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The nonexistence of finite linear spaces with $v = n^2$ points and $b = n^2 + n + 2$ lines

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Abstract

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We show that any finite linear space on $v = n^2$ points and $b = n^2 + n + 2$ lines has $n \leqslant 4$. We also describe all such spaces.

1. Introduction

In [5], Stinson considered the case of a finite linear space on v points and b lines, $n^2 < v \leqslant b = n^2 + n + 2$, and showed that it was necessarily the case that $n \leqslant 3$. (For $n \leqslant 3$, it is easy to compute those linear spaces satisfying the above conditions on v and b.)

In this article, we consider the missing case $v = n^2$.

A *finite linear space* is a pair $S = (P, L)$, where P is a finite set of elements, called *points*, and L a set of subsets of P of size at least two, called *lines*, such that any two distinct points p and q are in (on) a unique line, denoted by pq.

The letters v and b denote $|P|$ and $|L|$, respectively. For any point p and line l, $b(p)$ and $v(l)$ denote the number of lines on p and the number of points on l. A point on r lines and a line on k points are also referred to as an *r-point* and a *k-line*.

A *projective plane of order n*, $n \geqslant 2$, is a linear space on $n^2 + n + 1$ points in which each line has $n + 1$ points. It follows that each point is on $n + 1$ lines and $b = v$.

An *affine plane of order n*, $n \geqslant 2$, is a linear space on n^2 points in which each line has n points. It follows that each point is on $n + 1$ lines and $b = n^2 + n$.

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A *near-pencil* is a linear space in which $v=b$ and, for some line, l, $v(l)=v-1$.

Let S be a linear space, and n the unique integer such that $n^2 \leqslant v < (n+1)^2$.

In the early 1970s de Witte proved, in a manuscript remaining unpublished due to his death, that $b \leqslant n^2+n+1$ if and only if S is a near-pencil or embeds in a projective plane of order n. This was reproved in 1982 by Stinson [4].

In 1983, Stinson considered, as mentioned above, the case $b=n^2+n+2$. Since $v \leqslant b$ in any finite linear space [1], there was only one case not covered by Stinson: $v=n^2$ and $b=n^2+n+2$. This is the case we handle in this paper, obtaining the following result.

Theorem 1.1. *Let S be a finite linear space with $v=n^2$ points and $b=n^2+n+2$ lines, $n \geqslant 1$. Then $n \leqslant 4$ and S is one of the following:*

(i) *The unique linear space on 9 points and 14 lines, with intersecting 7-point and 3-point lines ($n=3$). All other lines have just two points.*

(ii) *An affine plane of order $n=3$ or 4 less one line and all (if $n=3$) or all but one (if $n=4$) of its points, but with, in addition, three points at infinity joined by 2-lines, no point at infinity corresponding to the missing line if $n=4$.*

(iii) *The unique linear space on 9 points and 14 lines, with a single 3-point on two 4-lines and one 3-line, and with precisely three 5-points, one on each line on the 3-point, joined to each other by 2-point lines; all other points are 4-points ($n=3$).*

(iv) *The unique linear space on 16 points and 22 lines, with a single 4-point on three 5-lines and one 4-line, and with precisely three 6-points, one on each 5-line, joined to each other by 2-point lines; all other points are 5-points ($n=4$).*

2. Results used

We shall use the results listed below.

Theorem 2.1 (de Witte [2] or Stinson [4]). *Let S be a finite linear space, not a near-pencil, on $n^2 \leqslant v < (n+1)^2$ points. Then S embeds in a projective plane of order n if and only if $b \leqslant n^2+n+1$.*

Theorem 2.2 (Metsch [3]). *Let S be a finite linear space in which $b \leqslant v+b(p)-2$ for some point p. Then one of the following cases occurs:*

(a) *S is a projective plane of order $b(p)-1$ less $b-v$ points, and p is any point.*

(b) *S is a near-pencil.*

(c) *S is a projective plane or a near-pencil with one additional point p added to some line, and the necessary 2-point lines added.*

(d) *S is an affine plane of order $b(p)-1$ with one 'point at infinity', p not the point at infinity.*

(e) *S is an 's-fold inflated projective plane of order n', some $1 \leqslant s < n$. (See [3] for the definition.)*

Lemma 2.1. $\sum_l v(l)(v(l)-1)=v(v-1)$.

This is an easy counting argument, taking pairs of points in two ways.

Lemma 2.2. $\sum_p b(p)(b(p)-1)\leqslant b(b-1)$.

This also is easy, counting line pairs in two ways.

3. Proof of Theorem 1.1.

If for some point p, $b-v\leqslant b(p)-2$, then $n+4\leqslant b(p)$. We apply Metsch' theorem (Theorem 2.2). In cases (a) and (b), the numbers of points and lines are the same, which is not our situation. The parameters of case (c) do not match ours except for $n=3$, which yields (i) of Theorem 1.1. In case (d), S has both a square, and one plus a square, number of points. This is not possible. Case (e) yields (ii) of Theorem 1.1.

