On Meromorphically Normal Families of Meromorphic Mappings of Several Complex Variables into $P^N(C)$

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We shall prove some meromorphic normality criteria for families of meromorphic mappings of several complex variables into $P^N(C)$, the complex $N$-dimensional projective space, related to Nochka's Picard type theorems. Some related results e.g., extension theorems, improved normality criteria, and quasi-normality criteria will be obtained also. The technique in this paper mainly depends on Stoll's normality criteria for families of non-negative divisors on a domain of $C^n$.

Key Words: complex projective spaces, extension theorems, hyperplanes in general position, meromorphic mappings, non-negative divisors, normal families, Picard type theorems.

1. INTRODUCTION

Montel [13] defined the notion of a quasi-normal family of meromorphic functions in one complex variable and obtained several results related to this notion. Afterwards, Rutishauser [16] generalized some of them to the case of meromorphic functions in several complex variables. By definition, a quasi-normal family of meromorphic functions on a domain $D$ in $C^n$ is a family $F$ such that any sequence in $F$ has a compactly convergent subsequence outside a nowhere dense analytic subset of $D$. Fujimoto [7] introduced the notion of a meromorphically normal family of meromorphic mappings into $P^N(C)$, the $N$-dimensional complex projective space, which improves the notion of quasi-normal family in [16], and gave some
sufficient conditions for a family of meromorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$ to be meromorphically normal.

By using the technique from [2], recently the author [18, 19] got some normality criteria for families of holomorphic mappings of several complex variables into $P^N(C)$. Inspired by the idea in [18, 19], we shall give some meromorphic normality criteria for families of meromorphic mappings in several complex variables into $P^N(C)$ related to Nochka’s Picard type theorem. Many examples will be given to complement our theory.

2. STATEMENT OF MAIN RESULTS

For the general reference of this paper, see [4, 7, 15, 19].

Let $A$ be a non-empty open subset of a domain $D$ in $C^n$ such that $S = D - A$ is an analytic set in $D$. Let $f: A \to P^N(C)$ be a holomorphic mapping. Let $U$ be a non-empty connected open subset of $D$. A holomorphic mapping $f$ from $U$ into $C$ is said to be a representation of $f$ on some neighborhood of $z$ in $D$.

Define the polydisc

$$\Delta_n := \Delta_n(z_0, r) = \{ (z_1, \ldots, z_n) \in C^n; |z_i - z_i^0| < r, i = 1, \ldots, n \}$$

for $z_0 = (z_1^0, \ldots, z_n^0) \in C^n$ and $r = (r_1, \ldots, r_n)$ with $r_i > 0 (i = 1, \ldots, n)$. Let $f$ be a meromorphic mapping from $\Delta_n(z_0, r)$ into $P^N(C)$. Then $f$ always has a representation $f(z) = (f_1(z), \ldots, f_{n+1}(z))$ on $\Delta_n(z_0, r)$ with

$$\dim \{ z \in \Delta_n(z_0, r); f_1(z) = \ldots = f_{n+1}(z) = 0 \} \leq n - 2.$$ 

A representation of $f$ satisfying this condition is called a reduced representation of $f$ on $\Delta_n(z_0, r)$.

Let $f$ be a meromorphic mapping of a domain $D$ in $C^n$ into $P^N(C)$. Then for any $z \in D$, $f$ always has a reduced representation on some neighborhood of $z$ in $D$. We denote by $I(f)$ the set of all points of indetermination of $f$ on $D$. Then $I(f)$ is an analytic set in $D$ with $\dim I(f) \leq n - 2$. Obviously a meromorphic mapping from $D$ into $P^N(C)$ is a holomorphic mapping from $D$ into $P^N(C)$ if and only if $I(f) = \emptyset$.

DEFINITION 1. Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into a compact complex manifold $M$. $F$ is said to be a normal family on $D$ if any sequence in $F$ contains a subsequence which converges
uniformly on compact subsets of $D$ to a holomorphic mapping of $D$ into $M$.

**Definition 2.** A sequence $\{f^{(p)}(z)\}$ of meromorphic mappings from a domain $D$ in $\mathbb{C}^n$ into $P^N(\mathbb{C})$ is said to **meromorphically converge** on $D$ to a meromorphic mapping $f(z)$ if and only if, for any $z \in D$, each $f^{(p)}(z)$ has a reduced representation

$$\tilde{f}^{(p)}(z) = (f^{(p)}_1(z), \ldots, f^{(p)}_{N+1}(z))$$

on some fixed neighborhood $U$ of $z$ such that $(f^{(p)}_i(z))_{p=1}^\infty$ converges uniformly on compact subsets of $U$ to a holomorphic function $f_i(z)$ $(i = 1, \ldots, N+1)$ on $U$ with the property that

$$\tilde{f}(z) := (f_1(z), \ldots, f_{N+1}(z))$$

is a representation of $f(z)$ on $U$, where $f_{i_0}(z) \neq 0$ on $U$ for some $i_0$.

For a detailed discussion on meromorphic convergence, see [7].

**Definition 3.** Let $F$ be a family of meromorphic mappings of a domain $D$ in $\mathbb{C}^n$ into $P^N(\mathbb{C})$. $F$ is said to be a **meromorphically normal family** on $D$ if any sequence in $F$ has a meromorphically convergent subsequence on $D$.

**Definition 4.** A sequence $\{f^{(p)}(z)\}$ of meromorphic mappings from a domain $D$ in $\mathbb{C}^n$ into $P^N(\mathbb{C})$ is said to be **quasi-regular** on $D$ if and only if any $z \in D$ has a neighborhood $U$ with the property that $\{f^{(p)}(z)\}$ converges compactly on $U$ outside a nowhere dense analytic subset $S$ of $U$; i.e., for any domain $G \subset U - S$ (the closure $\overline{G}$ of $G$ is a compact subset of $U - S$), there is some $p_0$ such that $\Delta(f^{(p)}) \cap G = \emptyset$ $(p \geq p_0)$ and $\{f^{(p)}|G; p \geq p_0\}$ converges uniformly on $G$ to a holomorphic mapping of $G$ into $P^N(\mathbb{C})$.

**Remark.** Obviously a meromorphically convergent sequence on $D$ is always a quasi-regular sequence on $D$. But a quasi-regular sequence on $D$ need not imply meromorphic convergence on $D$ (see [7, (3.4)]).

**Definition 5.** Let $F$ be a family of meromorphic mappings of a domain $D$ in $\mathbb{C}^n$ into $P^N(\mathbb{C})$. $F$ is said to be a **quasi-normal family** on $D$ if any sequence in $F$ has a subsequence so as to be quasi-regular on $D$.

