointed out to the authors, there is a (unique, up to isomorphism) two-dimensional real division algebra with automorphism group dihedral of order 6, and the property of having an automorphism group of order 2 is shared by a 2-parameter family of pairwise non-isomorphic two-dimensional real division algebras that includes $\mathbb{C}$ (see [8]).

alternative nucleus $\mathrm{N_{alt}}(A)$ of an algebra A is defined as $\mathrm{N_{alt}}(A)=\{a \in A \mid (a, x, y) = -(x, a, y) = (x, y, a)\, \forall x, y \in A\}$. Alternative algebras are those algebras, as the octonions, for which $\mathrm{N_{alt}}(A) = A$. The defining conditions of a being in $\mathrm{N_{alt}}(A)$ are equivalent to $(L_a, T_a, -L_a), (R_a, -R_a, T_a) \in \mathrm{Tder}(A)$ with $T_a = L_a + R_a$ the sum of the left and right multiplication operators by a. Thus, a large $\mathrm{N_{alt}}(A)$ leads to a large $\mathrm{Tder}(A)$, as for instance in the case of the octonions. It is worth mentioning that, for any nonassociative algebra A, $\mathrm{N_{alt}}(A)$ is always a Malcev algebra with the commutator product, and conversely, any Malcev algebra over a field of characteristic different from 2 or 3 is a Malcev subalgebra of $\mathrm{N_{alt}}(A)$ for an adequate A [22,24]. There are some open problems in the theory of Malcev algebras concerning the existence or not of Malcev algebras satisfying certain properties [9, Part 1. Problem 81]. Finding algebras with large Lie algebra of ternary derivations when investigating families of nonassociative algebras might eventually lead to new examples of Malcev algebras. Ternary derivations might fail to detect many kinds of symmetry in nonassociative structures since they are very much oriented towards alternative algebras. However, they constitute a generic tool well suited to deal with certain nonassociative phenomena even after deep alterations as those performed by the group $\mathscr{G}$.

The goal of this paper is to explore the family of finite-dimensional real division algebras using ternary derivations. Our methods rely on the representation theory of simple Lie algebras so we can only consider real division algebras with non-abelian Lie algebra of ternary derivations. The family of eight-dimensional real division algebras is rather complex however, so in that case we have to restrict ourselves to those algebras whose Lie algebra of ternary derivations has a simple subalgebra of toral rank at least two. A similar exploration under a weaker assumption such as the existence of a simple subalgebra of ternary derivations of toral rank one seems to be too complex with the present methods.

The paper is organized as follows. In Section 2 we prove that the toral rank of $\mathrm{Tder}(A)$ for a finite-dimensional real division algebra A is bounded from above by 2, 4, 5 or 6, and the dimension by 2, 4, 11 or 30 respectively depending on the dimension 1, 2, 4 or 8 of A. These maximal dimensions are reached in the orbits of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively (Proposition 9). Real division algebras of dimension 1 and 2 are isotopes of $\mathbb{R}$ and $\mathbb{C}$, so the whole theory becomes trivial in these cases. In Section 3 we deal with division algebras of dimension four. Theorem 10 shows that two types of algebras model this case. Section 4 is devoted to division algebras of dimension eight, where four types of algebras appear to be natural models (Theorem 11). All the algebras provided are division algebras indeed. In Section 5 we explore the family of nonassociative algebras given by (1). Not only the Okubo algebra shows up an exceptional behavior but also the algebra constructed from $sl(4, F)$ with commutative product $xy + yx - \frac{1}{2}t(xy)I$ has an unusual large algebra of ternary derivations (Proposition 17).

The traceless quaternions and octonions will be denoted by $\mathbb{H}_0$ and $\mathbb{O}_0$ respectively, $x \in \mathbb{O}$ decomposes as $x = \frac{1}{2}t(x) + x_0$ with $x_0 \in \mathbb{O}_0$, $t(x) = x + \bar{x}$ the trace of x and $x \mapsto \bar{x}$ the usual involution on $\mathbb{O}$. $\mathbb{C}$ will be identified with $\mathrm{span}\langle 1, i\rangle \subseteq \mathbb{O}$ and $\mathbb{H}$ with $\mathrm{span}\langle 1, i, j, ij\rangle \subseteq \mathbb{O}$. Although in general we will try to avoid it, sometimes we will use the notation xPy for the product of A. When not using this notation, the opposite and adjoint of A will be denoted by A^{op} and A^* respectively. Although in some places of the paper A^* will also stand for the dual vector space of A, however the meaning will be clear from the context. When a particular symbol such as $\circ$ denotes the product on A, we will try to reflect it on the multiplication operators by writing L_a° (resp. R_a°) instead of L_a (resp. R_a). Decompositions of tensor product of modules have been obtained from [5].

2. Ternary derivations

In this section A will denote a finite-dimensional real division algebra. It will be convenient to have in mind that this implies that $xA = A = Ax$ and $\dim xS = \dim S = \dim Sx$ for any $0 \neq x \in A$ and any subspace S. Also recall that for unital algebras the components of any ternary derivation (d_1, d_2, d_3) are related by

$$d_1 = d_2 + R_{d_3(1)} \quad \text{and} \quad d_1 = d_3 + L_{d_2(1)}. \tag{2}$$

The ternary derivations of any associative unital algebra B are $\{(d, d, d)|d \in \mathrm{Der}(B)\} + \mathrm{span}\langle(L_b, L_b, 0), (R_b, 0, R_b)|b \in B\rangle$. In particular, $\mathrm{Tder}(\mathbb{R})$ is a two-dimensional abelian Lie algebra, $\mathrm{Tder}(\mathbb{C})$ is a four-dimensional abelian Lie algebra and $\mathrm{Tder}(\mathbb{H}) \cong su(2) \oplus su(2) \oplus su(2) \oplus \mathrm{span}\langle(\mathrm{id}, \mathrm{id}, 0), (0, \mathrm{id}, \mathrm{id})\rangle$ with $su(2)$ the compact form of $sl(2, \mathbb{C})$, which is isomorphic to the traceless quaternions with the commutator product. The ternary derivations of any alternative unital algebra B over a field of characteristic $\neq 3$ are $(\{d, d, d)|d \in \mathrm{Der}(B)\} + \mathrm{span}\langle(L_b, T_b, -L_b), (R_b, -R_b, T_b)|b \in B\rangle$. In the case of the octonions this gives $\mathrm{Tder}(\mathbb{O}) \cong D_4 \oplus \mathrm{span}\langle(\mathrm{id}, \mathrm{id}, 0), (\mathrm{id}, 0, \mathrm{id})\rangle$ with D_4 the compact Lie algebra of type D_4 [20].

From any finite-dimensional division algebra we can obtain a unital isotope by

$$x \circ y = R_v^{-1}(x)L_u^{-1}(y), \tag{3}$$

the unit being uv. Since, up to isomorphism, the only unital real division algebras of dimension 1 or 2 are $\mathbb{R}$ or $\mathbb{C}$, then any real division algebra of dimension 1 or 2 is an isotope of $\mathbb{R}$ or $\mathbb{C}$. Its algebra of ternary derivations is an abelian Lie algebra of dimension 2 or 4.

2.1. *The structure of* $\mathrm{Tder}(A)$

Compare the following results with [3].

Proposition 1. *Any element in* $\mathrm{Tder}(A)$ *is semisimple.*

Proof. Since $\mathrm{Tder}(A)$ contains the semisimple and nilpotent parts of any of its elements, we have to prove that $(0, 0, 0)$ is the only nilpotent ternary derivation of A.

Let $(d_1, d_2, d_3) \in \mathrm{Tder}(A)$ with $d_i^{n_i} = 0$ and $d_i^{n_i-1} \neq 0$ for some $n_i \geqslant 1$ $(i = 1, 2, 3)$. In case that $n_2 = 1$ then for any $0 \neq y \in \ker(d_3)$ (recall that d_i are nilpotent) and any $x \in A$, $d_1(xy) = xd_3(y) = 0$ so $d_1 = 0$ and consequently $(d_1, d_2, d_3) = (0, 0, 0)$. The same argument applies when $n_3 = 1$. So we can assume that $n_2, n_3 \geqslant 2$. In this case,

$$d_1^{n_2+n_3-2}(xy) = \binom{n_2 + n_3 - 2}{n_2 - 1} d_2^{n_2-1}(x)d_3^{n_3-1}(y)$$

implies that $A = Ay \subseteq \ker(d_1^{n_2+n_3-2})$ for any $0 \neq y \in \ker(d_3^{n_3-1})$ so $d_1^{n_2+n_3-2} = 0$. Therefore, $0 = d_2^{n_2-1}(x)d_3^{n_3-1}(y)$ for any $x, y \in A$. Choosing $x \notin \ker(d_2^{n_2-1})$ and $y \notin \ker(d_3^{n_3-1})$ we obtain a contradiction. $\square$

Corollary 2. *Either* $\mathrm{Tder}(A)$ *is abelian or* $\mathrm{Tder}(A) = \mathrm{Tder}(A)' \oplus \mathscr{Z}$ *with* $\mathrm{Tder}(A)' = [\mathrm{Tder}(A), \mathrm{Tder}(A)]$ *a compact semisimple Lie algebra and* $\mathscr{Z}$ *the center of* $\mathrm{Tder}(A)$.

The space $\mathscr{Z}_0 = \mathrm{span}\langle(\mathrm{id}, \mathrm{id}, 0), (\mathrm{id}, 0, \mathrm{id})\rangle$ always lies inside the center of $\mathrm{Tder}(A)$. We will retain this notation in the future.

2.2. Triality

The projection of $\mathrm{Tder}(A)$ onto the ith component

$$\pi_i: \mathrm{Tder}(A) \to \mathrm{End}(A),$$
$$(d_1, d_2, d_3) \mapsto d_i$$

is a representation of $\mathrm{Tder}(A)$. A then becomes a $\mathrm{Tder}(A)$-module in three probably nonisomorphic ways A_1, A_2 and A_3. The product on A is a homomorphism

$$A_2 \otimes A_3 \to A_1,$$
$$x \otimes y \mapsto xy$$

of $\mathrm{Tder}(A)$-modules. The reader should compare this situation with the local principle of triality on $\mathbb{O}$, where these modules correspond to the natural and spin representations of D_4 [19].

Fix a subalgebra $S \leqslant \mathrm{Tder}(A)$ and assume that $A_2 \cong A_1 \cong A_3$ as S-modules, then fixed isomorphisms $\varphi_2: A_1 \to A_2$ and $\varphi_3: A_1 \to A_3$, we have $\varphi_2(d_1(x)) = d_2(\varphi_2(x))$ and $\varphi_3(d_1(x)) = d_3(\varphi_3(x))$ for any $(d_1, d_2, d_3) \in S$. By $(*)$, we can change A by $(A, \circ)$ with $x \circ y = \varphi_2(x)\varphi_3(y)$ where an isomorphic copy of S acts as derivations. Since derivations of real division algebras are well known, this trick will be useful.

To use concrete models for S, A_1, A_2 and A_3, these manipulations must be pushed further. Again fix a subalgebra $S \leqslant \mathrm{Tder}(A)$. Given a vector space B with three structures of S-module, say B'_1, B'_2 and B'_3, corresponding representations ρ_1, ρ_2 and ρ_3, and isomorphisms $\varphi_i: B'_i \to A_i$ as S-modules, we can define a product on B by $x \circ y = \varphi_1^{-1}(\varphi_2(x)\varphi_3(y))$. S is isomorphic to the subalgebra $\{(\rho_1(\underline{d}), \rho_2(\underline{d}), \rho_3(\underline{d})) | \underline{d} \in S\}$ of $\mathrm{Tder}(B, \circ)$ and the three structures of module induced on the algebra B by the projections of this subalgebra are exactly $B_1 = B'_1$, $B_2 = B'_2$ and $B_3 = B'_3$. Thus, up to isotopy we can identify A_i with B_i and S with $\{(\rho_1(\underline{d}), \rho_2(\underline{d}), \rho_3(\underline{d})) | \underline{d} \in S\}$.

We will base our study on the existence of large subalgebras $S \subseteq \mathrm{Tder}(A)$ and the possible decompositions of A_1, A_2 and A_3 as S-modules. In order to reduce the number of possibilities we will invoke a permutation argument that establishes that the role played by A_1, A_2 and A_3 can be permuted by considering another algebra in the orbit of A under the action of the corresponding $\mathscr{G}$. For instance, S is isomorphic to $\{(d_1, d_3, d_2) | (d_1, d_2, d_3) \in S\}$, a subalgebra of A^{op}. Hence, a subalgebra isomorphic to S acts as ternary derivations on A^{op} and the three structures of module for this subalgebra induced on A^{op} are $A_1^{\mathrm{op}} = A_1$, $A_2^{\mathrm{op}} = A_3$ and $A_3^{\mathrm{op}} = A_2$. To switch the role of A_1 and A_3 we assume that S is semisimple and consider any nondegenerate symmetric bilinear form $(,)$ on A. S is isomorphic to $\{(-d_1^*, d_2, -d_3^*) | (d_1, d_2, d_3) \in S\}$, a subalgebra of $\mathrm{Tder}(A^*)$. The three structures of module for this subalgebra induced on A^* are $A_1^* \cong (A_3)^*$ the dual of A_3, $A_2^* = A_2$ and $A_3^* \cong (A_1)^*$ the dual of A_1. Since any finite-dimensional module for a compact semisimple Lie algebra is selfdual [23] then, up to isotopy, $A_1^* = A_3$, $A_2^* = A_2$ and $A_3^* = A_1$.

The kernel of the projections π_i is also relevant. The kernel of, lets say π_1, is $\{(0, d_2, d_3) \in \mathrm{Tder}(A)\}$. The isomorphism induced on ternary derivations when moving from (A, P) to an isotope $(A, P^{\underline{\varphi}})$ with $\underline{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ sends $(0, d_2, d_3)$ to $(0, \varphi_2^{-1} d_2 \varphi_2, \varphi_3^{-1} d_3 \varphi_3)$, so the kernel of the first projection of $\mathrm{Tder}((A, P^{\underline{\varphi}}))$ is isomorphic to that of the first projection of $\mathrm{Tder}((A, P))$. Therefore, by (3) in order to study this kernel we may assume that A is unital. By (2), $(0, d_2, d_3) \in \mathrm{Tder}(A)$ if and only if $(0, d_2, d_3) = (0, R_a, -L_a)$ with $a \in \mathrm{N}_m(A)$ the middle associative nucleus of A, certain associative subalgebra of A. $\ker \pi_1$ is then isomorphic to either $\mathbb{R}^-$, $\mathbb{C}^-$ or $\mathbb{H}^-$, where D^- stands for the Lie algebra obtained from D when the product is taken to be $[x, y] = xy - yx$. In particular, A_i is always a faithful module for any semisimple subalgebra of $\mathrm{Tder}(A)$ not containing an ideal isomorphic to $su(2)$. If $\dim A = 4$ and $\mathrm{Tder}(A)' \cap \ker \pi_i \neq 0$ then $\ker \pi_i \cong su(2)$ and A is isotopic to $\mathbb{H}$.