So, suppose that, for all points p, $b(p)\leqslant n+3$.Then also $v(l)\leqslant n+3$ for all lines l. If, for some point p, $b(p)\leqslant n-2$, then $v\leqslant(n-2)(n+2)+1=n^2-3$, a contradiction. So, we may assume $n-1\leqslant b(p)\leqslant n+3$ for all p.

Suppose there is an $(n-1)$-point p. Then for all lines l not on p, $v(l)\leqslant n-1$. Using Lemma 2.1, we get

$$(n-1)(n+3)(n+2)+(n^2+3)(n-1)(n-2)\geqslant n^2(n^2-1),$$

yielding $n\leqslant 4$. The cases $n=1,2,3$ are not possible. If $n=4$, $v=16$, $b=22$, there is a unique 3-point p, $v(l)\leqslant 7$ for all lines l and $4\leqslant b(q)\leqslant 7$ for all points $q\neq p$. Let l_i, $1\leqslant i\leqslant 3$, be the lines on p. There are three possibilities here for $\{v(l_i)\}$. They are $\{4,7\}$, $\{5,6,7\}$, $\{6\}$. Let a and c be the number of 2-lines and of 3-lines, respectively. So, $a+c=19$. By Lemma 2.1,

$$\sum_{i=1}^{3} v(l_i)\,[v(l_i)-1]=16\cdot 15-38-4c.$$

The first two possibilities for $\{v(l_i)\}$ lead to the contradiction $4\,|\,38$. The third case yields $c=28$, which is too large. Thus, $n=4$ cannot hold.

Hence, $n\leqslant b(p)\leqslant n+3$ for all points p.

Let l be an $(n+3)$-line. Counting lines meeting l yields $b\geqslant(n+3)(n-1)+1=n^2+2n-2$ and, so, $n\leqslant 4$. The cases $n=1,2$ are trivially excluded, and $n=3$ is quickly seen to be impossible. If $n=4$, each line meets l, and each point of l is a 4-point. It follows that every line other than l is a 4-line. But any point off l is a 7-point, and counting v at such a point gives a contradiction.

Let l be an $(n+2)$-line. If each point of l has $\geqslant n+1$ lines on it, then $b\geqslant(n+2)n+1$, a contradiction. In fact, this counting argument shows that l has at least two n-points for $n\geqslant 3$. Clearly, n cannot be 2. So, let p and q be distinct n-points of l. If some line $\neq l$ on p has $>n$ points, we contradict q an n-point. Thus, $v\leqslant n+2+(n-1)^2=n^2-n+3$, implying $n=3$, $v=9$, $b=14$ and precisely two points of l are 3-points. This is not possible. So, $(n+2)$-lines do not exist.

Suppose p is an n-point. Counting v at p, we see that p is on $n-1$ $(n+1)$-lines and 1 n-line. The cases $n=1,2$ are not possible, and, if $n\geqslant 3$, p is unique. At least n^2+n lines meet an $(n+1)$-line and it follows that $(n+2)$- or $(n+3)$-points exist. Let x be such a point, and let l be an $(n+1)$-line missing x. So, there is a line h on x missing l, and no point of h is an n- or $(n+1)$-point.

If $|h|\geqslant 3$, then h meets an $(n+1)$-line l' in a point y, say, and each point of $h\backslash\{y\}$ is on a line missing l'. Counting lines meeting and missing l' yields $b\geqslant n^2+n+3$, a contradiction. So, $|h|=2$. Set $h=\{x, y\}$ and now suppose that x is on an $(n+1)$-line l. There is a line h' on y missing l. Arguing as above, $h'=\{y,z\}$ for some point z. Similarly, $v(xz)=2$. So, the non-n- or non-$(n+1)$-points form a triangle of just three points, and, counting b, each must be an $(n+2)$-point and $n=3$ or 4. Then S must be as in (iii) or (iv) of Theorem 1.1.

We may now assume that $n+1\leqslant b(p)\leqslant n+3$ and $v(l)\leqslant n+1$ for all points p and lines l.

Also, by Lemma 2.1, $(n+1)$-lines exist. Let l be an $(n+1)$-line. At least n^2+n+1 lines meet it; so, either there is a single $(n+2)$-point in S and it is on l, or there is a single line h of $(n+2)$-points missing l. (In either case, $(n+3)$-points do not exist.)

Suppose x is a unique $(n+2)$-point and $x\in l$. If there is an $(n+1)$-line not on x, then there is a line of $(n+2)$-points through x missing l, a contradiction. So, all $(n+1)$-lines are on x. Let p be a point on no $(n+1)$-line. Then p is on $n+1$ n-lines and, so, in particular, px is an n-line. Hence, all lines on x are n- or $(n+1)$-lines. Counting v at x now yields a contradiction.