Let $f(z) \neq 0$ be a holomorphic function on the connected open neighborhood $D$ of $a \in \mathbb{C}^n$. Then $f(z) = \sum_{m=0}^\infty p_m(z - a)$, where the series converges uniformly to $f$ on an open neighborhood of $a \in \mathbb{C}^n$ and the term $p_m$ is either identically zero or a homogeneous polynomial of degree $m$. The number $\mu_f(a) := \min\{m; p_m \neq 0\}$ is said to be the zero-multiplicity of $f$ at $a$. By definition, a non-negative divisor on a domain $D$ in $\mathbb{C}^n$ is
a non-negative integer-valued function $\nu$ on $D$ such that for every $a \in D$ there exists a holomorphic function $f(z) \neq 0$ on a neighborhood $U$ of $a$ with $\nu(z) = \nu_f(z)$ on $U$. Furthermore we define the support $\text{supp } \nu$ of the divisor $\nu$ on $D$ by $\text{supp } \nu := \{z \in D; \nu(z) \neq 0\}$.

Let $X$ be an analytic subset of a domain $D$ in $C^n$ and denote by $\text{Reg}(X)$ the set of all regular points of $X$. Let $\text{Reg}(X) = \bigcup_{\lambda \in \Lambda} X_\lambda$ be the decomposition of $\text{Reg}(X)$ into connected components. Then $X_\lambda$ are locally closed complex submanifolds of $D$, the closures $\overline{X}_\lambda$ are irreducible analytic subsets of $D$, and $X = \bigcup_{\lambda \in \Lambda} \overline{X}_\lambda$ is the irreducible decomposition of $X$. An irreducible analytic subset of $D$ is of pure dimension and the irreducible decomposition is unique (see [15, p. 141, 4, p. 124] for references).

Let $\nu$ be a non-negative divisor on a domain $D$ in $C^n$. Then $\text{supp } \nu$ is either empty or an analytic subset of pure dimension $n-1$ in $D$. Rewrite $\nu$ as the formal sum $\nu = \sum_{\lambda \in \Lambda} n_\lambda X_\lambda$, where $X_\lambda$ are the irreducible components of $\text{supp } \nu$ and $n_\lambda$ is $\nu(z)$ on $X_\lambda \cap \text{Reg}(\text{supp } \nu)$. ($\nu(z)$ is constant on $X_\lambda \cap \text{Reg}(\text{supp } \nu)$). We define $\sum_{\lambda \in \Lambda} \nu(X_\lambda \cap E)$ as the $(2n-1)$-dimensional Lebesgue area of $\nu \cap E$ regardless of multiplicities for any compact set $E \subset D$ and define $\sum_{\lambda \in \Lambda} n_\lambda \nu(X_\lambda \cap E)$ as the $(2n-1)$-dimensional Lebesgue area of $\nu \cap E$ with counting multiplicities for any compact set $E \subset D$, where $\nu(X_\lambda \cap E)$ is the $(2n-1)$-dimensional Lebesgue area of $X_\lambda \cap E$.

A subset $H$ of $P^N(C)$ is called a hyperplane if there is a $N$-dimensional linear subspace $\tilde{H}$ of $C^{N+1}$ with $\rho(\tilde{H} - \{0\}) = H$. If we write $(C^{N+1})^*$ for the dual space of $C^{N+1}$, then there is a $\alpha \in (C^{N+1})^* - \{0\}$ such that $\tilde{H} = \{\alpha = 0\} := \{x \in C^{N+1}; \alpha(x) = 0\}$. Let $B^*$ be the set of Euclidean unit vectors of $(C^{N+1})^*$. Then $\alpha, \beta \in B^*$ satisfy $\tilde{H} = \{\alpha = 0\} = \{\beta = 0\}$ if and only if $\alpha = c \beta$ with $c \in C$ and $|c| = 1$. Let $H_1, \ldots, H_{N+1}$ be hyperplanes in $P^N(C)$ and let $\alpha_i = (\alpha_i^{(i)}, \ldots, \alpha_i^{(N+1)}) \in B^*$ with $H_i = \{\alpha_i = 0\}$ $(i = 1, \ldots, N+1)$. Define

$$
D(H_1, \ldots, H_{N+1}) := \det \begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_{N+1}
\end{pmatrix},
$$

which is determined independently of a choice of $\alpha_i \in B^*$ with $H_i = \{\alpha_i = 0\}$ $(i = 1, \ldots, N+1)$. When $N = 1$, $D(a, b)$ is just the spherical distance between $a$ and $b$ in $P(C) = C \cup \{\infty\}$.

**Definition 6.** Let $H_1, \ldots, H_q$ $(q \geq N + 1)$ be hyperplanes in $P^N(C)$. Define

$$
D(H_1, \ldots, H_q) := \prod D(H_1, \ldots, H_{N+1}),
$$
where the product $\prod$ is taken over all $\{i_1, \ldots, i_{N+1}\}$ with $1 \leq i_1 < i_2 < \cdots < i_{N+1} \leq q$. We say that the hyperplane family $H_1, \ldots, H_q$ ($q \geq N + 1$) in $\mathbb{P}^N(\mathbb{C})$ is located in general position if $D(H_1, \ldots, H_q) > 0$.

Let $f$ be a meromorphic mapping from a domain $D$ in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Take a hyperplane $H$ in $\mathbb{P}^N(\mathbb{C})$ defined by

$$\tilde{H} := \{(z_1, \ldots, z_{N+1}) \in \mathbb{C}^{N+1}; a_1z_1 + \cdots + a_{N+1}z_{N+1} = 0\}.$$  

For $a \in D$, taking a reduced representation

$$\tilde{f}(z) = (f_1(z), \ldots, f_{N+1}(z))$$

on a neighborhood $U$ of $a$, we consider the holomorphic function

$$F(z) := a_1f_1(z) + \cdots + a_{N+1}f_{N+1}(z).$$

Then the divisor $\nu(f, H)(z) := \nu_f(z)$ ($z \in U$) is determined independently of a choice of reduced representations and hence is well defined on the totality of $D$ and obviously $\text{Supp} \nu(f, H)$ is either empty or a pure $(n - 1)$-dimensional analytic set in $D$ if $f(D) \not\subset H$ (i.e., $F(z) \neq 0$ on $U$). We define $\nu(f, H) := \infty$ on $D$ and $\text{Supp} \nu(f, H) = D$ if $f(D) \subset H$. Sometimes we identify $f^{-1}(H)$ with the divisor $\nu(f, H)$ on $D$. Rewrite $\nu(f, H)$ as the formal sum $\nu(f, H) = \sum_{\lambda \in \Lambda} \lambda X_{\lambda}$, where $X_{\lambda}$ are the irreducible components on $\text{Supp} \nu(f, H)$ and $n_{\lambda}$ are the constant $\nu(f, H)(z)$ on $X_{\lambda} \cap \text{Reg}(\text{Supp} \nu(f, H))$. For any positive integer or infinite $m$, the closure

$$\{z \in \text{Supp} \nu(f, H); \nu(f, H)(z) < m\} = \bigcup_{\lambda: n_{\lambda} < m} X_{\lambda}$$

is either empty or a pure $(n - 1)$-dimensional analytic set in $D$ and the $2(n - 1)$-dimensional Lebesque areas of the two sets

$$\{z \in \text{Supp} \nu(f, H); \nu(f, H)(z) < m\}$$

and

$$\{z \in \text{Supp} \nu(f, H); \nu(f, H)(z) < m\}$$

coincide. This concept is important in proving Theorems 1 and 4.

