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Uniqueness and structure of solutions to the Dirichlet problem for an elliptic system

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ABSTRACT

In this paper, we consider the Dirichlet problem for an elliptic system on a ball in $\mathbf{R}^2$. By investigating the properties for the corresponding linearized equations of solutions, and adopting the Pohozaev identity and Implicit Function Theorem, we show the uniqueness and the structure of solutions.

1. Introduction and main results

In this paper, we consider the following elliptic system

$$\begin{cases} \Delta u + e^v(1 - e^u) = 4\pi N_1 \delta_0, \\ \Delta v + e^u(1 - e^v) = 4\pi N_2 \delta_0 \end{cases} \tag{1.1}$$

in B_R with boundary condition

$$(u, v) = (0, 0) \quad \text{on } \partial B_R, \tag{1.2}$$

where $\Delta = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$, $B_R \subset \mathbf{R}^2$ is the ball centered at the origin with radius R, N_1 and N_2 are two positive constants, and δ_0 is the Dirac measure at the origin. System (1.1) arises from the relativistic Abelian Chern–Simons model with two Higgs particle. The related Chern–Simons problem with one

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Higgs particle has been intensively studied in the past twenty years, e.g., see [1–6,8–16,18–24]. For deriving (1.1), we refer the readers to [7,15,17] and the references therein. Recently, Lin, Ponce and Yang [17] has shown that for each $R>0$, (1.1)–(1.2) possesses a solution.

Theorem A. *(See [17].) For any $R>0$, (1.1)–(1.2) possesses a solution (u_R, v_R). Furthermore, there exists a sequence $R_i \to \infty$ such that $(u_{R_i}, v_{R_i}) \to (u_0, v_0)$ in $L^1_{loc}(\mathbf{R}^2) \times L^1_{loc}(\mathbf{R}^2)$, where (u_0, v_0) is a topological solution of (1.1) in $\mathbf{R}^2$.*

We note that an entire solution pair (u, v) of (1.1) satisfying the boundary condition

$$u(x) \to 0, \qquad v(x) \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

is called a topological solution of (1.1). The purpose of this paper is to study the uniqueness and structures of solutions for (1.1)–(1.2). Let (u, v) be a $C^2(B_R \setminus \{O\})$ solution of (1.1)–(1.2). Then by maximum principle we will have $u(x)<0$, $v(x)<0$ for $x \in B_R \setminus \{O\}$. Applying the method of moving planes on system equations, we can show that (u, v) is radially symmetric with respect to the origin. The proof is standard, and for the reader's convenience, we present it in Appendix A after Section 2. Therefore, we now study the structure of radial solutions for (1.1)–(1.2), i.e., $(u(r), v(r))$ satisfies the following ODE system

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) + e^{v(r)}\big(1 - e^{u(r)}\big) = 4\pi N_1 \delta_0, \\ v''(r) + \frac{1}{r}v'(r) + e^{u(r)}\big(1 - e^{v(r)}\big) = 4\pi N_2 \delta_0, \end{cases} \quad r>0, \tag{1.4}$$

with the boundary condition

$$u(R) = v(R) = 0. \tag{1.5}$$

We note that if (u, v) is a solution of (1.4), then it is easy to get

$$\begin{cases} u(r) = 2N_1 \log r + \alpha_1 + o(1), \\ v(r) = 2N_2 \log r + \alpha_2 + o(1) \end{cases} \quad \text{as } r \to 0^+ \tag{1.6}$$

for some $\alpha = (\alpha_1, \alpha_2) \in \mathbf{R}^2$. Denote the solution of (1.4) and (1.6) by $(u(r, \alpha), v(r, \alpha))$ or simply $(u(r), v(r))$ when there is no confusion. Let

$$D = \big\{\alpha \;\big|\; \big(u(r, \alpha), v(r, \alpha)\big) \text{ is a solution of (1.4)–(1.5) for some } R>0\big\}. \tag{1.7}$$

Our main result is the following.

Theorem 1.1. *For any positive numbers N_1, N_2 and R, (1.1)–(1.2) possesses one and only one solution (u_R, v_R) and the following properties are valid.*

(i) *u_R and v_R are radially symmetric satisfying*

$$u_R(r)<0, \quad v_R(r)<0, \quad u'_R(r)>0 \quad \textit{and} \quad v'_R(r)>0 \quad \forall r \in (0, R].$$

(ii) *For each* $r \in (0, R]$ *we have*

$$\begin{cases} u_R(r) > v_R(r) & \text{if } N_1 < N_2, \\ u_R(r) = v_R(r) & \text{if } N_1 = N_2, \\ u_R(r) < v_R(r) & \text{if } N_1 > N_2. \end{cases}$$

(iii) *There exist two strictly monotone* C^1 *functions* $\gamma_1, \gamma_2 : (0, \infty) \to \mathbf{R}$ *such that*

$$\begin{cases} D = \{\alpha(R) \mid \alpha(R) = (\gamma_1(R), \gamma_2(R))\ \forall R \in (0, \infty)\}, \\ \gamma_1'(R) \cdot \gamma_2'(R) > 0 \quad \forall R \in (0, \infty) \quad \text{and} \\ \lim_{R \to \infty} \alpha(R) = (\alpha_{10}, \alpha_{20}) = \alpha_0, \end{cases}$$

where D *is defined in* (1.7) *and* $(u(r, \alpha_0), v(r, \alpha_0))$ *is a topological solution of* (1.1)*.*

The paper is organized as follows. First, based on the investigations of linearized equations, and adopting the Pohozaev identity and Implicit Function Theorem, we show the uniqueness and the structure of solutions in Section 2. Finally, by applying the method of moving plane on system equations, we prove that for each $R > 0$ every solution pair of (1.1)–(1.2) is negative and radially symmetric in Appendix A.

