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# Homogeneous coordinates for algebraic varieties

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## Abstract

We associate to every divisorial (e.g., smooth) variety  $X$  with only constant invertible global functions and finitely generated Picard group a  $\text{Pic}(X)$ -graded homogeneous coordinate ring. This generalizes the usual homogeneous coordinate ring of the projective space and constructions of Cox and Kajiwara for smooth and divisorial toric varieties. We show that the homogeneous coordinate ring defines in fact a fully faithful functor. For normal complex varieties  $X$  with only constant global functions, we even obtain an equivalence of categories. Finally, the homogeneous coordinate ring of a locally factorial complete irreducible variety with free finitely generated Picard group turns out to be a Krull ring admitting unique factorization.

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## Introduction

The principal use of homogeneous coordinates is that they relate the geometry of algebraic varieties to the theory of graded rings. The classical example is the projective  $n$ -space: its homogeneous coordinate ring is the polynomial ring in  $n + 1$  variables, graded by the usual degree. Cox [6] and Kajiwara [13] introduced homogeneous coordinate rings for toric varieties. Cox's construction is meanwhile a standard instrument in toric geometry; for example, it is used in [5] to prove an equivariant Riemann–Roch Theorem, and in [20] for a description of  $\mathcal{D}$ -modules on toric varieties.

In this article, we construct homogeneous coordinates for a fairly general class of algebraic varieties: Let  $X$  be a divisorial variety—e.g.,  $X$  is  $\mathbb{Q}$ -factorial or quasiprojective [3]—such that  $X$  has only constant globally invertible functions and the Picard group  $\text{Pic}(X)$  is

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finitely generated. If the (algebraically closed) ground field  $\mathbb{K}$  is of characteristic  $p > 0$ , then we require that the multiplicative group  $\mathbb{K}^*$  is of infinite rank over  $\mathbb{Z}$ , and that  $\text{Pic}(X)$  has no  $p$ -torsion. Examples of such varieties are complete smooth rational complex varieties. Moreover, all Calabi–Yau varieties fit into this framework.

To define the homogeneous coordinate ring of  $X$ , consider a family of line bundles  $L$  on  $X$  such that the classes  $[L]$  generate  $\text{Pic}(X)$ . Choosing a common trivializing cover  $\mathfrak{U}$  for the bundles  $L$ , one can achieve that they form a finitely generated free abelian group  $\Lambda$ , which is isomorphic to a subgroup of the group of cocycles  $H^1(\mathcal{O}^*, \mathfrak{U})$ . The sheaves of sections  $\mathcal{R}_L$ , where  $L \in \Lambda$ , then fit together to a sheaf  $\mathcal{R}$  of  $\Lambda$ -graded  $\mathcal{O}_X$ -algebras. Such sheaves  $\mathcal{R}$  and their global sections  $\mathcal{R}(X)$  are often studied. For example, in [12] they have been used to characterize when Mori’s program can be carried out, and in [11] they are the starting point for quotient constructions in the spirit of Mumford’s Geometric Invariant Theory.

A first important observation is that we can pass from the above  $\Lambda$ -graded  $\mathcal{O}_X$ -algebras  $\mathcal{R}$  to a universal  $\mathcal{O}_X$ -algebra  $\mathcal{A}$ , which is graded by the Picard group  $\text{Pic}(X)$ . This solves in particular the ambiguity problem mentioned in [12, Remark, p. 341]. More precisely, we introduce in Section 3 the concept of a *shifting family* for the  $\mathcal{O}_X$ -algebra  $\mathcal{R}$ . This enables us to identify in a systematic manner two homogeneous parts  $\mathcal{R}_L$  and  $\mathcal{R}_{L'}$  if  $L$  and  $L'$  define the same class in  $\text{Pic}(X)$ . The result is a projection  $\mathcal{R} \rightarrow \mathcal{A}$  onto a  $\text{Pic}(X)$ -graded  $\mathcal{O}_X$ -algebra  $\mathcal{A}$ .

The *homogeneous coordinate ring* of  $X$  then is a pair  $(A, \mathfrak{A})$ . The first part  $A$  is the  $\text{Pic}(X)$ -graded  $\mathbb{K}$ -algebra of global sections  $\mathcal{A}(X)$ . The meaning of the second part  $\mathfrak{A}$  is roughly speaking the following: It turns out that  $A$  is the algebra of functions of a quasiaffine variety  $\widehat{X}$ . Such algebras need not be of finite type over  $\mathbb{K}$ , and  $\mathfrak{A}$  is a datum describing all the possible affine closures of  $\widehat{X}$ . From the algebraic point of view, the homogeneous coordinate ring is a *freely graded quasiaffine algebra*; the category of such algebras is introduced and discussed in Sections 1 and 2.

The first main result of this article is that the homogeneous coordinate ring is indeed functorial, that means that given a morphism  $X \rightarrow Y$  of varieties, we obtain a morphism of the associated freely graded quasiaffine algebras, see Section 5. In fact, we prove much more, see Theorem 5.7:

**Theorem.** *The assignment  $X \mapsto (A, \mathfrak{A})$  is a fully faithful functor from the category of divisorial varieties  $X$  with finitely generated Picard group and  $\mathcal{O}^*(X) = \mathbb{K}^*$  to the category of freely graded quasiaffine algebras.*

Note that this statement generalizes in particular the description of the set  $\text{Hom}(X, Y)$  of morphisms of two divisorial toric varieties  $X, Y$  obtained by Kajiwara in [13, Corollary 4.9]. In the toric situation,  $\mathcal{O}^*(X) = \mathbb{K}^*$  is a usual nondegeneracy assumption: it just means that  $X$  has no torus factors.

Having proved Theorem 5.7, the task is to translate geometric properties of a given variety  $X$  to algebraic properties of its homogeneous coordinate ring  $(A, \mathfrak{A})$ . In Section 6, we do this for basic properties of  $X$ , like smoothness and normality. In the latter case, the  $\mathbb{K}$ -algebra  $A$  is a normal Krull ring. Moreover, we discuss quasicoherent sheaves, and we give descriptions of affine morphisms and closed embeddings.

In our second main result, we restrict to normal divisorial varieties  $X$  with finitely generated Picard group and  $\mathcal{O}(X) = \mathbb{K}$ . We call such varieties *tame*. The homogeneous coordinate ring  $(A, \mathfrak{A})$  of a tame variety  $X$  is *pointed* in the sense that  $A$  is normal with  $A_0 = \mathbb{K}$  and  $A^* = \mathbb{K}^*$ . Moreover,  $(A, \mathfrak{A})$  is *simple* in the sense that the corresponding quasiaffine variety  $\widehat{X}$  admits only trivial “linearizable” bundles, see Section 7 for the precise definition. In Theorem 7.3, we show

**Theorem.** *The assignment  $X \mapsto (A, \mathfrak{A})$  defines an equivalence of the category of tame varieties with the category of simple pointed algebras.*

Specializing further to the case of a free Picard group gives the class of *very tame* varieties, see Section 8. Examples are the Grassmannians and all smooth complete toric varieties. For this class, we obtain a nice description of products in terms of homogeneous coordinate rings, see Proposition 8.5. The possibly most remarkable observation is that very tame varieties open a geometric approach to unique factorization conditions for multigraded Krull rings, see Proposition 8.4:

**Proposition.** *A very tame variety is locally factorial if and only if its homogeneous coordinate ring is a unique factorization domain.*

We conclude the article with an example underlining this principle: Let  $X$  be the projective line with the points  $0, 1$  and  $\infty$  doubled, that means that  $X$  is nonseparated. Nevertheless,  $X$  is very tame and its Picard group is isomorphic to  $\mathbb{Z}^4$ . As mentioned before,  $A = \mathcal{A}(X)$  is a unique factorization domain. It turns even out to be a classical example of a factorial singularity, namely

$$A = \mathbb{K}[T_1, \dots, T_6] / \langle T_1^2 + \dots + T_6^2 \rangle.$$

The quasiaffine variety  $\widehat{X}$  corresponding to the homogeneous coordinate ring of  $X$  is an open subset of  $\text{Spec}(A)$ . The prevariety  $X$  is a geometric quotient of  $\widehat{X}$  by a free action of a four-dimensional algebraic torus. In particular,  $\widehat{X}$  is locally isomorphic to the toric variety  $\mathbb{K} \times (\mathbb{K}^*)^4$ . That means that  $\widehat{X}$  is toroidal, even with respect to the Zariski Topology, but not toric.

## 1. Quasiaffine algebras and quasiaffine varieties

Throughout the whole article we work in the category of algebraic varieties following the setup of [14]. In particular, we work over an algebraically closed field  $\mathbb{K}$ , and the word point always refers to a closed point. Note that in our setting a variety is reduced but it need neither be separated nor irreducible.

The purpose of this section is to provide an algebraic description of the category of quasiaffine varieties. The idea is very simple: Every quasiaffine variety  $X$  is an open subset of an affine variety  $X'$  and hence is described by the inclusion  $\mathcal{O}(X') \subset \mathcal{O}(X)$  and the vanishing ideal of the complement  $X' \setminus X$  in  $\mathcal{O}(X')$ .

However, in general the algebra of functions  $\mathcal{O}(X)$  of a quasiaffine variety  $X$  is not of finite type, see, for example, [21]. Thus there is no canonical choice of an affine closure  $X'$  for a given  $X$ . To overcome this ambiguity, we have to treat all possible affine closures at once.

We introduce the necessary algebraic notions. By a  $\mathbb{K}$ -algebra we always mean a reduced commutative algebra  $A$  over  $\mathbb{K}$  having a unit element. We write  $\langle I \rangle$  for the ideal generated by a subset  $I \subset A$ . The set of nonzero divisors of a  $\mathbb{K}$ -algebra  $A$  is denoted by  $\text{nzd}(A)$ . Recall that we have a canonical inclusion  $A \subset \text{nzd}(A)^{-1}A$  into the algebra of fractions.

**Definition 1.1.** Let  $A$  be a  $\mathbb{K}$ -algebra.

- (i) A *closing subalgebra* of  $A$  is a pair  $(A', I')$  where  $A' \subset A$  is a subalgebra of finite type over  $\mathbb{K}$  and  $I' \subset A'$  is an ideal in  $A'$  with

$$I' = \sqrt{\langle I' \cap \text{nzd}(A) \rangle}, \quad A = \bigcap_{f \in I' \cap \text{nzd}(A)} A_f, \quad A'_f = A_f \text{ for all } f \in I'.$$

- (ii) Two closing subalgebras  $(A', I')$  and  $(A'', I'')$  of  $A$  are called *equivalent* if there is a closing subalgebra  $(A''', I''')$  of  $A$  such that

$$A' \cup A'' \subset A''', \quad I''' = \sqrt{\langle I' \rangle} = \sqrt{\langle I'' \rangle}.$$

Note that Definition 1.1(ii) does indeed define an equivalence relation. In terms of these notions, the algebraic data to describe quasiaffine varieties are the following

**Definition 1.2.**

- (i) A *quasiaffine algebra* is a pair  $(A, \mathfrak{A})$ , where  $A$  is a  $\mathbb{K}$ -algebra and  $\mathfrak{A}$  is the equivalence class of a closing subalgebra  $(A', I')$  of  $A$ .
- (ii) A *homomorphism* of quasiaffine algebras  $(B, \mathfrak{B})$  and  $(A, \mathfrak{A})$  is a homomorphism  $\mu: B \rightarrow A$  such that there exist  $(B', J') \in \mathfrak{B}$  and  $(A', I') \in \mathfrak{A}$  with

$$\mu(B') \subset A', \quad I' \subset \sqrt{\langle \mu(J') \rangle}.$$

We show now that the category of quasiaffine varieties is equivalent to the category of quasiaffine algebras by associating to every variety  $X$  an equivalence class  $\mathfrak{D}(X)$  of closing subalgebras of  $\mathcal{O}(X)$ . We use the following notation: Given a variety  $X$  and a regular function  $f \in \mathcal{O}(X)$ , let

$$X_f := \{x \in X; f(x) \neq 0\}.$$

**Definition 1.3.** Let  $X$  be a quasiaffine variety. Let  $A' \subset \mathcal{O}(X)$  be a subalgebra of finite type and  $I' \subset A'$  a radical ideal. We call  $(A', I')$  a *natural pair* on  $X$ , if for every  $f \in I'$  the set  $X_f$  is affine with  $\mathcal{O}(X_f) = A'_f$  and the sets  $X_f, f \in I'$ , cover  $X$ . We define  $\mathfrak{D}(X)$  to be the collection of all natural pairs on  $X$ .

So, our first task is to verify that the collection  $\mathfrak{D}(X)$  is in fact an equivalence class of closing subalgebras of  $\mathcal{O}(X)$ . This is done in two steps.

**Lemma 1.4.** *Let  $X$  be a quasiaffine variety. Let  $(A', I')$  be a natural pair on  $X$ , and set  $X' := \text{Spec}(A')$ .*

- (i) *The morphism  $X \rightarrow X'$  defined by  $A' \subset \mathcal{O}(X)$  is an open embedding,  $I'$  is the vanishing ideal of  $X' \setminus X$ , and  $(A', I')$  is a closing subalgebra of  $\mathcal{O}(X)$ .*
- (ii) *For a subalgebra  $A'' \subset \mathcal{O}(X)$  of finite type with  $A' \subset A''$ , consider the ideal  $I'' := \sqrt{\langle I' \rangle}$  of  $A''$ . Then  $(A'', I'')$  is a natural pair on  $X$ .*

**Proof.** Recall that for any  $f \in \mathcal{O}(X)$  we have  $\mathcal{O}(X_f) = \mathcal{O}(X)_f$ . In particular,  $X \rightarrow X'$  is locally given by isomorphisms  $X_f \rightarrow X'_f$ ,  $f \in I'$ . This implies that  $X \rightarrow X'$  is an open embedding and that  $I' \subset A'$  is the vanishing ideal of  $X' \setminus X$ . Finally,  $(A', I')$  is a closing subalgebra, because up to passing to the radical,  $I'$  is generated by the  $f \in I'$  that are nontrivial on each irreducible component of  $X$ .

We turn to assertion (ii). Let  $X'' := \text{Spec}(A'')$ . It suffices to verify that the morphism  $X \rightarrow X''$  defined by  $A'' \subset \mathcal{O}(X)$  is an open embedding and that  $I'' \subset A''$  is the vanishing ideal of the complement  $X'' \setminus X$ . Again this holds, because for every  $f \in I'$  the map  $X \rightarrow X''$  restricts to an isomorphism  $X_f \rightarrow X''_f$ .  $\square$

**Lemma 1.5.** *The collection  $\mathfrak{D}(X)$  of all natural pairs on a quasiaffine variety  $X$  is an equivalence class of closing subalgebras of  $\mathcal{O}(X)$ .*

**Proof.** First note that there exist natural pairs  $(A', I')$  on  $X$ , because for every affine closure  $X \subset X'$  we obtain such a pair by setting  $A' := \mathcal{O}(X')$  and defining  $I' \subset A'$  to be the vanishing ideal of the complement  $X' \setminus X$ . Moreover, by Lemma 1.4(i), we know that every natural pair is a closing subalgebra of  $\mathcal{O}(X)$ .

We show that any two natural pairs  $(A', I')$  and  $(A'', I'')$  on  $X$  are equivalent closing subalgebras of  $\mathcal{O}(X)$ . Let  $A''' \subset \mathcal{O}(X)$  be any subalgebra of finite type containing  $A' \cup A''$ . Define an ideal in  $A'''$  by  $I''' := \sqrt{\langle I' \rangle}$ . Then Lemma 1.4 tells us that the pair  $(A''', I''')$  is a closing subalgebra.