The following result is not used in the paper, however we incorporate it since to the knowledge of the authors, it is not known whether there exist eight-dimensional real division algebras with some (left, middle or right) associative nucleus isomorphic to $\mathbb{H}$. This result suggests a negative answer.

Proposition 3. *There are no eight-dimensional real division algebras with two associative nuclei (left, middle or right) isomorphic to* $\mathbb{H}$.

Proof. Let $A = (A, P)$ be an eight-dimensional real division algebra with two associative nuclei isomorphic to $\mathbb{H}$. Moving to an isotope we may assume that A is unital. Moving to (A, P^{op}) if necessary we also may assume that $\mathrm{N}_r(A) \cong \mathbb{H}$. Finally, moving to (A, P^*) and to an isotope again if necessary we assume that $\mathrm{N}_l(A) \cong \mathbb{H}$ and that A is unital.

A is a left $\mathrm{N}_l(A)$-module and a right $\mathrm{N}_r(A)^{\mathrm{op}}$-module. In fact, it is a bimodule, so it must decompose as $A = Q_1 \oplus Q_2$ for some irreducible four-dimensional sub-bimodules Q_1 and Q_2. By dimensions, for any nonzero elements $x_1 \in Q_1$ and $x_2 \in Q_2$, $Q_i = \mathrm{N}_l(A)x_i = x_i\mathrm{N}_r(A)$. Thus, there exist isomorphisms $\sigma_1, \sigma_2 : \mathrm{N}_l(A) \to \mathrm{N}_r(A)$, as algebras indeed, verifying that $ax_1 = x_1\sigma_1(a)$ and $ax_2 = x_2\sigma_2(a)$ for any $a \in \mathrm{N}_l(A)$. The map $\sigma_2^{-1}\sigma_1$ is an automorphism of $\mathrm{N}_l(A)$, so it is inner. Let $e \in \mathrm{N}_l(A)$ with $\sigma_2^{-1}\sigma_1(a) = eae^{-1}$. Thus $\sigma_1(a)\sigma_2(e) = \sigma_2(e)\sigma_2(a)$. Choosing $x_1\sigma_2(e)$ instead of x_1 allows us to assume that $\sigma_1 = \sigma_2$. That is, there exists an isomorphism $\sigma : \mathrm{N}_l(A) \to \mathrm{N}_r(A)$ such that $ax_i = x_i\sigma(a)$ for any $a \in \mathrm{N}_l(A)$ $(i = 1, 2)$.

Consider now $u, v \in A$ with $x_1 = uv$. The algebra $(A, \circ)$ with $x \circ y = R_v^{-1}(x)L_u^{-1}(y)$ is unital, with unit x_1, and Q_1 becomes its left associative nucleus as well as its right associative nucleus. Therefore, we may assume that $\mathrm{N}_l(A) = Q_1 = \mathrm{N}_r(A)$ and $x_1 = 1$.

The map σ becomes an automorphism of $\mathrm{N}_l(A)$ so it is inner. Let $e \in \mathrm{N}_l(A)$ be such that $\sigma(a) = eae^{-1}$ for any $a \in \mathrm{N}_l(A)$. So, $ax_2 = x_2eae^{-1}$. Choosing x_2e instead of x_2 we may assume that $\sigma = \mathrm{id}$.

After all these manipulations, we end up with an eight-dimensional real division algebra A with a subalgebra $Q = \mathrm{N}_l(A) = \mathrm{N}_r(A)$ isomorphic to $\mathbb{H}$ and such that A decomposes as $A = Q \oplus vQ$ with $Qv = vQ$ and $av = va$ for any $a \in Q$.

Let $v^2 = a + vb$ with $a, b \in Q$ (notice that $a = 0$ implies that $v(v - b) = 0$ a contradiction, so $a \neq 0$). The quadratic quaternionic equation $z^2 + zb - a = 0$ has always nonzero solutions in Q [16]. Let a' be such a root and $b' = -(a')^{-1}a$. We have

$$(a' + v)(b' + v) = (-a + a) + (a' + b' + b)v = (a')^{-1}((a')^2 + a'b - a)v = 0,$$

so A is not a division algebra. $\square$

2.3. *Upper bounds to the toral rank of* $\mathrm{Tder}(A)$

In this section T will denote a toral subalgebra of $\mathrm{Tder}(A)$ containing the ternary derivations $(\mathrm{id}, \mathrm{id}, 0)$ and $(\mathrm{id}, 0, \mathrm{id})$.

As learnt in Section 2.2, the kernel of the projection on the second component of $\mathrm{Tder}(A)$ is isomorphic to either $\mathbb{R}^-$, $\mathbb{C}^-$ or $\mathbb{H}^-$. The dimension of any subspace of commuting elements in these algebras is at most 2, so $\dim T \cap \ker(\pi_2) \leqslant 2$. Defining

$$T^2 = \{d_2 \in \mathrm{End}(A) | \exists d_1, d_3 \in \mathrm{End}(A) \text{ such that } (d_1, d_2, d_3) \in T\},$$

we get that

$$\dim T \leqslant \dim T^2 + 2.$$

Since T^2 consists of commuting semisimple linear maps then we can find a basis of A such that the coordinate matrix of any element d in T^2 has the form

$$\begin{pmatrix} \alpha_1(d) & & & & & & & & \\ & \ddots & & & & & & & \\ & & \alpha_r(d) & & & & & & \\ & & & \beta_1(d) & \lambda_1(d) & & & & \\ & & & -\lambda_1(d) & \beta_1(d) & & & & \\ & & & & & \ddots & & & \\ & & & & & & \beta_s(d) & \lambda_s(d) \\ & & & & & & -\lambda_s(d) & \beta_s(d) \end{pmatrix} \tag{4}$$

for linear maps $\alpha_1, \ldots, \alpha_r,\ \beta_1, \ldots, \beta_s,\ \lambda_1, \ldots, \lambda_s : T^2 \to \mathbb{R}$ with $\lambda_1, \ldots, \lambda_s \neq 0$. Let $\{v_1, \ldots, v_r, w'_1, w''_1, \ldots, w'_s, w''_s\}$ be such a basis.

Lemma 4. *There exists* $0 \neq e \in A$ *such that the subspace* $T_0^2 = \{d_2 \in T^2 | d_2(e) = 0\}$ *satisfies*

$$\dim T \leqslant \dim T_0^2 + 4.$$

Proof. We distinguish two cases depending on r. In case that $r > 0$, then the dimension of the kernel of α_1 is at least $\dim T^2 - 1$ and any $d_2 \in \ker(\alpha_1)$ kills v_1. Defining $e = v_1$ we obtain that $\dim T \leqslant \dim T^2 + 2 \leqslant \dim T_0^2 + 3$. In case that $r = 0$, then we focus on $\ker(\beta_1)$ and $\ker(\lambda_1)$. The dimension of $\ker(\beta_1) \cap \ker(\lambda_1)$ is at least $\dim T^2 - 2$ and any of its elements kills w'_1 (and w''_1 in fact). With $e = w'_1$ we obtain the result. $\square$

2.3.1. Making e the new unit element

Without loss of generality we may assume that A is unital. Consider e and T_0^2 as in Lemma 4 and define

$$x \circ y = x L_e^{-1}(y).$$

The element e becomes the unit of $(A, \circ)$. The map

$$\begin{aligned} \psi &: \mathrm{Tder}(A) \to \mathrm{Tder}(A, \circ), \\ &(d_1, d_2, d_3) \mapsto (d_1, d_2, L_e d_3 L_e^{-1}) \end{aligned}$$

is an isomorphism of Lie algebras. The torus T corresponds to $\psi(T)$; the projection onto the second component T_2 of T equals the projection onto the second component of $\psi(T)$, denoted by $\psi(T)^2$ for coherence, and $T_0^2 = \{d_2 \in T^2 | d_2(e) = 0\} = \psi(T)_0^2$. Since our goal is to bound the dimension of T_0^2 then there is no loss of generality in the following

Assumption. The element e in Lemma 4 is the unit of A.

The main advantage of our assumption about e is that for any $d_2 \in T_0^2$ and d_1, d_3 with $(d_1, d_2, d_3) \in T$ Eq. (2) implies that $d_1 = d_3$.

2.3.2. Bounds to the toral rank of $\mathrm{Tder}(A)$

We will fix a basis $\{v_1, \dots, v_r, w'_1, w''_1, \dots, w'_s, w''_s\}$ so that the coordinate matrix of the elements of T share the canonical form (4).

Proposition 5. *We have that*

(i) $\dim A = 4$ *implies* $\dim T \leqslant 5$,
(ii) $\dim A = 8$ *implies* $\dim T \leqslant 7$.

Proof. The statement will follow once we had proved that the restriction of $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$ to T_0^2 vanishes. In that case, the map $T_0^2 \to \mathbb{R}^s$ $d \mapsto (\lambda_1(d), \dots, \lambda_s(d))$ is injective, so $\dim T_0^2 \leqslant s \leqslant 2$ ($\dim A = 4$) or $\dim T_0^2 \leqslant s \leqslant 4$ ($\dim A = 8$). The possibilities $\dim T_0^2 = 2$ or $\dim T_0^2 = 4$ implies that there exist elements in T_0^2 with no real eigenvalues, which is false (e belongs to the kernel of any element in T_0^2). The bound follows from Lemma 4.

Let us prove that $\alpha_1(d_2) = \cdots = \alpha_r(d_2) = 0$ for any $d_2 \in T_0^2$. On the contrary, consider $d_2 \in T_0^2$ with $\alpha_1(d_2) \neq 0$. Take d_1 with $(d_1, d_2, d_1) \in \mathrm{Tder}(A)$. Since $d_2 \neq 0$ then $d_1 \neq 0$ and we can choose an eigenvector $x \in \widehat{A} = \mathbb{C} \otimes_{\mathbb{R}} A$ of d_1 (we use the same notation for the natural extension of d_1 to the algebra $\widehat{A}$) with a nonzero eigenvalue $\lambda \in \mathbb{C}$. In $\widehat{A}$ we have

$$d_1(v_1 x) = d_2(v_1)x + v_1 d_1(x) = (\alpha_1(d_2) + \lambda)v_1 x.$$

Since the multiplication by v_1 is bijective then $v_1 x$ is an eigenvector of d_1 with eigenvalue $\alpha_1(d_2) + \lambda$. Iterating, we get that $m\alpha_1(d_2) + \lambda$ is an eigenvalue of d_1 for any natural m, which is not possible. Therefore, $\alpha_1(d_2) = 0$ for any $d_2 \in T_0^2$. The same argument works for $\alpha_2, \dots, \alpha_r$.

Finally, let us prove that $\beta_1(d_2) = \cdots = \beta_s(d_2) = 0$ for any $d_2 \in T_0^2$. Again, on the contrary we assume that there exists $d_2 \in T_0^2$ with $\beta_1(d_2) \neq 0$, and consider $0 \neq d_1$ with $(d_1, d_2, d_1) \in \mathrm{Tder}(A)$. The elements $w'_1 \pm \mathrm{i}w''_1 \in \widehat{A}$ are eigenvectors of d_2 of eigenvalues $\beta_1(d_2) \pm \mathrm{i}\lambda_1(d_2)$ respectively. Given an eigenvector x of d_1 with nonzero eigenvalue $\lambda \in \mathbb{C}$ then $d_1((w'_1 \pm \mathrm{i}w''_1)x) = (\beta_1(d_2) \pm \mathrm{i}\lambda_1(d_2) + \lambda)(w'_1 \pm \mathrm{i}w''_1)x$. Thus

$$0 \neq w'_1 x \in \widehat{A}_{\beta_1(d_2)+i\lambda_1(d_2)+\lambda} + \widehat{A}_{\beta_1(d_2)-i\lambda_1(d_2)+\lambda},$$

where $\widehat{A}_\zeta$ denotes the eigenspace of d_1 of eigenvalue ζ. Iterating, we get that for any natural m there exists k such that $\widehat{A}_{m\beta_1(d_2)+ki\lambda_1(d_2)+\lambda} \neq 0$, which is not possible. Therefore, $\beta_1(d_2) = 0$ for any $d_2 \in T_0^2$. The same argument works for $\beta_2, \dots, \beta_s$. $\square$

Theorem 6. *We have that*

(i) $\dim A = 4$ *implies that the toral rank of* $\mathrm{Tder}(A)$ *is at most* 5,
(ii) $\dim A = 8$ *implies that the toral rank of* $\mathrm{Tder}(A)$ *is at most* 6.

Proof. On the contrary, let us assume that the dimension of A is 8 and the toral rank of $\mathrm{Tder}(A)$ is 7. Then all the inequalities in the proof of Proposition 5 must be equalities. Therefore,

(1) $\dim T = \dim T^2 + 2$ and $\dim T \cap \ker(\pi_2) = 2$,
(2) $\dim T^2 = \dim T_0^2 + 2$ and
(3) $\dim T_0^2 = 3$.

Item (3) and the proof of Proposition 5 tell us that we can get a basis of A where the coordinate matrix of any $d \in T_0^2$ is

$$\begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 & \lambda_1(d) &&&& \\ && -\lambda_1(d) & 0 &&&& \\ &&&& 0 & \lambda_2(d) && \\ &&&& -\lambda_2(d) & 0 && \\ &&&&&& 0 & \lambda_3(d) \\ &&&&&& -\lambda_3(d) & 0 \end{pmatrix}$$

and the map $T_0^2 \to \mathbb{R}^3$ given by $d \mapsto (\lambda_1(d), \lambda_2(d), \lambda_3(d))$ is an isomorphism. Thus we can choose $d \in T_0^2$ with $\dim \ker(d) = 6$. We will show in some steps that this is not possible.

1. Let $0 \neq d'$ be such that $(d', d, d') \in T$. For any $\mu \in \mathbb{C}$ consider $A_\mu = \{v \in A | (d' - \mu\mathrm{id})(d' - \bar{\mu}\mathrm{id})(v) = 0\}$ where $\bar{\mu}$ is the complex conjugate of μ, so $A = \oplus_{\mu\in\mathbb{C}} A_\mu$. Now $d'(xy) = xd'(y)$ for any $x \in \ker(d)$ implies that $\ker(d)A_\mu \subset A_\mu$, thus $\dim A_\mu \geqslant 6$ for any $A_\mu \neq 0$. Counting dimensions we get that $A = A_\mu$ for some $\mu \neq 0$. By subtracting a real multiple of $(\mathrm{id}, 0, \mathrm{id})$ from (d', d, d') we can assume that $\mu \in \mathbb{R}i$, and dividing (d', d, d') by the norm of μ we can also assume that $\mu = i$, so $(d')^2 = -\mathrm{id}$.