2. Uniqueness and solution structures of the Dirichlet problem

In this section, we will prove the uniqueness and solution structures of (1.1)–(1.2). Let (u, v) be a solution of (1.1)–(1.2). Then, by Proposition A.1 in Appendix A, (u, v) is radially symmetric and satisfies (1.4)–(1.6). In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. *Let* $(u(r), v(r))$ *be a solution of* (1.4)–(1.5)*. Then* $u' > 0$ *and* $v' > 0$ *on* $(0, R]$*, and the following statements are valid.*

(i) *If* $N_1 < N_2$ *then* $u > v$ *on* $(0, R)$.
(ii) *If* $N_1 > N_2$ *then* $u < v$ *on* $(0, R)$.
(iii) *If* $N_1 = N_2$ *then* $u \equiv v$ *on* $(0, R]$.

Proof. By Proposition A.1 in Appendix A, we have $u(r) < 0$ and $v(r) < 0$ $\forall r \in (0, R)$. Then the maximum principle implies that both u and v cannot attain their local minima inside $(0, R)$. Since $u'(r) > 0$ and $v'(r) > 0$ for r near 0, we obtain $u'(r) > 0$, $v'(r) > 0$ on $(0, R)$.

By (1.1), we have

$$\Delta(u - v) = 4\pi(N_1 - N_2)\delta(0) + (e^u - e^v).$$

Let $N_1 < N_2$. Then, by (1.6), we have $u(r) > v(r)$ for r near 0. If $(u - v)(r_0) < 0$ at some $r_0 \in (0, R_0)$, then we choose r_0 satisfying $(u - v)(r_0) = \min_{(0, R_0]}(u - v) < 0$, and we have

$$0 \leqslant \Delta(u - v)(r_0) = e^{u(r_0)} - e^{v(r_0)} < 0,$$

a contradiction. Hence $u(r) \geqslant v(r)$ $\forall r \in (0, R]$. By the strong maximum principle, the strict inequality $u(r) > v(r)$ holds for $r \in (0, R)$. This proves (i). Similarly, (ii) and (iii) follow easily. □

Secondly, we need the following identity.

Lemma 2.2 *(Pohozaev identity). Let* $(u(r), v(r))$ *be a solution of* (1.4)–(1.5). *Then we have*

$$\left[ru'(r) \cdot rv'(r) + r^2\left(e^{u(r)} + e^{v(r)}\right) - r^2 e^{u(r)+v(r)}\right] - 2\int_0^r s\left(e^{u(s)} + e^{v(s)}\right) ds + 2\int_0^r s e^{u(s)+v(s)}\, ds$$
$$= 4N_1N_2 \quad \forall r \in (0, R]. \tag{2.1}$$

Proof. By multiplying rv' and ru' on both sides of the first and second equation of (1.4) respectively, we obtain

$$\begin{cases} rv'(ru')' + rv're^v\left(1 - e^u\right) = rv'\left(4\pi N_1 r\delta(0)\right), \\ ru'(rv')' + ru're^u\left(1 - e^v\right) = ru'\left(4\pi N_2 r\delta(0)\right) \end{cases} \quad \forall r \in (0, R].$$

Then adding these two equation together, and using the integration by parts and (1.6), we can get (2.1) easily. □

In the following, we investigate the properties for the corresponding linearized equation of (1.4). Let $(u(r,\alpha), v(r,\alpha))$ be a solution of (1.4) and (1.6). Denote

$$\begin{cases} U(r,\alpha) = u(r,\alpha) - 2N_1 \log r, \\ V(r,\alpha) = v(r,\alpha) - 2N_2 \log r, \end{cases} \tag{2.2}$$

and let, for $i = 1, 2$,

$$\begin{cases} \phi_i(r) = \dfrac{\partial U(r,\alpha)}{\partial \alpha_i}, \\ \psi_i(r) = \dfrac{\partial V(r,\alpha)}{\partial \alpha_i}. \end{cases} \tag{2.3}$$

Then (ϕ_i, ψ_i), $i = 1, 2$, satisfy the corresponding linearized equations

$$\begin{cases} \Delta\phi_i - e^{u+v}\phi_i + e^v\left(1 - e^u\right)\psi_i = 0, \quad r \in (0, R], \\ \Delta\psi_i - e^{u+v}\psi_i + e^u\left(1 - e^v\right)\phi_i = 0, \quad r \in (0, R], \\ \phi_1(0) = 1 = \psi_2(0), \quad \phi_2(0) = 0 = \psi_1(0), \quad \phi_i'(0) = 0 = \psi_i'(0). \end{cases} \tag{2.4}$$

We have the following monotone property.

