ON THE TOPOLOGY OF COMPLETE MANIFOLDS OF NON-NEGATIVE RICCI CURVATURE

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INTRODUCTION

In this paper, we study the fundamental group and first Betti number of complete Riemannian manifolds \( N \) of non-negative Ricci curvature \( \text{Ric}_N \geq 0 \). If \( N \) is compact (without boundary), many basic properties of \( \pi_1(N) \) are known. Thus, if \( N \) is compact and \( \text{Ric}_N > 0 \), then \( \pi_1(N) \) is finite \( b_1(N) = 0 \), [14]. If \( N \) is compact and \( \text{Ric}_N \geq 0 \), then \( \pi_1(N) \) contains a normal free abelian subgroup \( A = \bigoplus_i \mathbb{Z} \) with \( k \leq \dim N \), of finite index in \( \pi_1(N) \), [5]. Also, each \( \mathbb{Z} \subset \pi_1(N) \) gives rise to an isometric product splitting \( \tilde{N} = \tilde{M} \times \mathbb{R} \) of the universal cover \( \tilde{N} \) of \( N \). Further, \( b_1(N) \leq n \), [4]. Thus, the main questions concern the structure of finite subgroups or quotient groups \( \pi_1(N) \).

If \( N \) is non-compact, then the most basic question is whether \( \pi_1(N) \) is finitely generated. This was conjectured by Milnor [13] and remains open to this date. Milnor [13] proved that any finitely generated subgroup of \( \pi_1(N) \) is of polynomial growth of order at most \( n = \dim N \). If \( \pi_1(N) \) itself is finitely generated, then in light of Gromov's polynomial growth theorem [9], \( \pi_1(N) \) has a nilpotent subgroup of finite index. A solution of the conjecture would then give a basic understanding of \( \pi_1(N) \). Note however that there are 4-manifolds \( X \) with \( \pi_1(X) \subset \mathbb{Q} \), with \( X \) in fact a \( K(\mathbb{Q}, 1) \). (I am grateful to Shmuel Weinberger for pointing this out).

We note that these questions on the structure of \( \pi_1(N) \) are open only for \( \dim N \geq 4 \). In dimension 2, it is classical that either \( \pi_1(N^2) = \{e\} \) or \( \pi_1(N) = \mathbb{Z} \). In dimension 3, it is proved in [17] and [1] that \( \pi_1(N^3) \) is either \( \{e\}, \mathbb{Z} \), or has \( \mathbb{Z} \oplus \mathbb{Z} \) as a subgroup of finite index. In fact, there is a full topological classification in these dimensions. Partly in view of these results, Schoen–Yau [17] have made a stronger conjecture on the structure of \( \pi_1(N) \) in the case that the Ricci curvature of \( N \) is positive, namely: \( \pi_1(N) \) is a finitely generated almost nilpotent group with no free abelian subgroups of rank \( \geq \dim N - 2 \). This result would be sharp, since Nabonnand [15] and Berard-Bergery [2] have constructed complete metrics of positive Ricci curvature on \( N^4 = S^1 \times \mathbb{R}^3 \) and \( N^n = T^{n-3} \times \mathbb{R}^3 \) respectively, i.e. \( \pi_1(N) \) is free abelian of rank \( \dim N - 3 \).

The main results of this paper are that if \( N \) is a complete manifold of positive Ricci curvature, then \( b_1(N) \leq \dim N - 3 \) and the rank of any free abelian subgroup of \( \pi_1(N) \) is at most \( \dim N - 3 \). This resolves part of the Schoen–Yau conjecture above. More generally, if the Ricci curvature of \( N \) is non-negative and \( b_1(N) > \dim N - 3 \), then there is an isometric product splitting \( \tilde{N} = \tilde{M} \times \mathbb{R} \) in the universal cover of \( N \). A similar statement applies for the

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The rank of a free abelian subgroup; see Theorem 3.1 for precise statement and related results. In certain special cases, we are also able to show \( \pi_1(N) \) is finitely generated.

The approach to these results combines two different methods. On the one hand, we generalize the Milnor-Svarc method [13], [21] to show that if \( \pi_1(N) \) has a finitely generated subgroup of relatively large polynomial growth, say \( \geq r^p \), then the volume growth of geodesic balls \( B(r) \) in \( N \) is of small polynomial growth, namely \( \leq r^{n-4} \), where \( n = \text{dim} \, N \). On the other hand, using techniques from the theory of minimal hypersurfaces, we show that if \( N \) has non-trivial topology, then any complete metric of non-negative Ricci curvature and of small volume growth on \( N \) is very rigid. In particular, if \( b_1(N) \neq 0 \), then there is no complete metric of positive Ricci curvature on \( N \) satisfying \( \text{vol}(B(r)) \leq c \cdot r^2 \). These results taken together give the main conclusions.

We are not able to deal with higher Betti numbers or homotopy groups. In this regard, we note that Gromoll-Meyer [8] have constructed examples of complete Riemannian manifolds \( N \) of \( \text{Ric} \, > 0 \) which are not of the homotopy type of a closed manifold. Also recently, Sha-Yang [19] have constructed examples of manifolds with positive Ricci curvature of infinite topological type. Note that all these examples are simply connected.

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§1. VOLUME AND FUNDAMENTAL GROUP

In this section, we study relations between the growth of a finitely generated group and the relative volume growth of coverings of a Riemannian manifold. These relations were first studied by Milnor [13] and Svarc [21].

Let \( \Gamma \) be a finitely generated discrete group, with generators \( \{g_1, \ldots, g_s\}, g_i \in \Gamma \). Define the growth function \( \gamma: \mathbb{Z}^+ \to \mathbb{Z}^+ \) of \( \Gamma \) by \( \gamma(r) = \# U(r) \), where \( U(r) \) is the ball of radius \( r \) with respect to the word metric on \( \Gamma \). Thus, \( g \in U(r) \) if and only if \( g \) is a reduced word of length \( \leq r \) in the generators \( g_i \) and their inverses.

We will be interested only in groups \( \Gamma \) of polynomial growth, i.e. those for which the growth function \( \gamma \) satisfies \( \gamma(r) \leq c \cdot r^p \), for some \( c \in \mathbb{R}^+ \), \( T \in \mathbb{R}^+ \). It is well-known [13] that the property of polynomial growth is independent of the generating set \( \{g_i\} \) of \( \Gamma \). We will say \( \Gamma \) has polynomial growth of order \( \geq p \) if

\[
\lim_{r \to \infty} \frac{\gamma(r)}{r^p} > 0
\]

and \( \Gamma \) has polynomial growth of order \( \leq p \) if

\[
\lim_{r \to \infty} \frac{\gamma(r)}{r^p} < \infty
\]

If both equations above hold, we say \( \Gamma \) has polynomial growth of order \( p \). Again, it is easily verified (cf. Lemma 1 of [13]) that these notions are independent of the choice of generators.

We shall be considering actions of \( \Gamma \) on complete Riemannian manifolds \( N \) by covering transformations. For any \( x_0 \in N \), let \( B^N_{x_0}(r) \) denote the geodesic ball of radius \( r \) about \( x_0 \) and let \( v^N(r) = \text{volume} (B^N_{x_0}(r)) \). In analogy to the above, \( N \) has polynomial volume growth if there are constants \( c \in \mathbb{R}^+ \), \( T \in \mathbb{R}^+ \) such that \( v^N(r) \leq c \cdot r^T \), for \( r \geq 1 \). Similarly, \( N \) has polynomial volume growth of order \( \geq p \) (respectively \( \leq p \)) if the above equations hold with \( v^N(r) \) in place of \( \gamma(r) \). One may verify without difficulty that these notions do not depend on the choice of base point \( x_0 \in N \).
Milnor and Svarc discovered relations between the growth rate of finitely generated group $\Gamma$ and the volume growth of the universal cover $\tilde{N}$ of a compact manifold $N$ with $\pi_1(N) = \Gamma$. In the theorem below, we give a generalization of these results, in the case of polynomial growth, to coverings of non-compact manifolds.