2. $\ker(\pi_2) \cap T = \{(R_a, 0, R_a) \in T | a \in \mathrm{N}_r(A)\}$ and $\mathrm{N}_r(A) \cong \mathbb{C}$ or $\mathbb{H}$. Up to scalar multiples, subtracting a real multiple of $(\mathrm{id}, 0, \mathrm{id})$ from $(R_a, 0, R_a)$, by item (1) we can choose $(R_a, 0, R_a) \in T$ with $R_a^2 = -\mathrm{id}$.

3. Both (d', d, d') and $(R_a, 0, R_a)$ belong to the torus T, so $[d', R_a] = 0$. In particular, $(d' - R_a)(d' + R_a) = (d')^2 - R_a^2 = \mathrm{id} - \mathrm{id} = 0$. Up to change of a by $-a$, we can choose $0 \neq v \in A$ killed by $d'' = d' + R_a$. The map d'' shares in common with d' that $(d'', d, d'') \in T$ (so $d'' \neq 0$) but $\ker(d'') \neq 0$. Starting the first step with d'' instead of d' we get a contradiction. □

The ternary derivations $(\mathrm{id}, \mathrm{id}, 0)$ and $(\mathrm{id}, 0, \mathrm{id})$ always belong to any maximal toral subalgebra.

Corollary 7. *The toral rank of* $\mathrm{Tder}(A)' = [\mathrm{Tder}(A), \mathrm{Tder}(A)]$ *is at most* 3 $(\dim A = 4)$ *or* 4 $(\dim A = 8)$.

2.4. Algebras with maximal dimension on ternary derivations

There is no loss of generality in assuming that A is unital. To bound the dimension of $\mathrm{Tder}(A)$ let us define the map

$$\begin{aligned} \epsilon \colon \mathrm{Tder}(A) &\to A \times A, \\ (d_1, d_2, d_3) &\mapsto (d_2(1), d_3(1)), \end{aligned} \tag{5}$$

whose kernel $\ker(\epsilon) = \{(d_1, d_2, d_3) \in \mathrm{Tder}(A) | d_1 = d_2 = d_3\}$ is isomorphic to $\mathrm{Der}(A)$ the algebra of derivations of A.

Proposition 8. *We have that*

(1) $\dim A = 1$ *implies that* $\dim \mathrm{Tder}(A) = 2$,
(2) $\dim A = 2$ *implies that* $\dim \mathrm{Tder}(A) = 4$,
(3) $\dim A = 4$ *implies that* $\dim \mathrm{Tder}(A) \leqslant 11$,
(4) $\dim A = 8$ *implies that* $\dim \mathrm{Tder}(A) \leqslant 30$.

Proof. Clearly $\dim \mathrm{Tder}(A) \leqslant \dim(A \times A) + \dim \ker(\epsilon) = 2\dim A + \dim \mathrm{Der}(A)$. The bounds follow from the corresponding bounds for $\dim \mathrm{Der}(A)$ in [3]. □

Recall [4] that in any dimension $\equiv 0, 1, 3 \bmod 4$ there exists one, up to isomorphism, irreducible $su(2)$-module. That of dimension $2m+1$, $W(2m)$, is absolutely irreducible and $\mathbb{C} \otimes_{\mathbb{R}} W(2m) \cong V(2m)$. The one $V(2n-1)$ in dimension $4n$ – the complex $sl(2,\mathbb{C})$-module $V(2n-1)$ seen as a real $su(2)$-module – satisfies $\mathbb{C} \otimes_{\mathbb{R}} V(2n-1) \cong V(2n-1) \oplus V(2n-1)$ so $\mathrm{End}_{su(2)}(V(2n-1)) \cong \mathbb{H}$. Any faithful four-dimensional $su(2)$-module is isomorphic to either $W(0) \oplus W(2)$ or $V(1)$. Both cases are easily identified since $\mathrm{End}_{su(2)}(W(0) \oplus W(2)) \cong \mathbb{R} \oplus \mathbb{R}$ and $\mathrm{End}_{su(2)}(V(1)) \cong \mathbb{H}$.

Proposition 9. *We have that*

(i) *If* $\dim A = 4$ *then* $\dim \mathrm{Tder}(A) = 11$ *if and only if A is isotopic to the quaternions* $\mathbb{H}$.
(ii) *If* $\dim A = 8$ *then* $\dim \mathrm{Tder}(A) = 30$ *if and only if A is isotopic to the octonions* $\mathbb{O}$.

Proof. Let us assume that $\dim A = 4$ and $\dim \mathrm{Tder}(A) = 11$. Using this dimension, the bound on the toral rank of $\mathrm{Tder}(A)$ and the possible dimensions of the semisimple Lie algebras of toral rank $\leqslant 3$ it is easy to conclude that the only possibilities are $\mathrm{Tder}(A)' \cong su(2) \oplus su(2) \oplus su(2)$ and $\mathscr{Z} = \mathscr{Z}_0$ or $\mathrm{Tder}(A)' \cong su(3)$ and $\dim \mathscr{Z} = 3$. In the latter case $\mathrm{Tder}(A)' \cap \ker \pi_i = 0$ $i = 1, 2, 3$, however $su(3)$ has no nontrivial four-dimensional representations. In case that $\mathrm{Tder}(A)' \cong su(2) \oplus su(2) \oplus su(2)$, if $\ker \pi_i \cap \mathrm{Tder}(A)' \neq 0$ then A will be isotopic to $\mathbb{H}$. Otherwise, $\mathrm{Tder}(A)' \cap \pi_i = 0$ implies that A_i is a faithful $su(2) \oplus su(2) \oplus su(2)$ module. For one of these $su(2)$ A_i decomposes as either $W(0) \oplus W(2)$ or $V(1)$. The other two copies of $su(2)$ will lay inside of $\mathrm{End}_{su(2)}(A_i) \cong \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ which is not possible.

Now we will assume that $\dim A = 8$ and $\dim \mathrm{Tder}(A) = 30$. By comparing the dimensions of semisimple Lie algebras of toral rank $\leqslant 4$, the dimension and the toral rank of $\mathrm{Tder}(A)$ we get that either $\mathrm{Tder}(A)' \cong D_4$ or $\mathrm{Tder}(A)' \cong G_2 \oplus G_2$. In both cases $\ker \pi_i \cap \mathrm{Tder}(A)' = 0$. The only faithful eight-dimensional module for G_2 is the direct sum of an absolutely irreducible seven-dimensional module and a trivial one. The commuting algebra for this action $\mathbb{R} \oplus \mathbb{R}$ should contain a copy of G_2 which is not possible. Thus, $\mathrm{Tder}(A)' \cong D_4$. The irreducible modules of dimension $\leqslant 8$ for D_4 are the trivial one, the natural and the two spin modules: $V(0)$, V, V' and V''. Since in all the last three modules D_4 acts as skew-symmetric operators [19], they share the same invariant bilinear form (that of $\mathbb{O}$). In case that two of them are isomorphic we can change by $(*)$ to an algebra A with $A_2 \cong A_3$. However, neither $V \otimes V$, $V' \otimes V'$ nor $V'' \otimes V''$ has an eight-dimensional submodule. Therefore, A_1, A_2 and A_3 are not isomorphic. Since $\dim \mathrm{Hom}_{D_4}(A_i \otimes A_j, A_k) = 1$ ($\{i, j, k\} = \{1, 2, 3\}$) and up to permutation (see Section 2.2) of A_i, A_j and A_k this product is given by the product of the octonions (Principle of local triality). This shows that A lies in the orbit of $\mathbb{O}$ under the action of $\mathscr{G}_8$. It is easy to conclude that in this case A is isotopic to $\mathbb{O}$. □

3. The four-dimensional case

It will be convenient to have at hand concrete realizations of the faithful four-dimensional $su(2)$-modules. All of them are obtained from the quaternions $\mathbb{H}$. The traceless elements $\mathbb{H}_0$ of $\mathbb{H}$ form a Lie subalgebra of $\mathbb{H}^-$ isomorphic to $su(2)$. $\mathbb{H}$ is an $su(2)$-module in several ways:

$$L\colon (a,x)\mapsto ax,\quad R\colon (a,x)\mapsto -xa \quad\text{and}\quad \mathrm{ad}\colon (a,x)\mapsto [a,x] \quad (a\in\mathbb{H}_0, x\in\mathbb{H}).$$

The first two representations $\mathbb{H}_L$ and $\mathbb{H}_R$ are isomorphic to $V(1)$ while the third $\mathbb{H}_{\mathrm{ad}}$ is isomorphic to $W(0)\oplus W(2)$.

Theorem 10. *A real algebra A is a division algebra of dimension 4 with non-abelian Lie algebra of ternary derivations if and only if there exists $\sigma\in\mathscr{G}_4$ such that A^σ is isomorphic to $\mathbb{H}$ with one of the following products*:

(i) $xy-\frac{1-\beta}{2}t(xy_0)1$ *with* $\beta>0$,
(ii) $xy-t(xcy)1$ *with* $c\in\mathbb{H}$ *and* $t(c)\neq 1$,

where xy denotes the product on $\mathbb{H}$, $t(\,)$ denotes its trace form and $y_0=y-\frac{1}{2}t(y)$ is the imaginary part of y.

Proof. Consider $(\mathbb{H},\circ)$ with $x\circ y=xy-\frac{1-\beta}{2}t(xy_0)$ and $\beta>0$. Clearly $(\mathrm{ad}_a,\mathrm{ad}_a,\mathrm{ad}_a)\in\mathrm{Tder}(\mathbb{H},\circ)$ for any $a\in\mathbb{H}$, so this Lie algebra is not abelian. Let us check that $(\mathbb{H},\circ)$ is a division algebra. On the contrary, assume that $x\circ y=0$ with $x\neq 0\neq y$. Then, $0\neq xy\in\mathbb{R}1$ implies that $y=\alpha\bar{x}$ for some $\alpha\in\mathbb{R}$ and $\bar{x}$ the conjugate of x. Since $x\bar{x}=n(x)1$, the norm of x, then $xy=\frac{1-\beta}{2}t(xy_0)$ implies that $n(x)=(1-\beta)n(x_0)$ with x_0 the imaginary part of x. Finally, $n(x)\geqslant n(x_0)$ leads to $\beta\leqslant 0$ which is not possible. In a similar way it can be proved that the algebra in part (ii) is a division algebra and that $(\mathrm{ad}_a,L_a,-R_a)$ is a ternary derivation for any $a\in\mathbb{H}$, so the Lie algebra of ternary derivations of this division algebra is not abelian.

Now let us prove the converse. Since $\mathrm{Tder}(A)$ is not abelian then it contains a subalgebra isomorphic to $su(2)$. We will distinguish two cases depending on the decomposition of A_i as an $su(2)$-module.

(i) Assume that there are two occurrences of $W(0)\oplus W(2)$ in $\{A_1,A_2,A_3\}$. Since $W(0)\otimes W(0)\cong W(0)$ and $W(0)\otimes V(1)\cong V(1)\cong V(1)\otimes W(0)$ then $A_1\cong A_2\cong A_3\cong W(0)\oplus W(2)$ and up to isotopy we may assume that $\mathrm{Der}(A)$ contains a subalgebra isomorphic to $su(2)$ (see Section 2.2). This implies that the multiplication table of $(A,\circ)$ is given by table (6.2) in [4], which corresponds to the product (i) in the statement.

(ii) The tensor product $V(1)\otimes V(1)$ does not contain a submodule isomorphic to $V(1)$, so in case that $A_2\cong V(1)\cong A_3$ then $A_1\cong W(0)\oplus W(2)$. Thus, if two modules A_i, A_j are isomorphic to $V(1)$ then the other is isomorphic to $W(0)\oplus W(2)$. By permutation we may assume that $A_1\cong V(1)\cong A_3$ (thus $A_2\cong W(0)\oplus W(2)$).

As seen is Section 2.2, up to isotopy we can look at A as $(\mathbb{H},\circ)$ for some product $\circ$, A_1 as $\mathbb{H}_L$ and $su(2)$ as $\{(L_a,\mathrm{ad}_a,L_a)|a\in\mathbb{H}_0\}$ where the multiplication operators are relative to $\mathbb{H}$. Thus, $\circ$ satisfies $a(x\circ y)=[a,x]\circ y+x\circ(ay)$ for all $a\in\mathbb{H}_0$. The left multiplication operator $L^\circ_x\colon y\mapsto x\circ y$ satisfies $L_aL^\circ_x=L^\circ_{[a,x]}+L^\circ_xL_a$ so $[L_a,L^\circ_x]=L^\circ_{[a,x]}$.

Let $\{a_1,\dots,a_4\}$ be a basis of $\mathbb{H}$. The well-known isomorphism $\mathrm{End}_{\mathbb{R}}(\mathbb{H})\cong\mathbb{H}\otimes\mathbb{H}^{\mathrm{op}}$ allows us to write in a unique way

$$L_x^\circ = \sum_{i=1}^{4} L_{\varphi_i(x)} R_{a_i}$$

for some $\varphi_i : \mathbb{H} \to \mathbb{H}$. Hence

$$\sum_{i=1}^{4} L_{\varphi_i([a,x])} R_{a_i} = L^\circ_{[a,x]} = [L_a, L_x^\circ] = \sum_{i=1}^{4} [L_a, L_{\varphi_i(x)}] R_{a_i} = \sum_{i=1}^{4} L_{[a,\varphi_i(x)]} R_{a_i},$$

which implies that $\varphi_i([a, x]) = [a, \varphi_i(x)]$ for any $a, x \in \mathbb{H}$ and $i = 1, 2, 3, 4$. The modules $W(0)$ and $W(2)$ are absolutely irreducible so the endomorphisms of $\mathbb{H}_{\mathrm{ad}}$ as a module form a two dimensional vector space spanned by the identity and the map $x \mapsto t(x)$. The maps φ_i then admit the friendly description $\varphi_i(x) = \alpha_i t(x) + \beta_i x$ for some $\alpha_i, \beta_i \in \mathbb{R}$. The operator L_x° can be written as $L_x^\circ = \sum_{i=1}^{4} L_{\alpha_i t(x)+\beta_i x} R_{a_i}$. Therefore, $L_x^\circ = t(x)R_a + L_x R_b$ and $x \circ y = t(x)ya + xyb$ with $a, b \in \mathbb{H}$ and $b \neq 0$. Changing $\circ$ by $x \circ R_b^{-1}(y)$ we obtain

$$x \circ y = xy - t(x)yc,$$

where $c = -b^{-1}a$. Being A a division algebra $(\mathbb{H}, \circ)$ must be a division algebra too, so $c \circ c = (1 - t(c))c^2 \neq 0$ if $c \neq 0$ implies that $t(c) \neq 1$.