Lemma 2.3. *Let* $(u(r), v(r))$ *be a solution of* (1.4)–(1.5). *Then the corresponding* (ϕ_i, ψ_i) *satisfy*

$$\begin{cases} \phi_1(r) > 0, \quad \phi_1'(r) > 0, \quad \phi_2(r) < 0, \quad \phi_2'(r) < 0, \\ \psi_1(r) < 0, \quad \psi_1'(r) < 0, \quad \psi_2(r) > 0, \quad \psi_2'(r) > 0 \end{cases} \quad \forall r \in (0, R]. \tag{2.5}$$

Proof. By (2.4) and (1.6), there exists $r_0 \in (0, R)$ such that

$$\begin{aligned} r\psi_1'(r) &= -\int_0^r s\left[e^{u(s)}\left(1 - e^{v(s)}\right)\phi_1(s) - e^{u(s)+v(s)}\psi_1(s)\right] ds \quad \forall r > 0 \\ &\leqslant -\int_0^r s\left[C_1 s^{2N_1}\left(1 - C_2 s^{2N_2}\right)\phi_1(s) - C_3 s^{2N_1+2N_2}\psi_1(s)\right] ds \quad \forall r \in (0, r_0) \\ &\leqslant -Cr^{2N_1+2} < 0 \quad \forall r \in (0, r_0). \end{aligned} \tag{2.6}$$

By $\psi_1(0)=0$, $\psi_1'(0)=0$ and (2.6), we have $\psi_1(r)<0$ and $\psi_1'(r)<0$ $\forall r\in(0,r_0)$. Also, by (2.4), (1.6), and the above result, we get

$$\begin{aligned} r\phi_1'(r) &= \int_0^r s\big[e^{u(s)+v(s)}\phi_1(s)+e^{v(s)}\big(e^{u(s)}-1\big)\psi_1(s)\big]\,ds \quad \forall r>0 \\ &\geqslant \int_0^r C_4 s\cdot s^{2N_1+2N_2}\phi_1(s)\,ds \quad \forall r\in(0,r_0) \\ &\geqslant Cr^{2N_1+2N_2+2}>0 \quad \forall r\in(0,r_0). \end{aligned} \tag{2.7}$$

By $\phi_1(0)=1$, $\phi_1'(0)=0$ and (2.7), we have $\phi_1(r)>0$ and $\phi_1'(r)>0$ $\forall r\in(0,r_0)$. These prove that the first inequality of (2.5) holds for $r\in(0,r_0)$. However (2.6) and (2.7) hold as long as the first inequality of (2.5) is true. This shows that the first inequality of (2.5) holds. The proof for the second inequality of (2.5) is similar. The proof is complete. □

To prove the uniqueness of solutions for (1.1)–(1.2), the following lemma is a key.

Lemma 2.4. *Let $(u(r),v(r))$ be a solution of* (1.4)–(1.5) *on* $(0,R_0]$ *for some* $R_0>0$. *If* $(\phi_i(r),\psi_i(r))$, $i=1,2$, *is the solution pair for respectively linearized equation* (2.4) *associated with* $(u(r),v(r))$, *then*

$$\det\begin{pmatrix}\phi_1(r) & \phi_2(r)\\ \psi_1(r) & \psi_2(r)\end{pmatrix}\neq 0 \quad \forall r\in[0,R_0]. \tag{2.8}$$

Proof. Let $M_A(r)=-\frac{\phi_1(r)}{\phi_2(r)}$ and $M_B(r)=-\frac{\psi_1(r)}{\psi_2(r)}$. Then, by (2.4), we have $\lim_{r\to0^+}M_A(r)=\infty$, $\lim_{r\to0^+}M_B(r)=0$, and thus $M_A(r)>M_B(r)$ $\forall r\in(0,r_1)$ for some $r_1\in(0,R_0]$. We divide the proof into the following steps.

Step 1. If $M_A(r)>M_B(r)$ $\forall r\in(0,r_0)$ for some $r_0\leqslant R_0$, then $M_A'(r)<0$ and $M_B'(r)>0$ $\forall r\in(0,r_0)$.

We prove Step 1 by contradiction. Suppose $M_A'(r)<0$ $\forall r\in(0,r_0)$ is not true. Then there exist $0<r_1<r_2\leqslant r_0$ such that

$$\begin{aligned} &M_A'(r_1)<0,\quad M_A'(r_2)>0,\quad M_A(r_1)=M_A(r_2)(\equiv C_0),\quad \text{and} \\ &\qquad 0<M_B(r)<M_A(r)<C_0 \quad \forall r\in(r_1,r_2). \end{aligned} \tag{2.9}$$

For any $c>0$ and $r\in(0,R_0]$, we define

$$A_c(r)=\phi_1(r)+c\cdot\phi_2(r) \quad \text{and} \quad B_c(r)=\psi_1(r)+c\cdot\psi_2(r). \tag{2.10}$$