**Theorem 1.1.** Let $N$ be a complete Riemannian manifold of polynomial volume growth of order $\geq k$. Let $\pi: \tilde{N} \to N$ be a normal covering, with covering group $\Gamma$ a finitely generated discrete group. If the volume growth of $\tilde{N}$ is polynomial of order $\leq k + 1$, then $\Gamma$ has polynomial volume growth of order $\leq k$. In the same way, if $\Gamma$ has polynomial growth of order $\geq 1$ and $\tilde{N}$ has polynomial volume growth of order $\leq k + 1$, then $N$ has polynomial volume growth of order $\leq k$.

**Proof.** Fix a point $\tilde{x}_0 \in \tilde{N}$ and let $D_g = \{ x \in \tilde{N} : \text{dist}(x, \tilde{x}_0) < \text{dist}(x, g \tilde{x}_0) \}$ for $g \in \Gamma$. Define the Dirichlet domain $F$ associated to $\tilde{x}_0$ by

$$F = \bigcap_{g \neq e} D_g.$$  

It is easily verified that $F$ is a fundamental domain for the action of $\Gamma$, i.e. $F \cap gF = \emptyset$ for $g \neq e$ and $\tilde{N} = \cup gF$.

Let $B^\tilde{N}(r)$ be the geodesic ball of radius $r$ in $\tilde{N}$ about $\tilde{x}_0$. We claim that $\pi$ maps $B^N(r) \cap \tilde{F}$ onto $B^N(r)$, where $B^N(r)$ is the geodesic ball of radius $r$ about $x_0 = \pi(\tilde{x}_0)$ in $N$. To see this, let $\gamma: [0, 1] \to N$ be a length minimizing geodesic in $N$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$ and $l(\gamma) \leq r$. Let $\tilde{\gamma}$ be the unique lift of $\gamma$ to $\tilde{N}$ with $\tilde{\gamma}(0) = \tilde{x}_0$. Then $l(\tilde{\gamma}) = l(\gamma)$ and $\tilde{x}_1 = \tilde{\gamma}(1)$ is a lift of $x_1$ in $B^\tilde{N}(r)$. Now there is a lift $x'_1$ of $x_1$ with $x'_1 \in \tilde{F}$ and so we may write $\tilde{x}_1 = g_1x'_1$, for some $g_1 \in \Gamma$. We have

$$r \geq l(\tilde{\gamma}) = \text{dist}(\tilde{x}_1, \tilde{x}_0) = \text{dist}(g_1x'_1, x_0) = \text{dist}(x'_1, g_1^{-1} \tilde{x}_0) \geq \text{dist}(x'_1, x_0).$$

Thus, $x'_1 \in B^\tilde{N}(r) \cap \tilde{F}$ is a lift of $x_1$, which proves the claim. Next note that $\partial F = \tilde{F} \setminus F$ has Lebesgue measure zero in $\tilde{N}$, since it is contained in a countable union of sets $\partial D_g$, each of measure zero. Since $\pi$ is a local isometry and injective on $F$, we see

$$\text{vol } B^\tilde{N}(r) = \text{vol } (B^\tilde{N}(r) \cap F).$$

Let $U(r) = \{ g \in \Gamma : |g| \leq r \}$, where $|g|$ is the length of the reduced word $g$ in the generators $\{g_i\}$ and their inverses. For $g \in U(r)$, we consider the sets $\Omega_g = B^\tilde{N}_{x_0}(r) \cap gF = g(B^\tilde{N}_{x_0}(r) \cap F)$. Clearly, $\Omega_g \cap \Omega_{g'} = \emptyset$ for $g \neq g'$. Note there is a constant $c > 0$, independent of $r$, such that for $r$ large

$$B^\tilde{N}_{g\tilde{x}_0}(r) \subset B^\tilde{N}_{\tilde{x}_0}(cr)$$

since dist$(g\tilde{x}_0, \tilde{x}_0) \leq mu$, where $\mu = \max \{ \text{dist}(g_i\tilde{x}_0, \tilde{x}_0) \}$. Thus

$$\bigcup_{g \in U(r)} \Omega_g \subset B^\tilde{N}_{\tilde{x}_0}(cr). \quad (1.1)$$

Taking the volume of both sets in (1.1), we find

$$\# U(r) \cdot \text{vol } B^N(r) \leq \text{vol } B^\tilde{N}(cr).$$
By assumption, \( \text{vol} B^N(r) \geq c_1 r^k \), for some \( c_1 > 0 \) and \( r \) sufficiently large, and \( \text{vol} B^N(cr) \leq c_2 (cr)^{k+1} \), for \( r \) sufficiently large. Thus
\[
\# U(r) \leq \frac{c_2 c^{k+1}}{c_1} r^k,
\]
for all \( r \) sufficiently large, i.e. \( \Gamma \) has polynomial growth of order \( \leq l \). In the same way one proves the second statement of the theorem.

It is easy to see that finitely generated groups of zero polynomial growth are finite. Thus if \( l = 0 \) in Theorem 1.1, it follows that \( \pi : \bar{N} \to N \) is a finite cover. If \( \pi : \bar{N} \to N \) is any regular cover (with possibly infinitely generated) covering group \( \Gamma \) and the volume growths of \( N \) and \( \bar{N} \) are polynomial of order \( k \), we would like to assert that \( \pi \) is a finite cover. This does not follow from Theorem 1.1, since for instance \( \Gamma \) may be an infinitely generated torsion group, e.g. \( \mathbb{Q}/\mathbb{Z} \). However, we have

**Proposition 1.2.** Let \( N \) be a complete Riemannian manifold of polynomial volume growth of order at least \( k \). Let \( \pi : \bar{N} \to N \) be a normal covering, with covering group \( \Gamma \), where \( \bar{N} \) has volume growth of order at most \( k \). Then \( \Gamma \) is a finite group.

**Proof:** Let \( \Gamma' \) be a finitely generated subgroup of \( \Gamma \), and consider the covering \( \pi' : \bar{N} \to N' \) with covering group \( \Gamma' \), so that \( N' = \bar{N}/\Gamma' \). Applying Theorem 1.1 to this cover, it follows that \( |\Gamma'| \) is finite. We will find an upper bound on \( |\Gamma'| \). Since \( N \) has polynomial volume growth of order \( \geq k \), there is a \( c_N > 0 \) such that \( \text{vol} B^N(r) \geq c_N r^k \), for \( r \) large. From this, it easily follows that \( \text{vol} B^{N'}(r) \geq c_N r^k \). As in the proof of Theorem 1.1, there is a Dirichlet fundamental domain \( F \subset \bar{N} \) for \( \Gamma' \) centered at \( x_0 \in \bar{N} \), such that
\[
\text{vol} B^N_{x_0}(r) = \text{vol}(B^N_{x_0}(r) \cap F)
\]
where \( x_0 = \pi'(\bar{x}_0) \). Since \( \pi' \) is a finite covering, there is a \( \mu > 0 \) such that
\[
B^N_{x_0}(r) \subset B^N_{x_0}(r + \mu),
\]
for all \( g \in \Gamma' \). (Of course \( \mu \) depends on \( \Gamma' \)). Since \( \bar{N} \) has polynomial volume growth of order \( \leq k \), \( \text{vol} (B^\bar{N}_{\bar{x}_0}(r)) \leq c_{\bar{N}} r^k \), for some constant \( c_{\bar{N}} > 0 \) and \( r \) large. We then have
\[
|\Gamma'| \text{vol} B^N_{x_0}(r) = \sum_{\gamma \in \Gamma'} \text{vol} (B^\bar{N}_{\bar{x}_0}(r) \cap gF)
\]
\[
\leq \sum_{\gamma \in \Gamma'} \text{vol} (B^\bar{N}_{\bar{x}_0}(r + \mu) \cap gF) \leq \text{vol} B^\bar{N}_{\bar{x}_0}(r + \mu) \leq c_{\bar{N}} (r + \mu)^k.
\]
Thus
\[
|\Gamma'| c_N r^k \leq c_{\bar{N}} (r + \mu)^k
\]
or
\[
|\Gamma'| \leq \frac{c_{\bar{N}}}{c_N} \lim_{r \to \infty} \left( \frac{r + \mu}{r} \right)^k = \frac{c_{\bar{N}}}{c_N}.
\]