So, suppose that a line h of $(n+2)$-points misses l, and all other points are $(n+1)$-points. We may then assume that h misses all $(n+1)$-lines. Counting v at a point of h, we see that any point of h is on at least three n-lines. Considering an n-line meeting h, and counting b, we have $|h|\leqslant n/2+1$. If some point of h is on a 2-line different from h, then $n/2+2+n(n-1)\geqslant v$ implies $n\leqslant 4$. The cases $n=1,2$ are trivially excluded. The case $n=3$ leads to (ii) of Theorem 1.1. If $n=4$, then $|h|=2$ or 3. Consider first the case $h=\{x,y,z\}$ and $|xp|=2$. Then p is on two 5-lines. Label these $\{p, 1, 2, 3, 4\}$ and $\{p, 5, 6, 7, 8\}$. The remaining lines on x may be labelled $\{x, 1, 5, 12\}$, $\{x, 2, 6, 11\}$, $\{x, 3, 7, 10\}$, $(x, 4, 8, 9\}$. Since py and pz must be 4-lines, we may label these $\{p, 9, 10, y\}$ $\{p, 11, 12, z\}$. The lines joining y to 11 and 12 must be 4-point lines. There are essentially three different situations:

(i) $11\in 1y$, $12\in 2y$. Consider lines on 9. We must have $3,5\in 911$ and $3,6\in 912$, a contradiction.

(ii) $11\in 1y$, $12\in 3y$. Consider lines on 9 and 10. We must have $3,5\in 911$ and $4,5\in 1011$, a contradiction.

(iii) $11\in 3y$, $12\in 4y$. It follows after some computation that the following sets must be lines: $\{y, 12, 4, 6\}$, $\{10, 12, 2, 8\}$, $\{10, 11, 5, 4\}$, $\{y, 11, 3, 8\}$, $\{9, 11, 1, 7\}$, $\{9, 12, 3, 6\}$, $\{z, 9, 2, 5\}$, $\{z, 10, 1, 6\}$, $\{y, 2, 7\}$, $\{y, 1, 8\}$. Now z and 3 must be on a line with 5, 6, or 8, which is not possible.

So, consider the case $n=4$ and $|h|=2$; $h=\{x, y\}$ and xp a 2-line. There are two subcases: (a) p is on three 5-lines and $|py|=3$; (b) p is on two 5-lines and two 4-lines.

In case (a), let q be the third point on py. Then each line on q meets each of the 5-lines and, so, q is on four 4-lines. But q is also on distinct lines qp and qx, a contradiction to $b(q)=5$.

In case (b), let q and z be the other two points on py, and let l be the 4-line on p distinct from py. Then xy misses l and, so, one of xq, xz meets l, say xq. But then xq also meets the 5-lines on p and, so, is itself a 5-line, a contradiction.

We may henceforth assume that no point of h is on a 2-line different from h.

We prove now, by induction on $2 \leqslant |h| \leqslant n/2+1$, that $S\backslash h$ embeds in a projective plane of order n.

$|h|=2$. Delete one point of h. The induced linear space has n^2-1 points and n^2+n+1 lines. In addition, it contains n-lines. Use one n-line to induce a spread and to introduce a new point, but no new line. By de Witte's theorem (Theorem 2.1), this new structure, and hence $S\backslash h$, embeds in a projective plane of order n.

$2<|h| \leqslant n/2+1$. Suppose some n-line l misses h. Then each point of S is on a unique line, other than h, missing l. Since $(n+1)$-lines exist, there are $n+1$ such lines, and we introduce a new $(n+1)$-point while at the same time deleting a point of h. The number of lines in the new structure remains at n^2+n+2. A counting argument (relative to lines) shows that no $(n+1)$-line in the new system meets h less its point. Hence, by induction, $S\backslash h$ embeds in a projective plane of order n.

So, suppose all n-lines meet h. If all n-lines meet each other, a given n-line l induces a partition of $S\backslash h$ into $n+1$ point sets, giving

$$v-|h| \leqslant (n+1)(n-2)+n+1-2(|h|-1),$$

yielding a contradiction. So, there exist disjoint n-lines l and l'. Two such n-lines induce a partition of $S\backslash h$ comprising precisely $n+1$ lines. However, each of l, l' meets precisely n^2+2 lines and together they meet n^2 lines. So, precisely $n-2$ lines miss both. Now any line missing l or l' is in the partition, while along with lines of the partition meeting l or l' at h, there are, therefore, $n+2$ lines in the partition, a contradiction. This completes the inductive proof.

The linear space $S'=S\backslash h$ has $b'=n^2+n+1$ lines and $v'=n^2-|h|$ points. An $(n+2)$-point of h, therefore, corresponds to a partition of $n+1$ lines in S'. Since S' contains n^2+n+1 lines, any two such partitions are already on a common line and, so, S does not exist.

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