Then A_c and B_c satisfy

$$\begin{cases} \Delta A_c-e^{u+v}A_c=e^v\big(e^u-1\big)B_c & \forall r\in(0,R_0],\\ \Delta B_c-e^{u+v}B_c=e^u\big(e^v-1\big)A_c & \forall r\in(0,R_0],\\ A_c(0)=1,\quad B_c(0)=c>0. \end{cases} \tag{2.11}$$

From (2.9) and (2.10), we easily obtain

$$A_{C_0}(r)<0<B_{C_0}(r) \quad \forall r\in(r_1,r_2) \quad \text{and} \quad A_{C_0}(r_1)=0=A_{C_0}(r_2), \tag{2.12}$$

which imply that A_{C_0} has a local minimum at some $\bar{r} \in (r_1, r_2)$ and $\Delta A_{C_0}(\bar{r}) \geqslant 0$. But, from (2.11) and (2.12), we get

$$\Delta A_{C_0}(\bar{r}) = e^{u(\bar{r})+v(\bar{r})} A_{C_0}(\bar{r}) + e^{v(\bar{r})}\big(e^{u(\bar{r})} - 1\big) B_{C_0}(\bar{r}) < 0. \tag{2.13}$$

This contradiction proves $M'_A(r) < 0$ $\forall r \in (0, r_0)$.

Similarly, suppose $M'_B(r) > 0$ $\forall r \in (0, r_0)$ is not true. Then there exist $0 < r_1 < r_2 \leqslant r_0$ such that

$$M'_B(r_1) > 0, \quad M'_B(r_2) < 0, \quad M_B(r_1) = M_B(r_2) (\equiv C_0), \quad \text{and}$$
$$C_0 < M_B(r) < M_A(r) \quad \forall r \in (r_1, r_2). \tag{2.14}$$

By (2.14) and (2.10), we easily obtain

$$B_{C_0}(r) < 0 < A_{C_0}(r) \quad \forall r \in (r_1, r_2) \quad \text{and} \quad B_{C_0}(r_1) = 0 = B_{C_0}(r_2), \tag{2.15}$$

and hence B_{C_0} has a local minimum at some $\bar{r} \in (r_1, r_2)$ with $\Delta B_{C_0}(\bar{r}) \geqslant 0$. However, from (2.11) and (2.12) we get

$$\Delta B_{C_0}(\bar{r}) = e^{u(\bar{r})+v(\bar{r})} B_{C_0}(\bar{r}) + e^{u(\bar{r})}\big(e^{v(\bar{r})} - 1\big) A_{C_0}(\bar{r}) < 0. \tag{2.16}$$

This contradiction proves Step 1.

Step 2. There does not exist $R \in (0, R_0)$ such that $M_A(R) = M_B(R)$.

Suppose Step 2 is not true. Then there exists a smallest $R \in (0, R_0]$ such that $M_A(R) = M_B(R) (\equiv C)$ and $M_A(r) > M_B(r) > 0$ $\forall r \in (0, R)$. Let A_c and B_c be defined in (2.10). Then, in this case, by Step 1 we obtain

$$\begin{gathered} A_C(r) > 0, \qquad B_C(r) > 0 \quad \forall r \in (0, R), \\ A_C(R) = B_C(R) = 0, \\ A'_C(R) < 0, \qquad B'_C(R) < 0 \quad \text{if } R < \infty. \end{gathered} \tag{2.17}$$

Taking the differentiation w.r.t. α_i, $i = 1, 2$, on the both sides of Pohozaev identity, (2.1), then, for any $c > 0$ and $r \in (0, R_0]$, we obtain

$$\begin{aligned} &r^2 A'_c(r) v'(r) + r^2 B'_c(r) u'(r) + r^2\big[e^{u(r)} A_c(r) + e^{v(r)} B_c(r)\big] - r^2 e^{u(r)+v(r)}\big(A_c(r) + B_c(r)\big) \\ &\quad - 2\int_0^r s\big[e^u A_c + e^v B_c\big]\,ds + 2\int_0^r s e^{u+v}(A_c + B_c)\,ds = 0. \end{aligned} \tag{2.18}$$

By replacing c and r with C and R in (2.18) respectively, we easily have

$$\begin{aligned} 0 &= \big[R^2 A'_C(R) v'(R) + R^2 B'_C(R) u'(R)\big] + \big[R^2 B_C(R) e^{v(R)}\big(1 - e^{u(R)}\big) + R^2 A_C(R) e^{u(R)}\big(1 - e^{v(R)}\big)\big] \\ &\quad + 2\left[\int_0^R r A_C e^u\big(e^v - 1\big)\,dr + \int_0^R r B_C e^v\big(e^u - 1\big)\,dr\right]. \end{aligned} \tag{2.19}$$

Then, combining (i) of Lemma 2.1, (2.17) and (2.19), we deduce

$$0 > R^2 A_C'(R) v'(R) + R^2 B_C'(R) u'(R)$$
$$= 2\left[\int_0^R r A_C e^u\left(1-e^v\right)dr + \int_0^R r B_C e^v\left(1-e^u\right)dr\right] > 0,$$

which yields a contradiction. This proves Step 2.

