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Coupled damage and plasticity models derived from energy and dissipation potentials

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Abstract

A theoretical framework is defined that allows plasticity and damage models of inelastic behaviour to be combined within a consistent approach. Much emphasis is placed on the fact that, within this framework, the entire constitutive response is specified through two potential functions, with no additional assumptions or evolution equations being necessary. Both plastic strain and damage parameter have roles as internal variables within the theory. Two classes of models are derived: involving respectively uncoupled and coupled plasticity and damage. Examples of application of the theory are presented.

Keywords: Continuum damage mechanics; Elastoplasticity; Hyperelasticity; Internal variables; Hyperplasticity; Geomaterials

1. Introduction

The inelastic behaviour of materials has been successfully modelled using two distinct approaches: plasticity and damage mechanics. Plasticity theory is very widely used in the modelling of ductile metals, and has also been successfully applied to geomaterials. It is based on the concept of additive elastic and plastic strains, the latter only occurring once a yield criterion is satisfied. Many authors have applied thermodynamic principles to plastic materials, and we have had considerable success in applying a method we term "hyperplasticity", which is rooted in thermodynamics, to derive plasticity theories (Houlsby and Puzrin, 2000). Continuum damage mechanics (CDM) was pioneered by Kachanov (1958). The damage of materials is the progressive process by which they break and thus lose strength and stiffness, and this process is represented in CDM by introducing a "damage internal variable". Damage theories are successfully used for modelling materials as diverse as polymers or brittle rocks. Whilst some approaches presented in the literature have a purely phenomenological

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basis, others have based the formulation of CDM taking into account thermodynamic principles. In this paper, we express CDM within the same framework as hyperplasticity, thus encompassing the two concepts of plasticity and damage within a single theory. It is shown that the entire constitutive knowledge of a model that undergoes plasticity and damage can be expressed through definition of two potentials. This allows the constitutive response to be derived directly in a way that ensures consistency with the laws of thermodynamics.

The "hyperplasticity" framework (Houlsby and Puzrin, 2000) allows the development of plasticity theories, within the framework of Generalized Thermodynamics, or Thermodynamics of Internal Variables (TIV), and has much in common with the work of Lubliner (1972), Halphen and Nguyen (1975), Ziegler (1977), and Maugin (1992). The roots of this work are found in that of Ziegler (1977) as developed by Houlsby (1981), Collins and Houlsby (1997), Houlsby and Puzrin (2000) and Einav (2002). A special feature of this approach is an emphasis on the fact that the entire constitutive response of a material can be derived from definition of only two potential functions: an energy potential and a dissipation potential.

It is demonstrated here that the hyperplasticity formulation can be used to develop a damage model, without plasticity. We call this a "damage hyperelastic model". The difference between this type of model and a plasticity model arises from the physical role the internal variables, which in turn derives from the functional nature of the potentials. Pure damage models have been presented by many authors including Krajcinovic (1983), Ortiz (1985), Kattan and Voyiadjis (1990), Maugin (1992) and Lemaitre (1992). For rate independent processes it is customary to assume (effectively) the existence of a yield surface for damage. Here instead the yield surface is derived from the assumed existence of a dissipation potential function, and the evolution of the damage internal variable is defined from further properties of the dissipation potential. In some models the evolution of damage is postulated as a separate evolution law: no such additional assumptions are necessary here.

Many damage models involve the use of an isotropic (scalar) measure of damage. Others have employed two scalar measures of damage for concrete in tension and compression (e.g. Mazars and Pijaudier-Cabot (1989), Fremond and Nedjar (1995), Lee and Fenves (1998), and Nguyen (2005) who used a separate measure of damage for concrete in tension and compression). However, there are good reasons, e.g. based on analysis of microscopic crack distributions, to make use of a tensorial damage variable (e.g., Ladeveze, 1983; Ju, 1989, 1990; Murakami and Kamiya, 1997). Most of those damage-plasticity approaches are based on stress criteria which do not usually give crack orientations in accordance with experimental data at the onset of damage. As a consequence, the direction of damage propagation may not be correct and the advantage of tensorial damage therefore can be lost. Furthermore, the calibration of such models against experimental data is not straightforward, and isotropic damage models are therefore often preferred for routine use. Our purpose here is to explore the combination of damage and plasticity theories, and so we deliberately keep the damage theory as simple as possible and use only isotropic damage.

It is also demonstrated how the concept of multiple surface hyperplasticity, proposed by Puzrin and Houlsby (2001), can allow description of models which undergo damage as well as plasticity (e.g. Lemaitre, 1985; Maugin, 1992; Hansen and Schreyer, 1994; Chaboche, 1997; Li, 1999). We term these "damage hyperplastic" models, and two classes of these are introduced. The first are *uncoupled* damage hyperplastic models in which damage and plasticity are independent processes, although the two processes can (under certain conditions) occur simultaneously. The second class are *coupled* damage hyperplastic models, in which damage and plasticity always occur simultaneously.

Even in the "uncoupled" models described below, plasticity and damage can on occasions occur simultaneously, and are implicitly linked at this stage. Alternative approaches have been made in the past to the coupling between plasticity and damage. Firstly the coupling can be implicitly embedded in the yield and damage criteria (Luccioni et al., 1996; Nguyen and Houlsby, 2004; Salari et al., 2004; Nguyen, 2005), with the material strength being a decreasing function with respect to the damage variable. This implicit coupling characterizes the strength reduction due to the material deterioration and is equivalent to introducing effective instead of nominal stress into the yield function (Lemaitre and Chaboche, 1990; Lemaitre, 1992). This way of introducing coupling enables the constitutive modelling to use separate yield and damage criteria, both of which can be derived from the dissipation function. The corresponding internal variables (damage variable and plastic strains for the coupled model) of the model do not explicitly depend on each other.

