Characterization of distributions by conditional expectation of record values

A.H. Khan, Ziaul Haque *, Mohd. Faizan

Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh 202002, India

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Abstract
A family of continuous probability distributions has been characterized by two conditional expectations of record statistics conditioned on a non-adjacent record value. Besides various deductions, this work extends the result of Lee [8] in which Pareto distribution has been characterized.

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1. Introduction

Characterizations of distributions through conditional expectations of record values have been considered among others by Nagaraja [1], Franco and Ruiz [2], Wu and Lee [3], Raqab [4], Athar et al. [5], Gupta and Ahsanullah [6].

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed continuous random variables with the distribution function $(df)$ $F(x)$ and the probability density function $(pdf)$ $f(x)$. Let $X_{a(s)}$ be the $s$-th upper record value, then the conditional $pdf$ of $X_{a(s)}$ given $X_{a(r)} = x$, $1 \leq r < s$ is Ahsanullah [7]

$$f(X_{a(s)}|X_{a(r)} = x) = \frac{1}{F(s - r)} \left[ - \ln T(y) + \ln F(x) \right]^{s-r-1} \frac{f(y)}{F(x)},$$

where $T(x) = 1 - F(x)$.

Lee [8] has characterized Pareto distribution by conditional expectation of two records $X_{a(s)}$ and $X_{a(r)}$ conditioned on $X_{a(m)}$ for all $s > r \geq m$, where $s = r + 1$, $r + 2$ and $r + 3$. In this paper we have characterized a general class of distributions $T(x) = ax + b$ by the conditional expectation of $X_{a(s)}$ and $X_{a(r)}$ conditioned on $X_{a(m)}$ for all $s > r \geq m$, thus extending the results of Lee [8].

2. Characterization results

**Theorem 2.1.** Let $X$ be an absolutely continuous random variable with the df $F(x)$ and the pdf $f(x)$ on the support $(\alpha, \beta)$, where $\alpha$ and $\beta$ may be finite or infinite. Then for $m \leq r < s$

...
\[ E[h(X_{(i)})|X_{(m)} = x] = a'E[h(X_{(i)})|X_{(m)} = x] + b' \]  
(2.1)  
if and only if  
\[ \mathcal{T}(x) = [ah(x) + b]^r, \]  
(2.2)  
where \( a' = \left(\frac{r}{r-1}\right) \alpha' \) and \( b' = -\frac{a}{r}(1 - a'). \)

**Proof.** In view of the Athar et al. [5], we have  
\[ E[h(X_{(i)})|X_{(m)} = x] = a'_ih(x) + b'_i, \]  
(2.3)  
where,  
\[ a'_i = \left(\frac{c}{c+1}\right)^{x-m} \text{ and } b'_i = -\frac{b}{a}(1 - a'_i) \]
and  
\[ E[h(X_{(i)})|X_{(m)} = x] = a'_2h(x) + b'_2 \]  
(2.4)  
where  
\[ a'_2 = \left(\frac{c}{c+1}\right)^{x-m} \text{ and } b'_2 = -\frac{b}{a}(1 - a'_2). \]

Using (2.3) and (2.4), it is easy to establish (2.1).

For sufficiency part, we have  
\[ \frac{1}{T(s-m)} \int_x^\beta h(y) \left[ -\ln \mathcal{T}(y) + \ln \mathcal{T}(x) \right]^{s-m-1}f(y)dy \]
\[ = a' \frac{1}{T(r-m)} \int_x^\beta h(y) \left[ -\ln \mathcal{T}(y) + \ln \mathcal{T}(x) \right]^{r-m-1}f(y)dy \]
\[ + b' \mathcal{T}(x) \]  
(2.5)  
Differentiate both the sides of (2.5) w.r.t. \( x \), to get  
\[ -\frac{(s-m-1)}{T(s-m)} \int_x^\beta h(y) \left[ -\ln \mathcal{T}(y) + \ln \mathcal{T}(x) \right]^{s-m-2}f(y)dy \]
\[ = -a' \frac{(r-m-1)}{T(r-m)} \int_x^\beta h(y) \left[ -\ln \mathcal{T}(y) + \ln \mathcal{T}(x) \right]^{r-m-2} \]
\[ \times f(y)\mathcal{T}(x)dy - b'f(x). \]

after noting that if \( B = \int_a^x f(x,y)dy \) then  
\[ \frac{\partial B}{\partial x} = f(x,v) \frac{\partial v}{\partial x} - f(x,u) \frac{\partial u}{\partial x} + \int_a^x \frac{\partial f(x,y)}{\partial x} dy. \]
Therefore,  
\[ \frac{1}{T(s-m-1)} \int_x^\beta h(y) \left[ -\ln \mathcal{T}(y) + \ln \mathcal{T}(x) \right]^{s-m-2}f(y)dy \]
\[ = a' \frac{1}{T(r-m-1)} \int_x^\beta h(y) \left[ -\ln \mathcal{T}(y) + \ln \mathcal{T}(x) \right]^{r-m-2}f(y)dy \]
\[ + b' \mathcal{T}(x). \]

Similarly, differentiating \( (r - m - 1) \) times both the sides w.r.t. \( x \), we get  
\[ \frac{1}{T(s-r)} \int_x^\beta h(y) \left[ -\ln \mathcal{T}(y) + \ln \mathcal{T}(x) \right]^{r-1}f(y)dy \]
\[ = a'h(x) + b' = g_{d'}(x). \]
Using the result (Khan et al. [9]),  
\[ E[h(X_{(i)})|X_{(m)} = x] = g_{d'}(x) \]

we get,  
\[ \mathcal{T}(x) = e^{-\int_x^\beta A(t)dt} \]

where  
\[ A(t) = \frac{g_{d'}(t)}{g_{d'}(t) - g_{d'+1}(t)} = \frac{ach(t)}{ah(t) + b} \]

and \( \lim_{s \to \beta} \int_s^\beta A(t)dt = \infty. \)

Thus,  
\[ \mathcal{T}(x) = [ah(x) + b]^r \]

and hence the theorem. \( \square \)

**Remark 2.1.** At \( r = m, h(x) = x \), we get the result as obtained by Franco and Ruiz [2,10], Athar et al. [5], Ahsanullah and Wesolowski [11], Dembinska and Wesolowski [12], Khan and Alzaid [13].

**Remark 2.2.** Lee [8] has obtained characterization result for Pareto distribution  
\[ \mathcal{T}(x) = x^{-\theta}, \quad x > 1, \quad \theta > 0, \quad \theta \neq 1, \]

which can be obtained by putting \( a = 1, b = 0, c = -\theta, h(x) = x \) at \( s = r + 1, r + 2 \) and \( r + 3 \) in the Theorem 2.1.

**Remark 2.3.** At \( a = -\frac{r}{2}, b = 1, c \to \infty \)  
\[ a' = 1, \quad b' = \frac{(s-r)}{a}, \]
\[ \mathcal{T}(x) = e^{-ab(x)}, \quad a > 0, \]

reduces to the result as obtained by Khan et al. [14].

3. **Examples based on the distribution function**  
\[ F(x) = 1 - [ah(x) + b]^r \]

Proper choice of \( a, b \) and \( h(x) \) characterize the distributions as given below:
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