Step 3. Suppose $\det\begin{pmatrix} \phi_1(R) & \phi_2(R) \\ \psi_1(R) & \psi_2(R)\end{pmatrix} = 0$ for some $R \in [0, R_0]$. Then, w.l.o.g., there exists $C_0 > 0$ such that

$$\begin{pmatrix} \phi_1(R) \\ \psi_1(R)\end{pmatrix} + C_0 \begin{pmatrix} \phi_2(R) \\ \psi_2(R)\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}. \tag{2.20}$$

By (2.20) we obtain $M_A(R) = C_0 = M_B(R)$. This contradicts to Step 2 and proves Step 3. The proof is complete. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into the following steps.

Step 1. By Proposition A.1 in Appendix A and Lemma 2.1 we obtain the results of parts (i) and (ii).

Step 2. Let $(u(r,\alpha_1,\alpha_2), v(r,\alpha_1,\alpha_2))$ be a solution of (1.4) and (1.6). Define the function F by

$$F(r,\alpha_1,\alpha_2) = \begin{pmatrix} u(r,\alpha_1,\alpha_2) \\ v(r,\alpha_1,\alpha_2)\end{pmatrix} \equiv \begin{pmatrix} F_1(r,\alpha_1,\alpha_2) \\ F_2(r,\alpha_1,\alpha_2)\end{pmatrix} \quad \forall r > 0,\ \forall(\alpha_1,\alpha_2) \in \mathbf{R}^2. \tag{2.21}$$

Denote the zero set of F by

$$\Theta = \left\{(r,\alpha_1,\alpha_2) \,\middle|\, F(r,\alpha_1,\alpha_2) = \begin{pmatrix} 0 \\ 0\end{pmatrix}\right\}. \tag{2.22}$$

Then if $(R_0, \alpha_1^0, \alpha_2^0) \in \Theta$, we have $\bar{\alpha} = (\alpha_1^0, \alpha_2^0) \in D$ and $(u(r,\bar{\alpha}), v(r,\bar{\alpha}))$ is a radial solution of (1.1)–(1.2) on B_{R_0}, where D is defined in (1.7). By Lemma 2.4, we obtain

$$\det\left(\frac{\partial F_i}{\partial \alpha_j}\left(R_0, \alpha_1^0, \alpha_2^0\right)\right)_{2\times 2} = \det\begin{pmatrix} \phi_1(R_0) & \phi_2(R_0) \\ \psi_1(R_0) & \psi_2(R_0)\end{pmatrix} \neq 0, \tag{2.23}$$

where $(\phi_i, \psi_i), i = 1, 2,$ is the respective solution of linearized equation (2.4) associated with $(u(r,\bar{\alpha}), v(r,\bar{\alpha}))$. Applying the Implicit Function Theorem, we obtain that there exist $\varepsilon = \varepsilon(R) > 0$ and a unique C^1 function curve $(\gamma_1, \gamma_2) : (R_0 - \varepsilon, R_0 + \varepsilon) \to \mathbf{R}^2$ such that $(R, \gamma_1(R), \gamma_2(R)) \in \Theta\ \forall R \in (R_0 - \varepsilon, R_0 + \varepsilon)$. Hence we have $(\gamma_1(R), \gamma_2(R)) \in D\ \forall R \in (R_0 - \varepsilon, R_0 + \varepsilon)$. Furthermore, by Proposition A.1, we deduce that γ_1 and γ_2 are defined on the whole domain $(0, \infty)$.

Step 3. $\gamma_1'(R) \cdot \gamma_2'(R) > 0,\ R \in (0, \infty)$.
Suppose Step 3 is not true. Then there exists $R_0 > 0$ such that one of the following cases occur:

(a) $\gamma_1'(R_0) \leqslant 0$ and $\gamma_2'(R_0) \geqslant 0$,
(b) $\gamma_1'(R_0) \geqslant 0$ and $\gamma_2'(R_0) \leqslant 0$.

By (i) and (2.21) we respectively obtain

$$\begin{cases} u'\bigl(R_0, \gamma_1(R_0), \gamma_2(R_0)\bigr) > 0, \\ v'\bigl(R_0, \gamma_1(R_0), \gamma_2(R_0)\bigr) > 0, \end{cases} \tag{2.24}$$

and

$$\begin{cases} u'\bigl(R, \gamma_1(R), \gamma_2(R)\bigr) + \phi_1(R)\gamma_1'(R) + \phi_2(R)\gamma_2'(R) = 0, \\ v'\bigl(R, \gamma_1(R), \gamma_2(R)\bigr) + \psi_1(R)\gamma_1'(R) + \psi_2(R)\gamma_2'(R) = 0 \end{cases} \quad \forall R > 0. \tag{2.25}$$

Suppose case (a) happens. Then, by (2.24), the second equation of (2.25) and $\psi_2(R_0) > 0$, we obtain that

$$0 < v'\bigl(R_0, \gamma_1(R_0), \gamma_2(R_0)\bigr) = -\psi_1(R)\gamma_1'(R) - \psi_2(R_0)\gamma_2'(R_0) \leqslant 0.$$

If case (b) happens, then, by (2.24), the first equation of (2.25) and $\phi_2(R_0) < 0$ we obtain that

$$0 < u'\bigl(R_0, \gamma_1(R_0), \gamma_2(R_0)\bigr) = -\phi_1(R)\gamma_1'(R) - \phi_2(R_0)\gamma_2'(R_0) < 0.$$

We all get a contradiction. This proves Step 3.