We note that $I(f) \subset \text{Supp} \nu(f, H)$ always holds for any hyperplane $H$ in $\mathbb{P}^N(\mathbb{C})$ and $I(f) = \emptyset$ if $f(D) \cap H = \emptyset$ for some hyperplane $H$ in $\mathbb{P}^N(\mathbb{C})$.

We say that a meromorphic mapping $f$ intersects $H$ with multiplicity at least $m$ on $D$ if $f(D) \subset H$, $f(D) \cap H \neq \emptyset$, and $\nu(f, H)(z) \geq m$ for all $z \in \text{Supp} \nu(f, H)$ and that $f$ intersects $H$ with multiplicity $\infty$ on $D$ if $f(D) \subset H$ or $f(D) \cap H = \emptyset$. 
Recently, by using the technique from [2], the author [18, 19] gave the following result related to the Picard type theorem given by Nochka [14].

**THEOREM A.** Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. Suppose that for each $f \in F$, there exist $q \geq 2N + 1$ hyperplanes $H_1(f), \ldots, H_q(f)$ (which may depend on $f$) in $P^N(C)$ such that $f$ intersects $H_j(f)$ with multiplicity at least $m_j$ ($j = 1, \ldots, q$), where $m_j$ ($j = 1, \ldots, q$) are fixed positive integers and may be infinite, with

$$\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q - (N + 1)}{N}$$

and

$$\inf \{ D(H_1(f), \ldots, H_q(f)) ; f \in F \} > 0.$$  

Then $F$ is a normal family on $D$.

Fujimoto [7] introduced the notion of meromorphic convergence and gave the following result.

**THEOREM B.** Let $F$ be a family of meromorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$ and let $H_i$ ($i = 1, \ldots, 2N + 1$) be $2N + 1$ hyperplanes in $P^N(C)$ located in general position such that for each $f \in F$, $f(D) \not\subset H_i$ ($i = 1, \ldots, 2N + 1$) and for any fixed compact subset $K$ of $D$, the $2(n - 1)$-dimensional Lebesque areas of $f^{-1}(H_i) \cap K$ ($i = 1, \ldots, 2N + 1$) with counting multiplicities for all $f$ in $F$ are bounded above. Then $F$ is a meromorphically normal family on $D$.

In this paper, we shall give the following improvement of Theorem B related to Theorem A.

**THEOREM 1.** Let $F$ be a family of meromorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. Suppose that for each $f \in F$, there exist $q \geq 2N + 1$ hyperplanes $H_1(f), \ldots, H_q(f)$ (which may depend on $f$) in $P^N(C)$ with

$$\inf \{ D(H_1(f), \ldots, H_q(f)) ; f \in F \} > 0$$

and

$$f(D) \not\subset H_i(f) \quad (i = 1, \ldots, N + 1)$$

such that the following two conditions are satisfied.

1. For any fixed compact subset $K$ of $D$, the $2(n - 1)$-dimensional Lebesque areas of $f^{-1}(H_i(f)) \cap K$ ($i = 1, \ldots, N + 1$) with counting multiplicities for all $f$ in $F$ are bounded above.
2. There exists a nowhere dense analytic set $S$ in $D$ such that for any fixed compact subset $K$ of $D - S$, the $2(n - 1)$-dimensional Lebesgue areas of
\[
\{ z \in \text{supp } \nu(f, H_j(f)); \nu(f, H_j(f))(z) < m_j \} \cap K
\]
\[(j = N + 2, \ldots, q),
\]
regardless of multiplicities for all $f$ in $F$, are bounded above, where $\{m_j\}_{j=N+2}^q$ are fixed positive integers and may be $\infty$ with $\sum_{j=N+2}^q (1/m_j) < \frac{2 - (N + 1)}{N}$.

Then $F$ is a meromorphically normal family on $D$.

Remark. If $f(D) \subset H$, then $\nu(f, H)(z) = \infty$ on $D$ and hence $\{ z \in D; \nu(f, H)(z) < \infty \} = \emptyset$ in condition 2 of Theorem 1.

We shall derive the following conclusion from Theorem 1.

**Corollary 2.** Let $F$ be a family of holomorphic mappings of a domain $D$ in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that for each $f \in F$, there exist $q \geq 2N + 1$ hyperplanes $H_1(f), \ldots, H_q(f)$ (which may depend on $f$) in $\mathbb{P}^N(\mathbb{C})$ with
\[
\inf \{ D(H_1(f), \ldots, H_q(f)); f \in F \} > 0
\]
such that the following two conditions are satisfied.

1. $f(D) \cap H_i(f) = \emptyset$ $(i = 1, \ldots, N + 1)$ for any $f$ in $F$.
2. There exists a nowhere dense analytic set $S$ in $D$ such that for any fixed compact subset $K$ of $D - S$, the $2(n - 1)$-dimensional Lebesgue areas of
\[
\{ z \in \text{supp } \nu(f, H_j(f)); \nu(f, H_j(f))(z) < m_j \} \cap K
\]
\[(j = N + 2, \ldots, q),
\]
regardless of multiplicities for all $f$ in $F$, are bounded above, where $\{m_j\}_{j=N+2}^q$ are fixed positive integers and may be $\infty$ with $\sum_{j=N+2}^q (1/m_j) < \frac{2 - (N + 1)}{N}$.

Then $F$ is a normal family on $D$.

Remark. A weakened version of Corollary 2 was announced in [18].

In order to obtain an insight into the version of Theorem A for meromorphic mappings, we shall prove the following extension theorem related to Nochka's Picard theorem.

**Theorem 3.** Let $S$ be an analytic subset of a domain $D$ in $\mathbb{C}^n$ with $\dim S \leq n - 2$. Let $f$ be a holomorphic mapping from $D - S$ into $\mathbb{P}^N(\mathbb{C})$. If there exist $q \geq 2N + 1$ hyperplanes $H_1, \ldots, H_q$ in $\mathbb{P}^N(\mathbb{C})$ in general position such that $f$ intersects $H_j$ with multiplicity at least $m_j$ $(j = 1, \ldots, q)$ on $D - S$,
where \( m_j ( j = 1, \ldots, q) \) are positive integers and may be \( \infty \), with \( \sum_{j=1}^{q} \frac{1}{m_j} < \frac{2-(N+1)}{N} \), then the holomorphic mapping \( f \) from \( D - S \) into \( \mathbb{P}^N(\mathbb{C}) \) extends to a holomorphic mapping from \( D \) into \( \mathbb{P}^N(\mathbb{C}) \).

Theorem 3 will play a key role in proving the following result.