We have to show that  $I'''$  equals  $\sqrt{\langle I'' \rangle}$ . By Lemma 1.4, the morphism  $X \rightarrow X'''$  defined by the inclusion  $A''' \subset \mathcal{O}(X)$  is an open embedding and  $I''' \subset A'''$  is the vanishing ideal of  $X''' \setminus X$ . For every  $f \in I''$ , the map  $X \rightarrow X'''$  restricts to an isomorphism  $X_f \rightarrow X'''_f$ . Hence the desired identity of ideals follows from

$$X = \bigcup_{f \in I''} X_f.$$

Finally, we show that if a closing subalgebra  $(A'', I'')$  is equivalent to a natural pair  $(A', I')$ , then also  $(A'', I'')$  is natural. Choose  $(A''', I''')$  as in Definition 1.1(ii). By Lemma 1.4(ii), the pair  $(A''', I''')$  is natural. In particular,  $X_f$  is affine for every  $f \in I''$ . Moreover,  $X$  is covered by these  $X_f$ , because  $I'''$  equals  $\sqrt{\langle I'' \rangle}$ .  $\square$

We are ready for the main result of this section. Given a quasiaffine variety  $X$ , we denote as before by  $\mathfrak{D}(X)$  the collection of all natural pairs on  $X$ . For a morphism  $\varphi : X \rightarrow Y$  of varieties, we denote by  $\varphi^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  the pullback of functions.

**Proposition 1.6.** *The assignments  $X \mapsto (\mathcal{O}(X), \mathfrak{D}(X))$  and  $\varphi \mapsto \varphi^*$  define a contravariant equivalence of the category of quasiaffine varieties with the category of quasiaffine algebras.*

**Proof.** First of all, we check that the above assignment is in fact well defined on morphisms. Let  $\varphi : X \rightarrow Y$  be any morphism of quasiaffine varieties. Choose a closing subalgebra  $(B', J')$  in  $\mathfrak{D}(Y)$ . By Lemma 1.4(ii), we can construct a closing subalgebra  $(A', I')$  in  $\mathfrak{D}(X)$  such that  $\varphi^*(B') \subset A'$ .

Now, consider the affine closures  $X' := \text{Spec}(A')$  and  $Y' := \text{Spec}(B')$  of  $X$  and  $Y$ . The morphism  $\varphi' : X' \rightarrow Y'$  defined by the restriction  $\varphi^* : B' \rightarrow A'$  maps  $X$  to  $Y$ . Since  $I'$  and  $J'$  are precisely the vanishing ideals of the complements  $X' \setminus X$  and  $Y' \setminus Y$ , we obtain the condition required in Definition 1.2(ii):

$$I' \subset \sqrt{\langle \varphi^*(J') \rangle}.$$

Thus  $\varphi \mapsto \varphi^*$  is in fact well defined. Moreover,  $X \mapsto (\mathcal{O}(X), \mathfrak{D}(X))$  and  $\varphi \mapsto \varphi^*$  clearly define a contravariant functor, and this functor is injective on morphisms.

For surjectivity, let  $\mu : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  be a homomorphism of quasiaffine algebras. Let  $(A', I') \in \mathfrak{D}(X)$  and  $(B', J') \in \mathfrak{D}(Y)$  as in Definition 1.2(ii). Then  $\mu$  defines a morphism  $\varphi'$  from  $\text{Spec}(A')$  to  $\text{Spec}(B')$ . The condition on the ideals and Lemma 1.4(i) ensure that  $\varphi'$  restricts to a morphism  $\varphi : X \rightarrow Y$ . Clearly, we have  $\varphi^* = \mu$ .

It remains to show that up to isomorphism, every quasiaffine algebra  $(A, \mathfrak{A})$  arises from a quasiaffine variety. Let  $(A', I') \in \mathfrak{A}$ , set  $X' := \text{Spec}(A')$ , and let  $X \subset X'$  be the open subvariety obtained by removing the zero set of  $I'$ . Then  $\mathcal{O}(X) = A$ , and  $(A', I')$  is a natural pair on  $X$ . Lemma 1.5 gives  $\mathfrak{D}(X) = \mathfrak{A}$ .  $\square$

We conclude this section with the observation, that restricted on the category of quasiaffine varieties  $X$  with  $\mathcal{O}(X)$  of finite type, our algebraic description collapses in a very convenient way:

**Remark 1.7.** For any quasiaffine algebra  $(A, \mathfrak{A})$  we have

- (i) The algebra  $A$  is of finite type over  $\mathbb{K}$  if and only if  $(A, I) \in \mathfrak{A}$  holds with some radical ideal  $I \subset A$ .
- (ii) The quasiaffine algebra  $(A, \mathfrak{A})$  arises from an affine variety if and only if  $(A, A) \in \mathfrak{A}$  holds.

## 2. Freely graded quasiaffine algebras

In this section, we introduce the formal framework of homogeneous coordinate rings, namely freely graded quasiaffine algebras and their morphisms. The geometric

interpretation of these notions amounts to an equivariant version of the equivalence of categories presented in the preceding section.

**Definition 2.1.** Let  $(A, \mathfrak{A})$  be a quasiaffine algebra, and let  $\Lambda$  be a finitely generated abelian group. We say that  $(A, \mathfrak{A})$  is *freely graded* by  $\Lambda$  (or *freely  $\Lambda$ -graded*) if there is a grading

$$A = \bigoplus_{L \in \Lambda} A_L,$$

and there exists a closing subalgebra  $(A', I') \in \mathfrak{A}$  admitting homogeneous elements  $f_1, \dots, f_r \in I'$  such that  $I'$  equals  $\sqrt{\langle f_1, \dots, f_r \rangle}$  and every localization  $A_{f_i}$  has in each degree  $L \in \Lambda$  a homogeneous invertible element.

**Example 2.2.** For  $n \geq 2$ , the polynomial ring  $\mathbb{K}[T_1, \dots, T_n]$  together with the usual  $\mathbb{Z}$ -grading can be made into a freely graded quasiaffine algebra: Let  $\mathfrak{A}$  be the class of  $(A, I)$ , where  $I := \langle T_1, \dots, T_n \rangle$ .

The *weight monoid* of an integral domain  $A$  graded by a finitely generated abelian group  $\Lambda$  is the submonoid  $\Lambda^* \subset \Lambda$  consisting of all weights  $L \in \Lambda$  with  $A_L \neq \{0\}$ . For the weight monoid of a freely graded quasiaffine algebra, we have

**Remark 2.3.** Let  $(A, \mathfrak{A})$  be a freely  $\Lambda$ -graded quasiaffine algebra. Then the weight monoid  $\Lambda^* \subset \Lambda$  of  $A$  generates  $\Lambda$  as a group.

We turn to homomorphisms. The final notion of a morphism of freely graded quasiaffine algebras will be given below. First we have to consider homomorphisms that are compatible with the structure:

**Definition 2.4.** Let the quasiaffine algebras  $(A, \mathfrak{A})$  and  $(B, \mathfrak{B})$  be freely graded by  $\Lambda$  and  $\Gamma$ , respectively. A homomorphism  $\mu: (B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$  of quasiaffine algebras is called *graded*, if there is a homomorphism  $\tilde{\mu}: \Gamma \rightarrow \Lambda$  with

$$\mu(B_E) \subset A_{\tilde{\mu}(E)} \quad \text{for all } E \in \Gamma. \quad (2.4.1)$$

By Remark 2.3, a graded homomorphism  $\mu: (B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$  of freely graded quasiaffine algebras uniquely determines its accompanying homomorphism  $\tilde{\mu}: \Gamma \rightarrow \Lambda$ . Moreover, the composition of two graded homomorphisms is again graded.

For the subsequent treatment of our homogeneous coordinate rings we need a coarser concept of a morphism of freely graded quasiaffine algebras than the notion of a graded homomorphism would yield. This is the following

**Definition 2.5.** Let the quasiaffine algebras  $(A, \mathfrak{A})$  and  $(B, \mathfrak{B})$  be freely graded by finitely generated abelian groups  $\Lambda$  and  $\Gamma$ , respectively.

- (i) Two graded homomorphisms  $\mu, \nu : (B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$  are called *equivalent* if there is a homomorphism  $c : \Gamma \rightarrow A_0^*$  such that for every  $E \in \Gamma$  and every  $g \in B_E$  we have

$$\nu(g) = c(E)\mu(g).$$

- (ii) A *morphism*  $(B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$  of the freely graded quasiaffine algebras  $(B, \mathfrak{B})$  and  $(A, \mathfrak{A})$  is the equivalence class  $[\mu]$  of a graded homomorphism  $\mu : (B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$ .

In the setting of (i) we shall say that  $\mu$  and  $\nu$  *differ by a character*  $c : \Gamma \rightarrow A_0^*$ . Since equivalence of graded homomorphisms is compatible with composition, this definition makes the freely graded quasiaffine algebras into a category.

We give now a geometric interpretation of the above notions. We assume for the rest of this section that if  $\mathbb{K}$  is of characteristic  $p > 0$ , then our finitely generated abelian groups  $\Lambda$  have no  $p$ -torsion, i.e.,  $\Lambda$  contains no elements of order  $p$ . Under this assumption, each  $\Lambda$  defines a diagonalizable algebraic group

$$H := \text{Spec}(\mathbb{K}[\Lambda]).$$

Recall that the characters of this group  $H$  are precisely the canonical generators  $\chi^L$ ,  $L \in \Lambda$ , of the group algebra  $\mathbb{K}[\Lambda]$ . In fact, the assignment  $\Lambda \mapsto H$  defines a contravariant equivalence of categories, see, for example, [4, Section III. 8].

Now, suppose that a diagonalizable group  $H = \text{Spec}(\mathbb{K}[\Lambda])$  acts by means of a regular map  $H \times X \rightarrow X$  on a (not necessarily affine) variety  $X$ . A function  $f \in \mathcal{O}(X)$  is called *homogeneous* with respect to a character  $\chi^L : H \rightarrow \mathbb{K}^*$  if for every  $(t, x) \in H \times X$  we have

$$f(t \cdot x) = \chi^L(t)f(x).$$

For  $L \in \Lambda$ , let  $\mathcal{O}(X)_L \subset \mathcal{O}(X)$  denote the subset of all  $\chi^L$ -homogeneous functions. It is well known, use, for example, [15, p. 67, Lemma], that the action of  $H$  on  $X$  defines a grading

$$\mathcal{O}(X) = \bigoplus_{L \in \Lambda} \mathcal{O}(X)_L.$$

Recall that one obtains in this way a canonical correspondence between affine  $H$ -varieties and  $\Lambda$ -graded affine algebras (the arguments presented in [7, p. 11] for the case  $\Lambda = \mathbb{Z}$  also work in the general case).

We are interested in free  $H$ -actions on quasiaffine varieties  $X$ , where *free* means that every orbit map  $H \rightarrow H \cdot x$  is an isomorphism. In this situation, we have

**Lemma 2.6.** *Let the diagonalizable group  $H = \text{Spec}(\mathbb{K}[\Lambda])$  act freely by means of a regular map  $H \times X \rightarrow X$  on a quasiaffine variety  $X$ . Then the associated  $\Lambda$ -grading of  $\mathcal{O}(X)$  makes  $(\mathcal{O}(X), \mathfrak{D}(X))$  into a freely graded quasiaffine algebra.*

**Proof.** Let  $(A'', I'')$  be any natural pair on  $X$ , and let  $g_1, \dots, g_s$  be a system of generators of  $A''$ . Let  $A' \subset \mathcal{O}(X)$  denote the subalgebra generated by all the homogeneous



components of the  $g_j$ . Then  $A'$  is graded, and according to Lemma 1.4(ii), we obtain a natural pair  $(A', I')$  on  $X$  by defining  $I' := \sqrt{\langle I'' \rangle}$ .

Now, the  $\Lambda$ -grading of  $A'$  comes from an  $H$ -action on  $X' := \text{Spec}(A')$ . This  $H$ -action extends the initial  $H$ -action on  $X$ . In particular, the ideal  $I' \subset A'$  is graded, because it is the vanishing ideal of the invariant set  $X' \setminus X$ , see Lemma 1.4(i). This fact enables us to verify the condition of Definition 2.1 for  $I'$ :

Choose generators  $L_1, \dots, L_k$  of  $\Lambda$ . Consider  $x \in X$ , and choose a homogeneous  $h \in I'$  with  $h(x) \neq 0$ . Since  $H$  acts freely, the orbit map  $H \rightarrow H \cdot x$  is an isomorphism. Thus we find for every  $i$  a  $\chi^{L_i}$ -homogeneous regular function  $h_i$  on  $H \cdot x$  with  $h_i(x) \neq 0$ . Since  $H \cdot x$  is closed in  $X_h$ , the  $h_i$  extend to  $\chi^{L_i}$ -homogeneous regular functions on  $X_h$ .

For a suitable  $r > 0$ , the product  $f := h^r h_1 \cdots h_k$  is a regular function on  $X'$  with  $f \in \langle h \rangle$  and hence  $f \in I'$ . By construction,  $f$  is homogeneous, and we have  $f(x) \neq 0$ . Moreover, the Laurent monomials in  $h_1, \dots, h_k$  provide for each degree  $L \in \Lambda$  a  $\chi^L$ -homogeneous invertible function on  $X_f$ . Since finitely many of the  $X_f$  cover  $X$ , this gives the desired property on the ideal  $I' \subset A'$ .  $\square$

In order to give the equivariant version of Proposition 1.6, we have to fix the notion of a morphism of quasiaffine varieties with an action of a diagonalizable group. This is the following

**Definition 2.7.** Let  $G \times X \rightarrow X$  and  $H \times Y \rightarrow Y$  be algebraic group actions. A morphism  $\varphi: X \rightarrow Y$  is called *equivariant* if there is a homomorphism  $\tilde{\varphi}: G \rightarrow H$  of algebraic groups such that for all  $(g, x) \in G \times X$  we have

$$\varphi(g \cdot x) = \tilde{\varphi}(g) \cdot \varphi(x).$$

This notion of an equivariant morphism makes the quasiaffine varieties with a free action of a diagonalizable group into a category. We obtain the following equivariant version of Proposition 1.6:

**Proposition 2.8.** *The assignments  $X \mapsto (\mathcal{O}(X), \mathfrak{D}(X))$  and  $\varphi \mapsto \varphi^*$  define a contravariant equivalence from the category of quasiaffine varieties with a free diagonalizable group action to the category of freely graded quasiaffine algebras and graded homomorphisms.*

**Proof.** By Lemma 2.6, the assignment  $X \mapsto (\mathcal{O}(X), \mathfrak{D}(X))$  is well defined. From Proposition 1.6 and the observation that equivariant morphisms of quasiaffine varieties correspond to graded homomorphisms of quasiaffine algebras we infer functoriality and bijectivity on the level of morphisms.

In order to see that up to isomorphism any quasiaffine algebra  $(A, \mathfrak{A})$  which is freely graded by some  $\Lambda$  arises in the above manner from a quasiaffine variety with free diagonalizable group action, we repeat the corresponding part of the proof of Proposition 1.6 in an equivariant manner:

Let  $(A', I')$  be as in Definition 2.1. Let  $A'' \subset A$  be any graded subalgebra of finite type with  $A' \subset A''$ , and let  $I'' := \sqrt{\langle I' \rangle}$ . Then  $(A'', I'')$  belongs to  $\mathfrak{A}$ , and the ideal  $I''$  still satisfies the condition of Definition 2.1.

The affine variety  $X'' := \text{Spec}(A'')$  comes along with an action of the diagonalizable group  $H := \text{Spec}(\mathbb{K}[\Lambda])$  such that the corresponding grading of  $\mathcal{O}(X'') = A''$  gives back the original  $\Lambda$ -grading of the algebra  $A''$ . Removing the  $H$ -invariant zero set of  $I''$  from  $X''$ , gives a quasiaffine  $H$ -variety  $X$ .