Step 4. By Steps 1–3, we obtain the solution of (1.1)–(1.2) is unique, and (iii) holds. This completes the proof. □

Appendix A

In this appendix, we will prove the radial symmetry of solutions for (1.1)–(1.2).

Proposition A.1. *For each $R > 0$, every solution (u_R, v_R) of (1.1)–(1.2) is radially symmetric and satisfies $u_R < 0$, $v_R < 0$ in $B_R \setminus \{O\}$. Furthermore, there exists a sequence $R_i \to \infty$ as $i \to \infty$ such that $(u_{R_i}, v_{R_i}) \to (u_0, v_0)$ uniformly on any compact subset of $(0, \infty)$, where (u_0, v_0) is a radially symmetric pair and a topological solution of (1.1).*

Proof. Let $R > 0$ be any given number and $B_R = B_R(O)$ be the ball centered at O with radius R. Then, by Theorem 3.1 in [17], (1.1)–(1.2) possesses a solution pair (u_R, v_R). We divide the proof into the following steps.

Step 1. $u_R < 0$ and $v_R < 0$ in $B_R \setminus \{O\}$.

We use the maximum principle to prove Step 1. Suppose $u_R(x_0) = \max_{x \in B_R \setminus \{O\}} u_R(x) > 0$. Then $\Delta u_R(x_0) \leqslant 0$ and thus

$$0 = \Delta u_R(x_0) + e^{v_R(x_0)}\bigl(1 - e^{u_R(x_0)}\bigr) < 0,$$

which yields a contradiction. Hence, $u_R \leqslant 0$ in $B_R \setminus \{O\}$. The strong maximum principle implies $u_R < 0$ in $B_R \setminus \{O\}$. Similarly, it holds for v_R.

Step 2. For any fixed $R > 0$, $(u, v) = (u_R, v_R)$ is a radially symmetric pair.

We will apply the method of moving plane with some modifications to prove Step 2. It suffices to prove u and v are increasing when the point $x = (x_1, x_2)$ changes its position along the x_1-axis from the point O to point $(R, 0)$. Let $A_R = \{x\colon\ 0 < |x| < R\}$. For $0 < \sigma < R$, define the sets $\Sigma_\sigma =$

$\{x \in B_R\colon\ x_1 > \sigma\}$, $T_\sigma = \{x \in B_R\colon\ x_1 = \sigma\}$, and $u_\sigma(x) = u(x^\sigma)$, $v_\sigma(x) = v(x^\sigma)$ for $x \in \Sigma_\sigma$, where x^σ is the reflection of x with respect to the line $x_1 = \sigma$, i.e., $x^\sigma = (2\sigma - x_1, x_2)$.

Set $w_\sigma(x) = u(x) - u_\sigma(x)$ and $z_\sigma(x) = v(x) - v_\sigma(x)$ for $x \in \Sigma_\sigma$. Then w_σ and z_σ satisfy the following equations respectively:

$$\begin{cases} \Delta w_\sigma - e^{v_\sigma}\left(e^u - e^{u_\sigma}\right) = -\left(e^v - e^{v_\sigma}\right)\left(1 - e^u\right) - 4\pi N_1 \delta(2\sigma, 0) & \text{in } \Sigma_\sigma, \\ w_\sigma \geqslant 0 & \text{on } \partial\Sigma_\sigma, \end{cases} \tag{A.1}$$

and

$$\begin{cases} \Delta z_\sigma - e^{u_\sigma}\left(e^v - e^{v_\sigma}\right) = -\left(e^u - e^{u_\sigma}\right)\left(1 - e^v\right) - 4\pi N_2 \delta(2\sigma, 0) & \text{in } \Sigma_\sigma, \\ z_\sigma \geqslant 0 & \text{on } \partial\Sigma_\sigma, \end{cases} \tag{A.2}$$

where $\delta(2\sigma, 0)$ is the Dirac measure at the point $(2\sigma, 0)$. Define

$$\begin{cases} S_u = \left\{\rho \in (0, R)\colon\ w_\sigma > 0 \text{ in } \Sigma_\sigma \text{ for } \sigma \in (\rho, R)\right\}, \\ \rho_u = \inf\limits_{\rho \in S_u} \{\rho\}, \end{cases} \tag{A.3}$$

and

$$\begin{cases} S_v = \left\{\rho \in (0, R)\colon\ z_\sigma > 0 \text{ in } \Sigma_\sigma \text{ for } \sigma \in (\rho, R)\right\}, \\ \rho_v = \inf\limits_{\rho \in S_v} \{\rho\}. \end{cases} \tag{A.4}$$

First, we show that $S_u \neq \emptyset$ and $S_v \neq \emptyset$. By Step 1 and the Hopf Boundary Lemma, we have

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}(x) > 0 \quad \text{for } |x| = R, \tag{A.5}$$

where ν is the unit outer normal to ∂B_R at x. In particular,

$$u(x) > u\left(x^\sigma\right) = u_\sigma(x) \quad \text{and} \quad v(x) > v\left(x^\sigma\right) = v_\sigma(x) \quad \text{for } x \in \Sigma_\sigma \tag{A.6}$$

if σ is sufficiently close to R. This shows that $w_\sigma(x) > 0$ and $z_\sigma(x) > 0$ in Σ_σ if σ is sufficiently close to R. Hence the sets S_u and S_v are all nonempty.