**THEOREM 4.** Let \( F \) be a family of meromorphic mappings of a domain \( D \) in \( \mathbb{C}^n \) into \( \mathbb{P}^N(\mathbb{C}) \). Suppose that for each \( f \in F \), there exist \( q \geq 2N + 1 \) hyperplanes \( H_1(f), \ldots, H_q(f) \) (which may depend on \( f \)) in \( \mathbb{P}^N(\mathbb{C}) \) with

\[
\inf \{ D(H_1(f), \ldots, H_q(f)); \ f \in F \} > 0
\]

such that for any fixed compact subset \( K \) of \( D \), the \( 2(n-1) \)-dimensional Lebesque areas of

\[
\{ z \in \text{supp} \nu(f, H_j(f)); \nu(f, H_j(f))(z) < m_j \} \cap K \quad (j = 1, \ldots, q),
\]

regardless of multiplicities for all \( f \) in \( F \), are bounded above, where \( \{ m_j \}_{j=1}^{q} \) are fixed positive integers and may be \( \infty \) with \( \sum_{j=1}^{q} \frac{1}{m_j} < \frac{2-(N+1)}{N} \). Then \( F \) is a quasi-normal family on \( D \).

Theorem 4 greatly improves [16, Theorem 18]; cf. [7, Theorem 8.1].

### 3. SOME EXAMPLES

Here we give some examples to complement our theory in this paper.

**EXAMPLE 1.** Let \( D = \{ z \in \mathbb{C}; |z| < 1 \} \) and \( f_n(z) = nz \ (z \in D) \). Then \( \{ f_n(z) \} \) is normal on \( D \). Since \( f_n(z) \) has a reduced representation \( f_n(z) = (z, \bar{z}) \ (n = 1, 2, \ldots) \), \( \{ f_n \} \) is meromorphically normal on \( D \). Obviously \( \{ f_n \} \) satisfies the assumption of Theorem 1. Hence the assumption of Theorem 1 need not imply that \( F \) is normal on \( D \) even if \( F \) is a family of holomorphic mappings of \( D \) into \( \mathbb{P}^N(\mathbb{C}) \).

**EXAMPLE 2.** Let \( f_n(z) = (nz)^n \) be defined on \( D = \{ z \in \mathbb{C}; |z| < 1 \} \). Then \( \{ f_n(z) \} \) is not meromorphically normal on \( D \) but \( \{ f_n(z) \} \) is quasi-normal on \( D \) (see [7, p. 28] for references). For any compact subset \( K \) of \( D \), the numbers of points of \( f_n^{-1}(1) \cap K \) for all \( n \in \mathbb{N} \) are bounded above. Since \( \text{supp} f_n^{-1}(\infty) = \emptyset \) and \( \text{supp} f_n^{-1}(0) = \{ 0 \} \), the multiplicities required in condition 1 of Theorem 1 cannot be removed.

**EXAMPLE 3.** Let \( f_n(z) = e^{nz} \) be defined on \( D = \{ z \in \mathbb{C}; |z| < 1 \} \). Since \( \{ f_n \} \) has no subsequence which is convergent at any point in the subset \( \{ z = y\sqrt{-1}; 0 < y < 1 \} \) of \( D \), \( \{ f_n \} \) is not quasi-normal on \( D \) and hence \( \{ f_n \} \) is not meromorphically normal on \( D \). Obviously \( f_n^{-1}(0) = f_n^{-1}(\infty) = \emptyset \).
Then \( \{f_n\} \) satisfies condition 1 but does not satisfy condition 2 in the assumption of Theorem 1.

**Example 4.** Let \( \{f_n(z)\}_{n=1}^\infty \) be a sequence of holomorphic mappings of \( D := \{z \in \mathbb{C}; |z| < 1\} \) into \( \mathbb{P}^N(\mathbb{C}) \) which have reduced representations

\[
\tilde{f}_n(z) := (z, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}) \quad (n = 1, 2, \ldots)
\]

Then \( \{f_n(z)\} \) is only meromorphically convergent on \( D \) but is not normal on \( D \). Obviously \( f_n(z) \) intersects any a hyperplane \( H \) in \( \mathbb{P}^N(\mathbb{C}) \) with at most a point in \( D \) and has \( N \) exceptional hyperplanes \( H_i = \rho(\tilde{H}_i - \{0\}) \) where \( \tilde{H}_i := \{(z_0, z_1, \ldots, z_N) \in C^{N+1}; z_i = 0 \} \) \( (i = 1, \ldots, N) \). Then the \( N + 1 \) exceptional hyperplanes in the assumption of Corollary 2 cannot be reduced.

**Example 5.** Let \( f_n(z) = nz \) be defined on \( D = \{z \in \mathbb{C}; 0 < |z| < 1\} \). Then \( \{f_n(z)\} \) is a normal family on \( D \). It is easy to verify that for any \( a_n, b_n, c_n \in P(\mathbb{C}) - \{nz; z \in D\} \), we have \( D(a_n, b_n, c_n) \rightarrow 0 \) as \( n \rightarrow \infty \) (see Definition 6 for \( D(a, b, c) \)). Hence \( \{f_n(z)\} \) does not satisfy the assumption of Corollary 2 in [19] but \( \{f_n(z)\} \) obviously satisfies the assumption of Corollary 2.

**Example 6.** Let

\[
B_r = \{ z = (z_1, \ldots, z_{n+1}) \in C^{n+1}; |z_1|^2 + \cdots + |z_{n+1}|^2 < r^2 \} \quad (r > 0)
\]

and

\[
E = \{ (u_1, \ldots, u_{n+1}) \in C^{n+1}; 1 \leq |u_1|^2 + \cdots + |u_{n+1}|^2 \leq 2 \}.
\]

Let \( u^{(j)} = (u_1^{(j)}, \ldots, u_{n+1}^{(j)}) \) \( (j = 1, 2, \ldots) \) be a countable and dense subset of \( E \). Define a holomorphic mapping \( f_{(i,j)}(z) \) of \( B_{1/2} \) into \( \mathbb{P}^n(\mathbb{C}) \) which has a reduced representation

\[
\tilde{f}_{(i,j)}(z) := (z_1^i + u_1^{(j)}, \ldots, z_{n+1}^i + u_{n+1}^{(j)})
\]

from \( z \in B_{1/2} \) into \( C^{n+1} - \{0\} \) \( (i, j = 1, 2, \ldots) \). Obviously \( \{f_{(i,j)}\}_{i, j=1}^\infty \) is a normal family on \( B_{1/2} \) and by [19, Theorem 4] there exist \( 2n + 1 \) hyperplanes \( H_1^{(i,j)}, \ldots, H_{2n+1}^{(i,j)} \) located in \( \mathbb{P}^n(\mathbb{C}) \) such that

\[
\inf \{D(H_1^{(i,j)}, \ldots, H_{2n+1}^{(i,j)}); i, j = 1, 2, \ldots \} > 0
\]

and

\[
f_{(i,j)}(B_{r_0}) \cap H_k^{(i,j)} = \emptyset \quad (k = 1, \ldots, 2n + 1; i, j = 1, 2, \ldots)
\]
for some fixed small $r_0$ with $0 < r_0 < \frac{1}{2}$. Hence $(f_{i,j})$ restricted on $B_{r_0}$ satisfies the assumption of Theorem 1.