Thus, there is a uniform bound on the order of finitely generated subgroups \( \Gamma' \) of \( \Gamma \). It follows that \( \Gamma \) is a finite group of order \( |\Gamma| \leq c_{\bar{N}}/c_N \). \( \square \)

Next, we show that the 1st Betti number of a manifold \( N \) may be estimated in terms of the growth of finitely generated subgroups of \( \pi_1(N) \). In fact, the following result is purely group-theoretical.
Theorem 1.3. Let N be a manifold such that every finitely generated subgroup \( \Gamma \subset \pi_1(N) \) is of polynomial growth of order at most \( n \), for some fixed \( n \in \mathbb{Z}^+ \cup \{0\} \). Then
\[ b_1(N) \leq n. \]

Proof. Consider the sequence of homomorphisms
\[
\pi_1(N) \xrightarrow{p_1} \pi_1(N) / [\pi_1(N), \pi_1(N)] = H_1(N, \mathbb{Z}) \xrightarrow{p_2} H_1(N, \mathbb{Z}) / T,
\]
where \( T \) is the torsion subgroup of the abelian group \( H_1(N, \mathbb{Z}) \). Clearly, the composite \( p_2 \circ p_1 \) is surjective. Now \( (H_1(N, \mathbb{Z})/T) \) is a torsion-free abelian group of torsion-free rank (c.f. [11], p. 138) \( r - b_1(N) \) (\( r \) may be infinite). The rank of a torsion-free abelian group \( A \) may be defined as the dimension of the vector space \( \mathbb{Q} \otimes A \), considering \( A \) as a \( \mathbb{Z} \)-module.

Now it is well-known (c.f. [11], p. 207) that torsion free abelian groups \( A \) of rank \( r \) can be embedded as subgroups of \( \bigoplus_1^s \mathbb{Q} \), where \( \mathbb{Q} \) is the additive group of rationals. Further, such \( A \) cannot be embedded in \( \bigoplus_1^s \mathbb{Q} \) for \( s < r \), or for \( s \) finite if \( r \) is infinite. Thus, we have an embedding
\[
\frac{H_1(N, \mathbb{Z})}{T} \hookrightarrow \mathbb{Q}^{b_1(N)}.
\]

Suppose \( b_1(N) \geq n + 1 \). Then we may choose \((n + 1)\) elements \( \{\alpha_i\}, i = 1, \ldots, n + 1 \in \) the image \( i(H_1(N, \mathbb{Z})/T) \) which are linearly independent over \( \mathbb{Q} \). Let \( \mathbb{Q}^{b_1(N)+1} \) be the subspace spanned by \( \{\alpha_i\} \) in \( \mathbb{Q}^{b_1(N)} \) and let \( P \) be the projection homomorphism of \( \mathbb{Q}^{b_1(N)} \) onto \( \mathbb{Q}^{b_1(N)+1} \). Further, let
\[
\phi: \pi_1(N) \to \mathbb{Q}^{b_1(N)+1}
\]
be the composite \( \phi = P \circ i \circ p_2 \circ p_1 \). Now let \( q_i \in \pi_1(N) \) be elements such that \( \phi(q_i) = \alpha_i \) and let \( \Gamma \) be the subgroup of \( \pi_1(N) \) generated by \( \{q_i\}, i = 1, \ldots, n + 1 \). Thus there is a surjective homomorphism
\[
F: \Gamma \to \mathbb{Z}^{n+1}
\]
onoonto \( \mathbb{Z}^{n+1} \) given by the restriction of \( \phi \) to \( \Gamma \).

By hypothesis, \( \Gamma \) has polynomial growth of order at most \( n \). We claim that such \( \Gamma \) have no surjective homomorphisms onto \( \mathbb{Z}^{n+1} \). This will clearly prove the Theorem. Thus, given \( F \) as in (1.3), let \( F = (f_1, f_2, \ldots, f_{n+1}), f_i: \Gamma \to \mathbb{Z} \) be the component homomorphisms onto \( \mathbb{Z} \) and let \( K_i = \ker f_i \). We then have exact sequences
\[
0 \to K_i \to \Gamma \to \mathbb{Z} \to 0.
\]

Note that each \( K_i \) is finitely generated. In fact, \( \{q_j\}_{j \neq i} \) is a finite generating set for \( K_i \). We now use the Splitting Lemma of Gromov [9], which states that if \( G' \subset G \) is a finitely generated subgroup of infinite index in a finitely generated group \( G \), then
\[
growth(G') \leq \growth(G) - 1,
\]
where \( \growth(G) \) is the infimum of numbers \( d \geq 0 \) such that \( \# U(r) \leq c \cdot r^d, r \geq 1 \). It follows that
\[
growth(K_i) \leq \growth(\Gamma) - 1.
\]

By the 2nd homomorphism theorem,
\[
\frac{K_i \cdot K_j}{K_i} \cong \frac{K_j}{K_i \cap K_j} \quad i \neq j.
\]
It is easily seen that $K_i \cdot K_j = \Gamma$, since each generator $q_i$ of $\Gamma$ is in $K_i \cdot K_j$. Further, $K_i \cap K_j$ is generated by the elements \( \{q_k\} \) for $k \neq i, j$. Thus, the Splitting Lemma again gives

$$\text{growth}(K_i \cap K_j) \leq \text{growth} K_j - 1 \leq \text{growth} \Gamma - 2.$$ 

Proceeding inductively, we find

$$\text{growth}(K_1 \cap \ldots \cap K_n) \leq \text{growth} \Gamma - n < 0$$

so that $K_1 \cap \ldots \cap K_n$ is a finitely generated group of zero polynomial growth and thus finite. However, $q_{n+1} \in K_1 \cap \ldots \cap K_n$ generates an infinite cyclic group, which gives the required contradiction.

The following fact, although a trivial consequence of the definitions, will be of use later.

**Proposition 1.4.** Let $\Gamma$ be a finitely generated group of polynomial growth of order $\leq p$. Then any finitely generated subgroup $\Gamma' \subset \Gamma$ is also of polynomial growth of order at most $p$. In particular, $\Gamma$ contains no subgroups isomorphic to $\mathbb{Z}^{p+1}$.

*Proof.* The first statement follows immediately from the definitions. One easily calculates that $\mathbb{Z}^n$ has polynomial volume growth of order equal to $n$ so the second statement follows.

In a similar fashion, one may bound the rank of a finitely generated, torsion free nilpotent group in terms of the growth of the group.