By construction, the  $\Lambda$ -graded algebras  $\mathcal{O}(X)$  and  $A$  coincide, and  $(A'', I'')$  is a natural pair on  $X$ . Moreover, the local existence of invertible homogeneous functions in each degree implies that for every  $x \in X$  the orbit map  $H \mapsto H \cdot x$  is an isomorphism. In other words, the action of  $H$  on  $X$  is free.  $\square$

**Example 2.9.** The standard  $\mathbb{K}^*$ -action on  $\mathbb{K}^{n+1} \setminus \{0\}$  has  $(A, \mathfrak{A})$  of Example 2.2 as associated freely graded quasiaffine algebra.

The remaining task is to translate the notion of equivalence of graded homomorphisms. For this let  $X$  and  $Y$  be quasiaffine varieties with actions of diagonalizable groups  $H := \text{Spec}(\mathbb{K}[\Lambda])$  and  $G := \text{Spec}(\mathbb{K}[\Gamma])$ . Denote by  $(A, \mathfrak{A})$  and  $(B, \mathfrak{B})$  the freely graded quasiaffine algebras associated to  $X$  and  $Y$ .

**Remark 2.10.** Two graded homomorphisms  $\mu, \nu : (B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$  are equivalent if and only if there is an  $H$ -invariant morphism  $\gamma : X \rightarrow G$  such that the morphisms  $\varphi, \psi : X \rightarrow Y$  corresponding to  $\mu$  and  $\nu$  always satisfy  $\psi(x) = \gamma(x) \cdot \varphi(x)$ .

### 3. Picard graded sheaves of algebras

Let  $X$  be an algebraic variety and denote by  $\text{Pic}(X)$  its Picard group. In this section we prepare the definition of a graded ring structure on the vector space

$$\bigoplus_{[L] \in \text{Pic}(X)} H^0(X, L).$$

More generally, we even need a ring structure for the corresponding sheaves of vector spaces. The problem is easy, if  $\text{Pic}(X)$  is free: Then we can realize it as a group  $\Lambda$  of line bundles as in [10, Section 2], and we can work with the associated  $\Lambda$ -graded  $\mathcal{O}_X$ -algebra  $\mathcal{R}$ .

If  $\text{Pic}(X)$  has torsion, then we can at most expect a surjection  $\Lambda \rightarrow \text{Pic}(X)$  with a free group  $\Lambda$  of line bundles. Thus the problem is to identify in a suitable manner isomorphic homogeneous components of the  $\Lambda$ -graded  $\mathcal{O}_X$ -algebra  $\mathcal{R}$ . This is done by means of shifting families and their associated ideals  $\mathcal{I} \subset \mathcal{R}$ , see Definitions 3.1 and 3.4. The quotient  $\mathcal{A} := \mathcal{R}/\mathcal{I}$  then will realize the desired ring structure.

To begin, let us recall the necessary constructions from [10]. Consider an open cover  $\mathfrak{U} = (U_i)_{i \in I}$  of  $X$ . This cover gives rise to an additive group  $\Lambda(\mathfrak{U})$  of line bundles on  $X$ : For each Čech cocycle  $\xi \in Z^1(\mathfrak{U}, \mathcal{O}_X^*)$ , let  $L_\xi$  denote the line bundle obtained by gluing the products  $U_i \times \mathbb{K}$  along the maps

$$(x, z) \mapsto (x, \xi_{ij}(x)z).$$

Define the sum  $L_\xi + L_\eta$  of two such line bundles to be  $L_{\xi\eta} = L_{\eta\xi}$ . This makes the set  $\Lambda(\mathfrak{U})$  consisting of all the bundles  $L_\xi$  into an abelian group, which is isomorphic to the cocycle group  $Z^1(\mathfrak{U}, \mathcal{O}_X^*)$ .

When we speak of a *group of line bundles* on  $X$ , we think of a finitely generated free subgroup of some group  $\Lambda$  as above. Note that for any such group  $\Lambda$  of line bundles, we have a canonical homomorphism  $\Lambda \rightarrow \text{Pic}(X)$  to the Picard group.

We come to the graded  $\mathcal{O}_X$ -algebra associated to a group  $\Lambda$  of line bundles on a variety  $X$ . For each line bundle  $L \in \Lambda$ , let  $\mathcal{R}_L$  denote its sheaf of sections. In the sequel, we shall identify  $\mathcal{R}_0$  with the structure sheaf  $\mathcal{O}_X$ . The *graded  $\mathcal{O}_X$ -algebra* associated to  $\Lambda$  is the quasicohereant sheaf

$$\mathcal{R} := \bigoplus_{L \in \Lambda} \mathcal{R}_L,$$

where the multiplication is defined as follows: The sections of a bundle  $L_\xi \in \Lambda$  over an open set  $U \subset X$  are described by families  $f_i \in \mathcal{O}_X(U \cap U_i)$  that are compatible with the cocycle  $\xi$ . For any two sections  $f \in \mathcal{R}_L(U)$  and  $f' \in \mathcal{R}_{L'}(U)$ , the product  $(f_i f'_i)$  of their defining families  $(f_i)$  and  $(f'_i)$  gives us a section  $ff' \in \mathcal{R}_{L+L'}(U)$ .

In the sequel, we fix an open cover  $\mathfrak{U} = (U_i)_{i \in I}$  of  $X$  and a group  $\Lambda \subset \Lambda(\mathfrak{U})$  of line bundles. Let  $\mathcal{R}$  denote the associated  $\Lambda$ -graded  $\mathcal{O}_X$ -algebra. Here comes the notion of a *shifting family* for  $\mathcal{R}$ :

**Definition 3.1.** Let  $\Lambda_0 \subset \Lambda$  be any subgroup of the kernel of  $\Lambda \rightarrow \text{Pic}(X)$ . By a  *$\Lambda_0$ -shifting family* for  $\mathcal{R}$  we mean a family  $\varrho = (\varrho_E)$  of  $\mathcal{O}_X$ -module isomorphisms  $\varrho_E: \mathcal{R} \rightarrow \mathcal{R}$ , where  $E \in \Lambda_0$ , with the following properties:

- (i) for every  $L \in \Lambda$  and every  $E \in \Lambda_0$  the isomorphism  $\varrho_E$  maps  $\mathcal{R}_L$  onto  $\mathcal{R}_{L+E}$ ,
- (ii) for any two  $E_1, E_2 \in \Lambda_0$  we have  $\varrho_{E_1+E_2} = \varrho_{E_2} \circ \varrho_{E_1}$ ,
- (iii) for any two homogeneous sections  $f, g$  of  $\mathcal{R}$  and every  $E \in \Lambda_0$  we have  $\varrho_E(fg) = f\varrho_E(g)$ .

If  $\Lambda_0$  is the full kernel of  $\Lambda \rightarrow \text{Pic}(X)$ , then we also speak of a *full shifting family* for  $\mathcal{R}$  instead of a  $\Lambda_0$ -shifting family.

The first basic observations are existence of shifting families and a certain uniqueness statement:

**Lemma 3.2.** Let  $\Lambda_0 \subset \Lambda$  be a subgroup of the kernel of  $\Lambda \rightarrow \text{Pic}(X)$ . Then there exist  $\Lambda_0$ -shifting families for  $\mathcal{R}$ , and any two such families  $\varrho, \varrho'$  differ by a character  $c: \Lambda_0 \rightarrow \mathcal{O}^*(X)$  in the sense that  $\varrho'_E = c(E)\varrho_E$  holds for all  $E \in \Lambda_0$ .

**Proof.** For the existence statement, fix a  $\mathbb{Z}$ -basis of the subgroup  $\Lambda_0 \subset \Lambda$ . For any member  $E$  of this basis choose a bundle isomorphism  $\alpha_E: 0 \rightarrow E$  from the trivial bundle  $0 \in \Lambda$

onto  $E \in \Lambda$ . With respect to the cover  $\mathcal{U}$ , this isomorphism is fibrewise multiplication with certain  $\alpha_i \in \mathcal{O}^*(U_i)$ ; so, on  $U_i \times \mathbb{K}$  it is of the form

$$(x, z) \mapsto (x, \alpha_i(x)z). \tag{3.2.1}$$

If  $\alpha_{E'} : 0 \rightarrow E'$  denotes the isomorphism for a further member of the basis of  $\Lambda_0$ , then the products  $\alpha_i \alpha'_i$  of the corresponding local data define an isomorphism  $\alpha_{E+E'} : 0 \rightarrow E + E'$ . Similarly, by inverting local data, we obtain isomorphisms  $\alpha_{-E} : 0 \rightarrow -E$ . Proceeding this way, we obtain an isomorphism  $\alpha_E : 0 \rightarrow E$  for every  $E \in \Lambda_0$ .

The local data  $\alpha_i$  of an isomorphism  $\alpha_E : 0 \rightarrow E$  as constructed above define as well an isomorphism  $L \rightarrow L + E$  for any  $L \in \Lambda$ . By shifting homogeneous sections according to  $f \mapsto \alpha_E \circ f$ , one obtains  $\mathcal{O}_X$ -module isomorphisms  $\varrho_E : \mathcal{R} \rightarrow \mathcal{R}$  mapping each  $\mathcal{R}_L$  onto  $\mathcal{R}_{L+E}$ . Properties (ii) and (iii) in Definition 3.1 are then clear by construction.

We turn to the uniqueness statement. Let  $\varrho, \varrho'$  be two  $\Lambda_0$ -shifting families for  $\mathcal{R}$ . Using property (iii) in Definition 3.1 we see that for every  $E \in \Lambda_0$  and every homogeneous section  $f$  of  $\mathcal{R}$ , we have

$$\varrho_E^{-1} \circ \varrho'_E(f) = \varrho_E^{-1} \circ \varrho'_E(f \cdot 1) = f \cdot \varrho_E^{-1} \circ \varrho'_E(1).$$

Thus, setting  $c(E) := \varrho_E^{-1} \circ \varrho'_E(1)$  we obtain a map  $c : \Lambda_0 \rightarrow \mathcal{O}^*(X)$  such that  $\varrho'_E$  equals  $c(E)\varrho_E$ . Properties (ii) and (iii) in Definition 3.1 show that  $c$  is a homomorphism:

$$\begin{aligned} c(E_1 + E_2) &= \varrho_{E_1+E_2}^{-1} \circ \varrho'_{E_1+E_2}(1) \\ &= \varrho_{-E_1} \circ \varrho_{-E_2} \circ \varrho'_{E_2} \circ \varrho'_{E_1}(1) \\ &= \varrho_{-E_1} \circ \varrho_{-E_2} \circ \varrho'_{E_2}(\varrho'_{E_1}(1) \cdot 1) \\ &= \varrho_{-E_1}(\varrho'_{E_1}(1)c(E_2)) \\ &= c(E_1)c(E_2). \quad \square \end{aligned}$$

We shall now associate to any shifting family an ideal in the  $\mathcal{O}_X$ -algebra  $\mathcal{R}$ . First we remark that for any subgroup  $\Lambda_0 \subset \Lambda$  the algebra  $\mathcal{R}$  becomes  $\Lambda/\Lambda_0$ -graded by defining the homogeneous component of a class  $[L] \in \Lambda/\Lambda_0$  as

$$\mathcal{R}_{[L]} := \sum_{L' \in [L]} \mathcal{R}_{L'}.$$

**Lemma 3.3.** *Let  $\Lambda_0$  be a subgroup of the kernel of  $\Lambda \rightarrow \text{Pic}(X)$ , and let  $\varrho$  be a  $\Lambda_0$ -shifting family. For each given open subset  $U \subset X$  consider the ideal*

$$\mathfrak{I}(U) := \langle f - \varrho_E(f); f \in \mathcal{R}(U), E \in \Lambda_0 \rangle \subset \mathcal{R}(U).$$

*Let  $\mathcal{I}$  denote the sheaf associated to the presheaf  $U \mapsto \mathfrak{I}(U)$ . Then  $\mathcal{I}$  is a quasicoherent ideal of  $\mathcal{R}$ , and we have:*

- (i) Every  $\mathcal{I}(U)$  is homogeneous with respect to the  $\Lambda/\Lambda_0$ -grading of  $\mathcal{R}(U)$ .  
(ii) For every  $L \in \Lambda$  we have  $\mathcal{R}_L(U) \cap \mathcal{I}(U) = \{0\}$ .

**Proof.** First note that the ideal sheaf  $\mathcal{I}$  is indeed quasicoherent, because it is a sum of images of quasicoherent sheaves.

We check (i). Using property (iii) in Definition 3.1, we see that each ideal  $\mathcal{I}(U)$  is generated by the elements  $1 - \varrho_E(1)$ , where  $E \in \Lambda_0$ . Consequently, each stalk  $\mathcal{I}_x$  is a  $\Lambda/\Lambda_0$ -homogeneous ideal in  $\mathcal{A}_x$ . This implies that the associated sheaf  $\mathcal{I}$  is a  $\Lambda/\Lambda_0$ -homogeneous ideal in  $\mathcal{A}$ .

We turn to (ii). By construction, it suffices to consider local sections  $f \in \mathcal{R}_L(U) \cap \mathcal{I}(U)$ . By the definition of  $\mathcal{I}(U)$  and property (iii) in Definition 3.1, there exist homogeneous elements  $f_i \in \mathcal{R}_{L_i}(U)$  such that we can write  $f$  as

$$f = \sum_{i=1}^r f_i - \varrho_{E_i}(f_i). \quad (3.3.1)$$

Since  $\mathcal{I}(U)$  is  $\Lambda/\Lambda_0$ -graded, all the  $L_i$  belong to the class  $[L]$  in  $\Lambda/\Lambda_0$ . Moreover, we can achieve in the representation (3.3.1) of  $f$  that all  $f_i$  are of degree  $L \in [L]$ . Namely, we can use property (ii) in Definition 3.1 to write  $f_i - \varrho_{E_i}(f_i)$  in the form

$$\begin{aligned} f_i - \varrho_{E_i}(f_i) &= \varrho_{L-L_i}(f_i) - \varrho_{E_i+L_i-L}(\varrho_{L-L_i}(f_i)) \\ &+ (-\varrho_{L-L_i}(f_i)) - \varrho_{L_i-L}(-\varrho_{L-L_i}(f_i)). \end{aligned}$$

Moreover we can choose the representation (3.3.1) minimal in the sense that  $r$  is minimal with the property that every  $f_i$  is of degree  $L$ . Then the  $E_i$  are pairwise different from each other, because otherwise we could shorten the representation by gathering. But this implies  $\varrho_{E_i}(f_i) = 0$  for every  $i$ . Hence we obtain  $f = 0$ .  $\square$

**Definition 3.4.** Let  $\Lambda_0$  be a subgroup of the kernel of  $\Lambda \rightarrow \text{Pic}(X)$ , and let  $\varrho$  be a  $\Lambda_0$ -shifting family for  $\mathcal{R}$ . The ideal associated to  $\varrho$  is the  $\Lambda/\Lambda_0$ -graded ideal sheaf  $\mathcal{I}$  of  $\mathcal{R}$  defined in Lemma 3.3.

With the aid of the ideal associated to a shifting family, we can pass from  $\mathcal{R}$  to more coarsely graded  $\mathcal{O}_X$ -algebras:

**Lemma 3.5.** Let  $\Lambda_0 \subset \Lambda$  be a subgroup, and let  $\varrho$  be a  $\Lambda_0$ -shifting family with associated ideal  $\mathcal{I}$ . Set  $\mathcal{A} := \mathcal{R}/\mathcal{I}$ , and let  $\pi : \mathcal{R} \rightarrow \mathcal{A}$  denote the projection.

- (i) The  $\mathcal{O}_X$ -algebra  $\mathcal{A}$  is quasicoherent, and it inherits a  $\Lambda/\Lambda_0$ -grading from  $\mathcal{R}$  as follows:

$$\mathcal{A} = \bigoplus_{[L] \in \Lambda/\Lambda_0} \mathcal{A}_{[L]} := \bigoplus_{[L] \in \Lambda/\Lambda_0} \pi(\mathcal{R}_{[L]}).$$

(ii) For any  $L \in \Lambda$  the induced map  $\pi_L : \mathcal{R}_L \rightarrow \mathcal{A}_{[L]}$  is an isomorphism of  $\mathcal{O}_X$ -modules. In particular, we obtain

$$\mathcal{A}(X) \cong \mathcal{R}(X)/\mathcal{I}(X).$$

(iii) The  $\mathcal{O}_X$ -algebra  $\mathcal{A}$  is locally generated by finitely many invertible homogeneous elements.