But $(f_{i,j})$ does not satisfy the assumption of Theorem B on any neighborhood of $z = 0$ in $B_{1/2}$. In fact we shall verify that the $2n$-dimensional Lebesque areas of $f_{i,j}^{-1}(H) \cap B_r (i, j = 1, 2, \ldots)$ with counting multiplicities are not bounded above for any fixed $r (0 < r < \frac{1}{2})$ and any fixed hyperplane $H$ in $P^n(C)$. Let $H$ be defined by

$$H = \{(z_1, \ldots, z_{n+1}) \in C^{n+1}; \ a_1 z_1 + \cdots + a_{n+1} z_{n+1} = 0\}.$$ Then $\nu(f_{i,j}, H)$ is defined by the holomorphic function

$$\left(a_1 z_1^i + \cdots + a_{n+1} z_{n+1}^i\right) + \left(a_1 u_1^{(j)} + \cdots + a_{n+1} u_{n+1}^{(j)}\right).$$

Since $\{u^{(j)}\}$ is a dense subset of $E$, we can choose a subsequence $(f_{j_k})_{k=1}^\infty$ with

$$\lim_{k \to \infty} (a_1 u_1^{(j_k)} + \cdots + a_{n+1} u_{n+1}^{(j_k)}) = 0.$$ Hence $\lim_{k \to \infty} \nu(f_{i,j_k}, H) = \nu_0$, as a sequence of divisors on $B_{1/2}$ (see Definition 8 in Section 4 for the convergence of divisors), where $g(z) = a_1 z_1^i + \cdots + a_{n+1} z_{n+1}^i$. The $2n$-dimensional Lebesque area of $\nu_0 \cap B_r (0 < r < \frac{1}{2})$ with counting multiplicities is at least $c_0 r^{2n}$ for a fixed positive constant $c_0$ (see [3, Lemma 3, p. 140]). Thus the $2n$-dimensional Lebesque areas of $\nu_0 \cap B_r (i = 1, 2, \ldots)$ with counting multiplicities are not bounded above for any $0 < r < \frac{1}{2}$ and hence by [17, Lemma 2.23] the $2n$-dimensional Lebesque areas of $\nu(f_{i,j_k}, H) \cap B_r (i, k = 1, 2, \ldots)$ with counting multiplicities are not bounded above for any $0 < r < \frac{1}{2}$. Then the $2n$-dimensional Lebesque areas of $f_{i,j_k}^{-1}(H) \cap B_r (i, j = 1, 2, \ldots)$ with counting multiplicities are not bounded above for any fixed $r (0 < r < \frac{1}{2})$ and any fixed hyperplane $H$ in $P^n(C)$.

4. SOME LEMMAS

To prove our results, we need some preparations.

We define the limit of a sequence $(F_k)_{k=1}^\infty$ of closed subsets of a locally compact Hausdorff space $M$ as follows:

**DEFINITION 7.** A point $x$ of $M$ is called a limit point of $(F_k)$ if there exist an integer $k_0$ and points $a_k \in F_k (k > k_0)$ such that $x = \lim a_k$. A point of $M$ is called a cluster point of $(F_k)$ if it is a limit point of some subsequence of $(F_k)$. If the set of limit points coincides with the set of cluster points, $(F_k)$ is said to converge to this set $F$, and write $\lim F_k = F$. (For a detailed discussion of this convergent concept, see [17, pp. 196–201]).
LEMMA 1. Let \( \{N_i\} \) be a sequence of pure \((n-1)\)-dimensional analytic subsets of a domain \( D \) in \( \mathbb{C}^n \). Suppose that the \( 2(n-1) \)-dimensional Lebesgue areas of \( N_i \cap K \) regardless of multiplicities \((i = 1, 2, \ldots)\) are bounded above for any fixed compact subset \( K \) of \( D \) and \( \{N_i\} \) converges to \( N \) as a sequence of closed subsets of \( D \). Then \( N \) is either empty or a pure \((n-1)\)-dimensional analytic subset of \( D \). (See [17, Proposition 4.11, 3, Theorem 1] for more general analytic subsets.)

LEMMA 2. Let \( \{N_i\} \) be a sequence of pure \((n-1)\)-dimensional analytic subsets of a domain \( D \) in \( \mathbb{C}^n \). If the \( 2(n-1) \)-dimensional Lebesgue areas of \( N_i \cap E \) regardless of multiplicities \((i = 1, 2, \ldots)\) are bounded above for any fixed compact set \( E \) of \( D \), then \( \{N_i\} \) is normal as a family of closed subsets of \( D \) (see [17, Proposition 4.12]).

Stoll [17] introduced the concept of convergence of a net of divisors. In the special case which we use, his definition reduces to the following:

DEFINITION 8. Let \( \{v_i\}_{i=1}^\infty \) be a sequence of non-negative divisors on a domain \( D \) in \( \mathbb{C}^n \). It is said to converge to a non-negative divisor \( v \) on \( D \) if and only if any \( a \in D \) has a neighborhood \( U \) such that there exist holomorphic functions \( h_i(z) (\neq 0) \) and \( h(z) (\neq 0) \) on \( U \) with \( v_i(z) = v_i^h(z) \) and \( v(z) = v^h(z) \) on \( U \) such that \( h_i(z) \) converges to \( h(z) \) uniformly on compact subsets of \( U \).

LEMMA 3. A sequence \( \{v_i\} \) of non-negative divisors on a domain \( D \) in \( \mathbb{C}^n \) is normal in the sense of the convergence of divisors on \( D \) if and only if the \( 2(n-1) \)-dimensional Lebesgue areas of \( v_i \cap E \) \((i = 1, 2, \ldots)\) with counting multiplicities are bounded above for any fixed compact set \( E \) of \( D \) (see [17, Theorem 2.24]).

LEMMA 4. Let \( \{f_i\} \) be a sequence of meromorphic mappings of a domain \( D \) in \( \mathbb{C}^n \) into \( \mathbb{P}^N(\mathbb{C}) \) and let \( S \) be a nowhere dense analytic subset in \( D \). Suppose that \( \{f_i\} \) meromorphically converges on \( D - S \) to a meromorphic mapping \( f \) of \( D \) into \( \mathbb{P}^N(\mathbb{C}) \). If there exists a hyperplane \( H \) in \( \mathbb{P}^N(\mathbb{C}) \) such that \( f(D - S) \not\subset H \) and \( (v_i(f_i), H) \) is a convergent sequence of divisors on \( D \), then \( \{f_i\} \) is meromorphically convergent on \( D \) (see [7, Proposition 3.5]).

LEMMA 5. Let \( \{f_i\} \) be a sequence of meromorphic mappings of a domain \( D \) in \( \mathbb{C}^n \) into \( \mathbb{P}^N(\mathbb{C}) \) and let \( S \) be a nowhere dense analytic subset in \( D \). Suppose that \( \{f_i\} \) meromorphically converges on \( D - S \) to a meromorphically mapping \( f \) of \( D - S \) into \( \mathbb{P}^N(\mathbb{C}) \). If for each \( f_i \) there exist \( N + 1 \) hyperplanes \( H_i(f_i), \ldots, H_{N+1}(f_i) \) in \( \mathbb{P}^N(\mathbb{C}) \) with

\[
\inf\{D(H_i(f_i), \ldots, H_{N+1}(f_i)) ; \ i = 1, 2, \ldots \} > 0
\]
such that $2(n - 1)$-dimensional Lebesgue areas of $f_i^{-1}(H_k(f_i)) \cap E$ ($k = 1, \ldots, N + 1; i = 1, 2, \ldots$) with counting multiplicities are bounded above for any fixed compact subset $E \subset D$, then $\{f_i\}$ has a meromorphically convergent subsequence on $D$.