By considering the opposite product to $\overline{\bar{x} \circ \bar{y}}$ we get a product $xy - t(y)\bar{c}x$. Using the bilinear form of $\mathbb{H}$, the adjoint product to this is $\bar{x}y - t(\bar{x}cy)$ that up to isotopy corresponds with the product given in (ii). □

It can be proved that if $\beta \neq 1$ in part (i) of the theorem, then $\mathrm{Tder}(A) = su(2) \oplus \mathscr{L}_0$. For the algebras in part (ii), if $c \in \mathbb{R}$ then A is an isotope of the quaternions; if $c \notin \mathbb{R}$ then $\mathrm{Tder}(A) = su(2) \oplus \mathscr{L}_0$.

4. The eight-dimensional case

In this section we will assume that $\dim A = 8$ and that $\mathrm{Tder}(A)$ is not abelian. We will prove

Theorem 11. *A real algebra A is a division algebra of dimension 8 with a simple subalgebra of toral rank $\geqslant 2$ contained in $\mathrm{Tder}(A)$ if and only if there exits $\sigma \in \mathscr{G}_8$ such that the product on A^σ is either given from the product on $\mathbb{O}$ by one of the formulae*

(i) $xy - \frac{1-\beta}{2} t(xy_0)1$ *with* $\beta > 0$,
(ii) $xy + (a, x)b(cy)$ *for some* $a, b, c \in \mathrm{span}\langle 1, i, j\rangle$ *with* $(a, bc) \neq -1$,
(iii) $xy + \sigma(x, y)$ *with* $\sigma : \mathbb{O} \times \mathbb{O} \to \mathbb{C}$ *an $\mathbb{R}$-bilinear map with* $\sigma(a, b) \neq -\lambda ab$ *for any nonzero* $a, b \in \mathbb{C}$ *and* $\lambda \geqslant 1$ *and* $\sigma(\mathbb{C}^\perp, \mathbb{O}) + \sigma(\mathbb{O}, \mathbb{C}^\perp) = 0$, *or*
(iv) *it is given from* $su(3)$ *by* $\alpha[x, y] + \sqrt{3}\mathrm{i}(xy + yx - \frac{2}{3}t(xy)I)$ *with xy the usual product of matrices and* $0 \neq \alpha \in \mathbb{R}$.

The products in the statement of Theorem 11 provide division algebras. This follows from [4] in the case (iv), and from the proof of Theorem 10, Step 9 in the proof of Proposition 14 and Lemma 16 in cases (i)–(iii) respectively. Any derivation of $\mathbb{O}$ is a derivation of the product in case (i). Derivations of $\mathbb{O}$ that kill $\mathbb{C}$ (they form a Lie algebra isomorphic to $su(3)$) remain derivations for the product in (iii). The adjoint representation of $su(3)$ gives a subalgebra of derivations isomorphic to $su(3)$ in case (iv). Ternary derivations of $\mathbb{O}$ of the form (d, d', d) with $d'(\mathbb{C}) = 0$

(which contain a Lie subalgebra of type B_2) remain ternary derivations of the product in part (ii). Hence, in all cases the Lie algebra of ternary derivations of the algebras in Theorem 11 contains a simple subalgebra of toral rank $\geqslant 2$.

The formulation of Theorem 11 aims for a simple description of the families involved. For instance, in (iii) another description is possible. One only has to develop $\sigma(x, y)$ as $\sigma(x, y) = a_1xy + a_2\bar{x}y + a_3x\bar{y} + a_4\bar{x}\bar{y}$ for some $a_1, \ldots, a_4 \in \mathbb{C}$. Since for any nonzero a the product of the octonions verifies the relations $xy = a^{-1}((axa)(a^{-1}y))$ and $xy = ((xa^{-1})(aya))a^{-1}$ then σ can be modified by $a^{-1}\sigma(axa, a^{-1}y)$ or $\sigma(xa^{-1}, aya)a^{-1}$. This would allow us to assume that two out of $\{a_2, a_3, a_4\}$ are scalars, which reduces the number of parameters involved in the definition of σ to six.

By some manipulations with elements in $\mathcal{G}_8$ the product in (ii) gives rise to $xy + t(((yb)c)x)e$ for some e. We can then use that $xy = e^{-1}((exe)(e^{-1}y))$ to obtain $xy + t((exe)(((e^{-1}y)b)c))1$ in analogy with (ii) in Theorem 10.

This theorem misses the cases where $\mathrm{Tder}(A)' = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$, $su(2) \oplus su(2) \oplus su(2)$, $su(2) \oplus su(2)$, $su(2)$ or 0. In fact, the first case is not possible although we will not present the proof. The second and the others are likely to be possible from minor modifications of the Cayley–Dickson process when constructing the octonions from the quaternions.

The result also suggests that from a division algebra A with product xPy other division algebras are obtained by $xQy = xPy - (x, y)a$ for some $a \in A$ and a bilinear form (,). We have not make attempt to incorporate this transformations as elements of $\mathcal{G}$ since they do not preserve the Lie algebra of ternary derivations. The element a and the bilinear form (,) making (A, Q) a division algebra probably will also strongly depend on the particular P.

The toral rank of the semisimple part of $\mathrm{Tder}(A)$ is $\leqslant 4$. Hence, it contains a compact subalgebra from the following list:

$$su(5), B_4, C_4, D_4, F_4, su(4), B_3, C_3, su(3), B_2, G_2, su(2).$$

When no danger of confusion we will denote an irreducible module by its dimension. Observe that for any real Lie algebra L and any irreducible L-modules V and W with $\mathbb{C} \otimes V \cong \mathbb{C} \otimes W$ as $\mathbb{C} \otimes L$-modules, since as L-modules $V \oplus V \cong \mathbb{C} \otimes_{\mathbb{R}} V \cong \mathbb{C} \otimes_{\mathbb{R}} W \cong W \oplus W$ then $V \cong W$. Thus, at most one irreducible real form of a complex module is possible.

Proposition 12. *The Lie algebra of ternary derivations of a finite-dimensional real division algebra does not contain any subalgebra isomorphic to F_4, B_4, C_4, $su(5)$ or C_3.*

Proof. The dimension of the smallest irreducible non trivial module for an split algebra of type F_4 is 26, so F_4 is dismissed. In the case of an algebra of type B_4 the dimension is 9, so this case does not appear. Some algebras of type C_4 admit an eight-dimensional representation. However, this representation possess an alternate invariant bilinear form which in turns implies that the algebra is split. The $su(5)$-irreducible modules of dimension $\leqslant 8$ have dimension 1 so this algebra does not appear. We are left with algebras of type C_3. The irreducible modules of dimension $\leqslant 8$ for this algebra are 1 and 6. Therefore, the three modules A_1, A_2 and A_3 are isomorphic. Up to isotopy this shows that $C_3 \subseteq \mathrm{Der}(A)$ which is not possible [3]. $\square$

This proposition reduces the cases under analysis to $D_4, su(4), B_3, su(3), B_2, G_2$ and $su(2)$. The largest cases, namely D_4, $su(4)$ and B_3 provide algebras isotopic to the octonions. In particular, the semisimple part of $\mathrm{Tder}(A)$ is the compact D_4.

Proposition 13. *If* $\mathrm{Tder}(A)$ *contains a subalgebra isomorphic to* D_4, B_3 *or* $su(4)$ *then* A *is isotopic to the octonions.*

Proof. The irreducible modules of dimension $\leqslant 8$ for B_3 are $V(0)$, $V(\lambda_1)$ and $V(\lambda_3)$ of dimensions 1, 7 and 8 respectively. In case that $A_1 \cong A_2 \cong A_3$, then up to isotopy B_3 will act as derivations, which is not possible [3]. Up to permutation we may assume that $A_1 \cong 8$, $A_3 \cong 8$, both of them absolutely irreducible, and $A_2 \cong 1 + 7$. Since $\dim \mathrm{Hom}_{B_3}(V(\lambda_1) \otimes V(\lambda_3), V(\lambda_3)) = 1$ then up to scalars the product $7 \otimes 8 \to 8$ can be taken the one induced by the product on the octonions where B_3 is contained inside $\mathrm{Tder}(\mathbb{O})$ as $\{(L_a + 2R_a, L_a - R_a, L_a + 2R_a) | a \in \mathbb{O}_0\}$. Thus, for any $x \perp 1$, the left multiplication operator by x is αL_x for some $\alpha \in \mathbb{R}$, where L_x denotes the corresponding multiplication operator on the octonions. The left multiplication operator by 1 is $\beta\mathrm{id}$ for some $\beta \in \mathbb{R}$. By defining $\varphi\colon 1 \mapsto \beta 1$ and $x \mapsto \alpha x$ if $x \perp 1$ we get that A is isomorphic to the isotope $\varphi(x)y$ of the octonions.

The irreducible modules of dimension $\leqslant 8$ for $su(4)$ have dimension 1, 6 or 8. The eight-dimensional irreducible module is not absolutely irreducible. It splits as the direct sum of two 4-dimensional non isomorphic dual modules. Up to permutation, $A_2 \cong 1 + 1 + 6$, $A_3 \cong 8$ and $A_1 \cong 8$. Up to isotopy we can look at A as the octonions $\mathbb{O}$ with a new product $\circ$. Then $su(4) \subseteq \mathrm{Tder}(\mathbb{O})$ as $su(4) = \{(d_1, d_2, d_1) | d_2(\mathbb{C}) = 0\}$. Clearly,

$$\{L^\circ_a | a \in \mathbb{C}\} + \{L_a | a \in \mathbb{C}\} \subseteq \mathrm{End}_{su(4)}(8) \cong \mathbb{C}$$

so $\{L^\circ_a | a \in \mathbb{C}\} = \{L_a | a \in \mathbb{C}\}$. Since $\dim \mathrm{Hom}_{su(4)}(6 \otimes 8, 8) = 2$ then there exists $e \in \mathbb{C}$ such that $x \circ y = e(xy)$ for all $x \perp \mathbb{C}$. That is, $L^\circ_x = L_e L_x$ for all $x \perp \mathbb{C}$. Moreover, $L^\circ_a = L_e L_{\varphi(a)}$ with $\varphi\colon \mathbb{C} \to \mathbb{C}$ bijective. Thus, if we extend φ by $\varphi|_{\mathbb{C}^\perp} = \mathrm{id}$ then $L^\circ_x = L_e L_{\varphi(x)}$ for all $x \in \mathbb{O}$. This implies that $L_e^{-1}(\varphi^{-1}(x) \circ y) = L_e^{-1} L_e(xy) = xy$, so A is isotopic to the octonions. $\square$

The only possibilities for a simple subalgebra of the ternary derivations of an eight-dimensional real division algebra not isotopic to the octonions are

$$G_2, su(3), B_2 \quad \text{and} \quad su(2).$$

4.1. Case B_2

Let L be a compact Lie algebra of type B_2 contained in $\mathrm{Tder}(A)$. The irreducible $\mathbb{C} \otimes_{\mathbb{R}} L$ modules have dimension:

$$\dim V(m_1\lambda_1 + m_2\lambda_2) = \frac{1}{3!}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)(2m_1 + m_2 + 3).$$

The only possibilities for dimension $\leqslant 8$ are $V(0)$, $V(\lambda_1)$ (five-dimensional) and $V(\lambda_2)$ (four-dimensional). In fact, any L-module V with $\mathbb{C} \otimes_{\mathbb{R}} V \cong V(\lambda_2)$ will have a nondegenerate alternating invariant bilinear form so L will be isomorphic to the split C_2 which is not possible. Therefore, a faithful eight-dimensional L-module will decompose as either 8 or $1 + 1 + 1 + 5$. Moreover, A_1, A_2 and A_3 are not isomorphic since B_2 does not appear acting as derivations [3]. Up to permutation, the only possibility is $A_1 \cong 8$, $A_2 \cong 1 + 1 + 1 + 5$ and $A_3 \cong 8$.

We will first determine some models for L and the modules involved. Let $T = \mathrm{span}\langle 1, i, j\rangle \subseteq \mathbb{O}$. We have

$$\mathbb{O} = T \oplus T^\perp \quad \text{with } \dim T = 3 \quad \text{and} \quad \dim T^\perp = 5.$$

The algebra

$$L' = \{(d, d', d) \in \mathrm{Tder}(\mathbb{O}) | d'|_T = 0, \mathrm{tr}(d) = 0\},$$

where tr() denotes the usual trace, is isomorphic to L, and the decomposition of $\mathbb{O}$ for the different projections of L' are $\mathbb{O}_2 \cong 1 + 1 + 1 + 5$ and $\mathbb{O}_1 \cong \mathbb{O}_3 \cong 8$. Thus, up to isotopy we can perform the following identification: $A = (\mathbb{O}, \circ)$ for some product $\circ$, $L = \{(d, d', d) \in \mathrm{Tder}(\mathbb{O}) | d'|_T = 0, \mathrm{tr}(d) = 0\}$, $1 + 1 + 1 = T$, $5 = T^{\perp}$, $\mathbb{O}_1 \cong \mathbb{O}_3 \cong 8$ and $\mathbb{O}_2 \cong 1 + 1 + 1 + 5$.

Proposition 14. *Let A be a finite-dimensional real division algebra with $B_2 \subseteq \mathrm{Tder}(A)$. Then there exist $a, b, c \in \mathrm{span}\langle 1, i, j\rangle \subseteq \mathbb{O}$ and $\sigma \in \mathscr{G}_8$ such that the product on A^{σ} is given from the product xy of $\mathbb{O}$ by*

$$xy + (a, x)b(cy)$$

with $(a, bc) \neq -1$.

Proof. The proof will follow several steps:

1. $D = \mathrm{End}_L(\mathbb{O}_3)$ *is a quaternion division algebra*: $\mathbb{O}_3$ is an irreducible L-module, so D is a division algebra. Since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}_3 \cong V(\lambda_2) \oplus V(\lambda_2)$ then $\mathbb{C} \otimes_{\mathbb{R}} D = M_2(\mathbb{C})$. Thus, D is a division quaternion algebra.

2. $\mathrm{span}\langle L_a | a \in T\rangle + \mathrm{span}\langle L_a^{\circ} | a \in T\rangle \subseteq D$: Since $d(xy) = d'(x)y + xd(y)$ then $[d, L_x] = L_{d'(x)}$, so $[d, L_a] = 0 = [d, L_a^{\circ}]$ for all $a \in T$. Thus, $L_a, L_a^{\circ} \in D$ for all $a \in T$.