**Proof.** The first assertion follows directly from the fact that we have a commutative diagram where the lower arrow is an isomorphism of sheaves:

$$\begin{array}{ccc} & \mathcal{R} & \\ \pi \swarrow & & \searrow \\ \mathcal{A} & \xrightarrow{\quad} & \bigoplus_{[L] \in \Lambda/\Lambda_0} \mathcal{R}_{[L]}/\mathcal{I}_{[L]} \end{array}$$

To prove (ii), note that  $\pi_L : \mathcal{R}_L \rightarrow \mathcal{A}_{[L]}$  is injective by Lemma 3.3(ii). For bijectivity, we have to show that  $\pi_L$  is stalkwise surjective. Let  $h$  be a local section of  $\mathcal{A}_{[L]}$  near some  $x \in X$ . Since  $\mathcal{A}_{[L]}$  equals  $\pi(\mathcal{R}_{[L]})$ , we may assume that  $h = \pi(f)$  with a local section  $f$  of  $\mathcal{R}_{[L]}$  near  $x$ . Write  $f$  as the sum of its  $\Lambda$ -homogeneous components:

$$f = \sum_{L' \in [L]} f_{L'}.$$

For every  $L' \neq L$ , we subtract  $f_{L'} - \varrho_{L-L'}(f_{L'})$  from  $f$ . The result is a local section  $g$  of  $\mathcal{R}_L$  near  $x$  which still projects onto  $h$ . This proves bijectivity of  $\pi_L : \mathcal{R}_L \rightarrow \mathcal{A}_{[L]}$ . The isomorphism on the level of global sections then is due to left exactness of the section functor.

To prove assertion (iii), note that the analogous statement holds for  $\mathcal{R}$ . In fact, for small  $U \subset X$ , the algebra  $\mathcal{R}(U)$  is even a Laurent monomial algebra over  $\mathcal{O}(U)$ . Together with assertion (ii), this observation gives statement (iii).  $\square$

**Definition 3.6.** Let  $\Lambda_0 \subset \Lambda$  be a subgroup of the kernel  $\Lambda \rightarrow \text{Pic}(X)$ , and let  $\varrho$  be a  $\Lambda_0$ -shifting family for  $\mathcal{R}$  with associated ideal  $\mathcal{I}$ . We call the  $\Lambda/\Lambda_0$ -graded  $\mathcal{O}_X$ -algebra  $\mathcal{A} := \mathcal{R}/\mathcal{I}$  of Lemma 3.5 the *Picard graded algebra* associated to  $\varrho$ .

If every global invertible function on  $X$  is constant, then the Picard graded algebras associated to different  $\Lambda_0$ -shifting families are isomorphic (a graded homomorphism of sheaves is defined by requiring (2.4.1) on the level of sections).

**Lemma 3.7.** Suppose  $\mathcal{O}^*(X) = \mathbb{K}^*$ . Let  $\Lambda_0 \subset \Lambda$  be a subgroup, and let  $\varrho, \varrho'$  be  $\Lambda_0$ -shifting families for  $\mathcal{R}$  with associated ideals  $\mathcal{I}$  and  $\mathcal{I}'$ . Then there is a graded automorphism of  $\mathcal{R}$  having the identity of  $\Lambda$  as accompanying homomorphism and mapping  $\mathcal{I}$  onto  $\mathcal{I}'$ .

**Proof.** By Lemma 3.2, there exists a homomorphism  $c: E \rightarrow \mathbb{K}^*$  such that  $\varrho'_E = c(E)\varrho_E$  holds. By Lemma 3.8 stated below, this homomorphism extends to a homomorphism  $c: \Lambda \rightarrow \mathbb{K}^*$ . Thus we can define the desired automorphism  $\mathcal{R} \rightarrow \mathcal{R}$  by mapping a section  $f \in \mathcal{R}_L(U)$  to  $c(L)f \in \mathcal{R}_L(U)$ .  $\square$

In the proof of this lemma, we made use of the following standard property of lattices:

**Lemma 3.8.** *Let  $\Lambda_0 \subset \Lambda$  be an inclusion of lattices. Then any homomorphism  $\Lambda_0 \rightarrow \mathbb{K}^*$  extends to a homomorphism  $\Lambda \rightarrow \mathbb{K}^*$ .*

Let us give a geometric interpretation of Picard graded algebras. Let  $\Lambda$  be a group of line bundles on  $X$  with associated  $\Lambda$ -graded  $\mathcal{O}_X$ -algebra  $\mathcal{R}$ . Fix a subgroup  $\Lambda_0$  of the kernel of  $\Lambda \rightarrow \text{Pic}(X)$  and a  $\Lambda_0$ -shifting family  $\varrho$  for  $\mathcal{R}$ .

Similar to the preceding section, we assume for the rest of this section that in the case of a ground field  $\mathbb{K}$  of characteristic  $p > 0$ , the group  $\Lambda/\Lambda_0$  has no  $p$ -torsion. Under this hypothesis, we can show that the quotient  $\mathcal{A} := \mathcal{R}/\mathcal{I}$  by the ideal associated to the shifting family  $\varrho$  is reduced:

**Lemma 3.9.** *For every open  $U \subset X$ , the ideal  $\mathcal{I}(U)$  is a radical ideal in  $\mathcal{R}(U)$ .*

**Proof.** First note that we may assume that  $U$  is a small affine open set such that  $\mathcal{R}(U)$  is of finite type. Consider the affine variety  $Z := \text{Spec}(\mathcal{R}(U))$ . Then the  $\Lambda/\Lambda_0$ -grading of  $\mathcal{R}(U) = \mathcal{O}(Z)$  defines an action of the diagonalizable group  $H := \text{Spec}(\mathbb{K}[\Lambda/\Lambda_0])$  on  $Z$ . Let  $Z_0 \subset Z$  denote the zero set of the ideal  $\mathcal{I}(U) \subset \mathcal{R}(U)$ .

Now we can enter the proof of the assertion. Let  $f \in \mathcal{O}(Z)$  with  $f^n \in \mathcal{I}(U)$ . We have to show that  $f \in \mathcal{I}(U)$  holds. Consider the decomposition of  $f$  into homogeneous parts:

$$f = \sum_{[L] \in \Lambda/\Lambda_0} f_{[L]}.$$

Since  $f$  vanishes along the  $H$ -invariant zero set  $Z_0$  of the  $\Lambda/\Lambda_0$ -graded ideal  $\mathcal{I}(U)$ , also every homogeneous component  $f_{[L]}$  has to vanish along  $Z_0$ .

We show that every  $f_{[L]}$  belongs to  $\mathcal{I}(U)$ . Since the  $f_{[L]}$  vanish along  $Z_0$ , Hilbert's Nullstellensatz tells us that for every degree  $[L]$  some power  $f_{[L]}^m$  lies in  $\mathcal{I}(U)$ . Now consider

$$g := \sum_{L' \in [L]} f_{L'} - (f_{L'} - \varrho_{L-L'}(f_{L'})).$$

Then  $g$  is  $\Lambda$ -homogeneous of degree  $L$ . Moreover, by explicit multiplication, we see  $g^m \in \mathcal{I}(U)$ . But any  $\Lambda$ -homogeneous element of  $\mathcal{I}(U)$  is trivial. Thus  $g^m = 0$ . Hence  $g = 0$ , which in turn implies  $f_{[L]} \in \mathcal{I}(U)$ .  $\square$

In our geometric interpretation, we use the global “Spec”-construction, see, for example, [9]. Moreover, for any homogeneous section  $f \in \mathcal{A}(U)$ , we denote its zero set

in  $X$  by  $Z(f)$ . This is well defined, because the components  $\mathcal{A}_{[L]}$  are locally free due to Lemma 3.5(ii).

**Proposition 3.10.** *Let  $\widehat{X} := \text{Spec}(\mathcal{A})$ , and let  $q : \widehat{X} \rightarrow X$  be the canonical map.*

- (i)  $\widehat{X}$  is a variety,  $q : \widehat{X} \rightarrow X$  is an affine morphism, and we have  $\mathcal{A} = q_* \mathcal{O}_{\widehat{X}}$ .
- (ii) For a homogeneous section  $f \in \mathcal{A}_{[L]}(X)$  we obtain  $q^{-1}(Z(f)) = V(\widehat{X}; f)$ , where  $V(\widehat{X}; f)$  is the zero set of the function  $f \in \mathcal{O}(\widehat{X})$ .
- (iii) If  $f_i \in \mathcal{A}(X)$  are homogeneous sections such that the sets  $X \setminus Z(f_i)$  are affine and cover  $X$ , then  $\widehat{X}$  is a quasiaffine variety.

**Proof.** To check (i), note that  $\widehat{X}$  is indeed a variety, because by Lemmas 3.5(iii) and 3.9, the algebra  $\mathcal{A}$  is reduced and locally of finite type. The rest of (i) are standard properties of the global “Spec”-construction for sheaves of  $\mathcal{O}_X$ -algebras.

Assertion (ii) is clear in the case  $[L] = 0$ , because then we have  $\mathcal{A}_0 = \mathcal{O}_X$ . For a general  $[L]$ , we may reduce to the previous case by multiplying  $f$  locally with invertible sections of degree  $-[L]$ . Note that invertible sections exist locally by Lemma 3.5(iii).  $\square$

#### 4. The homogeneous coordinate ring

In this section, we give the precise definition of the homogeneous coordinate ring of a given variety, see Definition 4.4. Moreover, we show in Proposition 4.6 that the homogeneous coordinate ring is unique up to isomorphism.

In order to fix the setup, recall from [3] that a (neither necessarily separated nor irreducible) variety  $X$  is said to be *divisorial* if every  $x \in X$  admits an affine neighbourhood of the form  $X \setminus Z(f)$  where  $Z(f)$  is the zero set of a global section  $f$  of some line bundle  $L$  on  $X$ .

**Remark 4.1.** Every separated irreducible  $\mathbb{Q}$ -factorial variety is divisorial, and every quasiprojective variety is divisorial.

Here is the setup of this section: We assume that the multiplicative group  $\mathbb{K}^*$  is of infinite rank over  $\mathbb{Z}$ , e.g.,  $\mathbb{K}$  is of characteristic zero or it is uncountable. The variety  $X$  is divisorial and satisfies  $\mathcal{O}^*(X) = \mathbb{K}^*$ . Moreover,  $\text{Pic}(X)$  is finitely generated and, if  $\mathbb{K}$  is of characteristic  $p > 0$ , then  $\text{Pic}(X)$  has no  $p$ -torsion.

**Lemma 4.2.** *There exists a group  $\Lambda$  of line bundles on  $X$  mapping onto  $\text{Pic}(X)$ . For any such  $\Lambda$  the associated  $\Lambda$ -graded  $\mathcal{O}_X$ -algebra  $\mathcal{R}$  admits homogeneous global sections  $h_1, \dots, h_r$  such that the sets  $X \setminus Z(h_i)$  are affine and cover  $X$ .*

**Proof.** Only for the first statement there is something to show. For this, we may assume that  $\text{Pic}(X)$  is not trivial. Write  $\text{Pic}(X)$  as a direct sum of cyclic groups  $\Pi_1, \dots, \Pi_m$  and fix a generator  $P_l$  for each  $\Pi_l$ . Choose a finite open cover  $\mathcal{U}$  of  $X$  such that each  $P_l$  is



represented by a cocycle  $\xi^{(l)} \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ . Choose members  $U_i, U_j$  of  $\mathfrak{U}$  such that  $U_i \neq U_j$  holds and there is a point  $x_0 \in U_i \cap U_j$ .

We adjust the  $\xi^{(l)}$  as follows: By the assumption on the ground field  $\mathbb{K}$ , we find  $a_1, \dots, a_m \in \mathbb{K}^*$  which are linearly independent over  $\mathbb{Z}$ . Define a locally constant cochain  $\eta^{(l)}$  by setting  $\eta^{(l)} := a_l/\xi_{ij}^{(l)}(x_0)$  on  $U_i$  and  $\eta^{(l)} := 1$  on the  $U_k$  different from  $U_i$ . Let  $\zeta^{(l)} \in Z^1(\mathfrak{U}, \mathcal{O}^*)$  be the product of  $\xi^{(l)}$  with the coboundary of  $\eta^{(l)}$ .

Let  $\Lambda \subset \Lambda(\mathfrak{U})$  be the subgroup generated by the line bundles arising from  $\zeta^{(1)}, \dots, \zeta^{(m)}$ . By construction  $\Lambda$  maps onto  $\text{Pic}(X)$ . Moreover, we have

$$((\zeta_{ij}^{(1)})^{n_1} \cdots (\zeta_{ij}^{(m)})^{n_m})(x_0) = a_1^{n_1} \cdots a_m^{n_m}$$

for the cocycle corresponding to a general element of  $\Lambda$ . By the choice of the  $a_l$ , this cocycle can only be trivial if all exponents  $n_l$  vanish. It follows that  $\Lambda$  is free.  $\square$

We fix a group  $\Lambda$  of line bundles on  $X$  as provided by Lemma 4.2, and a full shifting family  $\varrho$  for the  $\Lambda$ -graded  $\mathcal{O}_X$ -algebra  $\mathcal{R}$  associated to  $\Lambda$ . Let  $\mathcal{I}$  denote the ideal associated to the shifting family  $\varrho$ . As seen in Lemma 3.5(i), the  $\mathcal{O}_X$ -algebra  $\mathcal{A} := \mathcal{R}/\mathcal{I}$  is graded by  $\text{Pic}(X)$ . In particular, we have a grading

$$\mathcal{A}(X) = \bigoplus_{[L] \in \text{Pic}(X)} \mathcal{A}_{[L]}(X).$$

According to Lemmas 3.5(ii) and 4.2, there are homogeneous  $f_1, \dots, f_r \in \mathcal{A}(X)$  such that the sets  $X \setminus Z(f_i)$  are affine and cover  $X$ . Hence Proposition 3.10(iii) tells us that the variety  $\widehat{X} := \text{Spec}(\mathcal{A})$  is quasiaffine. Thus we have the collection  $\mathfrak{A}(X)$  of natural pairs on  $\widehat{X}$  as closing subalgebras for  $\mathcal{A}(X) = \mathcal{O}(\widehat{X})$ , see Lemma 1.5.

**Proposition 4.3.** *The pair  $(\mathcal{A}(X), \mathfrak{A}(X))$  is a freely graded quasiaffine algebra.*

**Proof.** We have to show that there is a natural pair  $(A', I') \in \mathfrak{A}(X)$  with the properties of Definition 2.1.

Choose homogeneous  $f_1, \dots, f_r \in \mathcal{A}(X)$  such that the sets  $X \setminus Z(f_i)$  form an affine cover of  $X$ . Let  $q: \widehat{X} \rightarrow X$  be the canonical map. By Proposition 3.10(ii), each  $\widehat{X}_{f_i}$  equals  $q^{-1}(X \setminus Z(f_i))$  and thus is affine. Consequently the algebras

$$\mathcal{A}(X)_{f_i} = \mathcal{O}(\widehat{X})_{f_i} = \mathcal{O}(\widehat{X}_{f_i})$$

are of finite type. Thus we find a subalgebra  $A' \subset \mathcal{A}(X)$  of finite type satisfying  $A'_{f_i} = \mathcal{A}(X)_{f_i}$  for every  $i$ . Then  $\overline{X} := \text{Spec}(A')$  is an affine closure of  $\widehat{X}$ , and the vanishing ideal  $I' \subset A'$  of  $\overline{X} \setminus \widehat{X}$  is the radical of the ideal generated by  $f_1, \dots, f_r$ . It follows that  $(A', I')$  is a natural pair on  $\widehat{X}$ .