**Proof of Lemma 5.** Without loss of generality we take $D = \Delta_n$ (a polydisc).

**Case 1.** We assume that $H_k(f_i)$ is the same for all $f_i$; i.e., $H_k := H_k(f_i)$ ($k = 1, \ldots, N + 1$) is independent of $i$. We can take some $k_0$ ($1 \leq k_0 \leq N + 1$) with $f(\Delta_n - S) \not\subset H_{k_0}$ because $H_k$ ($k = 1, \ldots, N + 1$) are located in $P^N(C)$ in general position by the assumption of Lemma 5. On the other hand, the divisor sequence $\{\nu(f_i, H_{k_0})\}_{i=1}^\infty$ has a convergent subsequence $\nu(f_i, H_{k_0})_{i=1}^\infty$ on $\Delta_n$ by Lemma 3. Then by Lemma 4 $\{f_i\}$ has a meromorphically convergent subsequence $\{f_i\}_{i=1}^\infty$ on $\Delta_n$.

**Case 2.** Here we shall prove Lemma 5 for the general case by taking linear coordinate transformation and using the conclusion of Lemma 5 in Case 1.

Consider the hyperplanes $H_k(f_i)$ ($k = 1, \ldots, N + 1$ and $i = 1, 2, \ldots$). Then there exist $\alpha_k(f_i) = (\alpha_k^1(f_i), \ldots, \alpha_k^{N+1}(f_i)) \in B^*$, the set of Euclidean unit vectors of $(C^{N+1})^*$, such that

$$H_k(f_i) = \{(z_1, \ldots, z_{N+1}) \in C^{N+1}; \alpha_k^1(f_i)z_1 + \cdots + \alpha_k^{N+1}(f_i)z_{N+1} = 0\}$$

for $k = 1, \ldots, N + 1$ and $i = 1, 2, \ldots$. Thus we have

$$\inf_i \det \begin{pmatrix} \alpha_1^1(f_i) & \cdots & \alpha_1^{N+1}(f_i) \\ \vdots & \ddots & \vdots \\ \alpha_{N+1}^1(f_i) & \cdots & \alpha_{N+1}^{N+1}(f_i) \end{pmatrix}$$

$$= \inf_i \{D(H_1(f_i), \ldots, H_{N+1}(f_i)) \geq 0.$$

Since $|\alpha_k^j(f_i)| \leq 1$ ($k, j = 1, 2, \ldots, N + 1$ and $i = 1, 2, \ldots$), without loss of generality we assume that $\alpha_k^j(f_i) \to \alpha_k^j$ ($k, j = 1, \ldots, N + 1$) as $i \to \infty$ (otherwise we can find a required subsequence of $\{f_i\}$). Hence we have

$$\geq \liminf_{i \to \infty} D(H_1(f_i), \ldots, H_{N+1}(f_i)) > 0.$$
Let $\tilde{f}_i = (f_i^1, \ldots, f_i^{N+1})$ be a reduced representation of $f_i$ on $\Delta_n$ and define

$$
\tilde{g}_i(z) := (f_i^1(z), \ldots, f_i^{N+1}(z)) \left( \begin{array}{ccc}
\alpha_1^1(f_i) & \cdots & \alpha_1^{N+1}(f_i) \\
\vdots & \ddots & \vdots \\
\alpha_{N+1}^1(f_i) & \cdots & \alpha_{N+1}^{N+1}(f_i)
\end{array} \right)^T,
$$

where $A^T$ denotes the transposed matrix of a matrix $A$. Now consider the sequence $(g_i(z))$ of meromorphic mappings from $\Delta_n$ into $P^N(C)$ where $g_i(z)$ has the given reduced representation $\tilde{g}_i(z)$ on $\Delta_n$. Let

$$
H_i^0 := \rho\{(z_1, \ldots, z_{N+1}) \in C^{N+1} - \{0\}; z_i = 0\} \quad (i = 1, 2, \ldots, N+1).
$$

Then

$$
g_i^{-1}(H_k^0) = f_i^{-1}(H_k(f_i)) \quad (k = 1, \ldots, N+1; i = 1, 2, \ldots).
$$

By the assumption of Lemma 5 $(f_i(z))$ meromorphically converges on $\Delta_n - S$ to a meromorphic mapping $f(z)$ of $\Delta_n - S$ into $P^N(C)$ and thus $(g_i(z))$ meromorphically converges on $\Delta_n - S$ to a meromorphic mapping $g(z)$ of $\Delta_n - S$ into $P^N(C)$. Then by the conclusion of Lemma 5 in Case 1 there exists a subsequence $(g_i(z))_{i=1}^n$ such that the meromorphically converges on $\Delta_n$ to a meromorphic mapping $g_0(z)$ of $\Delta_n$ into $P^N(C)$ and hence $(f_i(z))_{i=1}^n$ meromorphically converges on $\Delta_n$ to a meromorphic mapping $f_0(z)$ of $\Delta_n$ into $P^N(C)$ which has a representation

$$
(f_0^1(z), \ldots, f_0^{N+1}(z)) := (g_0^1(z), \ldots, g_0^{N+1}(z)) \times \left( \begin{array}{ccc}
\alpha_1^1 & \cdots & \alpha_1^{N+1} \\
\vdots & \ddots & \vdots \\
\alpha_{N+1}^1 & \cdots & \alpha_{N+1}^{N+1}
\end{array} \right)^{-1}
$$

on $\Delta_n$, where $(g_0^1(z), \ldots, g_0^{N+1}(z))$ is a representation of $g_0(z)$ on $\Delta_n$ and $A^{-1}$ denotes the inverse matrix of a square matrix $A$. Hence $(f_i(z))_{i=1}^n$ has a meromorphically convergent subsequence on $\Delta_n$. The proof of Lemma 5 is completed.

**Lemma 6.** Let $(f_i)$ be a meromorphically convergent sequence of holomorphic mappings of a polydisc $\Delta_n$ in $C^n$ into $P^N(C)$. If for each $f_i$, there exist $N + 1$ hyperplanes $H_i^1(f_i), \ldots, H_{N+1}(f_i)$ in $P^N(C)$ with

$$
\inf\{D(H_i(f_i), \ldots, H_{N+1}(f_i)); i = 1, 2, \ldots\} > 0
$$
and

\[ f_i(\Delta_n) \cap H_0(f_i) = \emptyset \quad (k = 1, \ldots, N + 1; i = 1, 2, \ldots), \]

then \( \{f_i\} \) converges uniformly on compact subsets of \( \Delta_n \) to a holomorphic mapping of \( \Delta_n \) into \( P^N(C) \).