Next, we prove $\rho_u = 0 = \rho_v$. Suppose this is not true. Then, w.l.o.g., we can assume $0 \leqslant \rho_v \leqslant \rho_u$ and $\rho_u > 0$. Then $v(x) \geqslant v_{\rho_u}(x)$ in Σ_{ρ_u} and, by continuity, we have $w_{\rho_u}(x) \geqslant 0$ in Σ_{ρ_u}. Now, by (A.1), it is easy to see that

$$\begin{cases} \Delta w_{\rho_u} + C(x) w_{\rho_u} \leqslant -4\pi N_1 \delta(2\rho_u, 0) & \text{in } \Sigma_{\rho_u}, \\ w_{\rho_u} \geqslant 0 & \text{in } \Sigma_{\rho_u} \cup \partial\Sigma_{\rho_u}, \end{cases} \tag{A.7}$$

where $C(x) = -e^{v_{\rho_u}} \frac{e^u - e^{u_{\rho_u}}}{u - u_{\rho_u}} \leqslant 0$ $\forall x \in \Sigma_{\rho_u}$. Thus, if $w_{\rho_u}(x_1) = 0$ for some $x_1 \in \Sigma_{\rho_u}$, then by (A.7) and the strong maximum principle, we have $w_{\rho_u} \equiv 0$ in $\overline{\Sigma}_{\rho_u}$. However, this contradicts to the fact that $w_{\rho_u}(x) = u(x) - u(x^{\rho_u}) = -u(x^{\rho_u}) > 0$ for $x \in \partial\Sigma_{\rho_u} \setminus \{x_1 = \rho_u\}$. Therefore we obtain that

$$\begin{cases} w_{\rho_u}(x) > 0 & \text{for any } x \in \overline{\Sigma}_{\rho_u} \setminus T_{\rho_u}, \\ w_{\rho_u} = 0 & \text{on } \partial\Sigma_{\rho_u} \cap T_{\rho_u}. \end{cases} \tag{A.8}$$

By (A.7)–(A.8) and Hopf Boundary Lemma, we obtain

$$\frac{\partial w_{\rho_u}}{\partial x_1} > 0 \quad \text{on } \partial\Sigma_{\rho_u} \cap T_{\rho_u}. \tag{A.9}$$

On the other hand, since $\rho_u > 0$, there exists a positive sequence ε_k such that $\rho_u - \varepsilon_k > 0$ and $(\rho_u - \varepsilon_k) \to \rho_u$ as $k \to \infty$. By the definition of ρ_u, for each ε_k, we obtain that $w_{\rho_u-\varepsilon_k}$ is non-positive somewhere in $\Sigma_{\rho_u-\varepsilon_k}$. By the way, we have $w_{\rho_u-\varepsilon_k} > 0$ on $\partial\Sigma_{\rho_u-\varepsilon_k} \setminus T_{\rho_u-\varepsilon_k}$ and $w_{\rho_u-\varepsilon_k} = 0$ on $\partial\Sigma_{\rho_u-\varepsilon_k} \cap T_{\rho_u-\varepsilon_k}$. Hence, for each ε_k there exists $x_k \in \Sigma_{\rho_u-\varepsilon_k}$ such that

$$\begin{cases} w_{\rho_u-\varepsilon_k}(x_k) \leqslant 0, \\ \nabla w_{\rho_u-\varepsilon_k}(x_k) = (0,0). \end{cases} \tag{A.10}$$

Since $\{x_k\}$ is a bounded sequence, there exists a convergent subsequence, we still denote it by x_k, such that $x_k \to x_0$. By (A.10) we obtain that

$$0 \geqslant \lim_{k\to\infty} w_{\rho_u-\varepsilon_k}(x_k) = \lim_{k\to\infty}\left[u(x_k) - u\left(x^{\rho_u-\varepsilon_k}\right)\right] = u(x_0) - u\left(x_0^{\rho_u}\right) = w_{\rho_u}(x_0).$$

Hence, by the above inequality and (A.8), we conclude that $x_0 \in \partial\Sigma_{\rho_u} \cap T_{\rho_u}$ and, by (A.10),

$$0 = \lim_{k\to\infty} \frac{\partial w_{\rho_u-\varepsilon_k}}{\partial x_1}(x_k) = \lim_{k\to\infty}\left[\frac{\partial u}{\partial x_1}(x_k) - \frac{\partial u}{\partial x_1}\left(x_k^{\rho_u-\varepsilon_k}\right)\right] = \frac{\partial u}{\partial x_1}(x_0) - \frac{\partial u}{\partial x_1}\left(x_0^{\rho_u}\right) = \frac{\partial w_{\rho_u}}{\partial x_1}(x_0).$$

This contradicts to (A.9). Thus, $\rho_u = 0 = \rho_v$, and u and v are radially symmetric.