We verify the condition on the degrees. Given  $x \in \widehat{X}$ , choose an  $f_i$  with  $q(x) \in U := X \setminus Z(f_i)$ . By Lemma 3.5(iii), there is a small neighbourhood  $U_h \subset U$  of  $x$  defined by some  $h \in \mathcal{O}(U)$  such that every  $[L] \in \text{Pic}(X)$  admits an invertible section in  $\mathcal{A}_{[L]}(U_h)$ .

Now,  $U_h$  equals  $X \setminus Z(hf_i^n)$  for some large positive integer  $n$ . Since finitely many of such  $U_h$  cover  $X$ , we obtain the desired property of Definition 2.1 with finitely many of the homogeneous sections  $hf_i^n \in I'$ .  $\square$

**Definition 4.4.** We call  $(\mathcal{A}(X), \mathfrak{A}(X))$  the *homogeneous coordinate ring* of  $X$ .

We show now that homogeneous coordinate rings are unique up to isomorphism. This amounts to comparing Picard graded algebras arising from different groups of line bundles on  $X$ . As we shall need it later, we do this in a slightly more general setting:

**Lemma 4.5.** Let  $\Lambda$  and  $\Gamma$  be groups of line bundles on  $X$  with associated graded  $\mathcal{O}_X$ -algebras  $\mathcal{R}$  and  $\mathcal{S}$ . Suppose that the image of  $\Lambda \rightarrow \text{Pic}(X)$  contains the image of  $\Gamma \rightarrow \text{Pic}(X)$ , and let  $\varrho$  be a full shifting family for  $\mathcal{R}$ .

- (i) There exist a graded homomorphism  $\gamma : \mathcal{S} \rightarrow \mathcal{R}$  with accompanying homomorphism  $\tilde{\gamma} : \Gamma \rightarrow \Lambda$  and a full shifting family  $\sigma$  for  $\Gamma$  such that for every  $K \in \Gamma$  we have  $K \cong \tilde{\gamma}(K)$ , and, given an  $F$  from the kernel of  $\Gamma \rightarrow \text{Pic}(X)$ , there is a commutative diagram of  $\mathcal{O}_X$ -module isomorphisms

$$\begin{array}{ccc}
 \mathcal{S}_K & \xrightarrow{\gamma_K} & \mathcal{R}_{\tilde{\gamma}(K)} \\
 \sigma_F \downarrow & & \downarrow \varrho_{\tilde{\gamma}(F)} \\
 \mathcal{S}_{K+F} & \xrightarrow{\gamma_{K+F}} & \mathcal{R}_{\tilde{\gamma}(K)+\tilde{\gamma}(F)}
 \end{array}$$

- (ii) Given data as in (i), let  $\mathcal{B}$  and  $\mathcal{A}$  denote the Picard graded algebras associated to  $\sigma$  and  $\varrho$ . Then one has a commutative diagram

$$\begin{array}{ccc}
 \mathcal{S} & \xrightarrow{\gamma} & \mathcal{R} \\
 \downarrow & & \downarrow \\
 \mathcal{B} & \xrightarrow{\tilde{\gamma}} & \mathcal{A}
 \end{array}$$

of graded  $\mathcal{O}_X$ -algebra homomorphisms. The lower row is an isomorphism if  $\Gamma$  and  $\Lambda$  have the same image in  $\text{Pic}(X)$ .

**Proof.** Let  $\Gamma \subset \Lambda(\mathfrak{W})$  and  $\Lambda \subset \Lambda(\mathfrak{U})$ . Then  $\Lambda$  and  $\Gamma$  embed canonically into  $\Lambda(\mathfrak{W})$ , where  $\mathfrak{W}$  denotes any common refinement of the open covers  $\mathfrak{U}$  and  $\mathfrak{V}$ . Hence we may assume that  $\Lambda$  and  $\Gamma$  arise from the same trivializing cover.

Let  $K_1, \dots, K_m$  be a basis of  $\Gamma$  and choose  $E_1, \dots, E_m \in \Lambda$  in such a way that the isomorphism class of  $E_i$  equals the class of  $K_i$  in  $\text{Pic}(X)$ . Furthermore let  $\tilde{\gamma} : \Gamma \rightarrow \Lambda$  be the homomorphism sending  $K_i$  to  $E_i$ . For each  $i = 1, \dots, m$ , fix a bundle isomorphism  $\beta_{K_i} : K_i \rightarrow E_i$ .

By multiplying the local data of these homomorphisms, we obtain as in the proof of Lemma 3.2 a bundle isomorphism  $\beta_K : K \rightarrow \tilde{\gamma}(K)$  for every  $K \in \Gamma$ . Shifting sections via these  $\beta_K$  defines  $\mathcal{O}_X$ -module isomorphisms  $\gamma_K : \mathcal{S}_K \rightarrow \mathcal{R}_{\tilde{\gamma}(K)}$ . By construction, the  $\gamma_K$  fit together to a graded homomorphism  $\gamma : \mathcal{S} \rightarrow \mathcal{R}$  of  $\mathcal{O}_X$ -algebras.

Now it is clear how to define the full shifting family  $\sigma$ : Take an  $F$  from the kernel of  $\Gamma \rightarrow \text{Pic}(X)$ . Define  $\sigma_F : \mathcal{S} \rightarrow \mathcal{S}$  by prescribing on the homogeneous components the (unique) isomorphisms  $\mathcal{S}_K \rightarrow \mathcal{S}_{K+F}$  that make the above diagrams commutative. It is then straightforward to verify the properties of a shifting family for the maps  $\sigma_F$ . This settles assertion (i).

We prove (ii). By the commutative diagram of (i), the ideal associated to  $\sigma$  is mapped into the ideal associated to  $\varrho$ . Hence, we obtain the desired homomorphism  $\bar{\gamma} : \mathcal{B} \rightarrow \mathcal{A}$  of Picard graded algebras.

Now, assume that the images of  $\Gamma$  and  $\Lambda$  in  $\text{Pic}(X)$  coincide. Since every  $\gamma_K : \mathcal{S}_K \rightarrow \mathcal{R}_{\tilde{\gamma}(K)}$  is an isomorphism, we can use Lemma 3.5(ii) to see that  $\bar{\gamma}$  is an isomorphism in every degree. By assumption the accompanying homomorphism of  $\bar{\gamma}$  is bijective, whence the assertion follows.  $\square$

The uniqueness of homogeneous coordinate rings is a direct consequence of Lemmas 3.7 and 4.5.

**Proposition 4.6.** *Different choices of the group of line bundles and the full shifting family define isomorphic freely graded quasiaffine algebras as homogeneous coordinate rings for  $X$ .*

**Proof.** Let  $\Lambda$  and  $\Gamma$  be two groups of line bundles mapping onto  $\text{Pic}(X)$  and let  $\mathcal{A}$  and  $\mathcal{B}$  denote the Picard graded algebras associated to choices of full shifting families for the corresponding  $\Lambda$ - and  $\Gamma$ -graded  $\mathcal{O}_X$ -algebras. From Lemmas 3.7 and 4.5 we infer the existence of a graded  $\mathcal{O}_X$ -algebra isomorphism  $\mu : \mathcal{B} \rightarrow \mathcal{A}$ . In particular, we have  $\mathcal{B}(X) \cong \mathcal{A}(X)$ .

We show that  $\mu$  defines an isomorphism of quasiaffine algebras. Let  $(B', J') \in \mathfrak{B}(X)$  as in Definition 2.1. Then Lemma 1.5 ensures that  $(B', J')$  is a natural pair on  $\text{Spec}(\mathcal{B})$ . We have to show that  $(\mu(B'), \mu(J'))$  is a natural pair on  $\text{Spec}(\mathcal{A})$ . Since  $\mu$  is an  $\mathcal{O}_X$ -module isomorphism in every degree, we have  $Z(\mu(g)) = Z(g)$  for any homogeneous  $g \in J'$ . Thus Proposition 3.10(ii) tells us that  $(\mu(B'), \mu(J'))$  is a natural pair.  $\square$

## 5. Functoriality of the homogeneous coordinate ring

In this section, we present the first main result. It says that homogeneous coordinates are a fully faithful contravariant functor, see Theorem 5.7. But first we have to define the homogeneous coordinate ring functor on morphisms. The basic tool for this definition are Picard graded pullbacks, see Definition 5.1 and Proposition 5.3.

As in the preceding section, we assume that  $\mathbb{K}^*$  is of infinite rank over  $\mathbb{Z}$ . Moreover, in this section we assume all varieties to be divisorial and to have only constant invertible

global functions. Finally, we require that any variety has a finitely generated Picard group, and, if  $\mathbb{K}$  is of characteristic  $p > 0$ , this Picard group has no  $p$ -torsion.

For a variety  $X$  fix a group  $\Lambda$  of line bundles mapping onto  $\text{Pic}(X)$  and denote the associated  $\Lambda$ -graded  $\mathcal{O}_X$ -algebra by  $\mathcal{R}$ . Moreover, we fix a full shifting family  $\varrho$  for  $\mathcal{R}$  and denote the resulting Picard graded algebra by  $\mathcal{A}$ . For a further variety  $Y$  we denote the corresponding data by  $\Gamma$ ,  $\mathcal{S}$ ,  $\sigma$ , and  $\mathcal{B}$ . Let  $\varphi: X \rightarrow Y$  be a morphism of the varieties  $X$  and  $Y$ .

**Definition 5.1.** By a *Picard graded pullback* for  $\varphi: X \rightarrow Y$  we mean a graded homomorphism  $\mathcal{B} \rightarrow \varphi_*\mathcal{A}$  of  $\mathcal{O}_Y$ -algebras having the pullback map  $\varphi^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$  as its accompanying homomorphism.

Note that the property of being an  $\mathcal{O}_Y$ -algebra homomorphism means in particular, that in degree zero any Picard graded pullback is the usual pullback of functions. As a consequence, we remark:

**Lemma 5.2.** *Let  $\mu: \mathcal{B} \rightarrow \varphi_*\mathcal{A}$  be a Picard graded pullback for  $\varphi: X \rightarrow Y$ , and let  $g \in \mathcal{B}(Y)$  be homogeneous. Then the zero set  $Z(\mu(g)) \subset X$  is the inverse image  $\varphi^{-1}(Z(g))$  of the zero set  $Z(g) \subset Y$ .*

**Proof.** It suffices to prove the statement locally, over small open  $V \subset Y$ . But on such  $V$ , we may shift  $g$  by multiplication with invertible elements into degree zero. This does not affect zero sets, whence the assertion follows.  $\square$

The basic step in the definition of the homogeneous coordinate ring functor on morphisms is to show existence of Picard graded pullbacks and to provide a certain uniqueness property:

**Proposition 5.3.** *There exist Picard graded pullbacks for  $\varphi: X \rightarrow Y$ . Moreover, any two Picard graded pullbacks  $\mu, \nu: \mathcal{B} \rightarrow \varphi_*\mathcal{A}$  for  $\varphi$  differ by a character  $c: \text{Pic}(Y) \rightarrow \mathbb{K}^*$  in the sense that  $\nu_P = c(P)\mu_P$  holds for all  $P \in \text{Pic}(Y)$ .*

The proof of this statement is based on two lemmas. The first one is an extension property for shifting families:

**Lemma 5.4.** *Let  $\Pi$  be any group of line bundles on  $X$ , and let  $\Pi_0 \subset \Pi_1$  be two subgroups of the kernel of  $\Pi \rightarrow \text{Pic}(X)$ . Then every  $\Pi_0$ -shifting family  $\tau^0$  for the  $\Pi$ -graded  $\mathcal{O}_X$ -algebra  $\mathcal{T}$  associated to  $\Pi$  extends to a  $\Pi_1$ -shifting family  $\tau^1$  for  $\mathcal{T}$  in the sense that  $\tau_E^1 = \tau_E^0$  holds for all  $E \in \Pi_0$ .*

**Proof.** Let  $\vartheta$  be any  $\Pi_1$ -shifting family for  $\mathcal{T}$ . Then  $\vartheta$  restricts to a  $\Pi_0$ -shifting family. By Lemma 3.2, there is a character  $c: \Pi_0 \rightarrow \mathcal{O}^*(X)$  with  $\tau_E^0 = c(E)\vartheta_E$  for all  $E \in \Pi_0$ . As we assumed  $\mathcal{O}^*(X) = \mathbb{K}^*$ , Lemma 3.8 tells us that  $c$  extends to  $\Pi_1$ . Thus, setting  $\tau_E^1 := c(E)\vartheta_E$  for  $E \in \Pi_1$  gives the desired extension.  $\square$

The second lemma provides a pullback construction for shifting families. By pulling back cocycles, we obtain the (again free) pullback group  $\varphi^*\Gamma$ . We denote the associated  $\varphi^*\Gamma$ -graded  $\mathcal{O}_X$ -algebra by  $\varphi^*\mathcal{S}$ . Indeed  $\varphi^*\mathcal{S}$  is canonically isomorphic to the ringed inverse image of  $\mathcal{S}$ . Observe that we have a canonical sheaf homomorphism  $\mathcal{S} \rightarrow \varphi_*\varphi^*\mathcal{S}$ .

**Lemma 5.5.** *Let  $\Gamma_0 \subset \Gamma$  a subgroup, and let  $\sigma$  be a  $\Gamma_0$ -shifting family for  $\mathcal{S}$ .*

- (i) *The  $\mathcal{O}_X$ -module homomorphisms  $\varphi^*\sigma_F$  define a  $\varphi^*\Gamma_0$ -shifting family  $\varphi^*\sigma$  for  $\varphi^*\mathcal{S}$ .*
- (ii) *The ideal  $\mathcal{J}^*$  associated to  $\varphi^*\sigma$  equals the pullback  $\varphi^*\mathcal{J}$  of the ideal  $\mathcal{J}$  associated to  $\sigma$ .*

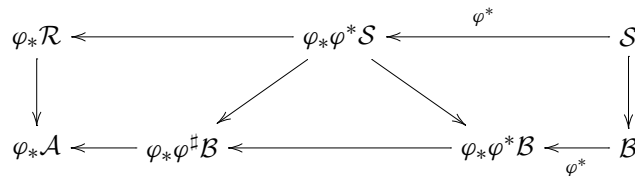
**Proof.** For (i), note that the isomorphisms  $\sigma_F : \mathcal{S}_K \rightarrow \mathcal{S}_{K+F}$  can be written as  $g \mapsto \beta_{K,F}(g)$  with unique line bundle isomorphisms  $\beta_{K,F} : K \rightarrow K + F$ . The family  $\varphi^*\sigma_F$  corresponds to the collection  $\varphi^*\beta_{K,F} : \varphi^*K \rightarrow \varphi^*K + \varphi^*F$ . The properties of a shifting family become clear by writing the  $\varphi^*\beta_{K,F}$  in terms of local data as in (3.2.1).

To prove (ii), we just compare the stalks of the two sheaves in question. By Property 3.2(iii), we obtain for any  $x \in X$

$$\begin{aligned} \mathcal{J}_x^* &= \langle 1_x - \varphi^*\sigma_F(1_x); F \in \Gamma_0 \rangle \\ &= \langle \varphi^*(1_{\varphi(x)}) - \varphi^*(\sigma_F(1_{\varphi(x)})); F \in \Gamma_0 \rangle \\ &= (\varphi^*\mathcal{J})_x. \quad \square \end{aligned}$$

**Proof of Proposition 5.3.** We establish the existence of Picard graded pullbacks: As usual, let  $\mathcal{I}$  and  $\mathcal{J}$  denote the respective ideals associated to the shifting families  $\varrho$  for  $\mathcal{R}$  and  $\sigma$  for  $\mathcal{S}$ . Thus the corresponding Picard graded algebras are  $\mathcal{A} = \mathcal{R}/\mathcal{I}$  and  $\mathcal{B} = \mathcal{S}/\mathcal{J}$ .