**Proof of Lemma 6.** Let \( z_0 \in \Delta_n \). By the assumption of Lemma 6, every \( f_i(z) \) has a reduced representation

\[ \tilde{f}_i(z) = (f_i^1(z), \ldots, f_i^{N+1}(z)) \quad (i = 1, 2, \ldots) \]

on a fixed neighborhood \( U(z_0) \) such that \( \{\tilde{f}_i(z)\}_{i=1}^{\infty} \) converges uniformly on compact subsets of \( U(z_0) \) to \( \tilde{f}_0(z) := (f_0^1(z), \ldots, f_0^{N+1}(z)) \) \( (\neq 0) \) on \( U(z_0) \). Now we shall prove \( \tilde{f}_0(z) \neq 0 \) everywhere on \( U(z_0) \) and hence \( \{f_i\} \) converges uniformly on compact subsets of \( \Delta_n \) to a holomorphic mapping of \( \Delta_n \) into \( P^N(C) \).

Let \( \alpha_k f_i = (\alpha^1_k(f_i), \ldots, \alpha^{N+1}_k(f_i)) \in B^* \), the set of Euclidean unit vectors of \( (C^{N+1})^* \), with \( H_k(f_i) = \{\alpha_k f_i = 0\} \). Without loss of generality we assume that \( \alpha^1_k(f_i) \to \alpha^1_k \) \( (k = 1, \ldots, N + 1) \) as \( i \to \infty \) (otherwise we can find a required subsequence). Define

\[
\tilde{g}_i(z) := \begin{pmatrix}
\alpha^1_1(f_i) & \cdots & \alpha^{N+1}_1(f_i) \\
\vdots & \ddots & \vdots \\
\alpha^1_{N+1}(f_i) & \cdots & \alpha^{N+1}_{N+1}(f_i)
\end{pmatrix}
\]

on \( U(z_0) \). Then \( \{\tilde{g}_i(z)\} \) converges uniformly on compact subsets of \( U(z_0) \) to a holomorphic mapping \( \tilde{g}_0(z) \) of \( U(z_0) \) into \( \mathbb{C}^{N+1} \), where

\[
\tilde{g}_0(z) := \begin{pmatrix}
g_0^1(z) & \cdots & g_0^{N+1}(z)
\end{pmatrix}
\]

\[
:= \begin{pmatrix}
f_0^1(z) & \cdots & f_0^{N+1}(z)
\end{pmatrix}
\]

Since \( (f_0^1(z), \ldots, f_0^{N+1}(z)) \neq 0 \) on \( U(z_0) \) and \( \det(\alpha^1_k)^{(N+1)\times(N+1)} \neq 0 \), we have at least one \( g_0^{k_0}(z) \neq 0 \) on \( U(z_0) \) for some \( 1 \leq k_0 \leq N + 1 \). Since \( \Sigma_{k=1}^{N+1} \alpha^k(f_i) f_i^k(z) \) \( (\neq 0) \) everywhere on \( U(z_0) \) by the assumption of Lemma 6 converges uniformly on compact subsets of \( U(z_0) \) to \( g_0^{k_0}(z) \) \( (\neq 0) \) on \( U(z_0) \) as \( i \to \infty \), by the Hurwitz theorem in several complex variables [15, Lemma 1.5.16] we have \( g_0^{k_0}(z) \neq 0 \) everywhere on \( U(z_0) \). Hence \( \tilde{f}_0(z) \neq 0 \) everywhere on \( U(z_0) \). We complete the proof of Lemma 6.
5. PROOF OF THEOREM 1

Without loss of generality we assume $D = \Delta_n$ (a polydisc).

Take any sequence $\{f_i\} \subset F$. By the assumption and Lemma 2 we can find a subsequence (again denoted by $\{f_i\}$) such that

$$\lim_{i \to \infty} f_i^{-1}(H_k(f_i)) = S_k \quad (k = 1, \ldots, N + 1)$$

as a sequence of closed subsets of $\Delta_n$, where $S_k$ are either empty or pure $(n - 1)$-dimensional analytic sets of $\Delta_n$ by Lemma 1 and

$$\lim_{i \to \infty} \left( \{z \in \supp \nu(f, H_k(f_i)); \nu(f, H_k(f_i))(z) < m_k\} - S \right) = S_k$$

$(k = N + 2, \ldots, q)$ as a sequence of closed subsets of $\Delta_n - S$, where $S_k$ are either empty or pure $(n - 1)$-dimensional analytic sets of $\Delta_n - S$ by Lemma 1. Let $E := \bigcup_{k = 1}^{q} S_k - S$. Then $E$ is either empty or a pure $(n - 1)$-dimensional analytic set of $\Delta_n - S$.

For any fixed point $z_0$ in $(\Delta_n - S) - E$, there exist an integer $i_0$ and a neighborhood $U(z_0)$ in $(\Delta_n - S) - E$ such that

$$f_i^{-1}(H_k(f_i)) \cap U(z_0) = \emptyset \quad (k = 1, 2, \ldots, N + 1)$$

and each $f_i(z)$ intersects $H_k(f_i)$ with multiplicities at least $m_k$ $(k = N + 2, \ldots, q)$ on $U(z_0)$ for $i \geq i_0$. Hence $\{f_i(z)\}_{i = i_0}^{\infty}$ is a sequence of holomorphic mappings of $U(z_0)$ into $P^N(C)$ and by Theorem A $\{f_i(z)\}$ is a normal family on $U(z_0)$. Therefore, by the usual diagonal argument, we can find a subsequence (again denoted by $\{f_i\}$) which converges uniformly on compact subsets of $(\Delta_n - S) - E$ to a holomorphic mapping $f$ of $(\Delta_n - S) - E$ into $P^N(C)$. By Lemma 5 $\{f_i\}$ has a meromorphically convergent subsequence (again denoted by $\{f_i\}$) on $\Delta_n - S$ and again by Lemma 5 $\{f_i\}$ has a meromorphically convergent subsequence on $\Delta_n$. Then $F$ is a meromorphically normal family on $\Delta_n$. The proof of Theorem 1 is completed.

6. PROOF OF COROLLARY 2

By Theorem 1 $F$ is a meromorphically normal family on $D$ and hence by Lemma 6 $F$ is a normal family on $D$. This proves Corollary 2.
7. PROOF OF THEOREM 3

DEFINITION 9. Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain and let $M$ be a complete complex Hermitian manifold with metric $ds^2_M$. A holomorphic mapping $f(z)$ from $\Omega$ into $M$ is said to be a normal holomorphic mapping from $\Omega$ into $M$ if and only if there exists a positive constant $c$ such that for all $z \in \Omega$ and all $\xi \in T_z(\Omega)$,

$$\left| ds^2_M(f(z), df(z)(\xi)) \right| \leq c K_\Omega(z, \xi),$$

where $df(z)$ is the mapping from $T_z(\Omega)$ into $T_{f(z)}(M)$ induced by $f$ and $K_\Omega$ denotes the infinitesimal Kobayashi metric on $\Omega$.