We now apply the results above to study the fundamental group of complete, $n$-dimensional manifolds $N$ of non-negative Ricci curvature $\text{Ric}_N \geq 0$. The well-known Bishop comparison theorem [3] implies that

$$\nu(r) \leq \omega_n r^n$$

(1.5)

where $\nu(r) = \text{vol}(B_N^n(r))$ and $\omega_n$ is the volume of the unit ball in Euclidean space $\mathbb{R}^n$. In particular, $N$ as well as all its coverings $\tilde{N}$ have polynomial volume growth of order at most $n$. The combination of (1.5) with Theorem 1.1 applied to the universal cover $\tilde{N}$ of $N$ immediately implies Milnor's Theorem: If $N$ is a complete manifold of non-negative Ricci curvature, then every finitely generated subgroup $\Gamma$ of $\pi_1(N)$ is of polynomial growth of order at most $n = \text{dim} N$. We also note the following result of Cheeger-Gromoll [5]: If there is a finitely generated subgroup $\Gamma_0$ of $\pi_1(N)$ with polynomial volume growth of order equal to $n$, then $\pi_1(N)$ is finitely generated and $N$ is a compact manifold with a flat Riemannian metric.

The results above may be combined in the following

**Corollary 1.5.** Let $N$ be a complete Riemannian manifold of non-negative Ricci curvature, $n = \text{dim} N$. Then

1. The first Betti number $b_1(N) = \text{dim} H_1(N, \mathbb{Q})$ satisfies $b_1(N) \leq n$, with equality only if $N$ is a compact manifold with flat Riemannian metric.

2. If the volume growth $\nu(r)$ of $N$ is polynomial of order $\geq k$, then every finitely generated subgroup of $\pi_1(N)$ has polynomial growth of order $\leq n - k$. In particular, $b_1(N) \leq n - k$ and the maximal rank of a free abelian subgroup of $\pi_1(N)$ is $n - k$.

3. If the volume growth $\nu(r)$ of $N$ satisfies $\nu(r) \geq c_n r^n$, for some $c_n > 0$, then $\pi_1(N)$ is finite and $|\pi_1(N)| \leq \frac{\omega_n}{c_n}$. 
Proof. (1) Applying Theorem 1.1 to the universal cover \( \tilde{N} \) of \( N \), (1.5) and Theorem 1.3 immediately show \( b_1(N) \leq n \). If \( b_1(N) = n \), the proof of Theorem 1.3 shows that \( \pi_1(N) \) has a finitely generated subgroup of polynomial growth order \( \geq n \). The Cheeger–Gromoll Theorem then proves the result.

(2) This follows in the same way as (1)

(3) This follows immediately from (1.5) and Proposition 1.2. \( \square \)

Remarks. (a) The examples of Nabonnand [15] \( N^4 = \mathbb{R}^3 \times S^1 \) of \( \text{Ric}_N > 0 \) satisfy \( e^N(r) \geq c \cdot r^3 \) and \( e^N(r) \geq c \cdot r^4 \). Thus, in a certain sense, (2) is sharp and there is no metric rigidity for the equality case \( b_1(N) = n - k \) as in (1). See §3 for further comments.

(b) (3) has recently been proved by Li [12], using global estimates for the heat kernel on \( N \).

\[ \section{COMPLETE AREA-MINIMIZING HYPERSURFACES IN N.} \]

In order to extend the results above further, we use some techniques from the theory of minimal varieties. Let \( N \) be complete (non-compact) Riemannian manifold of dimension \( n \). A complete area-minimizing hypersurface \( M \) in \( N \) is defined to be a multiplicity one integral \( (n - 1) \) current, without boundary in \( N \), such that if \( U \) is any domain with compact closure in \( N \), then

\[ V(M \upharpoonright U) \leq V(T) \]

where \( T \) is an integral \( (n - 1) \) current with \( \partial T = \partial (M \upharpoonright U) \). \( V \) denotes the mass (or volume in the case of multiplicity one) of a current and \( M \upharpoonright U \) is the restriction of \( M \) to \( U \). We refer to [20] and [7] for background material on integral currents. The regularity theory for area-minimizing hypersurfaces shows that there is a closed set sing \( M \subset \text{supp} M \) of Hausdorff codimension \( \geq 7 \) in \( M \) such that \( M \setminus \text{sing} M \) is a smooth properly embedded hypersurface in \( N \).

We now show that if \( N \) is a complete manifold of non-negative Ricci curvature and the volume growth of \( N \) is rather small, then an area-minimizing hypersurface in \( N \) gives rise to strong restrictions on the metric. The following result for \( \dim N = 3 \) was proved in [1]. Recall an oriented boundary of least area in a manifold \( N^* \) is an area-minimizing integral \( (n - 1) \) current \( M \) in \( N^* \) such that \( M - \partial T \), where \( T \) is an integral \( n \)-current in \( N^* \). We will also assume throughout the paper that \( T \) is of multiplicity one (so that \( T \) is the characteristic function of an open set in \( N^* \)).

**Theorem 2.1.** Let \( N \) be a complete oriented Riemannian manifold of volume growth

\[ e^N(r) \leq c_0 \cdot r^3 \]

for some constant \( c_0 = c_0(x) > 0 \), \( r \geq 1 \). If \( N \) has positive Ricci curvature, then there is no complete area-minimizing hypersurface \( M \) which is a boundary of least area in \( N \). Further, suppose \( \text{Ric}_N \geq 0 \), \( K_N \leq b < \infty \) and such an \( M \) exists. Then \( M \) is smooth (\( \text{sing} M = \emptyset \)) and there is an isometric splitting of the universal covers \( \tilde{N} = \tilde{M} \times \mathbb{R} \).

**Proof.** We suppose \( M = \partial T \) is an oriented boundary of least area in \( N \). If \( B_x(s) \) is the geodesic ball of radius \( s \) in \( N \) about \( x \), then the current \( \partial T \upharpoonright B_x(s) \) is homologous to the slice \( T \upharpoonright \partial B_x(s) \) (via \( T \upharpoonright B_x(s) \)). It follows that

\[ V(M \upharpoonright B_x(s)) \leq V(T \upharpoonright \partial B_x(s)) \leq \text{vol} S_x(s) \]

(2.2)
Further, by (2.1), it follows there is a sequence \{r_i\} \to \infty such that
\[ V(M \setminus B_{x_0}(r_i)) \leq c_1 \cdot r_i^2, \quad (2.3) \]
for a fixed \( x_0 \in N \).