3. $\mathrm{End}_{\mathbb{R}}(\mathbb{O}_3) \cong L^* \otimes_{\mathbb{R}} D^{\mathrm{op}}$ *with* $L^* = \mathrm{alg}_1\langle d | (d, d', d) \in L\rangle$: Since $\mathbb{O}_3$ is an L-module, it is also an L^*-module and $D = \mathrm{End}_{L^*}(\mathbb{O}_3)$. Thus, $\mathbb{O}_3$ is an $L^* \otimes_{\mathbb{R}} D^{\mathrm{op}}$-module in the natural way. The centralizer of this action is $\mathbb{R}$, the center of D. By the Jacobson Density Theorem, $L^* \otimes_{\mathbb{R}} D^{\mathrm{op}} \cong \mathrm{End}_{\mathbb{R}}(\mathbb{O}_3)$.

4. *As L-modules*, $\mathrm{End}_{\mathbb{R}}(\mathbb{O}_3) \cong L^* \otimes_{\mathbb{R}} D^{\mathrm{op}}$: The vector space $\mathrm{End}_{\mathbb{R}}(\mathbb{O}_3)$ is an L-module in the natural way

$$(d, d', d) \cdot \varphi = [d, \varphi].$$

L^* is an L-submodule of $\mathrm{End}_{\mathbb{R}}(\mathbb{O}_3)$ and D^{op} can be seen as a four-dimensional trivial L-module. Given $(d, d', d) \in L$, $\varphi \in L^*$ and $\gamma \in D^{\mathrm{op}}$ we have that

$$(d, d', d) \cdot (\varphi \otimes \gamma) = [d, \varphi] \otimes \gamma$$

corresponds by the previous isomorphism to $[d, \varphi]\gamma = (d, d', d) \cdot \varphi\gamma$ since d commutes with all the elements in D.

5. $L^* = L + 1 + 5$ *as an L-module*: The dimension of L^* is $64/4 = 16$. Since the dimension of L is 10 then the complement of L in L^* is six-dimensional. The identity id belongs to L^*, thus the only possibility is $L^* = L + 1 + 5$.

6. *There exist $\gamma \in D$ with $L_x^{\circ} = L_x\gamma$ for all $x \in T^{\perp}$*: The relation $[d, L_x^{\circ}] = L_{d'(x)}^{\circ}$ implies that $\{L_x^{\circ} | x \in T^{\perp}\}$ is an absolutely irreducible five-dimensional submodule of $\mathrm{End}_{\mathbb{R}}(\mathbb{O}_3) \cong L^* \otimes_{\mathbb{R}} D^{\mathrm{op}}$. These submodules are collected as $\{L_x | x \in T^{\perp}\}D$. Thus, it exists $\gamma \in D$ such that $L_x^{\circ} = L_x\gamma$ for all $x \in T^{\perp}$.

7. *Up to isotopy we may assume that there exists $0 \neq a \in T$ with $L_x^{\circ} = L_x$ for all $x \in (\mathbb{R}a)^{\perp}$*: Since $\mathrm{span}\langle L_a | a \in T\rangle + \mathrm{span}\langle L_a^{\circ} | a \in T\rangle \subseteq D$ and $\gamma \in D$ then $\dim(L_T \cap L_T^{\circ}\gamma^{-1}) \geqslant 2$. Let $S \subseteq T$ and $\varphi: S \to T$ such that $L_b = L_{\varphi(b)}^{\circ}\gamma^{-1}$ for all $b \in S$. We define $\varphi|_{T^{\perp}} = \mathrm{id}$. Let $a \in T$ with $(\mathbb{R}a)^{\perp} = S \oplus T^{\perp}$. We have that for any $x \in (\mathbb{R}a)^{\perp}$, $L_x = L_{\varphi(x)}^{\circ}\gamma^{-1}$. Define

$$x \diamond y = \varphi(x) \circ \gamma^{-1}(y).$$

Given $x \in (\mathbb{R}a)^{\perp}$

$$x \diamond y = \varphi(x) \circ \gamma^{-1}(y) = L^{\circ}_{\varphi(x)}(\gamma^{-1}(y)) = L_x\gamma\gamma^{-1}(y) = xy.$$

Since φ commutes with d', and γ commutes with d for any $(d, d', d) \in L$, then $L \subseteq \mathrm{Tder}(A, \diamond)$.

8. *Up to isotopy, there exist $a, b, c \in T$ such that $x \circ y = xy + (x, a)b(cy)$*: Since $L^{\circ}_a \in D$ and $D = \mathrm{alg}\langle L_x | x \in T\rangle$ ($\mathbb{H}$ has no three-dimensional subalgebras) then there exist $a', a'', a''' \in T$ with $L^{\circ}_a = L_{a'} + L_{a''}L_{a'''}$. We also may assume that $a'' \perp a'''$ and $a'', a''' \perp 1$. Thus,

$$a \circ y = a'y + a''(a'''y) \quad \text{and} \quad x \circ y = xy \text{ for all} \quad x \perp a.$$

Let us now analyze when this product gives a division algebra. Given $0 \neq y$, there exists $\alpha a + x_0$ with $x_0 \perp a$ such that

$$\begin{aligned}(\alpha a + x_0) \circ y = 0 &\Leftrightarrow \alpha a'y + \alpha a''(a'''y) + x_0 y = 0\\
&\Leftrightarrow \alpha a' + \alpha(a''(a'''y))y^{-1} = -x_0\\
&\Leftrightarrow (\alpha a' + \alpha(a''(a'''y))y^{-1}, a) = 0\\
&\Leftrightarrow \begin{cases}\alpha = 0 \text{ and } x_0 = 0 \text{(a trivial case that we omit)}\\ \text{or}\\ (a' + (a''(a'''y))y^{-1}, a) = 0\end{cases}\\
&\Leftrightarrow (a', a) + (a''(a'''y), ay/n(y)) = 0\\
&\Leftrightarrow (a', a) + (a''a'''y_0, ay_0/n(y)) + ((a'''a'')y_1, ay_1/n(y)) = 0\\
&\Leftrightarrow (a', a) + (a''a''', a)n(y_0)/n(y) + (a'''a'', a)n(y_1)/n(y) = 0\\
&\Leftrightarrow (a', a) = 0,\end{aligned}$$

where $n(y) = (y, y)$ is the usual norm of $\mathbb{O}$, we have written $y = y_0 + y_1$ with $y_0 \in \mathbb{H}$ and $y_1 \in \mathbb{H}^{\perp}$, and the last equivalence holds since $a'', a''' \perp 1$ and $a'' \perp a'''$ implies that $a''a''' \perp T$, thus $(a', a) \neq 0$. Now we write $a' = \alpha a + b_0$ with $b_0 \in T$, $b_0 \perp a$ and $\alpha \neq 0$. Since

$$\begin{aligned}L^{\circ}_{\alpha^{-1}(a-b_0)} &= \alpha^{-1}L^{\circ}_a - \alpha^{-1}L^{\circ}_{b_0} = \alpha^{-1}(L_{a'} + L_{a''}L_{a'''}) - \alpha^{-1}L_{b_0}\\
&= L_a + \alpha^{-1}L_{a''}L_{a'''}\end{aligned}$$

then the product $x \diamond y = \varphi(x) \circ y$ with $\varphi: a \mapsto \alpha^{-1}(a - b_0)$ and $\varphi|_{(\mathbb{R}a)^{\perp}} = \mathrm{id}$ satisfies that $x \diamond y = xy$ for any $x \perp a$ and $a \diamond y = ay + \alpha^{-1}a''(a'''y)$. Therefore, up to isotopy, there exist $a, b, c \in T$ such that $x \circ y = xy + (a, x)b(cy)$.

9. $(\mathbb{O}, \circ)$ *is a division algebra if and only if* $(a, bc) \neq -1$: As in the Step 8, given $0 \neq y = y_0 + y_1 \in \mathbb{O}$ with $y_0 \in \mathbb{H}$ and $y_1 \in \mathbb{H}^{\perp}$, we have that there exists $0 \neq \alpha a + x_0$ with $x_0 \perp a$ such that $(\alpha a + x_0) \circ y = 0$ if and only if $n(a) + \frac{n(a)}{n(y)}(b(cy), ay) = 0$. Since $a \neq 0$, this is equivalent to $(b(cy), ay/n(y)) = -1$. That is, $(bc, a)\frac{n(y_0)}{n(y)} + (cb, a)\frac{n(y_1)}{n(y)} = -1$. Since $[b, c] \perp T$ implies $(bc, a) = (cb, a)$ then this last condition is the same as $(bc, a) = -1$. $\square$

4.2. Case G_2

The irreducible G_2-modules of dimension $\leqslant 8$ have dimensions 1 and 7. Thus, in case that $G_2 \subseteq \mathrm{Tder}(A)$ then $A_1 \cong A_2 \cong A_3 \cong 1 + 7$ and, up to isotopy, G_2 acts by derivations. By [4] we can find a basis such that the multiplication table of A is given by Table 1. The left

Table 1
G_2 acts on A as derivations

	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
u	u	ηe_1	ηe_2	ηe_3	ηe_4	ηe_5	ηe_6	ηe_7
e_1	ζe_1	$-\beta u$	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	ζe_2	$-e_4$	$-\beta u$	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	ζe_3	$-e_7$	$-e_5$	$-\beta u$	e_6	e_2	$-e_4$	e_1
e_4	ζe_4	e_2	$-e_1$	$-e_6$	$-\beta u$	e_7	e_3	$-e_5$
e_5	ζe_5	$-e_6$	e_3	$-e_2$	$-e_7$	$-\beta u$	e_1	e_4
e_6	ζe_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	$-\beta u$	e_2
e_7	ζe_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	$-\beta u$

and right multiplication operators by u, L_u° and R_u°, are given by $\operatorname{diag}(1,\zeta,\zeta,\zeta,\zeta,\zeta,\zeta,\zeta)$ and $\operatorname{diag}(1,\eta,\eta,\eta,\eta,\eta,\eta,\eta)$. With the new product $x\diamond y=(R_u^\circ)^{-1}(x)\circ(L_u^\circ)^{-1}(y)$ we obtain a similar table where $\zeta=1=\eta$ and an appropriate $\beta>0$. This product can be recovered from the product xy of the octonions by $x\diamond y=xy-\frac{1-\beta}{2}t(xy_0)$ where $y_0=y-\frac{1}{2}t(y)$.

Proposition 15. *If* $\operatorname{Tder}(A)$ *contains a subalgebra of type* G_2 *then* A *is isotopic to* $\mathbb{O}$ *with the product given by*

$$x\circ y=xy-\frac{1-\beta}{2}t(xy_0),$$

where xy *denotes the product on the octonions*, $y_0=y-\frac{1}{2}t(y)$ *and* $\beta>0$. *Moreover, if* $\beta=1$ *then* $\operatorname{Tder}(A)=D_4\oplus\mathscr{L}_0$ *and if* $\beta\neq 1$ *then* $\operatorname{Tder}(A)=G_2\oplus\mathscr{L}_0$.

Proof. The complete proof is not a very illuminating computation. We will only present a sketch. First, observe that the involution of the octonions, $x\mapsto\bar{x}$, induces an involution on A. In turn, we have an automorphism $(d_1,d_2,d_3)\mapsto(\bar{d}_1,\bar{d}_3,\bar{d}_2)$ on $\operatorname{Tder}(A)$ where $\bar{d}(x)=\overline{d(\bar{x})}$. This automorphism decomposes $\operatorname{Tder}(A)$ as the direct sum of eigenspaces $S(1)$ and $S(-1)$ corresponding to eigenvalues 1 and -1. Note that

$$S(1)=\{(d_1,d_2,\bar{d}_2)\in\operatorname{Tder}(A)|\bar{d}_1=d_1\},$$
$$S(-1)=\{(d_1,d_2,-\bar{d}_2)\in\operatorname{Tder}(A)|\bar{d}_1=-d_1\}.$$

Any derivation d of $\mathbb{O}$ induces a derivation on A, so $G_2\cong\{(d,d,d)|d\in\operatorname{Der}(\mathbb{O})\}\subseteq S(1)$. We will prove that if $\beta\neq 1$ then $S(-1)=\operatorname{span}\langle(0,\mathrm{id},-\mathrm{id})\rangle$ and $S(1)=\{(d,d,d)|d\in\operatorname{Der}(\mathbb{O})\}+\operatorname{span}\langle(2\mathrm{id},\mathrm{id},\mathrm{id})\rangle$.

Given $(d_1,d_2,-\bar{d}_2)\in S(-1)$, $\bar{d}_1=-d_1$ implies that $d_1(1)\in\mathbb{O}_0$ and $d_1(\mathbb{O}_0)\subseteq\mathbb{R}1$. Using the definition of ternary derivation with $x\in\mathbb{O}_0$ and $y=x$ we obtain that $-\beta n(x)d_1(1)=d_2(x)x-\overline{d_2(x)x}$, where $n(x)$ denotes the usual norm of $\mathbb{O}$. Since $2y=(y-\bar{y})+t(y)$ then we have that $2d_2(x)x=(-\beta n(x)d_1(1))+t(d_2(x)x)$. Dividing by x in case that $x\neq 0$ we obtain $d_2(x)=\frac{1}{2}\beta d_1(1)x-\frac{1}{2}\frac{t(d_2(x)x)}{n(x)}x$ for any $x\in\mathbb{O}_0$. In particular, the map $x\mapsto-\frac{1}{2}\frac{t(d_2(x)x)}{n(x)}x$ must be linear so $-\frac{1}{2}\frac{t(d_2(x)x)}{n(x)}$ is a constant, say α, independent of x. The map d_2 reads as $d_2(x)=\frac{1}{2}\beta d_1(1)x+\alpha x$ for any $x\in\mathbb{O}_0$. Subtracting $(0,\alpha\mathrm{id},-\alpha\mathrm{id})$ we can assume that $\alpha=0$, so

$$d_2(x)=ax\quad\text{and}\quad d_3(x)=xa\quad\text{if } x\in\mathbb{O}_0$$

with $a = \frac{1}{2}\beta d_1(1)$, $t(a) = 0$. We now show that these formulas extend to the whole A. With $x, y \in \mathbb{O}_0$ and our knowledge of d_1, d_2 and d_3, since $(ax)y + x(ya) = a(xy) + (xy)a$, it is not hard to see that $d_1(xy) = a(xy) + (xy)a + \frac{1-\beta}{2}(t(xy)d_1(1) - t(a(xy) + (xy)a))$. Since any element in $\mathbb{O}$ is a linear combination of elements xy with $x, y \in \mathbb{O}_0$ then $d_1(z) = az + za + \frac{1-\beta}{2}(t(z)d_1(1) - t(az + za))$. Since $\beta \neq 0$ then we get

$$d_1(z) = \frac{1}{\beta}t(z)a - 2\beta(a, z).$$

With $y = 1$ and $x \perp 1, a, d_2(1)_0$ the definition of ternary derivation gives $d_2(1) = -\bar{a} = a$, so

$$d_2 = L_a \text{ and } d_3 = R_a.$$

By imposing the condition of ternary derivation and our formulas for d_1, d_2, d_3 and comparing scalar and vector parts we get that either $\beta = 1$ or $a = 0$. Therefore, $S(-1) = \operatorname{span}\langle(0, \mathrm{id}, -\mathrm{id})\rangle$.