By Lemma 5.5, we have the  $\varphi^*\Gamma_0$ -shifting family  $\varphi^*\sigma$  for  $\varphi^*\mathcal{S}$ . Lemma 5.4 enables us to choose a full shifting family  $\varphi^\sharp\sigma$  extending  $\varphi^*\sigma$ . We denote by  $\varphi^\sharp\mathcal{J}$  the ideal associated to  $\varphi^\sharp\sigma$ , and write  $\varphi^\sharp\mathcal{B} := \varphi^*\mathcal{S}/\varphi^\sharp\mathcal{J}$  for the quotient. In this notation, we have a commutative diagram of graded  $\mathcal{O}_Y$ -algebra homomorphisms such that the unlabelled arrows are isomorphisms in each degree:



Indeed, the right square is standard. To obtain the middle triangle, we only have to show that  $\varphi^\sharp\mathcal{J}$  contains the kernel of  $\varphi^*\mathcal{S} \rightarrow \varphi^*\mathcal{B}$ . But this follows from exactness of  $\varphi^*$  and Lemma 5.5(ii). Existence of the left square follows from combining Lemmas 3.7 and 4.5. Now the desired Picard graded pullback of  $\varphi : X \rightarrow Y$  is the composition of the lower horizontal arrows.

We turn to the uniqueness statement. Let  $P \in \text{Pic}(Y)$ . Since  $\mathcal{B}_P$  locally admits invertible sections, we can cover  $Y$  by open  $V \subset Y$  such that there exist invertible sections  $h \in \mathcal{B}_P(V)$ . We define

$$c(P, V) := \nu(h)/\mu(h) \in \mathcal{A}_0^*(\varphi^{-1}(V)).$$

This does not depend on the choice of  $h$ : For a further invertible  $g \in \mathcal{B}_P(V)$ , the section  $g/h$  is of degree zero. But in degree zero any Picard graded pullback is the usual pullback of functions. Thus we have  $\mu(h/g) = \nu(h/g)$ . Consequently,  $\nu(g)/\mu(g)$  equals  $\nu(h)/\mu(h)$ .

Similarly we see that for two open  $V, V' \subset Y$  as above, the corresponding sections  $c(P, V)$  and  $c(P, V')$  coincide on the intersection  $\varphi^{-1}(V) \cap \varphi^{-1}(V')$ . Thus, by gluing, we obtain a global section  $c(P) \in \mathcal{A}_0^*(X) = \mathcal{O}^*(X)$ . Then it is immediate to check, that  $P \mapsto c(P)$  has the desired properties.  $\square$

With the help of Picard graded pullbacks we can now make the homogeneous coordinate ring into a functor. We fix for any morphism  $\varphi : X \rightarrow Y$  a Picard graded pullback  $\mu_\varphi : \mathcal{B} \rightarrow \varphi_*\mathcal{A}$ , and denote the induced homomorphism on global sections again by  $\mu_\varphi : \mathcal{B}(Y) \rightarrow \mathcal{A}(X)$ .

For a graded homomorphism  $\nu$  of freely graded quasiaffine algebras, we denote by  $[\nu]$  its equivalence class in the sense of Definition 2.5.

**Proposition 5.6.** *The assignments  $X \mapsto (\mathcal{A}(X), \mathfrak{A}(X))$  and  $\varphi \mapsto [\mu_\varphi]$  define a contravariant functor into the category of freely graded quasiaffine algebras.*

**Proof.** By Proposition 4.3, the homogeneous coordinate rings  $(\mathcal{A}(X), \mathfrak{A}(X))$  and  $(\mathcal{B}(Y), \mathfrak{B}(Y))$  of  $X$  and  $Y$  are in fact freely graded quasiaffine algebras. The first task is to show that the homomorphism  $\mu_\varphi : \mathcal{B}(Y) \rightarrow \mathcal{A}(X)$  associated to a morphism  $\varphi : X \rightarrow Y$  is a graded homomorphism of freely graded quasiaffine algebras.

As a Picard graded pullback,  $\mu_\varphi$  is graded and has as accompanying homomorphism the pullback map  $\text{Pic}(Y) \rightarrow \text{Pic}(X)$ . Thus we are left with checking the conditions of Definition 1.2(ii) for  $\mu_\varphi$ . This is done geometrically in terms of the constructions of Proposition 3.10:

$$\widehat{X} := \text{Spec}(\mathcal{A}), \quad \widehat{Y} := \text{Spec}(\mathcal{B}), \quad q_X : \widehat{X} \rightarrow X, \quad q_Y : \widehat{Y} \rightarrow Y.$$

Let  $(B', J') \in \mathfrak{B}(Y)$  be a closing subalgebra as in Definition 2.1. Then Lemma 1.4 provides a closing subalgebra  $(A', I') \in \mathfrak{A}(X)$  such that  $\mu_\varphi(B') \subset A'$  holds. We have to verify the condition on the ideals  $I'$  and  $J'$  required in Definition 1.2(ii). For this, consider the affine closures of  $\widehat{X}$  and  $\widehat{Y}$ :

$$\overline{X} := \text{Spec}(A'), \quad \overline{Y} := \text{Spec}(B').$$

Then the restricted homomorphism  $\mu_\varphi : B' \rightarrow A'$  defines a morphism  $\overline{\varphi} : \overline{X} \rightarrow \overline{Y}$ . Recall from Section 1 that  $I'$  and  $J'$  are the vanishing ideals of the complements  $\overline{X} \setminus \widehat{X}$  and  $\overline{Y} \setminus \widehat{Y}$ .

Thus we have to show that  $\bar{\varphi}$  maps  $\widehat{X}$  to  $\widehat{Y}$ . For this, let  $g_1, \dots, g_r \in J'$  be homogeneous sections as in Definition 2.1. Using Lemma 5.2, we obtain

$$\begin{aligned} \widehat{X} &= \bigcup_{j=1}^r q_X^{-1}(\varphi^{-1}(Y \setminus Z(g_j))) = \bigcup_{j=1}^r q_X^{-1}(X \setminus Z(\mu_\varphi(g_j))) \\ &\subset \bigcup_{j=1}^r \bar{X}_{\mu_\varphi(g_j)} = \bar{\varphi}^{-1}(\widehat{Y}). \end{aligned}$$

Finally, we check that  $\varphi \mapsto [\mu_\varphi]$  is functorial. Note that by Proposition 5.3, the class  $[\mu_\varphi]$  does not depend on the choice of the Picard graded pullback  $\mu_\varphi$  of a given morphism.

From this we conclude that the identity morphism of a variety is mapped to the identity morphism of its homogeneous coordinate ring. Moreover, as the composition of two Picard graded pullbacks is a Picard graded pullback for the composition of the respective morphisms, the above assignment commutes with composition.  $\square$

In the sequel we shall speak of the homogeneous coordinate ring functor. We present the first main result of this article. It tells us that the morphisms of two varieties are in one-to-one correspondence with the morphisms of their coordinate rings.

**Theorem 5.7.** *The homogeneous coordinate ring functor  $X \mapsto (\mathcal{A}(X), \mathfrak{A}(X))$  and  $\varphi \mapsto [\mu_\varphi]$  is fully faithful.*

**Proof.** Let  $X, Y$  be varieties with associated Picard graded algebras  $\mathcal{A}$  and  $\mathcal{B}$ . We denote the respective homogeneous coordinate rings of  $X$  and  $Y$  for short by  $(A, \mathfrak{A})$  and  $(B, \mathfrak{B})$ . We construct an inverse to

$$\text{Mor}(X, Y) \rightarrow \text{Mor}((B, \mathfrak{B}), (A, \mathfrak{A})), \quad \varphi \mapsto [\mu_\varphi].$$

So, start with any graded homomorphism  $\mu : (B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$  of quasiaffine algebras. Then Lemma 1.4 provides closing subalgebras  $(A', \mathfrak{A}') \in \mathfrak{A}$  and  $(B', \mathfrak{B}') \in \mathfrak{B}$  such that  $(B', \mathfrak{B}')$  is as in Definition 2.1 and we have  $\mu(B') \subset A'$ .

Consider the affine closures  $\bar{X} := \text{Spec}(A')$  and  $\bar{Y} := \text{Spec}(B')$  of  $\widehat{X} := \text{Spec}(\mathcal{A})$  and  $\widehat{Y} := \text{Spec}(\mathcal{B})$ . Then  $\mu$  gives rise to a morphism  $\bar{\varphi} : \bar{X} \rightarrow \bar{Y}$ , and restricting this morphism to  $\widehat{X}$  yields a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\bar{\varphi}} & \widehat{Y} \\ q_X \downarrow & & \downarrow q_Y \\ X & \xrightarrow{\varphi} & Y \end{array}$$

where  $q_X$  and  $q_Y$  denote the canonical maps, and the morphism  $\varphi : X \rightarrow Y$  has as its pullbacks on the level of functions the maps obtained by restricting the localizations  $\mu_g : B_g \rightarrow A_{\mu(g)}$  to degree zero over the affine sets  $Y_g$  for homogeneous  $g \in J'$ .

Observe that applying the above procedure to a further graded homomorphism  $\nu: (B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$  yields the same induced morphism  $X \rightarrow Y$  if and only if the homomorphisms  $\mu$  and  $\nu$  are equivalent; the “only if” part follows from the uniqueness statement of Proposition 5.3 and the fact that  $\mu$  and  $\nu$  define Picard graded pullbacks via localizing. Thus  $[\mu] \mapsto \varphi$  defines an injection

$$\text{Mor}((B, \mathfrak{B}), (A, \mathfrak{A})) \rightarrow \text{Mor}(X, Y).$$

We check that this map is inverse to the one defined by the homogeneous coordinate ring functor. Start with a morphism  $\varphi: X \rightarrow Y$ , and let  $[\mu_\varphi]: B \rightarrow A$  be as before Proposition 5.6. Write shortly  $\mu := \mu_\varphi$ . Consider a homogeneous  $g \in B$  such that  $V := Y \setminus Z(g)$  is affine and let  $U := \varphi^{-1}(V)$ . Using Lemma 5.2, we obtain a commutative diagram

$$\begin{array}{ccc} A_{(\mu(g))} & \xleftarrow{\mu(g)} & B_{(g)} \\ \parallel & & \parallel \\ \mathcal{O}_X(U) & \xleftarrow{\varphi^*} & \mathcal{O}_Y(V) \end{array}$$

where the above horizontal map is the map on degree zero induced by the localized map  $\mu_g: B_g \rightarrow A_{\mu(g)}$ . Since  $Y$  is covered by open affine sets of the form  $V = Y \setminus Z(g)$ , we see that the morphism  $X \rightarrow Y$  associated to  $\mu = \mu_\varphi$  is again  $\varphi$ .  $\square$

So far, our homogeneous coordinate ring functor depends on the choice of the homogeneous coordinate ring for a given variety. By passing to isomorphism classes, the whole construction can even be made canonical:

**Remark 5.8.** If one takes as target category the category of isomorphism classes of freely graded quasiaffine algebras, then the homogeneous coordinate functor  $X \rightarrow (\mathcal{A}(X), \mathfrak{A}(X))$  and  $\varphi \mapsto [\mu_\varphi]$  becomes unique.

### 6. A first dictionary

We present a little dictionary between geometric properties of a variety and algebraic properties of its homogeneous coordinate ring. We consider separatedness, normality and smoothness. Moreover, we treat quasicoherent sheaves, and we describe affine morphisms and closed embeddings.

The setup is the same as in Sections 4 and 5: The multiplicative group  $\mathbb{K}^*$  of the ground field  $\mathbb{K}$  is supposed to be of infinite rank over  $\mathbb{Z}$ . Moreover,  $X$  is a divisorial variety with  $\mathcal{O}^*(X) = \mathbb{K}^*$  and its Picard group is finitely generated and has no  $p$ -torsion if  $\mathbb{K}$  is of characteristic  $p > 0$ .

Denote by  $(A, \mathfrak{A}) := (\mathcal{A}(X), \mathfrak{A}(X))$  the homogeneous coordinate ring of  $X$ . Recall that  $A$  is the algebra of global sections of a suitable Picard graded  $\mathcal{O}_X$ -algebra  $\mathcal{A}$ .



In the subsequent proofs, we shall often use the geometric interpretation provided by Propositions 3.10 and 2.8:

**Lemma 6.1.** Consider  $\widehat{X} := \text{Spec}(\mathcal{A})$ , the canonical map  $q : \widehat{X} \rightarrow X$  and the diagonalizable group  $H := \text{Spec}(\mathbb{K}[\text{Pic}(X)])$ .

- (i) There is a unique free action of  $H$  on  $\widehat{X}$  such that each  $\mathcal{A}_{[L]}(U)$  consists precisely of the  $\chi^{[L]}$ -homogeneous functions of  $q^{-1}(U)$ .
- (ii) The canonical map  $q : \widehat{X} \rightarrow X$  is a geometric quotient for the above  $H$ -action on  $X$ .

**Proof.** The first statement follows from Propositions 3.10 and 2.8. The second statement is due to the facts that  $\mathcal{O}_X = (\mathcal{A}_0)$  is the sheaf of invariants and the action of  $H$  is free.  $\square$

We begin with the dictionary. It is quite easy to characterize separatedness in terms of the homogeneous coordinate ring.

**Proposition 6.2.** The variety  $X$  is separated if and only if there exists a graded closing subalgebra  $(A', I') \in \mathfrak{A}$  and homogeneous  $f_1, \dots, f_r \in I'$  as in Definition 2.1 such that each of the maps  $A_{(f_i)} \otimes A_{(f_j)} \rightarrow A_{(f_i f_j)}$  is surjective.

**Proof.** First recall that the sets  $X_i := X \setminus Z(f_i)$  form an affine cover of  $X$ . The above condition means just that the canonical maps from  $\mathcal{O}(X_i) \otimes \mathcal{O}(X_j)$  to  $\mathcal{O}(X_i \cap X_j)$  is surjective. This is the usual separatedness criterion [14, Proposition 3.3.5].  $\square$

Next we show how normality of the variety  $X$  is reflected in its homogeneous coordinate ring (for us, a normal variety is in particular irreducible).

**Proposition 6.3.** The variety  $X$  is normal if and only if  $A$  is a normal ring.

**Proof.** We work in terms of the geometric data  $q : \widehat{X} \rightarrow X$  and  $H$  discussed in Lemma 6.1. First suppose that  $A = \mathcal{A}(X)$  is a normal ring. Then the quasiaffine variety  $\widehat{X}$  is normal. It is a basic property of geometric quotients that the variety  $X$  inherits normality from  $\widehat{X}$ , see, e.g., [7, p. 39].

Suppose conversely that  $X$  is normal. Luna's Slice Theorem tells us that  $q : \widehat{X} \rightarrow X$  is an  $H$ -principal bundle in the étale topology, see [17], and [2, Proposition 8.1]. Thus, up to étale maps,  $\widehat{X}$  looks locally like  $X \times H$ . Since normality of local rings is stable under étale maps [19, Proposition I.3.17], we can conclude that all local rings of  $\widehat{X}$  are normal.

It remains to show that  $\widehat{X}$  is connected. Assume the contrary. Then there is a connected component  $\widehat{X}_1 \subset \widehat{X}$  with  $q(\widehat{X}_1) = X$ . Let  $H_1 \subset H$  be the stabilizer of  $\widehat{X}_1$ , that means that  $H_1$  is the maximal subgroup of  $H$  with  $H_1 \cdot \widehat{X}_1 = \widehat{X}_1$ . Note that we have  $t \in H_1$  if  $t \cdot x \in \widehat{X}_1$  holds for at least one point  $x \in \widehat{X}_1$ . In particular,  $H_1$  is a proper subgroup of  $H$ .