Now since \( M \) is an area-minimizing hypersurface, the regular set \( M' = M \setminus \text{sing} \ M \) is a smooth, stable minimal hypersurface. Thus, for any Lipschitz function \( f \) of compact support on \( M' \), we have
\[ \int_{M'} |df|^2 - kf^2 \geq 0 \quad (2.4) \]
where \( k = |A|^2 + \langle \text{Ric}, v \rangle \), \( A \) is the 2nd fundamental form of \( M' \) in \( N \) and \( v \) is a unit normal of \( M' \) in \( N \). Choose a point \( x_0 \in M' \) and let \( r: M' \to \mathbb{R} \) be the distance function in \( M' \) from \( x_0 \). Define \( h: M' \to \mathbb{R} \) by
\[ h(x) = \begin{cases} 1 & r(x) \leq R_1 \\ \log r(x) - \log R_2 & R_1 \leq r(x) \leq R_2 \\ 0 & r(x) \geq R_2 \end{cases} \]
for \( R_2 > R_1 > 0 \). Let \( \psi \) be a Lipschitz function of compact support on \( M' \), with \( 0 \leq \psi \leq 1 \) and \( \psi \equiv 1 \) outside a small neighborhood of \( \text{sing} \ M \); \( \psi \) will be specified more precisely below. Setting \( f = \psi \cdot h \), one obtains
\[ \int_{M'} k(\psi h)^2 \leq \int_{M'} \psi^2 |dh|^2 + \int_{M'} |d\psi|^2. \quad (2.5) \]
Let \( A(R_1, R_2) = \{ x \in M' : R_1 < r(x) < R_2 \} \). Since \( r \) is a Lipschitz function on \( M' \) with \( |\nabla r| = 1 \) a.e., we have
\[ \int_{M'} \psi^2 |dh|^2 \leq \frac{1}{(\log R_1 - \log R_2)^2} \int_{A(R_1, R_2)} \frac{1}{r^2} \int_{R_1}^{R_2} v(S(r)) \frac{r^2}{r^2} dr \]
where the last equality follows from the coarea formula and \( v(S(r)) \) denotes the \((n-2)\)-volume of the sphere of radius \( r \), \( S(r) = \partial B(r) \) in \( M' \). Note that \( \int_0^R v(S(r)) dr = v(B(R)) \) and \( B(R) \subset B^N(R) \cap M' \). Integrating the latter integral by parts, one has
\[ \int_{R_1}^{R_2} v(S(r)) \frac{r^2}{r^2} dr \leq \frac{\text{vol}(M \setminus B_{x_0}(r))}{r^2} \bigg|_{R_1}^{R_2} + 2 \int_{R_1}^{R_2} \frac{v(B(r))}{r^3} dr \]
Since, by (2.2), \( \text{vol}(M \setminus B_{x_0}(r)) \leq \text{vol}S_{x_0}(r) \), we obtain
\[ \int_{R_1}^{R_2} v(M \setminus B_{x_0}(r)) \frac{r^2}{r^3} \leq \frac{\text{vol}B_{x_0}(r)}{r^3} \bigg|_{R_1}^{R_2} + 3 \int_{R_1}^{R_2} \frac{\text{vol}B_{x_0}^N(r)}{r^4} \]
If we use (2.1) and let \( R_1, R_2 \) be chosen from the sequence satisfying (2.3), it follows that
\[ \int_{R_1}^{R_2} \frac{\text{vol}(S(r))}{r^2} \leq c_2 + c_3 [\log R_2 - \log R_1] \]
so that
\[ \int_{M'} \psi^2 |dh|^2 \leq \frac{c_5}{(\log R_2 - \log R_1)}, \quad (2.6) \]
for $R_2 \gg R_1$. Next we estimate the term $\int_{M'} |d\psi|^2$. Let $D_\varepsilon = \{ x \in N : \text{dist}(x, \text{sing } M) < \varepsilon \}$ and let $K$ be a compact domain in $N$. Let $\Lambda_d(\text{sing } M \cap K)$ be the $d$-dimensional Hausdorff measure of $\text{sing } M \cap K$ in $N$. Thus $\Lambda_d(\text{sing } M \cap K) = 0$, for $d > n - 7$ by the regularity theory for area-minimizing currents. For convenience, we may assume $N$ is isometrically embedded as a submanifold of a large Euclidean space $\mathbb{R}^l$. Given $\varepsilon > 0$, one may choose a covering of $\text{sing } M \cap K$ by cubes $\{Q_k\}$ of side length $s_k \leq \varepsilon$ with $\cup 3/2Q_k \subset D_\varepsilon$ such that $\sum s_k^d \leq c_5 \cdot (\Lambda_d(\text{sing } M \cap K) + \varepsilon)$, for some constant $c_5$. By relabelling, we may assume $s_1 \geq s_2 \geq \ldots \geq s_p$. By Lemma 3.1 and 3.2 of [10], there is a function $\varphi_e : N \rightarrow \mathbb{R}$ with $\varphi_e \equiv 1$ on $\cup Q_k$, $\text{supp } \varphi_e \subset \cup 3/2Q_k$ and

$$|d\varphi_e|(x) \leq c_6 \cdot s_k^{-1}$$

for all $x \in T_k \equiv 3/2Q_k - \bigcup_{j \neq k} 3/2Q_j$. Note that $T_k = \cup 3/2Q_k \subset D_\varepsilon$. Set $\psi = \psi_e = 1 - \varphi_e$. We have

$$\int_{M' \cap K} |d\psi|^2 = \sum_{j=1}^p \int_{M' \cap T_j} |d\psi|^2 \leq c_7 \cdot \sum_{j} s_j^{-2} \text{vol}(M' \cap T_j).$$

By the area-minimizing property of $M'$, $\text{vol}(M' \cap T_j) \leq c_9 \cdot s_j^{n-1}$ for $s_j$ sufficiently small, so that

$$\int_{M' \cap K} |d\psi|^2 \leq c_9 (\Lambda_{n-3}(\text{sing } M \cap K) + \varepsilon) = c_9 \cdot \varepsilon,$$  \hspace{1cm} (2.7)

since $\Lambda_{n-3}(\text{sing } M \cap K) = 0$. From (2.5), (2.6) and (2.7), it follows that, given any $\delta > 0$ small and $R_1 > 0$ large, $R_2 \gg R_1$, there exists $\varepsilon > 0$ small so that

$$\int_{D(R_1) \setminus D_\varepsilon} k < \delta.$$ 

Since $k \geq 0$ pointwise on $M'$, it follows that $k \equiv 0$ on $M'$.

This completes the first part of Theorem 2.1. To complete the second part, we see that $M'$ is a totally geodesic hypersurface in $N$. We first claim that sing $M = \emptyset$, so that $M' = M$ is a smooth totally geodesic hypersurface in $N$. To see this, if $z \in \text{sing } M$, then since $M'$ is totally geodesic, one easily sees that a tangent cone $C$ of $M$ at $z$, obtained by ‘blowing up’ the regular set $M'$ in small neighborhoods of $z$, is a cone on a totally geodesic web $S \subseteq S^{n-1} \subseteq \mathbb{R}^n$, i.e. $S = \Sigma S_i$, where each $S_i$ is a domain in a totally geodesic $S^{n-2} \subseteq S^{n-1}$. If $C$ does not lie in a single $n$-plane $P$, then an elementary area-comparison argument (rounding off the corners of intersection) shows that one may reduce the area of $C$, which contradicts the fact that $C$ is a boundary of least area. Thus, any tangent cone of $M$ at $z$ is a flat $n$-plane, and for instance the Allard regularity theorem implies that $z$ is a regular point. Thus, $M$ is a smooth complete hypersurface and note also that $\text{Ric } v \equiv 0$, where $v$ is the unit normal of $M$ in $N$.

The remainder of the proof follows the proof in [1] in case $\dim N = 3$. We refer to [1] for some details. Fix a point $x_0 \in M$ as above and let $S_0$ be the geodesic sphere of radius $r$ about $x_0$ in $M$. Let $\tilde{S}_e(t) = \exp_{S_0} tv$ be the image of $S_0$ under the normal exponential map from $M$.

We claim that the integral $(n-1)$ current $A$ of least mass with $\partial A = S_R - \tilde{S}_e(t)$, for $R$ large, $\varepsilon$ small is indecomposable, i.e. cannot be decomposed as a sum $A = A_1 + A_2$, where the $A_i$ have no boundary in $A$. For if $A = A_1 + A_2$, then we must have $\partial A_1 = S_R$ and $\partial A_2 = \tilde{S}_e(t)$, so that by the area-minimizing property of $M$, $A_1$ is the geodesic ball $B_R$ of radius $R$ in $M$. Thus, the claim will be proved if there is a comparison surface $\Sigma = \Sigma_{1, \varepsilon, k}$ with
the given boundary such that, for all \( \varepsilon \leq \varepsilon_0 \), \( |t| \leq t_0 \), and \( R \geq R_0 \),

\[
\text{vol} \Sigma < \text{vol} B_R + \text{vol} \tilde{B}_t(t),
\]

(2.8)

where \( \tilde{B}_t(t) \) is the least area hypersurface with \( \partial \tilde{B}_t(t) = \tilde{S}_t(t) \).