Let $(d_1, d_2, \bar{d}_2) \in S(1)$. Since $\bar{d}_1 = d_1$ then $d_1(1) \in \mathbb{R}1$. Subtracting a scalar multiple of $(2\mathrm{id}, \mathrm{id}, \mathrm{id})$ we may assume that $d_1(1) = 0$, so $d_1(\mathbb{O}) \subseteq \mathbb{O}_0$. The condition of ternary derivation with $x = y = 1$ and $x = y \in \mathbb{O}_0$ implies $d_2(1) \in \mathbb{O}_0$ and $(d_2(x), y) + (x, d_2(y)) = 0$ for any $x, y \in \mathbb{O}_0$. The same condition with $y = 1$ implies $d_1(x) = d_2(x) - xd_2(1) + \frac{1-\beta}{2}t(xd_2(1))$. By taking traces we also get $t(d_2(x)) = \beta t(xd_2(1))$ for all $x \in \mathbb{O}$. Now write $d_2|_{\mathbb{O}_0} : \mathbb{O}_0 \to \mathbb{O}$ as $x \mapsto f(x) + \alpha(x)1$ with $f(x) \in \mathbb{O}_0$ and $\alpha(x) \in \mathbb{R}$. The skew-symmetry of d_2 forces the skew-symmetry of f. Therefore,

$$f = D + \mathrm{ad}_a$$

with $D \in \mathrm{Der}(\mathbb{O})$ and $\mathrm{ad}_a : x \mapsto ax - xa$ for some $a \in \mathbb{O}_0$. Subtracting (D, D, D) we can assume that $d_2(x) = [a, x] + \alpha(x)1$. Computing the trace of $d_2(x)$ we have $\beta t(xd_2(1)) = 2\alpha(x)$. Hence $\alpha(x) = \frac{\beta}{2}t(xd_2(1))$. A minor modification leads to

$$d_2(x) = [a, x] + \frac{\beta}{2}t(xd_2(1)) + \frac{t(x)}{2}d_2(1)$$

for any $x \in \mathbb{O}$. Easily we also get $\bar{d}_2(x) = [a, x] - \frac{\beta}{2}t(xd_2(1)) - \frac{t(x)}{2}d_2(1)$. With $y = 1$ in the condition of ternary derivation one obtains

$$d_1(x) = \left[a + \frac{1}{2}d_2(1), x\right].$$

After some computations, the condition of ternary derivation gives that

$$\frac{1-\beta}{4}[d_2(1), [x, y]] = \left(x, y, 3a + \frac{\beta}{2}d_2(1)\right),$$

where (x, y, z) stands for the usual associator. Define $\tilde{a} = 3a + \frac{\beta}{2}d_2(1) \neq 0$ ($\tilde{a} = 0$ implies that either $\beta = 1$ or $d_2(1) = 0$, so $d_1 = d_2 = d_3 = 0$). For any $b \perp \tilde{a}, b \perp 1$, the subalgebra generated by $\tilde{a}$ and b has dimension $\geqslant 3$, so it is a quaternion subalgebra. In particular, $d_2(1)$ must commute with $\tilde{a}$, 1 and any element orthogonal to them, so $d_2(1)$ lies in the commutative nucleus of $\mathbb{O}$. Therefore, $d_2(1) = 0$. This implies that $(x, y, \tilde{a}) = 0$ for any x, y, so $\tilde{a} \in \mathbb{R}$. Since $t(\tilde{a}) = 0$ then $\tilde{a} = a = 0$. This shows that $d_1 = d_2 = d_3 = 0$. □

4.3. Case $su(3)$ and A_i not irreducible

The irreducible $su(3)$-modules of dimension $\leqslant 8$ have dimensions 1, 6 and 8. In contrast with the eight-dimensional module, the six-dimensional one is not absolutely irreducible. In case that $su(3) \subseteq \mathrm{Tder}(A)$ then, up to permutation, the only possibilities for the triple (A_1, A_2, A_3) are

Table 2
$su(3)$ acts as derivations on A leaving trivial submodules

	u	v	z_1	z_2	z_3	z_4	z_5	z_6
u	$\eta_1 u+\theta_1 v$	$\eta_2 u+\theta_2 v$	z_1	z_2	z_3	z_4	z_5	z_6
v	$\eta_3 u+\theta_3 v$	$\eta_4 u+\theta_4 v$	z_3	z_6	$-z_1$	z_5	$-z_4$	$-z_2$
z_1	z_1	$-z_3$	$-u$	z_4	v	$-z_2$	z_6	$-z_5$
z_2	z_2	$-z_6$	$-z_4$	$-u$	z_5	z_1	$-z_3$	v
z_3	z_3	z_1	$-v$	$-z_5$	$-u$	z_6	z_2	$-z_4$
z_4	z_4	$-z_5$	z_2	$-z_1$	$-z_6$	$-u$	v	z_3
z_5	z_5	z_4	$-z_6$	z_3	$-z_2$	$-v$	$-u$	z_1
z_6	z_6	z_2	z_5	$-v$	z_4	$-z_3$	$-z_1$	$-u$

$(8, 8, 8)$, $(8, 1+1+6, 8)$ and $(1+1+6, 1+1+6, 1+1+6)$. In the second case, $\{L^{\circ}_a | a \in 1+1\} \subseteq \mathrm{End}_{su(3)}(8, 8)$. However, this space is one-dimensional. Therefore, we are left with $(8, 8, 8)$ and $(1+1+6, 1+1+6, 1+1+6)$. In both cases, up to isotopy, $su(3)$ acts by derivations.

We will consider the case $(1+1+6, 1+1+6, 1+1+6)$. A first approach to the product on A is given by table in Appendix A [4]. This table with $\eta_1=\theta_2=\theta_3=\sigma_1=\sigma_4=\tau_1=1$, $\eta_4=\tau_4=-1$ and $\eta_2=\eta_3=\theta_1=\theta_4=\sigma_2=\sigma_3=\tau_2=\tau_3=0$ gives the octonions. The usual bilinear form (,) of $\mathbb{O}$ is obtained by taking this basis to be orthonormal. We will denote by L°_x the left multiplication operator by x corresponding to this multiplication table and by L_x the one corresponding to the octonions. Let Z be the subspace spanned by $\{z_1, \ldots, z_6\}$. Clearly, for any $z \in Z$

$$L^{\circ}_x(z) = L_x(z) + L_{(x,u)((\sigma_1-1)u+\sigma_2 v)}(z) + L_{(x,v)(\sigma_3 u+(\sigma_4-1)v)}(z).$$

By defining $\varphi\colon x \mapsto x + (x, u)((\sigma_1-1)u+\sigma_2 v) + (x, v)(\sigma_3 u + (\sigma_4-1)v)$ we have $L^{\circ}_x(z) = L_{\varphi(x)}(z)$. In particular, $\varphi(x)$ is bijective and we can introduce a new product

$$x * y = \varphi^{-1}(x) \circ y,$$

so that $x * z = xz$ for any $x \in A$, $z \in Z$. It is also clear that $z * u = z \circ u = za$ and $z * v = z \circ v = zb$ with $a = \tau_1 u - \tau_2 v$ and $b = \tau_3 u - \tau_4 v$. This shows that $z * x = zx + (x, u)z(a-u) + (x, v)z(b-v)$. The map $\psi\colon x \mapsto x + (x, u)(a-u) + (x, v)(b-v)$ must be bijective. Thus we can define finally a new product $x \diamond y = x * \psi^{-1}(y)$. This product verifies

$$z \diamond y = zy \quad \text{and} \quad x \diamond z = xz$$

and $Z^{\perp} = \mathrm{span}\langle u, v\rangle$ is a two-dimensional subalgebra. In other words, up to isotopy we can assume that $\sigma_1=\sigma_4=\tau_1=1$, $\tau_4=-1$ and $\sigma_2=\sigma_3=\tau_2=\tau_3=0$ so the multiplication is given in Table 2. We can write the product on A as

$$x \circ y = xy + \sigma(x, y), \tag{6}$$

where $\sigma\colon \mathbb{O}\times\mathbb{O} \to \mathbb{C}$ is a bilinear map which vanishes on $\mathbb{O}\times Z$ and $Z\times\mathbb{O}$ and

$$\sigma(u, u) = (\eta_1-1)u + \theta_1 v, \quad \sigma(u, v) = \eta_2 u + (\theta_2-1)v,$$
$$\sigma(v, u) = \eta_3 u + (\theta_3-1)v, \quad \sigma(v, v) = (\eta_4+1)u + \theta_4 v.$$

Lemma 16. *$\mathbb{O}$ endowed with the product $x \circ y$ in (6) is a division algebra if and only if $\sigma(a, b) \neq -\lambda ab$ for any nonzero $a, b \in Z^{\perp}$ and $\lambda \geqslant 1$.*

Proof. Given $x \circ y = 0$ with $x \neq 0$ we write $x = x_0 + x_1$ with $x_0 \in Z^\perp$ and $x_1 \in Z$. The equality $x \circ y = 0$ implies that $xy = -\sigma(x, y)$ (so $x_0 \neq 0$). Let $c = \sigma(x, y) \in Z^\perp$. We have $y = -\frac{\bar{x}}{n(x)}c = -\frac{\bar{x}_0}{n(x)}c - \frac{\bar{x}_1}{n(x)}c$. In particular, $c = \sigma(x, y) = -\frac{1}{n(x)}\sigma(x_0, \bar{x}_0 c)$. With $a = x_0$ and $b = \frac{\bar{x}_0}{n(x_0)}c$ we have that

$$\sigma(a, b) = -\frac{n(x)}{n(x_0)}ab$$

with $\frac{n(x)}{n(x_0)} \geqslant 1$. The converse is similar. □

We can provide a more concrete criterion on when these algebras are division algebras. By the lemm, this is equivalent to the algebra $Z^\perp$ being a division algebra with the products $a \circ b + \lambda ab$ for all $\lambda \geqslant 0$. The determinant of the left multiplication operator by $\alpha u + \beta v$ is a quadratic form on α and β with coordinate matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ where

$$\begin{aligned} &\alpha_{11} = \lambda^2 + \lambda(\eta_1 + \theta_2) + \eta_1\theta_2 - \eta_2\theta_1, \\ &\alpha_{12} = \alpha_{21} = \tfrac{1}{2}\lambda(-\eta_2 + \eta_3 + \theta_1 + \theta_4) + \tfrac{1}{2}(-\eta_4\theta_1 + \eta_3\theta_2 - \eta_2\theta_3 + \eta_1\theta_4), \\ &\alpha_{22} = \lambda^2 + \lambda(-\eta_4 + \theta_3) + \eta_3\theta_4 - \eta_4\theta_3. \end{aligned}$$

In order to be a division algebra, for a given λ, the quadratic form should be either definite positive or negative. Since this must hold for any $\lambda \geqslant 0$, and for λ big enough it is definite positive, it must be definite positive for any $\lambda \geqslant 0$. This is equivalent to the nonexistence of roots $\lambda \geqslant 0$ of the quadratic and the quartic polynomials α_{11} and $\alpha_{11}\alpha_{22} - \alpha_{12}^2$. The polynomials corresponding to the octonions are $\alpha_{11} = (\lambda + 1)^2$ and $\alpha_{11}\alpha_{22} = (\lambda + 1)^4$.

5. Case $su(3)$ and A_i irreducible

By [4], in this case the product on A can be written from the 3×3 complex skew-Hermitian traceless matrices by

$$x{*}y = \alpha'[x, y] + \beta'\mathrm{i}\left(xy + yx - \frac{2}{3}t(xy)I\right),$$

where xy denotes the usual product of matrices, I the identity matrix and $\alpha'\beta' \neq 0$.

Since ternary derivations are preserved by extending scalars, we will work in a more general setting. We consider F an algebraically closed field of characteristic zero and $A = A_n (n \geqslant 2)$ the set of all $(n + 1) \times (n + 1)$ traceless matrices over F with the following product

$$x{*}y = \alpha xy + \beta yx - \frac{\alpha + \beta}{n + 1}t(xy)I,$$

where $(\alpha, \beta) \neq (0, 0)$. Clearly $\mathrm{ad}_a\colon x \mapsto ax - xa$ is a derivation of A, so $\mathrm{Der}(A, *)$ contains the subalgebra $\mathrm{span}\langle \mathrm{ad}_a | a \in A\rangle$, a Lie algebra of type A_n. Our main result in this section is

Proposition 17. *One of the following statements holds*:

(i) $\mathrm{Tder}(A) \cong A_n \oplus \mathscr{L}_0$.
(ii) *A is isotopic to an octonion algebra. In this case* $\alpha = -w^2\beta$ *with* $1 \neq w$ *a cubic root of the unit and* $\mathrm{Tder}(A) \cong D_4 \oplus \mathscr{L}_0$.
(iii) *A is isotopic to the traceless* 4×4 *matrices with product*

$$x{*}y = xy + yx - \frac{1}{2}t(xy)I.$$

In this case $\mathrm{Tder}(A) \cong A_5 \oplus \mathscr{L}_0$.

As a corollary

Corollary 18. *Let A be an eight-dimensional real division algebra obtained from the* 3×3 *complex skew-Hermitian traceless matrices with product given by*

$$x{*}y = \alpha'[x, y] + \beta'\mathrm{i}\left(xy + yx - \frac{2}{3}t(xy)I\right),$$

where xy denotes the usual product of matrices and $\alpha'\beta' \neq 0$. *Then either*

(i) $\mathrm{Tder}(A) \cong su(3) \oplus \mathscr{L}_0$ *or*
(ii) $\mathrm{Tder}(A) \cong D_4 \oplus \mathscr{L}_0$. *In this case* $\beta' = \pm\sqrt{3}\alpha'$ *and A is an isotope of* $\mathbb{O}$.