We claim that restricting the canonical map  $q : \widehat{X} \rightarrow X$  to  $\widehat{X}_1$  yields a geometric quotient for the action of  $H_1$  on  $\widehat{X}_1$ . Indeed,  $H_1$  acts freely on  $\widehat{X}_1$ . Hence we have a geometric quotient  $\widehat{X}_1 \rightarrow \widehat{X}_1/H_1$  and a commutative diagram

$$\begin{array}{ccc}
 \widehat{X}_1 & \xrightarrow{c} & \widehat{X} \\
 \downarrow /H_1 & & \downarrow q /H \\
 \widehat{X}_1/H_1 & \longrightarrow & X
 \end{array}$$

The map  $\widehat{X}_1/H_1 \rightarrow X$  is bijective, because the intersection of a  $q$ -fibre with  $\widehat{X}_1$  always is a single  $H_1$ -orbit. Since  $X$  is normal, we may apply Zariski’s Main Theorem to conclude that  $\widehat{X}_1/H_1 \rightarrow X$  is even an isomorphism. This verifies our claim.

Since  $H_1$  is a proper subgroup of  $H$ , we find a nontrivial class  $[L] \in \text{Pic}(X)$  such that the corresponding character  $\chi^{[L]}$  of  $H$  is trivial on  $H_1$ . We construct a defining cocycle for the class  $[L]$ : Cover  $X$  by small open sets  $U_i$  admitting invertible sections  $g_i \in \mathcal{A}_{[L]}(U_i)$ . Then the cocycle  $g_i/g_j$  defines a bundle belonging to the class  $[L]$ .

On the other hand, the  $g_i$  are  $\chi^{[L]}$ -homogeneous functions on  $q^{-1}(U_i)$ . So they restrict to  $H_1$ -invariant functions on  $q^{-1}(U_i) \cap \widehat{X}_1$ . As seen before,  $X$  is the quotient of  $\widehat{X}_1$  by the action of  $H_1$ . Thus we conclude that the  $g_i/g_j$  form in fact a coboundary on  $X$ . Consequently, the class  $[L]$  must be trivial. This contradicts the choice of  $[L]$ .  $\square$

Thus we see that if  $X$  is normal, then  $A$  is the ring of global functions of a normal variety. That means that  $A$  belongs to a intently studied class of rings.

**Corollary 6.4.** *Let  $X$  be normal. Then  $A$  is a Krull ring.*

As we did in Proposition 6.3 for normality, we can characterize smoothness in terms of the homogeneous coordinate ring.

**Proposition 6.5.**  *$X$  is smooth if and only if there is a closing subalgebra  $(A', I') \in \mathfrak{A}$  such that all localizations  $A_{\mathfrak{m}}$  are regular, where  $\mathfrak{m}$  runs through the maximal ideals with  $I' \not\subset \mathfrak{m}$ .*

**Proof.** Let  $\widehat{X} := \text{Spec}(A)$ , and consider the affine closure  $\overline{X} := \text{Spec}(A')$  defined by any closing subalgebra  $(A', I')$  of  $A$ . Recall from Lemmas 1.4 and 1.5 that  $I'$  is the vanishing ideal of the complement  $\overline{X} \setminus \widehat{X}$ . So, the regularity of the local rings  $A_{\mathfrak{m}}$ , where  $I' \not\subset \mathfrak{m}$ , just means smoothness of  $\widehat{X}$ .

The rest is similar to the proof of Proposition 6.3: The canonical map  $q : \widehat{X} \rightarrow X$  is an étale  $H$ -principal bundle for a diagonalizable group  $H$ . Thus, up to étale maps,  $\widehat{X}$  looks locally like  $X \times H$ . The assertion then follows from the fact that regularity of local rings is stable under étale maps, see [19, Proposition I.3.17].  $\square$

We give a description of quasicoherent sheaves. Consider a graded  $A$ -module  $M$ . Given  $f_1, \dots, f_r \in A$  as in Definition 2.1, set  $\mathcal{M}_i := M_{(f_i)}$ . Then these modules glue together to a quasicoherent  $\mathcal{O}_X$ -module  $\mathcal{M}$  on  $X$ . As in the toric case [1, Section 4], one obtains

**Proposition 6.6.** *The assignment  $M \mapsto \mathcal{M}$  defines an essentially surjective functor from the category of graded  $A$ -modules to the category of quasicoherent  $\mathcal{O}_X$ -modules.*

We come to properties of morphisms. Let  $Y$  be a further variety like  $X$ , and denote its homogeneous coordinate ring by  $(B, \mathfrak{B})$ . Let  $\varphi: X \rightarrow Y$  be any morphism. Denote by  $[\mu]: (B, \mathfrak{B}) \rightarrow (A, \mathfrak{A})$  the corresponding morphism of freely graded quasiaffine algebras.

**Proposition 6.7.** *The morphism  $\varphi: X \rightarrow Y$  is affine if and only if there are graded closing subalgebras  $(A', I') \in \mathfrak{A}$  and  $(B', J') \in \mathfrak{B}$  satisfying Definition 1.2 such that*

$$\sqrt{I'} = \sqrt{\langle \mu(J') \rangle}.$$

Moreover,  $\varphi: X \rightarrow Y$  is a closed embedding if and only if it satisfies the above condition and, given  $g_1, \dots, g_s \in B$  as in Definition 2.1, every induced map  $B_{(g_i)} \rightarrow A_{(\mu(g_i))}$  is surjective.

**Proof.** Let  $\mathcal{B}$  be a Picard graded  $\mathcal{O}_Y$ -algebra with  $B = \mathcal{B}(Y)$ . Consider the affine closures  $\overline{X} := \text{Spec}(A')$  of  $\widehat{X} := \text{Spec}(\mathcal{A})$  and  $\overline{Y} := \text{Spec}(B')$  of  $\widehat{Y} := \text{Spec}(\mathcal{B})$ . Then  $\mu: B \rightarrow A$  gives rise to a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\hat{\varphi}} & \widehat{Y} \\ q_X \downarrow & & \downarrow q_Y \\ X & \xrightarrow{\varphi} & Y \end{array}$$

The morphism  $\varphi$  is affine if and only if  $\hat{\varphi}$  is affine. The latter is equivalent to the condition of  $\sqrt{I'} = \sqrt{\langle \mu(J') \rangle}$  of the assertion. The supplement on embeddings is obvious.  $\square$

## 7. Tame varieties

In this section we shed some light on the question which freely graded quasiaffine algebras occur as homogeneous coordinate rings. As before, we assume that the multiplicative group  $\mathbb{K}^*$  of the ground field is of infinite rank over  $\mathbb{Z}$ . We consider varieties of the following type:

**Definition 7.1.** A *tame variety* is a normal divisorial variety  $X$  with  $\mathcal{O}(X) = \mathbb{K}$  and a finitely generated Picard group  $\text{Pic}(X)$  having no  $p$ -torsion if  $\mathbb{K}$  is of characteristic  $p > 0$ .

The prototype of a tame variety lives in characteristic zero, and is a smooth complete variety with finitely generated Picard group. Moreover, in characteristic zero, every Calabi–Yau variety is tame, and every  $\mathbb{Q}$ -factorial rational variety  $X$  with  $\mathcal{O}(X) = \mathbb{K}$  is tame. Finally, in characteristic zero every normal divisorial variety with finitely generated Picard group admits an open embedding into a tame variety.

In order to figure out the coordinate rings of tame varieties, we need some preparation. Suppose that an algebraic group  $G$  acts on a variety  $X$ . Recall that a  $G$ -linearization of a line bundle  $E \rightarrow X$  is a fibrewise linear  $G$ -action on  $E$  making the projection equivariant. By a *simple  $G$ -variety* we mean a  $G$ -variety for which any  $G$ -linearizable line bundle is trivial.

**Definition 7.2.** Let  $\Lambda$  be a finitely generated abelian group, and let  $(A, \mathfrak{A})$  be a freely  $\Lambda$ -graded quas affine algebra.

- (i) We say that  $(A, \mathfrak{A})$  is *pointed* if  $A$  is a normal ring,  $A_0 = \mathbb{K}$  holds, and the set  $A^* \subset A$  of invertible elements is just  $\mathbb{K}^*$ .
- (ii) We say that  $(A, \mathfrak{A})$  is *simple* if  $\Lambda$  has no  $p$ -torsion if  $\mathbb{K}$  is of characteristic  $p > 0$ , and the quas affine  $\text{Spec}(\mathbb{K}[\Lambda])$ -variety corresponding to  $(A, \mathfrak{A})$  is simple.

These two subclasses define full subcategories of the categories of divisorial varieties with finitely generated Picard group and freely graded quas affine algebras. The second main result of this article is the following

**Theorem 7.3.** *The homogeneous coordinate ring functor restricts to an equivalence from the category of tame varieties to the category of simple pointed algebras.*

**Proof.** Let  $X$  be a tame variety with Picard group  $\Pi := \text{Pic}(X)$ , and denote the associated homogeneous coordinate ring by  $(A, \mathfrak{A})$ . Then  $A$  is the algebra of global sections of some Picard graded  $\mathcal{O}_X$ -algebra  $\mathcal{A}$  on  $X$ . We shall use again the geometric data discussed in Lemma 6.1:

$$\widehat{X} := \text{Spec}(\mathcal{A}), \quad q : \widehat{X} \rightarrow X, \quad H := \text{Spec}(\mathbb{K}[\Pi]).$$

The first task is to show that  $(A, \mathfrak{A})$  is in fact pointed. From Proposition 6.3 we infer that  $A$  is a normal ring. Since we assumed  $\mathcal{O}(X) = \mathbb{K}$ , and  $\mathcal{O}(X)$  equals  $A_0$ , we have  $A_0 = \mathbb{K}$ . So we have to verify  $A^* = \mathbb{K}^*$ . For this, consider an arbitrary element  $f \in A^*$ .

Choose a direct decomposition of  $\Pi$  into a free part  $\Pi_0$  and the torsion part  $\Pi_t$ . This corresponds to a splitting  $H = H_0 \times H_t$  with a torus  $H_0$  and a finite group  $H_t$ . As an invertible element of  $\mathcal{O}(\widehat{X})$ , the function  $f$  is necessarily  $H_0$ -homogeneous, see, e.g., [18, Proposition 1.1]. Thus, there is a degree  $P \in \Pi_0$  such that

$$f = \sum_{G \in \Pi_t} f_{P+G}, \quad f^{-1} = \sum_{G \in \Pi_t} f_{-P+G}^{-1}.$$

From the identity  $ff^{-1} = 1$  we infer that  $f_{P+G}f_{-P-G}^{-1} \neq 0$  holds for at least one component  $f_{P+G}$  of  $f$ . Since  $\mathcal{O}(X) = \mathbb{K}$  holds, we see that the homogeneous section  $f_{P+G} \in A$  is invertible. Thus the homogeneous component  $\mathcal{A}_{P+G}$  is isomorphic to  $\mathcal{O}_X$ .

On the other hand we noted in Lemma 3.5(ii) that  $\mathcal{A}_{P+G}$  is isomorphic to the sheaf of sections of a bundle representing the class  $P + G$  in  $\Pi_0 \oplus \Pi_t$ . Thus  $P + G$  is trivial, and we obtain  $P = 0$ . Hence all homogeneous components of  $f$  have torsion degree. By  $\mathcal{O}(X) = \mathbb{K}$  this yields that  $f_G = 0$  if  $G \neq 0$ . Thus we have  $f \in A_0 = \mathbb{K}$ .

The next task is to show that  $\widehat{X}$  is a simple  $H$ -variety. For this, let  $\text{Pic}_H(\widehat{X})$  denote the group of equivariant isomorphy classes of  $H$ -linearized line bundles on  $\widehat{X}$ , compare [16, Section 2]. Moreover, let  $\text{Pic}_{\text{lin}}(\widehat{X}) \subset \text{Pic}(\widehat{X})$  denote the subgroup of the classes of all  $H$ -linearizable bundles. We have to show that  $\text{Pic}_{\text{lin}}(\widehat{X})$  is trivial.

First, we consider the possible linearizations of the trivial bundle  $\widehat{X} \times \mathbb{K}$ . Using  $\mathcal{O}^*(\widehat{X}) = \mathbb{K}^*$ , as verified before, one directly checks that any linearization of the trivial bundle is given by a character  $\chi$  of  $H$  as follows:

$$t \cdot (x, z) := (t \cdot x, \chi(t)z). \tag{7.3.1}$$

In particular, the character group  $\text{Char}(H)$  canonically embeds into the group  $\text{Pic}_H(X)$ . Since we obtain in (7.3.1) indeed any linearization of the trivial bundle, the map  $\text{Char}(H) \rightarrow \text{Pic}_H(\widehat{X})$  and the forget map  $\text{Pic}_H(\widehat{X}) \rightarrow \text{Pic}_{\text{lin}}(\widehat{X})$  fit together to an exact sequence, compare also [16, Lemma 2.2]:

$$0 \rightarrow \text{Char}(H) \rightarrow \text{Pic}_H(\widehat{X}) \rightarrow \text{Pic}_{\text{lin}}(\widehat{X}) \rightarrow 0. \tag{7.3.2}$$

Thus, to obtain  $\text{Pic}_{\text{lin}}(\widehat{X}) = 0$ , it suffices to split the map  $\text{Char}(H) \rightarrow \text{Pic}_H(\widehat{X})$  into isomorphisms as follows:

$$\begin{array}{ccc}
 \text{Char}(H) & \xrightarrow{\quad} & \text{Pic}_H(\widehat{X}) \\
 \searrow \cong & & \nearrow \cong \\
 \chi^P \mapsto P & \xrightarrow{\quad} & \Pi \\
 & & \nearrow q^*
 \end{array}
 \tag{7.3.3}$$

But this is not hard: The fact that  $q^*$  induces an isomorphism of  $\Pi = \text{Pic}(X)$  and  $\text{Pic}_H(\widehat{X})$  is due to [16, Proposition 4.2]. To obtain commutativity, consider  $P \in \Pi$ . Choose invertible sections  $g_i \in \mathcal{A}_P(U_i)$  for small open  $U_i$  covering  $X$ . Then the class of  $P$  is represented by the bundle  $P_\xi$  arising from the cocycle

$$\xi_{ij} := \frac{g_j}{g_i}. \tag{7.3.4}$$

So the pullback class  $q^*(P) \in \text{Pic}_H(\widehat{X})$  is represented by the trivially linearized bundle  $q^*(P_\xi)$ , which in turn arises from the cocycle

$$q^*(\xi_{ij}) := q^*\left(\frac{g_j}{g_i}\right) = \frac{g_j}{g_i}. \tag{7.3.5}$$

But on  $\widehat{X}$ , the  $g_i$  are ordinary invertible functions. So we obtain an isomorphism from the representing bundle  $q^*(P_\xi)$  onto the trivial bundle by locally multiplying with  $g_i$ . Obviously, the induced linearization on the trivial bundle is the linearization (7.3.1) for  $\chi = \chi^P$ .

Thus we proved that  $(A, \mathfrak{A})$  is in fact a simple pointed algebra. In other words, the homogeneous coordinate ring functor restricts to the subcategories in consideration. It remains to show that up to isomorphism, every simple pointed algebra is the homogeneous coordinate ring of some tame variety  $X$ .

So, let  $(A, \mathfrak{A})$  be a simple pointed algebra, graded by some finitely generated abelian group  $\Pi$ . According to Proposition 2.8, we may assume that  $(A, \mathfrak{A})$  equals  $(\mathcal{O}(\widehat{X}), \mathfrak{O}(\widehat{X}))$  for some normal quasiaffine variety  $\widehat{X}$  with a free action of a diagonalizable group  $H = \text{Spec}(\mathbb{K}[\Pi])$ .