For the moment, we assume (2.8) and proceed with the proof of Theorem 2.1. We may fix \( t \) and \( \varepsilon \) choose a sequence \( R_i \to \infty \) such that the least mass current \( A_t \) with boundary \( S_{R_i} - \tilde{S}_t(t) \) converges to a least mass current \( S = S_{t, \varepsilon} \) in \( N \) with \( \partial S = \tilde{S}_t(t) \). As above, it is easy to see that \( S \) is indecomposable. There is now a sequence \( \varepsilon_i \to 0 \) such that the currents \( S_{t, \varepsilon_i} \) converge to a least area current \( M_t \) in \( N - q_t \), where \( q_t = -\exp_{q_t(t)} t \). We claim that \( q_t \in \text{supp} M_t \).

For if this were not true, then there is a ball \( B = B_q(t, \delta) \), for \( 0 < \delta < t/2 \), such that \( \text{supp} M_t \cap B = \emptyset \). Thus,

\[
V(S_{t, \varepsilon} \cap B) \to 0, \quad \text{as} \quad i \to \infty.
\]

(2.9)

However, \( \partial S_{t, \varepsilon} \subset B \), for \( i \) sufficiently large, so that \( \text{supp} S_{t, \varepsilon} \cap \partial B \neq \emptyset \). If we let \( x_i \in \text{supp} S_{t, \varepsilon} \cap \partial B \), then by the well-known monotonicity formula [20] for stationary currents, one obtains

\[
V\left(B_{x_i} B_{x_i} \left(\frac{\delta'}{4}\right)\right) \geq \frac{\delta'}{4}^{-1},
\]

(2.10)

where \( c \) is a constant depending only on the local geometry of \( N \) near \( q_t \). This contradicts (2.9), so that \( q_t \in \text{supp} M_t \), as claimed.

Since \( \partial M_t = 0 \) in \( N - q_t \), it follows that \( \partial M_t = 0 \) in \( N \). Further, one may easily verify that \( M_t \) is of least mass in \( N \). For suppose \( M_t \) is not of least mass in \( N \). There there is a domain \( \Omega \subset \subset N \) and a hypersurface \( \Sigma \) in \( N \) such that \( V(\Sigma) + \varepsilon \leq V(M_t - \Omega) \) with \( \partial \Sigma = \partial \tilde{M}_t - \Omega \), for some \( \varepsilon > 0 \). Let \( T_\delta \) be the \( \delta \)-tubular neighborhood of a smooth curve \( \gamma(s) \) from \( q_t \) to a point \( \rho_0 \in \Sigma \) and consider the Lipschitz hypersurface \( (\Sigma - T_\delta) \cup \partial T_\delta = \Sigma' \). Then

\[
V(\Sigma') \leq V(\Sigma) + \delta',
\]

where \( \delta' \) is arbitrarily small depending on \( \delta \). This contradicts the mass-minimizing property of \( M_t \) in \( N - q_t \).

The results above, in the first part of the proof, show that each \( M_{t, \varepsilon} \), for \( |t| < t_0 \), is a complete, totally geodesic hypersurface in \( N \) and \( \text{Ric}_v \equiv 0 \), where \( v \) is the unit normal of \( M_t \) in \( N \). Note that, by construction, each current \( A_t \) has support locally on one side of \( M_t \), and thus \( M_t \) lies locally on one side of \( M_t \), for \( t \neq 0 \). By the maximum principle, \( M_t \cap M = \emptyset \) for all \( t \neq t_0 \). We also claim that \( M_t \cap M_{t_0} = \emptyset \), for all \( t \neq t' \). To see this, if \( M_t \) and \( M_{t'} \) intersect, again by the maximum principle, an associated sequence of least mass approximating currents \( A_t = A(e, t, R_t) \) and \( B_t = A(e, t', R_t) \) must also have intersecting supports away from their boundary. (Note that since \( M_t \) and \( M_{t'} \) are smooth, the Allard regularity theorem implies that \( A_t \cap K \) and \( B_t \cap K \) are also smooth hypersurfaces, for any compact set \( K \), for \( t \) sufficiently large). Since the boundary components of \( A_t \) and \( B_t \) either coincide or are disjoint, a straightforward area comparison argument shows that \( A_t \) and \( B_t \) cannot intersect, as required.

Thus, the family \( M_{t, \varepsilon} \), \( |t| < t_0 \) is a family of properly embedded disjoint minimal hypersurfaces. By the maximum principle, if \( t_i \to t \), then \( M_{t_i} \to M_t \) smoothly on compact sets of \( N \). This means exactly that \( \{ M_{t, \varepsilon} \}, |t| < t_0 \) is a \( C^0 \) foliation of a region of \( M \) by stable minimal hypersurfaces. In particular, the original area-minimizing hypersurface \( M - M_0 \) is a leaf of this foliation. Let \( J \) be the Jacobi field on \( M_t \) induced by the foliation. Then \( J = f v \) and \( f \) satisfies the Jacobi equation

\[
Af + kf = 0
\]
However, $k \equiv 0$ and $f > 0$, since the hypersurfaces never intersect. Thus, $f$ is a positive harmonic function on $M$. Since, for instance, $v^M(r) \leq c \cdot r^2$, using the result of [6], this forces $f$ to be constant. It now follows easily that $v$ is a Killing field on the region $R_0 = M \times [-t_0, t_0]$ defined by the foliation and the metric on $R_0$ is a product metric (c.f. [1] for further details). Thus, the theorem follows, once (2.8) is established.

To do this, we will consider comparison surfaces $\Sigma_f$ which are images of smooth (or Lipschitz) maps $F_f : \Omega \to \mathbb{N}$, $F_f(x) = \exp_{x} f(x) v$, where $f : \Omega \to \mathbb{R}$ is a smooth (or Lipschitz) function and $\Omega$ is a domain in $M$. Equivalently, $\Sigma_f$ may be considered as the current $(F_f)_{*}\{\Omega\}$.

We will consider $f$ of the form $f(x) = h(r(x))$, where $r$ is the distance function on $M$ from $x_0 \in M$. Let $A(\varepsilon, R) = \{x \in M : r(x) \leq R \leq \varepsilon \}$ as above. We need to find $h : [\varepsilon, R] \to \mathbb{R}$ such that $h(\varepsilon) = t$, $|t| \leq t_0$, $h(R) = 0$ and

$$V(\Sigma_f) < V(A(\varepsilon, R)) + V(B_{\varepsilon}) + V(\tilde{B}_{\varepsilon}(t)),$$

(2.11)

where $B_{\varepsilon} = \{x \in M : r(x) \leq \varepsilon \}$ and $\tilde{B}_{\varepsilon}(t)$ is the area-minimizing hypersurface bounding $\tilde{h}_t$. (2.11) must hold for any given $t$ with $|t| \leq t_0$, for all $\varepsilon \leq \varepsilon_0$ and all $R \geq R_0 = R_0(\varepsilon)$. Clearly, for $\varepsilon$ and $t_0$ sufficiently small

$$V(B_{\varepsilon}) + V(\tilde{B}_{\varepsilon}(t)) > \frac{3}{2} \omega_{n-1} \varepsilon^{n-1}$$

where $\omega_{n-1}$ is the volume of the unit ball in $\mathbb{R}^{n-1}$. Thus it suffices to show

$$V(\Sigma_f) - V(A(\varepsilon, R)) < \frac{3}{2} \omega_{n-1} \varepsilon^{n-1}.$$  (2.12)

Since $\text{Ric}_N \geq 0$, it is not difficult to show (cf. [1]) that

$$V(\Sigma_f) \leq \int_{A(\varepsilon, R)} \left[ 1 + \left( \frac{h'}{\lambda} \right)^2 \right]^{1/2} dV_M$$

where $\lambda = \cos(\sqrt{bt_0})$. (Recall $b$ is an upper bound on the sectional curvature). Consequently, (2.11) follows from

$$\int_{A(\varepsilon, R)} \left[ 1 + \left( \frac{h'}{\lambda} \right)^2 - 1 \right] dV_M \leq \frac{3}{2} \omega_{n-1} \varepsilon^{n-1}. $$  (2.13)