Step 3. Let $\{(u_R, v_R)\}_{R>0}$ be a sequence of solution pairs for (1.1)–(1.2). Then, by Theorem 3.1 in [17], there exists a subsequence $\{(u_{R_i}, v_{R_i})\}_{i=1}^{\infty}$ such that $(u_{R_i}, v_{R_i}) \to (u_0, v_0)$ uniformly on any compact subset of $\mathbf{R}^2 \setminus \{O\}$ as $i \to \infty$, and (u_0, v_0) is a topological solution of (1.1). By Step 2, each (u_{R_i}, v_{R_i}) is a radial symmetric pair, we have (u_0, v_0) is also a radial symmetric pair. This completes the proof of Proposition A.1. □

References

[1] L.A. Caffarelli, Y. Yang, Vortex condensation in the Chern–Simons Higgs model: An existence theorem, Comm. Math. Phys. 168 (1995) 321–336.
[2] D. Chae, O.Y. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory, Comm. Math. Phys. 215 (2000) 119–142.
[3] H. Chan, C.-C. Fu, C.-S. Lin, Non-topological multi-vortex solutions to the self-dual Chern–Simons–Higgs equation, Comm. Math. Phys. 231 (2002) 189–221.
[4] C.-C. Chen, C.-S. Lin, Uniqueness of the ground state solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, $n \geqslant 3$, Comm. Partial Differential Equations 16 (1991) 1549–1572.
[5] X. Chen, S. Hastings, J.B. Mcleod, Y. Yang, A nonlinear elliptic equation arising from gauge filed theory and cosmology, Proc. R. Soc. Lond. Ser. A 446 (1994) 453–478.
[6] G. Dunne, Self-Dual Chern–Simons Theories, Lecture Notes in Phys., vol. 36, Springer-Verlag, Berlin, 1995.
[7] J. Dziarmaga, Low energy dynamics of $[U(1)]^N$ Chern–Simons solitons, Phys. Rev. D 49 (1994) 5469–5479.
[8] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243.
[9] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren Math. Wiss., vol. 224, Springer-Verlag, Berlin, 1983.
[10] J. Hong, Y. Kim, P.Y. Pac, Multivortex solutions of the Abelian Chern–Simons–Higgs theory, Phys. Rev. Lett. 64 (1990) 2230–2233.
[11] R. Jackiw, S.-Y. Pi, Soliton solutions to the gauged nonlinear Schrödinger equation on the plane, Phys. Rev. Lett. 64 (1990) 2969–2972.
[12] R. Jackiw, E.J. Weinberg, Self-dual Chern–Simons vortices, Phys. Rev. Lett. 64 (1990) 2234–2237.
[13] A. Jaffe, C. Taubes, Vortices and Monopoles, Prog. Phys., vol. 2, Birkhäuser, Boston, MA, 1980.

[14] C.N. Kumar, A. Khare, Charged vortex of finite energy in nonabelian gauge theories with Chern–Simons term, Phys. Lett. B 178 (1986) 395–399.
[15] C. Kim, C. Lee, P. Ko, B.H. Lee, H. Min, Schrödinger fields on the plane with $[U(1)]^N$ Chern–Simons interactions and generalized self-dual solitons, Phys. Rev. 48 (1993) 1821–1840.
[16] C. Li, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations, Invent. Math. 123 (2) (1996) 221–231.
[17] C.-S. Lin, A.C. Ponce, Y. Yang, A system of elliptic equations arising in Chern–Simons field theory, J. Funct. Anal. 247 (2007) 289–350.
[18] J. Spruck, Y. Yang, The existence of non-topological solitons in the self-dual Chern–Simons theory, Comm. Math. Phys. 149 (1992) 361–376.
[19] J. Spruck, Y. Yang, Topological solutions in the self-dual Chern–Simons theory: Existence and approximation, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (1995) 75–97.
[20] G. Tarantello, Uniqueness of selfdual periodic Chern–Simons vortices of topological type, Calc. Var. Partial Differential Equations 29 (2007) 191–217.
[21] H.J. de Vega, F.A. Schaponsnilk, Electrically charged vortices in non-abelian gauge theories with Chern–Simons term, Phys. Rev. Lett. 56 (1986) 2564–2566.
[22] E. Yanagida, Mini-maximizers for reaction–diffusion systems with skew-gradient structure, J. Differential Equations 179 (2002) 311–335.
[23] Y. Yang, The relativistic non-abelian Chern–Simons equations, Comm. Math. Phys. 186 (1997) 199–218.
[24] Y. Yang, Solitons in Filed Theory and Nonlinear Analysis, Springer Monogr. Math., Springer-Verlag, New York, 2001.