For a detailed discussion of normal holomorphic mapping, see [1] and for the basic notation of hyperbolic space, see [10, 12, 15].

LEMMA 7. Let $\Omega \subset \mathbb{C}^n$ be a hyperbolic domain and let $M$ be a compact complex Hermitian manifold. Let $f: \Omega \rightarrow M$ be a holomorphic mapping such that for every sequence of holomorphic mappings $\varphi_j(z)$ from the unit disc $D$ in $\mathbb{C}$ into $\Omega$, the sequence $(f \circ \varphi_j(z))_{j=1}^\infty$ from $D$ into $M$ is a normal family on $D$. Then $f$ is a normal holomorphic mapping from $\Omega$ into $M$ (See [1, Proposition 1.14, 2, Proposition 2.9]).

LEMMA 8. Let $M$ be a complex manifold and let $S$ be a complex analytic subset of $M$ with $\text{codim} S \geq 2$. Then $K_{M-S} = K_M$ on $M - S$ (i.e., the infinitesimal Kobayashi metric $K_{M-S}$ is the restriction of $K_M$ to $M - S$) (See [15, Proposition 1.2.22]).

LEMMA 9. Let $f$ be a holomorphic mapping from a hyperbolic domain $\Omega$ in $\mathbb{C}^n$ into $P^N(\mathbb{C})$. If there exist $q \geq 2N + 1$ hyperplanes $H_1, \ldots, H_q$ in $P^N(\mathbb{C})$ in general position such that $f$ intersects $H_j$ with multiplicity at least $m_j$ ($j = 1, \ldots, q$) on $\Omega$, where $m_j$ ($j = 1, \ldots, q$) are positive integers and may be $\infty$, with $\sum_{j=1}^q (1/m_j) < \frac{4 - (N + 1)}{N}$, then $f$ is a normal holomorphic mapping from $\Omega$ into $P^N(\mathbb{C})$.

Proof of Lemma 9. For any sequence of holomorphic mappings $\varphi_k(z)$ from the unit disc $D$ in $\mathbb{C}$ into $\Omega$, every $f \circ \varphi_k(z)$ intersects $H_j$ with multiplicity at least $m_j$ ($j = 1, \ldots, q$) on $D$ by the definition of multiplicities. By Theorem A $(f \circ \varphi_k(z))_{k=1}^\infty$ is a normal family on $D$ and hence $f$ is a normal mapping by Lemma 7. This proves Lemma 8.

Proof of Theorem 3. Without loss of generality we assume that $D$ is a bounded domain of $\mathbb{C}^n$ (i.e., $D$ is hyperbolic). By Lemma 9 $f$ is a normal holomorphic mapping from $D - S$ into $P^N(\mathbb{C})$. Thus by Definition 9 and the definition of the integrated distance there exists a positive constant $c$
such that
\[ d_{\rho^N}(f(z), f(w)) \leq c d_{D-S}^K(z, w) \]
for all \( z, w \in D - S \), where \( d_{D-S}^K \) and \( d_{\rho^N} \) denote the Kobayashi distance on \( D - S \) and the Fubini–Study distance on \( P^N(C) \), respectively. For any \( z_0 \in S \), let \( \{z_i\}_{i=1}^\infty \) be a sequence of points of \( D - S \) so as to converge to \( z_0 \). By Lemma 8 we have
\[
 d_{\rho^N}(f(z_i), f(z_j)) \leq c d_{D-S}^K(z_i, z_j) = c d_{D}^K(z_i, z_j).
\]
Then \( \{f(z_i)\}_{i=1}^\infty \) is a Cauchy sequence of \( P^N(C) \) and hence \( \{f(z_i)\}_{i=1}^\infty \) converges to a point \( a_0 \in P^N(C) \). It is easy to check that \( a_0 \) is independent of the choice of \( \{z_i\} \) as far as it converges to \( z_0 \). Then \( f(z) \) has an extension \( \tilde{f}(z) \) on \( D \) so as to be holomorphic on \( D - S \) and continuous on \( \tilde{D} \) and hence \( \tilde{f}(z) \) is holomorphic on \( D \) by the Riemann extension theorem. The proof of Theorem 3 is completed.

8. PROOF OF THEOREM 4

The following lemma is a special conclusion of Theorem 3.

**Lemma 10.** Let \( f \) be a meromorphic mapping from a domain \( D \) in \( C^n \) into \( P^N(C) \). If there exist \( q \geq 2N + 1 \) hyperplanes \( H_1, \ldots, H_q \) in \( P^N(C) \) in general position such that \( f \) intersects \( H_j \) with multiplicity at least \( m_j \) \( (j = 1, \ldots, q) \) on \( D \), where \( m_j \) \( (j = 1, \ldots, q) \) are positive integers and may be \( \infty \), with \( \sum_{j=1}^q (1/m_j) < \frac{q - (N + 1)}{N} \), then \( f \) is actually a holomorphic mapping from \( D \) into \( P^N(C) \).

**Proof of Theorem 4.** Take any sequence \( \{f_i\} \subset F \). By the assumption and Lemma 2 we can find a subsequence (again denoted by \( \{f_i\} \)) such that
\[
 \lim_{i \to \infty} \{z \in \text{supp} \nu(f_i, H_k(f_i)) \mid \nu(f_i, H_k(f_i))(z) < m_k \} = S_k
\]
\((k = 1, \ldots, q)\) as a sequence of closed subsets of \( D \), where \( S_k \) are either empty or pure \((n - 1)\)-dimensional analytic sets of \( D \) by Lemma 1. Let \( E := \bigcup_{k=1}^q S_k \). Then \( E \) is either empty or a pure \((n - 1)\)-dimensional analytic set of \( D \) and hence \( E \) is a nowhere dense analytic set of \( D \).

Now we shall prove that \( \{f_i(z)\}_{i=1}^\infty \) has a compactly convergent subsequence on \( D - E \). For any fixed point \( z_0 \) in \( D - E \), there exist an integer \( i_0 \) and a neighborhood \( U(z_0) \) in \( D - E \) such that
\[
 \{z \in \text{supp} \nu(f_i, H_k(f_i)) \mid \nu(f_i, H_k(f_i))(z) < m_k \} \cap U(z_0) = \emptyset
\]
for $i \geq i_0$ and $k = 1, \ldots, q$. Hence by Lemma 10 $(f_i(z))_{i=i_0}^\infty$ is a sequence of holomorphic mappings of $U(z_0)$ into $P^N(C)$ and by Theorem A $(f_i(z))_{i=i_0}^\infty$ has a subsequence which converges uniformly on compact subsets of $U(z_0)$ to a holomorphic mapping of $U(z_0)$ into $P^N(C)$. Therefore, by the usual diagonal argument, we can find a subsequence $(f_{i_j}(z))$ so as to converge uniformly on compact subsets of $D - E$ to a holomorphic mapping of $D - E$ into $P^N(C)$ and hence $(f_{i_j}(z))$ is quasi-regular on $D$. The proof of Theorem 4 is completed.

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