Let $g(r) = \sqrt{1 + \left( \frac{h'}{\lambda} \right)^2} - 1$, $v'(r) = \text{vol}(S(r))$ and recall $v(r) = \text{vol}(B(r)) = \int_0^r v'(s) ds$. By the coarea formula, we have

$$\int_{A(\varepsilon, R)} g(r) dV_M = \int_{\varepsilon}^R g(r) v'(r) dr = g(r)v(r) \big|_{\varepsilon}^R - \int_{\varepsilon}^R g'(r)v(r).$$  (2.14)

As in [1], we choose the function $h = h(r)$ so that the surface of revolution it determines in $\mathbb{R}^3$ is a catenoid spanning a circle of radius $\varepsilon$ in the plane $z = t$ and a circle of radius $R$ in the $(x, y)$-plane. To simplify the $\lambda$-factor in $g(r)$, we set

$$h(r) = \frac{1}{c} \cosh^{-1}(c \lambda R) - \frac{1}{c} \cosh^{-1}(c \lambda r)$$  (2.15)

where

$$R = \frac{1}{c \lambda} \cosh [\cosh^{-1}(c \lambda \varepsilon) + c \varepsilon].$$  (2.16)

where $c = c(\lambda, \varepsilon, R)$ satisfies $c \to \infty$, as $R \to \infty$, for any fixed $\varepsilon$, $\lambda$. We will assume $R$ is chosen
large enough so that $c\lambda e \geq 2$. Then $h(\varepsilon) = t, h(R) = 0$ and

$$g(r) = \frac{c\lambda r}{\sqrt{(c\lambda r)^2 - 1}} - 1 \leq \frac{1}{(c\lambda r)^2}.$$

Now by (2.3), if we choose $R$ to belong to the sequence $\{r_i\}$, $v(R) \leq c_1 R^2$, so that $g(R)v(R) \leq \frac{c_1}{(c\lambda)^2}$. Further, $g'(r) = -c\lambda((c\lambda r)^2 - 1)^{-3/2}$, so that by (2.2)

$$- \int_{\varepsilon}^{R} g'(r)v(r) dr \leq c_1 c\lambda \int_{\varepsilon}^{R} \frac{\operatorname{vol} S^3(r)}{[(c\lambda r)^2 - 1]^{3/2}} dr.$$

Since $c\lambda e \geq 2$, we have $(c\lambda r)^2 - 1 \geq \frac{1}{2}(c\lambda r)^2$ for $r \geq \varepsilon$. Then, by essentially the same argument as that preceding (2.6), we find

$$- \int_{\varepsilon}^{R} g'(r)v(r) dr \leq \frac{2c_1}{(c\lambda)^2} [\ln R - \ln \varepsilon] \leq \frac{c_3}{(c\lambda)^2} [-\ln ec\lambda + \cosh^{-1}(c\lambda e) + cr]$$

Thus, from (2.14) and the estimates above, we obtain

$$V(\Sigma_f) - V(A(\varepsilon, R)) \leq \frac{c_4}{(c\lambda)^2} [1 + \cosh^{-1}(c\lambda e) + ct - \ln c\lambda e].$$

Since $\lambda$ is fixed, clearly by choosing $c$ sufficiently large, this can be made smaller than $\omega_{n-1} \varepsilon^{n-1}$, for a given $\varepsilon$. Note that as $R \to \infty$ (for $\varepsilon$ fixed), $c \to \infty$ by (2.16). This establishes (2.11) and completes the proof. \(\square\)

In order to apply Theorem 2.1, one seeks conditions under which a complete Riemannian manifold $N$ has complete area-minimizing hypersurfaces. The following, Lemma, gives a sufficient condition in terms of the topology of $N$.

**Lemma 2.2.** Let $N$ be a complete Riemannian manifold with finitely generated homology $H_1(N, \mathbb{Z})$. Then any non-zero line $\mathbb{R} \cdot x, x \in H_1(N, \mathbb{Z})$ gives rise to a complete homologically area-minimizing hypersurface $M_x$, which is a boundary of least area in a cover $\mathcal{Z} \to \tilde{N} \to N$. The volume growth of $M$ satisfies

$$\operatorname{vol}(M \pitchfork B^\mathcal{Z}(r)) \leq \operatorname{vol} S^\mathcal{Z}(r)$$

and $M$ has non-zero intersection number with the lift of $x$ to $\tilde{N}$.

**Proof.** There is a projection $\pi_1(N) \to H_1(N, \mathbb{Z}) \to \mathbb{Z} = \langle x \rangle \subset H_1(N, \mathbb{Z})$. Let $\Gamma$ be the kernel of this projection, so $\pi_1(N)/\Gamma \simeq \langle x \rangle$. Let $\tilde{N}$ be the covering of $N$ with group $\mathbb{Z}$, so that $\mathbb{Z}$ acts on $\tilde{N}$ with quotient $N$. Also, let $L = \mathbb{R}$ be a lift of a representative $\sigma \in [x] \in H_1(N, \mathbb{Z})$ to $\tilde{N}$. Then $L$ is $\mathbb{Z}$-invariant and $L/\mathbb{Z} = \sigma \subset N$. We define $C(r) = \{ x \in \tilde{N} : \text{dist}(x, L) \leq r \}$ to be the $r$-tubular neighborhood of $L$ in $\tilde{N}$.

Frist, let $H_\varepsilon$ be a properly embedded, connected hypersurface in $C(r)$, $\partial H_\varepsilon \subset \partial C(r)$ compact, which disconnects $C(r)$ into (necessarily) two components $\Omega^+, \Omega^-$ and such that $l(H_\varepsilon, L) \neq 0$. We will construct such an $H_\varepsilon$ below. We now modify $\{ H_\varepsilon \}$ to obtain a complete area-minimizing hypersurface $M$ in $\tilde{N}$ with the required properties.

We may alter the metric near $\partial C(r)$, so that $\partial C(r)$ has positive mean curvature w.r.t. the normal pointing toward $L$. If $B_\varepsilon = \partial(H_\varepsilon)$, then $B_\varepsilon$ is closed integral $(n-2)$ current of compact support and we let $\Sigma_\varepsilon$ be the solution to the Plateau problem, i.e.
the least mass integral $(n-1)$ current with $\partial \Sigma_\varepsilon = B_\varepsilon$, in the homology class $H_\varepsilon$. (The mean curvature condition implies we may solve the Plateau problem in $C(r)$).
Let $U$ be the homology between $H_r$ and $\Sigma_r$, i.e. $\partial U = H_r - \Sigma_r$. Then $U$ is an open set in $N$ and since $H_r$ is connected, $\partial U$ is also connected. Since $H_r$ is of multiplicity one, so is $\partial U$, and thus each component of $U$ is of multiplicity $+1$ or $-1$. Note that $\Sigma_r$ is the boundary of an open set $V_r = \Omega^+ \setminus U$, with components of multiplicity $+1$ or $-1$. Further, we have $I(\Sigma_r, L) \neq 0$.