Proof. We can rewrite the product as $x{*}y = (\alpha' + \beta'\mathrm{i})xy + (\beta'\mathrm{i} - \alpha')yx - \frac{2\beta'\mathrm{i}}{3}t(xy)I$. By extending scalars, the previous proposition tells us that the only two possibilities are those in (i) and (ii). The latter occurs if and only if $(\alpha' + \beta'\mathrm{i}) = -w^2(\beta'\mathrm{i} - \alpha')$, that is, $\beta' = \pm\sqrt{3}\alpha'$. □

5.1. *The projections of* $\mathrm{Tder}(A)$

As an ad_A-module, $A \cong A_n \cong V(\lambda_1 + \lambda_n)$. It is well known [2] that $\mathrm{End}(A) \cong A^* \otimes A \cong V(\lambda_1 + \lambda_n) \otimes V(\lambda_1 + \lambda_n)$ decomposes as

$$\mathrm{End}(A) \cong \begin{cases} V(4\lambda_1) \oplus V(2\lambda_1) \oplus V(0) \quad \text{if } n = 1, \\ V(2\lambda_1 + 2\lambda_2) \oplus V(3\lambda_1) \oplus V(3\lambda_2) \oplus 2V(\lambda_1 + \lambda_2) \oplus V(0) \quad \text{if } n = 2, \\ V(2\lambda_1 + 2\lambda_3) \oplus V(2\lambda_1 + \lambda_2) \oplus V(\lambda_2 + 2\lambda_3) \\ \quad \oplus V(2\lambda_2) \oplus 2V(\lambda_1 + \lambda_3) \oplus V(0) \quad \text{if } n = 3, \\ V(2\lambda_1 + 2\lambda_n) \oplus V(2\lambda_1 + \lambda_{n-1}) \oplus V(\lambda_2 + 2\lambda_n) \\ \quad \oplus V(\lambda_2 + \lambda_{n-1}) \oplus 2V(\lambda_1 + \lambda_n) \oplus V(0) \quad \text{if } n \geqslant 4. \end{cases}$$

The bilinear form $(x, y) = t(xy)$ is nondegenerate on A, $(x{*}y, z) = (x, y{*}z)$ and induces an involution on $\mathrm{End}(A)$. Let $\mathrm{Skew}(A)$ be the skew-symmetric operators for (,), and $\mathrm{Sym}(A)$ the symmetric ones. Then

$$\mathrm{Skew}(A) \cong \begin{cases} V(2\lambda_1) \quad \text{if } n = 1, \\ V(3\lambda_1) \oplus V(3\lambda_2) \oplus V(\lambda_1 + \lambda_2) \quad \text{if } n = 2, \\ V(2\lambda_1 + \lambda_2) \oplus V(\lambda_2 + 2\lambda_3) \oplus V(\lambda_1 + \lambda_3) \quad \text{if } n = 3, \\ V(2\lambda_1 + \lambda_{n-1}) \oplus V(\lambda_2 + 2\lambda_n) \oplus V(\lambda_1 + \lambda_n) \quad \text{if } n \geqslant 4, \end{cases}$$

while $\mathrm{Sym}(A)$ is the direct sum of the remaining submodules. In particular, any irreducible submodule not isomorphic to $V(\lambda_1 + \lambda_n)$ is composed either of symmetric or skew-symmetric maps.

Given a Lie subalgebra S of $\mathrm{End}(A)$ containing ad_A then S decomposes as a direct sum of irreducible submodules and $\mathrm{ad}_A \cong V(\lambda_1 + \lambda_n) \subseteq S$. Since ad_A are skew-symmetric operators then S must contain the symmetric and skew-symmetric components of all its elements.

Let $\pi_i\colon (d_1, d_2, d_3) \mapsto d_i$ and $L_i = \pi_i(\mathrm{Tder}(A))$. L_i is a Lie subalgebra of $\mathrm{End}(A)$ which contains ad_A so it contains the symmetric and skew-symmetric components of all its elements. Therefore, L_i is invariant under the involution $d \mapsto d^*$ induced by (,). Thus, given $(d_1, d_2, d_3) \in \mathrm{Tder}(A)$, since (,) is associative, then $(-d_2^*, d_3, -d_1^*) \in \mathrm{Tder}(A)$. The map

$$\theta\colon (d_1, d_2, d_3) \mapsto (-d_2^*, d_3, -d_1^*)$$

is an automorphism of $\mathrm{Tder}(A)$ with $\Theta^3 = \mathrm{id}$. In particular, $L_1 = L_2^* = L_2 = L_3$.

In the following we will study the Lie algebra $L = [L_i, L_i]$. Since $\mathrm{ad}_A \subseteq L$ acts irreducibly on A, the impressive work of Dynkin [11] suggests that only a few possibilities for L are allowed.

The transposition $x \mapsto x^{\mathrm{T}}$ is an involution of $(A, *)$ so

$$\tau\colon (d_1, d_2, d_3) \mapsto (\bar{d}_1, \bar{d}_3, \bar{d}_2),$$

where $\bar{d}(x) = d(x^{\mathrm{T}})^{\mathrm{T}}$, is an automorphism with $\tau^2 = \mathrm{id}$. Both automorphisms θ and τ are related by

$$\tau\theta\tau = \theta^2. \tag{7}$$

Lemma 19. *We have that either* $L \cap \mathrm{Skew}(A) = \mathrm{ad}_A$ *or* $\mathrm{Skew}(A) \subseteq L$.

Proof. Let $L_0 = L \cap \mathrm{Skew}(A)$, since $\mathrm{Skew}(A) = V(2\lambda_1 + \lambda_{n-1}) \oplus V(\lambda_2 + 2\lambda_n) \oplus V(\lambda_1 + \lambda_n)$ then either $L_0 = \mathrm{ad}_A$ or L_0 contains a copy of $V(\lambda_2 + 2\lambda_n)$ or $V(2\lambda_1 + \lambda_{n-1})$. In case that $L_0 = \mathrm{ad}_A \oplus V(\lambda_2 + 2\lambda_n)$ then, since A is a faithful irreducible L_0-module, L_0 must be semisimple as a Lie algebra. The Killing form of L_0 would be nondegenerate and invariant, so ad_A and $V(\lambda_2 + 2\lambda_n)$ would be orthogonal and $V(\lambda_2 + 2\lambda_n)$ would be isomorphic to its dual module. However, this is false. In the same vein we can rule out the possibility $L_0 = \mathrm{ad}_A \oplus V(2\lambda_1 + \lambda_{n-1})$, and we are left with $L_0 = \mathrm{Skew}(A)$. □

Lemma 20. *We have that either*

(i) *L is either* ad_A, $\mathrm{Skew}(A)$ *or* $sl(A)$, *or*
(ii) $n = 3$ *and* $L \cong A_5$.

Proof. In case that $L \cap \mathrm{Skew}(A) = \mathrm{Skew}(A)$, since $\mathrm{Sym}(A) \cap sl(A)$ is an irreducible $\mathrm{Skew}(A)$-module then $L = \mathrm{Skew}(A)$ or $L = sl(A)$. In the following we may assume that $L_0 = L \cap \mathrm{Skew}(A) = \mathrm{ad}_A$. As an ad_A-module, $\mathrm{Sym}(A)$ decomposes as

$$\mathrm{Sym}(A) \cong \begin{cases} V(4\lambda_1) \oplus V(0) & \text{if } n = 1,\\ V(2\lambda_1 + 2\lambda_2) \oplus V(\lambda_1 + \lambda_2) \oplus V(0) & \text{if } n = 2,\\ V(2\lambda_1 + 2\lambda_3) \oplus V(2\lambda_2) \oplus V(\lambda_1 + \lambda_3) \oplus V(0) & \text{if } n = 3,\\ V(2\lambda_1 + 2\lambda_n) \oplus V(\lambda_2 + \lambda_{n-1}) \oplus V(\lambda_1 + \lambda_n) \oplus V(0) & \text{if } n \geqslant 4, \end{cases}$$

We will distinguish some cases:

1. $V(2\lambda_1 + 2\lambda_n) \subseteq L \cap \mathrm{Sym}(A)$: Consider the maps $\epsilon_{a,b}\colon x \mapsto (x, a)b + (x, b)a$. For any $d \in \mathrm{Skew}(A)$ we have $[d, \epsilon_{a,b}] = \epsilon_{d(a),b} + \epsilon_{a,d(b)}$. Thus, $\epsilon_{E_{1,n+1},E_{1,n+1}}$ is a maximal weight vector of weight $2\lambda_1 + 2\lambda_n$ and $V(2\lambda_1 + 2\lambda_n)$ is the submodule generated by $\epsilon_{E_{1,n+1},E_{1,n+1}}$. The minimal weight of $V(2\lambda_1 + 2\lambda_n)$ is $-(2\lambda_1 + 2\lambda_n)$ and the corresponding weight vector is $\epsilon_{E_{n+1,1},E_{n+1,1}}$.

Let $\delta_{a,b}\colon x \mapsto (x, a)b - (x, b)a$. Clearly, $\delta_{a,b} \in \mathrm{Skew}(A)$. Moreover, given any $f \in \mathrm{Sym}(A)$ we have that $[f, \epsilon_{a,b}] = -\delta_{f(a),b} + \delta_{a,f(b)}$. In particular, $[\epsilon_{E_{1,n+1},E_{1,n+1}}, \epsilon_{E_{n+1,1},E_{n+1,1}}] = -4\delta_{E_{1,n+1},E_{n+1,1}} \in L_0$. By commuting this element with $\mathrm{ad}_{E_{ij}}$ $i \neq j$ we get that $\mathrm{Skew}(A) = \delta_{A,A} \subseteq L_0$, contradicting our assumption.

2. $V(\lambda_1+\lambda_n) \subseteq L \cap \mathrm{Sym}(A)$: The maps $T_a: x \mapsto ax + xa - \frac{2}{n+1}t(ax)$ are symmetric and they form a submodule isomorphic to $V(\lambda_1+\lambda_n)$. Moreover,

$$[T_a, T_b](x) = [[a,b],x] + \frac{4}{n+1}t(ax)b - \frac{4}{n+1}t(bx)a.$$

Thus, $V(\lambda_1+\lambda_n) \subseteq L \cap \mathrm{Sym}(A)$ implies that $\mathrm{ad}_{[a,b]} + \frac{4}{n+1}\delta_{a,b} \in L_0$, which in turns implies that $\delta_{a,b} \in L_0$, so $L_0 = \mathrm{Skew}(A)$. This contradicts our assumption.

Note that at this point we have proved the statement for the case $n = 2$, so we will assume that $n \geqslant 3$.

3. $V(\lambda_2+\lambda_{n-1}) \subseteq L \cap \mathrm{Sym}(A)$ and $n \geqslant 3$: The map $\epsilon = \epsilon_{E_{1,n},E_{2,n+1}} - \epsilon_{E_{1,n+1},E_{2,n}}$ is a maximal weight vector of weight $\lambda_2+\lambda_{n-1}$. Thus, $V(\lambda_2+\lambda_{n-1})$ is the submodule generated by ϵ. The weight vector of minimal weight is $\epsilon_{E_{n,1},E_{n+1,2}} - \epsilon_{E_{n+1,1},E_{n,2}}$. Since $[\epsilon, \epsilon_{E_{n,1},E_{n+1,2}} - \epsilon_{E_{n+1,1},E_{n,2}}] = -\delta_{E_{2,n+1},E_{n+1,2}} + \delta_{E_{n,1},E_{1,n}} - \delta_{E_{2,n},E_{n,2}} + \delta_{E_{n+1,1},E_{1,n+1}} \in L_0$ and $L_0 = \mathrm{ad}_A$ then this map must be ad_D for some diagonal matrix $D \in A$. When applied to $E_{i,j}$ with $(i,j) \notin \{(1,n),(n,1),(2,n+1),(n+1,2),(n+1,1),(1,n+1),(2,n),(n,2)\}$ we obtain 0. The elements $E_{1,n}$, $E_{2,n+1}$, $E_{1,n+1}$ and $E_{2,n}$ are eigenvectors of eigenvalue 1, and the elements $E_{n,1}$, $E_{n+1,2}$, $E_{n+1,1}$ and $E_{n,2}$ are eigenvectors of eigenvalue -1. If we denote by ϵ_i the map that associates to D the ith element of its diagonal, then $\epsilon_i(D) - \epsilon_j(D) = 0$ if $(i,j) \notin \{(1,n),(n,1),(2,n+1),(n+1,2),(n+1,1),(1,n+1),(2,n),(n,2)\}$, $\epsilon_1(D) - \epsilon_n(D) = 1$, $\epsilon_2(D) - \epsilon_{n+1}(D) = 1$, $\epsilon_1(D) - \epsilon_{n+1}(D) = 1$ and $\epsilon_2(D) - \epsilon_n(D) = 1$.

In case that $n > 3$ then $\epsilon_i(D) = \epsilon_3(D)$, so $D = \epsilon_3(D)\mathrm{id}$. Since the trace of D is zero then $D = 0$. This contradicts the equality $\epsilon_1(D) - \epsilon_n(D) = 1$.

In case that $n = 3$ the previous conditions about $\epsilon_i(D)$ are compatible. They imply that $D = \mathrm{diag}(1/2, 1/2, -1/2, -1/2)$. In this case $L = \mathrm{ad}_A \oplus V(2\lambda_2)$ should be isomorphic to a simple Lie algebra of dimension 35, namely A_5. □

5.2. *Proof of Proposition 17*

L is a simple Lie algebra and A is an irreducible L-module, thus $L_1 = L_2 = L_3 = L \oplus F\mathrm{id}$. The kernel of the projection $\pi_2: \mathrm{Tder}(A) \to L_i$ is $\{(d_1, 0, d_3) \in \mathrm{Tder}(A)\}$. Given two ternary derivations $(d_1, 0, d_3), (d_1', 0, d_3') \in \mathrm{Tder}(A)$ then $(-d_3^*, -d_1^*, 0), ([d_1', d_3^*], 0, 0)$ belong to $\mathrm{Tder}(A)$ too. The product $A * A$ is a nonzero submodule of A for the action of ad_A, so $A * A = A$ and $[d_1', d_3^*] = 0$. Both $\pi_1(\ker \pi_2)$ and $\pi_3(\ker \pi_2)^*$ are commuting ideals of $L \oplus F\mathrm{id}$, hence one of them is $F\mathrm{id}$. In case that $\pi_1(\ker \pi_2) = F\mathrm{id}$ then $d_1 = \lambda\mathrm{id}$ for some $\lambda \in F$ so $(-d_3^*, -\lambda\mathrm{id}, 0) \in \mathrm{Tder}(A)$ and $d_3^* = \lambda\mathrm{id}$, thus proving that $\ker \pi_2 = F(\mathrm{id}, 0, \mathrm{id})$. If $\pi_3(\ker \pi_2) = F\mathrm{id}$ then $\ker \pi_2 = F(\mathrm{id}, 0, \mathrm{id})$ too. Similar arguments prove that $\ker \pi_1 = F(0, \mathrm{id}, -\mathrm{id})$ and $\ker \pi_3 = F(\mathrm{id}, \mathrm{id}, 0)$. The projection π_2 provides an isomorphism from the ideal $\{(d_1, d_2, d_3) \in \mathrm{Tder}(A) | t(d_2) = t(d_3) = 0\}$ onto L, so $\mathrm{Tder}(A) \cong L \oplus \mathscr{L}_0$.