The action of  $H$  on  $\widehat{X}$  admits a geometric quotient  $q: \widehat{X} \rightarrow X$ : First divide by the finite factor  $H_t$  of  $H$  to obtain a normal quasiaffine variety  $\widehat{X}/H_t$ , and then divide by the induced action of the unit component  $H_0$  of  $H$  on  $\widehat{X}/H_t$ , see, for example, [7, Ex. 4.2] and [22, Corollary 3].

The candidate for our tame variety is  $X$ . Since the structure sheaf  $\mathcal{O}_X$  is the sheaf of invariants  $q_*(\mathcal{O}_{\widehat{X}})^H$  and  $A = \mathcal{O}(\widehat{X})$  is pointed, we have  $\mathcal{O}(X) = \mathbb{K}$ . Moreover, as a geometric quotient space of a normal quasiaffine variety by a free diagonalizable group action,  $X$  is again normal and divisorial, for the latter see [11, Lemma 3.3].

To conclude the proof, we have to realize the ( $\Pi$ -graded) direct image  $\mathcal{A} := q_*\mathcal{O}_{\widehat{X}}$  as a Picard graded algebra on  $X$ . First note that we have again the exact sequence (7.3.2). Since we assumed  $\text{Pic}_{\text{lin}}(\widehat{X}) = 0$ , the character group  $\text{Char}(H)$  maps isomorphically onto  $\text{Pic}_H(\widehat{X})$ .

Moreover, we have a canonical map  $\Pi \rightarrow \text{Pic}(X)$ : For a degree  $P \in \Pi$  choose invertible  $\chi^P$ -homogeneous functions  $g_i \in \mathcal{O}(q^{-1}(U_i))$  with small open  $U_i \subset X$  covering  $X$ , see Definition 2.1. As in (7.3.4), such functions define a cocycle  $\xi$  and hence we may map  $P$  to the class of the bundle  $P_\xi$ . In conclusion, we arrive again at a commutative diagram as in (7.3.3). In particular,  $\Pi \rightarrow \text{Pic}(X)$  is an isomorphism.

In fact, the construction (7.3.4) allows us to define a group  $\Lambda$  of line bundles on  $X$ : As in the proof of Lemma 4.2, we may adjust the sections  $g_i$  for a system of generators  $P$  of  $\Pi$ , such that the corresponding cocycles  $\xi$  generate a finitely generated free abelian group. Let  $\Lambda$  be the resulting group of line bundles, and denote the associated  $\Lambda$ -graded  $\mathcal{O}_X$ -algebra by  $\mathcal{R}$ .

We construct a graded  $\mathcal{O}_X$ -algebra homomorphism  $\mathcal{R} \rightarrow \mathcal{A}$ . The accompanying homomorphism will be the canonical map  $\Lambda \rightarrow \Pi$ , associating to  $L$  its class under the identification  $\Pi \cong \text{Pic}(X)$ . Now, the sections of  $\mathcal{R}_L$ , where  $L = P_\xi$ , are given by families  $(h_i)$  satisfying

$$h_j = \xi_j h_i = \frac{g_j}{g_i} h_i.$$

This enables us to define a map  $\mathcal{R}_L \rightarrow \mathcal{A}_P$  by sending  $(h_i)$  to the section obtained by patching together the  $h_i g_i$ . Note that this indeed yields a graded homomorphism  $\mathcal{R} \rightarrow \mathcal{A}$ . By construction, this homomorphism is an isomorphism in every degree. Thus we only have to show that its kernel is the ideal associated a shifting family for  $\mathcal{R}$ .

Let  $\Lambda_0 \subset \Lambda$  denote the kernel of the canonical map  $\Lambda \rightarrow \Pi$ . Then every bundle  $E \in \Lambda_0$  admits a global trivialization. In terms of the defining cocycle  $g_i/g_j$  of  $E$  this means that there exist invertible local functions  $\tilde{g}_i$  on  $X$  with

$$\frac{g_j}{g_i} = \frac{\tilde{g}_j}{\tilde{g}_i}.$$

The functions  $\tilde{g}_i$  can be used to define a shifting family: Let  $L \in \Lambda$  and  $E \in \Lambda_0$ . Then the sections of  $\mathcal{R}_L$  are given by families  $(h_i)$  of functions that are compatible with the defining cocycle. Thus we obtain maps

$$\varrho_E : \mathcal{R}_L \rightarrow \mathcal{R}_{L+E}, \quad (h_i) \mapsto \left( \frac{h_i}{\tilde{g}_i} \right).$$

By construction, the  $\varrho_E$  are homomorphisms, and they fit together to a shifting family  $\varrho$  for  $\mathcal{R}$ . It is straightforward to check that the ideal  $\mathcal{I}$  associated to  $\varrho$  is precisely the kernel of the homomorphism  $\mathcal{R} \rightarrow \mathcal{A}$ .  $\square$

## 8. Very tame varieties

Finally, we take a closer look to the case of a free Picard group. The only assumption in this section is that the multiplicative group  $\mathbb{K}^*$  is of infinite rank over  $\mathbb{Z}$ . But even this could be weakened, see the concluding Remark 8.8.

**Definition 8.1.** A *very tame* variety is a normal divisorial variety with finitely generated free Picard group and only constant functions.

Examples of very tame varieties are Grassmannians and all smooth complete toric varieties. On the algebraic side we work with the following notion:

**Definition 8.2.** A *very simple* algebra is a freely  $\Lambda$ -graded quasiaffine algebra  $(A, \mathfrak{A})$  such that

- (i) the grading group  $\Lambda$  of  $(A, \mathfrak{A})$  is free,
- (ii)  $A$  is normal, and we have  $A_0 = \mathbb{K}$  and  $A^* = \mathbb{K}^*$ ,
- (iii) the quasiaffine variety associated to  $(A, \mathfrak{A})$  has trivial Picard group.

Again, very tame varieties and very simple algebras form subcategories, and we have an equivalence theorem:

**Theorem 8.3.** *The homogeneous coordinate ring functor restricts to an equivalence of the category of very tame varieties with the category of very simple algebras.*

**Proof.** Let  $X$  be a very tame variety. We only have to show that the quasiaffine  $H$ -variety  $\hat{X}$  corresponding to the homogeneous coordinate ring of  $X$  has trivial Picard group.

Since  $\widehat{X}$  is normal and  $H$  is a torus, every line bundle on  $\widehat{X}$  is  $H$ -linearizable, see [15, Remark, p. 67]. But from Theorem 7.3, we know that every  $H$ -linearizable bundle on  $\widehat{X}$  is trivial.  $\square$

In the setting of very tame varieties, we can go further with the dictionary presented in Section 6. The first remarkable statement is that very tame varieties produce unique factorization domains.

**Proposition 8.4.** *Let  $X$  be a very tame variety with homogeneous coordinate ring  $(A, \mathfrak{A})$ . Then  $X$  is locally factorial if and only if  $A$  is a unique factorization domain.*

**Proof.** Let  $A = \mathcal{A}(X)$  with some Picard graded  $\mathcal{O}_X$ -algebra  $\mathcal{A}$ , and the geometric quotient  $q: \widehat{X} \rightarrow X$  provided by Lemma 6.1. Since  $\text{Pic}(X)$  is free we divide by a torus  $H$ . Thus  $q: \widehat{X} \rightarrow X$  is an  $H$ -principal bundle with respect to the Zariski topology. In particular,  $X$  is locally factorial if and only if  $\widehat{X}$  is so. But  $\widehat{X}$  is locally factorial if and only if  $A$  is a factorial ring, because we have  $\text{Pic}(\widehat{X}) = 0$ .  $\square$

Next we treat products. Let  $X$  and  $Y$  be very tame varieties with homogeneous coordinate rings  $(A, \mathfrak{A})$  and  $(B, \mathfrak{B})$ . Fix closing subalgebras  $(A', I') \in \mathfrak{A}$  and  $(B', J') \in \mathfrak{B}$ , as in Definition 2.1, and consider the algebra

$$A \boxtimes B := \bigcap_f (A' \otimes_{\mathbb{K}} B')_f = \bigcap_f (A \otimes_{\mathbb{K}} B)_f,$$

where the intersections are taken in the quotient field of  $A' \otimes_{\mathbb{K}} B'$  and  $f$  runs through the elements of the form  $g \otimes h$  with homogeneous  $g \in I'$  and  $h \in J'$ .

Now  $A$  and  $B$  are graded, say by  $\Lambda$  and  $\Gamma$ . These gradings give rise to a  $(\Lambda \times \Gamma)$ -grading of  $A \boxtimes B$ . Moreover,

$$\left( A' \otimes_{\mathbb{K}} B', \sqrt{I' \otimes_{\mathbb{K}} J'} \right)$$

is a closing subalgebra of  $A \boxtimes B$ . Let  $\mathfrak{A} \boxtimes \mathfrak{B}$  denote the equivalence class of this closing subalgebra. Then we obtain

**Proposition 8.5.** *Let  $X$  and  $Y$  be locally factorial very tame varieties. Then  $X \times Y$  is locally factorial and very tame with homogeneous coordinate ring  $(A \boxtimes B, \mathfrak{A} \boxtimes \mathfrak{B})$ . Moreover, if  $A$  and  $B$  are of finite type over  $\mathbb{K}$ , then  $A \boxtimes B$  equals  $A \otimes_{\mathbb{K}} B$ .*

**Proof.** First note that for any two quasiaffine varieties  $\widehat{X}$  and  $\widehat{Y}$  with free diagonalizable group actions, their product  $\widehat{X} \times \widehat{Y}$  is again such a variety. Moreover, if  $\widehat{X}$  and  $\widehat{Y}$  have only constant invertible functions, then so does  $\widehat{X} \times \widehat{Y}$ . If  $\widehat{X}$  and  $\widehat{Y}$  are additionally locally factorial with trivial Picard groups, then the same holds for  $\widehat{X} \times \widehat{Y}$ , use e.g., [8, Proposition 1.1].

Now, let  $\widehat{X} := \text{Spec}(\mathcal{A})$  and  $\widehat{Y} := \text{Spec}(\mathcal{B})$ . By Proposition 8.4 both are locally factorial. By construction  $(A \boxtimes B, \mathfrak{A} \boxtimes \mathfrak{B})$  is the freely graded quasiaffine algebra corresponding to



the product  $\widehat{X} \times \widehat{Y}$ . Thus the above observations and Proposition 2.8 tell us that it is a coproduct in the category of simple pointed algebras. Hence the assertion follows from Theorem 7.3. The second statement is an easy consequence of Remark 1.7(i).  $\square$

**Corollary 8.6.** *Let  $X$  and  $Y$  be locally factorial very tame varieties. Then  $\text{Pic}(X \times Y)$  is isomorphic to  $\text{Pic}(X) \times \text{Pic}(Y)$ .*

We give an explicit example emphasizing the role of Proposition 8.4. We assume that the ground field  $\mathbb{K}$  is not of characteristic two. Consider the prevariety  $X$  obtained by gluing two copies of the projective line  $\mathbb{P}_1$  along the common open subset  $\mathbb{K}^* \setminus \{1\}$ . We think of  $X$  as the projective line with three doubled points, namely

$$0, 0', \quad 1, 1', \quad \infty, \infty'.$$

Note that  $X$  is smooth and divisorial. Moreover,  $\text{Pic}(X)$  is isomorphic to  $\mathbb{Z}^4$ . Thus we obtain in particular that  $X$  is very tame. Let  $(\mathcal{A}(X), \mathfrak{A}(X))$  denote the homogeneous coordinate ring of  $X$ . We show

**Proposition 8.7.**  $\mathcal{A}(X) \cong \mathbb{K}[T_1, \dots, T_6] / \langle T_1^2 + \dots + T_6^2 \rangle$ .

Before giving the proof, let us remark that the ring  $\mathcal{A}(X)$  is a classical example of a singular factorial affine algebra. In view of our results, factoriality is a consequence of Proposition 8.4.

**Proof of Proposition 8.7.** First observe that we may realize  $\text{Pic}(X)$  as well as a subgroup  $\Lambda$  of the group of Cartier divisors of  $X$ . For example,  $\text{Pic}(X)$  is isomorphic to the group  $\Lambda$  generated by

$$D_0 := \{0\}, \quad D_1 := \{1\}, \quad D_{1'} := \{1'\}, \quad D_\infty := \{\infty\}.$$

For any Cartier divisor  $D$  on  $X$ , let  $\mathcal{A}_D$  denote its sheaf of sections. Then the homogeneous coordinate ring  $\mathcal{A}(X)$  is the direct sum of the  $\mathcal{A}_D(X)$ , where  $D \in \Lambda$ . Consider the following homogeneous elements of  $\mathcal{A}(X)$ :

$$\begin{aligned} f_1 &:= 1 \in \mathcal{A}_{D_0}(X), & f_2 &:= 1 \in \mathcal{A}_{D_1}(X), \\ f_3 &:= 1 \in \mathcal{A}_{D_{1'}}(X), & f_4 &:= 1 \in \mathcal{A}_{D_\infty}(X), \\ f_5 &:= \left( \frac{1}{z-1} \right) \in \mathcal{A}_{D_1+D_{1'}-D_\infty}(X), & f_6 &:= \left( \frac{z}{z-1} \right) \in \mathcal{A}_{D_1+D_{1'}-D_0}(X). \end{aligned}$$

Let  $\varphi$  be the algebra homomorphism  $\mathbb{K}[T_1, \dots, T_6] \rightarrow \mathcal{A}(X)$  sending  $T_i$  to  $f_i$ . It is elementary to check that  $\varphi$  is surjective. Since we assumed  $\mathbb{K}$  not to be of characteristic two, it suffices to show that the kernel of  $\varphi$  is the ideal generated by

$$Q := T_2T_3 + T_5T_4 - T_6T_1.$$

An explicit calculation shows that the  $f_i$  fulfill the claimed relation, that means that  $Q$  lies in the kernel of  $\varphi$ . Conversely, consider an arbitrary element  $R$  of the kernel of  $\varphi$ . Then there are  $r_j \in \mathbb{K}[T_1, \dots, T_5]$  such that  $R$  is of the form

$$R = \sum_{j=0}^s r_j T_6^j.$$

We proceed by induction on  $s$ . For  $s = 0$  the fact that  $f_1, \dots, f_5$  are algebraically independent implies  $R = 0$ . For  $s > 0$  note first that  $\varphi(r_j)$  is nonnegative in  $D_0$  in the sense that its component in a degree containing a multiple  $nD_0$  is trivial for negative  $n$ .

Since  $f_6$  is negative in  $D_0$ , and  $f_1$  is the only generator of  $\mathcal{A}(X)$  which is strictly positive in degree  $D_0$ , we can write  $r_j = \tilde{r}_j T_1^j$ . Hence we obtain a representation

$$R = \sum_{j=0}^s \tilde{r}_j T_1^j T_6^j.$$

The element  $\tilde{r}_s((T_1 T_6)^s - (T_2 T_3 + T_4 T_5)^s)$  is a multiple of  $Q$ . In particular, it belongs to the kernel of  $\varphi$ . Subtracting it from  $R$ , we obtain

$$R' = \sum_{j=0}^{s-1} r'_j T_1^j T_6^j,$$

with  $r'_j = \tilde{r}_j$  for  $j > 0$  and  $r'_0 = \tilde{r}_0 + \tilde{r}_s (T_2 T_3 + T_4 T_5)^j$ . Applying the induction hypothesis to  $R'$  yields that  $R$  is a multiple of  $Q$ .  $\square$

Finally, let us note that all our statements on very tame varieties hold under more general assumptions. This is due to the fact that free Picard groups always can be realized by (free) groups of line bundles. Hence in this case we don't need shifting families to define the homogeneous coordinate ring. This means

**Remark 8.8.** For very tame varieties  $X$ , the results of this article hold over any algebraically closed ground field  $\mathbb{K}$ , and one might weaken the assumption  $\mathcal{O}(X) = \mathbb{K}$  to  $\mathcal{O}^*(X) = \mathbb{K}^*$ .

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