We may then obtain mass bounds on $\Sigma_r$ in the usual way. Namely, if $p \in \Sigma_r \cap L$, then for $B_p(s) \subset C(r)$, $\Sigma_r \leftarrow B_p(s) = \partial V_r \leftarrow B_p(s)$ is homologous to the slice $V_r \leftarrow \partial B_p(s)$, so that

$$V(\Sigma_r \leftarrow B_p(s)) \leq \text{vol} S_p(s). \tag{2.18}$$

Any component of the hypersurfaces $\Sigma_r$ may now be translated back to a fixed fundamental domain $F$ by an element $g \in \Gamma$, so that $M_r \equiv g \Sigma_r$ satisfies $I(M_r, L) \neq 0$, $M_r \cap F \cap L \neq \emptyset$. The mass bounds (2.18) hold for $M_r$ also. Then choose a sequence $\{r_i\} \to \infty$ and perform this construction for each $i$. The mass of the area-minimizing hypersurfaces $M_r$, remains uniformly bounded, by (2.18), so that by the compactness theorem for integral currents, a subsequence converges to a complete area-minimizing hypersurface $M$ satisfying

$$V(M \leftarrow B_p(s)) \leq \text{vol} S_p(s)$$

for all $s > 0$ and some $p' \in M \cap L$. To see that $I(M, L) \neq 0$, we note that the regular set $M_r'$ of $M_r$ converges smoothly to the regular set $M'$ of $M$. Since intersection $I(., .)$ is a homotopy invariant, one may perturb $L$ so that $L$ intersects $M_r$ and $M$ only in their regular sets. Since $I(M, L) \neq 0$, it follows that $I(M, L) \neq 0$, by the smooth convergence.

It remains to construct the hypersurfaces $H_r$ with the required properties. For this, we note that $C(r)$ is $\mathbb{Z}$-invariant, so that it is the lift of a domain $\Omega = \Omega_r \subset N$. Let $\omega$ be the closed 1-form corresponding to $[\sigma]$ in deRham cohomology, thus $\int_{\sigma'} \omega = 1$, $\int_{\theta} \omega = 0$, for all $\theta$ orthogonal to $\sigma$ in $H_1(\Omega, \mathbb{R})$. Then, as usual, $\omega$ defines a smooth map $f: \Omega \to S^1$ given by $f(x) = \int_{x'} \omega$. The lift of $f$ to $C(r)$ gives a map $g:C(r) \to \mathbb{R}$ and we may choose $H_r$ to be a regular level set $g^{-1}(s) \subset C(r)$. Since $C(r)$ is connected, it is clear that there exist connected regular sets $H_r$, and we have $I(H_r, L) \neq 0$.

§3. MAIN RESULTS

The results obtained in the previous sections may be combined to give the following:

**Theorem 3.1.** Let $N$ be a complete Riemannian manifold of positive Ricci curvature or of non-negative Ricci curvature and sectional curvature bounded above, $n = \dim N$. Then either both conditions below hold:

(1) The first Betti number $b_1(N)$ satisfies

$$b_1(N) \leq n - 3$$

and

(2) No finitely generated subgroup of $\pi_1(N)$ is of polynomial growth of order $\geq (n - 2)$. In particular, there is no subgroup isomorphic to $\mathbb{Z}^n$ in $\pi_1(N)$

or, there is a covering $\tilde{N}$ of $N$ with $H_1(\tilde{N}, \mathbb{R}) \neq 0$ such that every non-trivial line $\mathbb{R} \cdot x$, $x \in H_1(\tilde{N}, \mathbb{R})$ gives rise to an isometric splitting $\tilde{N} = \tilde{M} \times \mathbb{R}$ of the universal cover.

**Proof.** By the proof of Theorem 1.3 (more precisely the existence of a surjective homomorphism $\pi_1(N) \to \mathbb{Z}^{b_1(N)}$), we see that (2) $\Rightarrow$ (1). Thus, if (2) does not hold, there is a
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finitely generated group $\Gamma' \subset \pi_1(N)$ of polynomial growth of order $\geq n - 2$. We claim there is a finitely generated subgroup $\Gamma \subset \Gamma'$ of finite index in $\Gamma'$ such that the 1st Betti number $b_1(\Gamma) \neq 0$. If $\Gamma'$ is (almost) abelian, this is clear. Otherwise, we may use Gromov's polynomial growth theorem [9] to assert that $\Gamma'$ contains a finitely generated nilpotent subgroup $\Gamma''$ of finite index in $\Gamma'$. It is then an exercise in group theory (or see e.g. Cor. 4, Ch. 1 [18]) to conclude that $\Gamma''$ has a finitely generated subgroup $\Gamma$ of finite index with $b_1(\Gamma) \neq 0$.

Let $\tilde{N}$ be the covering of $N$ with $\pi_1(\tilde{N}) = \Gamma$. Note that $\Gamma$ has polynomial growth of order $\geq n - 2$. From (1.5) and Theorem 1.1 (applied to $\pi: \tilde{N} \to \tilde{N}$), it follows that there is a constant $c_6 > 0$ such that $v^\tilde{N}(r) \leq c_6 r^2$. Lemma 2.2 implies that there is an oriented boundary of least area in the $\mathbb{Z}$-fold covering of $\tilde{N}$ corresponding to $n \in H_1(M, \mathbb{R})$ and by Theorem 1.1, the covering space has volume growth of order at most three. The result then follows from Theorem 2.1.

Theorem 3.1 answers in part the conjecture of Schoen-Yau [17], as mentioned in the Introduction. In fact, this conjecture is now reduced to the Milnor conjecture that $\pi_1(N)$ is finitely generated.

Note that Theorem 3.1 is sharp in all dimensions in the sense that there are metrics of positive Ricci curvature on $N = T^{n-3} \times \mathbb{R}^3$, so that $\pi_1(N)$ is a free abelian group of rank $n - 3$, cf. [15], [2]. (It is easily verified that these metrics are of bounded geometry; in fact, there are quasi-isometric to a standard flat metric on $T^{n-3} \times \mathbb{R}^3$).

Under special circumstances, we are able to show that $\pi_1(N)$ is finitely generated.

**Corollary 3.2.** Let $N$ be a complete Riemannian manifold of non-negative Ricci curvature, $n = \dim N$, with volume growth $v(r) \geq c_n r^{n-k}$. If either $b_1(N) = k$ or $\pi_1(N)$ has a finitely generated quotient group $\pi_1(N)/\Gamma$ of polynomial growth of order $\geq k$, then $\pi_1(N)$ is finitely generated.

**Proof.** Suppose $\pi_1(N)/\Gamma$ is a finitely generated quotient group of polynomial growth of order $\geq k$. Let $\pi: \tilde{N} \to N$ be the covering with $\pi_1(\tilde{N}) = \Gamma$, so that the covering group is $\pi_1(N)/\Gamma$. We have a short exact sequence

$$0 \to \pi_1(\tilde{N}) \to \pi_1(N) \to \pi_1(N)/\pi_1(N) \to 0.$$ 

By Theorem 1.1, Prop. 1.2 and (1.5), $\pi_1(\tilde{N})$ is a finite group. Thus $\pi_1(N)$ is an extension of a finitely generated group by a finite group, and so finitely generated.

If $b_1(N) = k$ we choose $\tilde{N}$ to be the homology covering of $N$, i.e. $\pi_1(\tilde{N}) = [\pi_1(N), \pi_1(N)]$, with $\pi: \tilde{N} \to N$ having covering group $H_1(N, \mathbb{Z})$. Let $\mathbb{Z}^k \subset H_1(N, \mathbb{Z})$ be a maximal free abelian subgroup and consider the intermediate covering $N' = \tilde{N}/\mathbb{Z}^k$.

On the one hand, by Theorem 1.1, it is easily seen that $\tilde{N}$ has maximal volume growth, i.e. $v^{\tilde{N}}(r) \geq c_6 r^2$. By Prop. 1.2, $\pi_1(\tilde{N})$ is finite. Similarly, $N'$ has volume growth $v^{N'}(r) \leq c_6 r^{n-k}$, so that again the covering $\pi': \tilde{N}/\mathbb{Z}^k \to N' = \tilde{N}/H_1(N, \mathbb{Z})$ is a finite covering. It follows that $H_1(N, \mathbb{Z})$ is finitely generated. Since $\pi_1(N)$ is an extension of a finite group $\pi_1(\tilde{N})$ by $H_1(N, \mathbb{Z})$, it follows that $\pi_1(N)$ is finitely generated.

**REFERENCES**


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