Let us now rule out the possibility $L = sl(A)$. On the contrary, since $L = sl(A)$ is simple the maps

$$\xi: d_1 \mapsto d_2 \quad \text{and} \quad \xi': d_1 \mapsto d_3$$

with $(d_1, d_2, d_3) \in \mathrm{Tder}(A)$ induce automorphisms on L. Therefore, for any $d \in L$

$$\xi(d) = \begin{cases} PdP^{-1} \\ \text{or} \\ -Pd^{\mathrm{T}}P^{-1} \end{cases}$$

for some invertible linear map P. Similarly $\xi'(d) = QdQ^{-1}$ or $-Qd^{\mathrm{T}}Q^{-1}$ for some invertible Q. Making a change by isotopy we obtain a product $\circ$ on A with one of the following properties:

(1) $\{(d, d, d)|d \in sl(A)\} \subseteq \mathrm{Tder}(A, \circ)$: In this case $\circ \in \mathrm{Hom}_L(A \otimes A, A)$,
(2) $\{(d, -d^{\mathrm{T}}, d)|d \in sl(A)\} \subseteq \mathrm{Tder}(A, \circ)$: In this case $\circ \in \mathrm{Hom}_L(A^* \otimes A, A)$,
(3) $\{(d, d, -d^{\mathrm{T}})|d \in sl(A)\} \subseteq \mathrm{Tder}(A, \circ)$: In this case $\circ \in \mathrm{Hom}_L(A \otimes A^*, A)$,
(4) $\{(d, -d^{\mathrm{T}}, -d^{\mathrm{T}})|d \in sl(A)\} \subseteq \mathrm{Tder}(A, \circ)$: In this case $\circ \in \mathrm{Hom}_L(A^* \otimes A^*, A)$,

where A^* denotes the dual of A. As a module for $sl(A)$, A is isomorphic to $V(\lambda_1)$, $A \otimes A^* \cong V(\lambda_1 + \lambda_n) \oplus V(0)$ so $\mathrm{Hom}_L(A \otimes A^*, A) = 0$. Since $\mathrm{Hom}_L(A \otimes A, A) \cong \mathrm{Hom}_L(A, A \otimes A^*)$ then $\mathrm{Hom}_L(A \otimes A, A) = \mathrm{Hom}_L(A^* \otimes A^*, A) = 0$.

Now we will pay attention to the case $L = \mathrm{Skew}(A)$. Let $w \in F$, $w^3 = 1$ a primitive root of the unit. Since $\theta^3 = \mathrm{id}$, the only possible eigenspaces for θ are of the form $S(w^i) = \{(d, w^i d, w^{2i} d) \in \mathrm{Tder}(A)\}$. In case that $S(w) = S(w^2) = 0$ then $\mathrm{Der}(A) = \mathrm{Skew}(A)$ but $A \otimes A$ does not contain a $\mathrm{Skew}(A)$-submodule isomorphic to A. In case that $S(w) \neq 0$, up to changing w by w^2 we can assume that $V(2\lambda_1 + \lambda_{n-1}) \subseteq S(w)$. Since the map $\varphi = \delta_{E_{1,n+1}, E_{1,n}} : x \mapsto (x, E_{1,n+1})E_{1,n} - (x, E_{1,n})E_{1,n+1}$ belongs to $V(2\lambda_1 + \lambda_{n-1})$ then

$$(\varphi, w\varphi, w^2\varphi) \in \mathrm{Tder}(A).$$

In particular,

$$\varphi(E_{n,1} * E_{n+1,1}) = \begin{cases} 0, \\ -wE_{1,n+1} * E_{n+1,1} + w^2 E_{n,1} * E_{1,n} = (w^2\beta - w\alpha)E_{11} \\ \quad +w^2\alpha E_{nn} - w\beta E_{n+1,n+1} + (w - w^2)\frac{\alpha+\beta}{n+1} I. \end{cases}$$

If $n > 2$ then $n + 1 > 3$ and $\alpha = 0 = \beta$. Therefore, $n = 1, 2$. In case that $n = 1$ then $\mathrm{Skew}(A) = \mathrm{ad}_A$. In case that $n = 2$ then we have that $\mathrm{diag}(-w\alpha + w^2\beta, w^2\alpha, -w\beta) = \frac{w^2 - w}{3}(\alpha + \beta)I$, so $\alpha = -w^2\beta$ and up to isotopy A is an Okubo algebra, that is, an isotope of an octonion algebra [13].

Now we will consider the exceptional case appeared in the previous lemma. The Lie algebra L decomposes as $L = A_3 \oplus V(2\lambda_2)$ and the maximal weight vector of $V(2\lambda_2)$ is $\varphi = \epsilon_{E_{1,3}, E_{2,4}} - \epsilon_{E_{1,4}, E_{2,3}}$. Since by (7) the automorphism τ permutes $S(w)$ and $S(w^2)$ then they are submodules of the same dimension. Thus, $S(w) = 0 = S(w^2)$ and

$$\mathrm{Tder}(A) = \{(d, -d^*, -d^*)|d \in L\} \oplus \mathscr{L}_0.$$

The condition of ternary derivation for φ reads as

$$\varphi(x * y) = -\varphi(x) * y - x * \varphi(y).$$

Evaluating on $E_{4,2} * E_{3,1}$ we have

$$\varphi(E_{4,2} * E_{3,1}) = \begin{cases} 0, \\ -\varphi(E_{4,2}) * E_{3,1} - E_{4,2} * \varphi(E_{3,1}) \\ \quad = -\alpha E_{1,1} - \beta E_{2,2} - \beta E_{3,3} - \alpha E_{4,4} + \frac{\alpha+\beta}{2} I, \end{cases}$$

so $\alpha = \beta$ and the algebra is isotopic to the one in part (iii). The existence of this case is considered in the next section.

5.3. The exceptional case

The algebra $(A, *)$ corresponding to $n = 3$ and $0 \neq \alpha = \beta$ is exceptional among the family of algebras A_n of $(n + 1) \times (n + 1)$ traceless matrices with product

$$\alpha xy + \beta yx - \frac{\alpha+\beta}{n+1} t(xy) I.$$

In this section we will check that $\mathrm{Tder}(A,*) \cong A_5 \oplus \mathscr{Z}_0$. Since we have proved that either $L = \mathrm{ad}_A$ or $(A,*)$ is isotopic to an octonion algebra or $L \cong A_5$ then it basically amounts to proving that in this exceptional case $L \neq \mathrm{ad}_A$. The quickest way is to check that the map $d = \epsilon_{E_{1,3},E_{2,4}} - \epsilon_{E_{1,4},E_{2,3}}$ above provides a ternary derivation $(d,-d,-d)$. This map, being symmetric with respect to $(x,y) = t(xy)$ lays outside ad_A. In fact, by linearizing twice the identity

$$x^4 - \frac{1}{2} t(x^2) x^2 - \frac{1}{3} t(x^3) x + \frac{1}{4}\left(\frac{1}{2} t(x^2)^2 - t(x^4)\right) I$$

valid for any matrix x in A_3 it is easy to prove that the symmetric maps $x \mapsto \sum_i a_i x b_i + b_i x a_i - (a_i,x) b_i - (b_i,x) a_i$ with $\sum_i a_i b_i + b_i a_i = 0, a_i, b_i \in A_3$ provide ternary derivations of $(A,*)$ of the kind $(d,-d,-d)$, the map $\epsilon_{E_{1,3},E_{2,4}} - \epsilon_{E_{1,4},E_{2,3}}$ being a particular instance. We will however present another approach.

To understand the way A_5 acts as ternary derivations on A let us consider a vector space W ($\dim W \geqslant 2$) endowed with a nondegenerate symmetric bilinear form (,). For any $\varphi \in \mathrm{End}(W)$ let us denote by φ^* the adjoint of φ relative to (,), so $(\varphi(w), w') = (w, \varphi^*(w'))$ for any $w, w' \in W$. The Lie algebra $sl(W)$ of traceless linear maps over W acts on the skew-symmetric maps $so(W) = \{\delta \in \mathrm{End}(W) \mid \delta^* = -\delta\}$ by

$$\varphi \cdot \delta = \varphi\delta + \delta\varphi^*. \tag{8}$$

The module $so(W)$ with this action is isomorphic to the second exterior power $\wedge^2 W$ of W with the natural action, the isomorphism is given by $w_1 \wedge w_2 \mapsto \delta_{w_1,w_2}: w \mapsto (w_1,w)w_2 - (w_2,w)w_1$. The bilinear form $(\delta_1,\delta_2) = t(\delta_1\delta_2)$ is nondegenerate on $so(W)$ and for any $\varphi \in sl(W)$ we have $(\varphi\cdot\delta_1,\delta_2) = t(\varphi\delta_1\delta_2 + \delta_1\varphi^*\delta_2) = t(\delta_1\delta_2\varphi + \delta_1\varphi^*\delta_2) = (\delta_1, \varphi^*\cdot\delta_2)$.

Let us consider now a four-dimensional vector space V over an algebraically closed field of characteristic zero F, and the natural action of A_3 on V. The second exterior power $W = \wedge^2 V$ is six-dimensional and the isomorphism $\wedge^4 V \cong F$ given by the determinant relative to any fixed basis provides a nondegenerate symmetric bilinear form (,) on W invariant under the action of A_3, i.e. A_3 acts on W as skew-symmetric maps relative to (,). Since $\dim so(W) = 15$ we obtain that $A_3 \cong so(W)$, so $A_5 \cong sl(W)$ acts on A_3 by (8). This will be the action we are looking for.

Since $\dim W = 6$ then $\wedge^6 W \cong F$ determines a nonzero symmetric trilinear form $(,,): so(W) \otimes so(W) \otimes so(W) \cong \wedge^2 W \otimes \wedge^2 W \otimes \wedge^2 W \to \wedge^6 W \cong F$ invariant under $sl(W)$. The bilinear and trilinear forms on $so(W)$ define a commutative product $*$ by

$$(\delta_1,\delta_2,\delta) = (\delta_1 * \delta_2, \delta)$$

for any $\delta_1,\delta_2,\delta \in so(W)$. This product is related to the action of $sl(W)$ by

$$\begin{aligned}(\varphi\cdot(\delta_1 * \delta_2), d) &= (\delta_1 * \delta_2, \varphi^*\cdot\delta) = (\delta_1,\delta_2,\varphi^*\cdot\delta)\\ &= -(\varphi^*\cdot\delta_1,\delta_2,\delta) - (\delta_1,\varphi^*\cdot\delta_2,\delta)\\ &= ((-\varphi^*\cdot\delta_1) * \delta_2 + \delta_1 * (-\varphi^*\cdot\delta_2), \delta),\end{aligned}$$

that is,

$$\varphi\cdot(\delta_1 * \delta_2) = (-\varphi^*\cdot\delta_1) * \delta_2 + \delta_1 * (-\varphi^*\cdot\delta_2). \tag{9}$$

In this way $sl(W)$ acts as ternary derivations on $(so(W), *)$.

Let us check that, up to scalar multiples, this product $*$ is precisely the product $\delta_1 * \delta_2 = \delta_1\delta_2 + \delta_2\delta_1 - \frac{1}{2} t(\delta_1\delta_2)\mathrm{id}$. To this end, observe that in case that $\varphi \in so(W)$ then $\varphi^* = -\varphi$ and

$\varphi \cdot \delta = \varphi\delta - \delta\varphi$, so recovering the adjoint action of $so(W)$. Therefore, by (9), $*$ may be understood as a nonzero homomorphism $\mathrm{Sym}(so(W)) \to so(W)$ between $so(W)$-modules. In general there is no such homomorphisms. Our exceptional situation arises from the fact that $so(W) \cong A_3$, a Lie algebra of type A, and then, up to scalar multiples, there is only one such homomorphism, namely $\delta_1 * \delta_2 = \delta_1\delta_2 + \delta_2\delta_1 - \frac{1}{2}t(\delta_1\delta_2)\mathrm{id}$.

As an $sl(W)$-module, $so(W) \cong \wedge^2 W \cong V(\lambda_2)$ so $\mathrm{End}(so(W)) \cong V(\lambda_2) \otimes V(\lambda_4) \cong V(0) \oplus V(\lambda_1 + \lambda_5) \oplus V(\lambda_2 + \lambda_4)$. Looking at A_5 as inside of $\mathrm{End}(so(W))$ by (8), any Lie subalgebra of $\mathrm{End}(so(W))$ containing A_5 and the identity I is either $A_5 \oplus V(0)$ or the whole $\mathrm{End}(so(W))$. In particular, the projection of $\mathrm{Tder}(A, *)$ on each component is either $A_5 \oplus V(0)$ or the whole $\mathrm{End}(so(W))$. As at the begining of Section 5.2 one can rule out the latter case to get $\mathrm{Tder}(A) \cong A_5 \oplus \mathscr{Z}_0$.

Appendix A

Multiplication table of real division algebras containing $su(3)$ as derivations leaving trivial submodules

	u	v	z_1	z_2	z_3	z_4	z_5	z_6
u	$\eta_1 u + \theta_1 v$	$\eta_2 u + \theta_2 v$	$\sigma_1 z_1 + \sigma_2 z_3$	$\sigma_1 z_2 + \sigma_2 z_6$	$\sigma_1 z_3 - \sigma_2 z_1$	$\sigma_1 z_4 + \sigma_2 z_5$	$\sigma_1 z_5 - \sigma_2 z_4$	$\sigma_1 z_6 - \sigma_2 z_2$
v	$\eta_3 u + \theta_3 v$	$\eta_4 u + \theta_4 v$	$\sigma_3 z_1 + \sigma_4 z_3$	$\sigma_3 z_2 + \sigma_4 z_6$	$\sigma_3 z_3 - \sigma_4 z_1$	$\sigma_3 z_4 + \sigma_4 z_5$	$\sigma_3 z_5 - \sigma_4 z_4$	$\sigma_3 z_6 - \sigma_4 z_2$
z_1	$\tau_1 z_1 + \tau_2 z_3$	$\tau_3 z_1 + \tau_4 z_3$	$-u$	z_4	v	$-z_2$	z_6	$-z_5$
z_2	$\tau_1 z_2 + \tau_2 z_6$	$\tau_3 z_2 + \tau_4 z_6$	$-z_4$	$-u$	z_5	z_1	$-z_3$	v
z_3	$\tau_1 z_3 - \tau_2 z_1$	$\tau_3 z_3 - \tau_4 z_1$	$-v$	$-z_5$	$-u$	z_6	z_2	$-z_4$
z_4	$\tau_1 z_4 + \tau_2 z_5$	$\tau_3 z_4 + \tau_4 z_5$	z_2	$-z_1$	$-z_6$	$-u$	v	z_3
z_5	$\tau_1 z_5 - \tau_2 z_4$	$\tau_3 z_5 - \tau_4 z_4$	$-z_6$	z_3	$-z_2$	$-v$	$-u$	z_1
z_6	$\tau_1 z_6 - \tau_2 z_2$	$\tau_3 z_6 - \tau_4 z_2$	z_5	$-v$	z_4	$-z_3$	$-z_1$	$-u$

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