An alternative type of coupling has been used by others (Lemaitre, 1985; Lee and Fenves, 1998; Faria et al., 1998), in which only one loading function is specified and used to control the dissipation process. This function can be a damage loading function (Voyiadjis and Kattan, 1992; Faria et al., 1998) or a yield function (Lemaitre, 1985; Lee and Fenves, 1998). In the first case with a damage loading function governing the dissipation process, an evolution law for the plastic strain is required (Faria et al., 1998). For the use of a yield function, the damage measures, activated by a simple damage criterion, are expressed explicitly in terms of other internal variables controlling the plastic flow process. Many ad hoc assumptions are usually used during the formulation of such constitutive models (see Lee and Fenves, 1998; Faria et al., 1998). As an example, we explore the structure of the coupled damage-plasticity von Mises model by Lemaitre (1985), and suggest an alternative and more systematic form. While the energy potential takes a similar form to that used by Lemaitre, the dissipative mechanisms are expressed differently, yet in a clear and consistent manner.

In Section 2, we present briefly the general formulation, introducing multiple internal variables, without giving a specific interpretation to their physical meaning. In Section 3, we use this framework to develop two theories with a single internal variable: a simple hyperplastic model and a damage hyperelasticity model. It is emphasized that the two theories are expressed within the same framework. In Section 4, we use the general formulation to derive damage hyperplasticity constitutive relations with two internal variables, damage and plastic strain. Both uncoupled and coupled damage hyperplasticity models are explored.

2. Theoretical background

2.1. Notation

This paper concerns processes that can be idealized as rate independent, isothermal and undergoing only small strain deformations. It is convenient to employ second order tensors, indicated by bold face $\boldsymbol{a}$. The inner product between two second order tensors is denoted by $\boldsymbol{a}:\boldsymbol{b}$. The tensor $\mathbf{1}$ denotes the second order identity tensor, i.e. $(\mathbf{1})_{ij} = \delta_{ij}$, also called the Kronecker delta. The sign '$\sim$' over a variable $\tilde{a}$ is added if its dimension is left undefined: it could be a tensor, vector or scalar, according to its physical interpretation. The sign "$\bullet$" denotes the inner product between two variables with the same unknown dimension, e.g. $\tilde{a} \bullet \tilde{b}$. The superposed dot over a variable $\dot{\tilde{a}}$ denotes a rate of change.

2.2. Formulation

The following description summarizes the approach for deriving thermomechanical models. The formulation of Houlsby and Puzrin (2000), which we follow here, was originally applied to derive elastoplastic models that we term "hyperplastic". In this paper we demonstrate how the same approach could be used to derive continuum damage models. It is convenient to present the theoretical concepts in terms of an arbitrary number of internal variables, thus allowing the framework to encompass the two concepts of plasticity and damage within a single theory. The local state of the material is assumed to be completely defined by knowledge of (a) strain tensor $\boldsymbol{\varepsilon}$, (b) a set of N internal variables $\tilde{\alpha}_i$, $\mathscr{A} \equiv \mathscr{A}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N)$, (c) the entropy, although this does not enter the formulation for the isothermal case.

The First Law of Thermodynamics effectively states that there is a function of the state, called the internal energy. In isothermal conditions this function can be replaced by the Helmholtz free energy $f = f(\boldsymbol{\varepsilon}, \mathscr{A})$, which depends only on the kinematic state variables. Alternatively, a Legendre transform can be made to express a Gibbs free energy $g = g(\boldsymbol{\sigma}, \mathscr{A})$, where $\boldsymbol{\sigma}$ is the Cauchy stress tensor. The two energies are related by

$$g(\boldsymbol{\sigma}, \mathscr{A}) = f(\boldsymbol{\varepsilon}, \mathscr{A}) - \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \tag{1}$$

The formalism of Houlsby and Puzrin (2000) requires that, if f is defined

$$\boldsymbol{\sigma} = \frac{\partial f}{\partial \boldsymbol{\varepsilon}} \tag{2}$$

or alternatively if g is defined

$$\boldsymbol{\varepsilon} = -\frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{3}$$

We define the generalized stress as

$$\bar{\tilde{\chi}}_i = -\frac{\partial f}{\partial \tilde{\alpha}_i} = -\frac{\partial g}{\partial \tilde{\alpha}_i} \tag{4a, b}$$

We next assume that the mechanical dissipation d is a strictly non-negative function of both the state of the material and of the rate of change of the internal variables $d = d^f(\boldsymbol{\varepsilon}, \mathscr{A}, \dot{\mathscr{A}}) = d^g(\boldsymbol{\sigma}, \mathscr{A}, \dot{\mathscr{A}}) \geqslant 0$. In the case of rate-independent processes the dissipation function is homogeneous first order function of the internal variable rates. When the internal variable set consists of a single internal variable, this homogeneity can be expressed by Euler's equation $d = \partial d / \partial \dot{\tilde{\alpha}} \bullet \dot{\tilde{\alpha}}$. For N internal variables, we write

$$d = \sum_{i=1}^{N} \frac{\partial d}{\partial \dot{\tilde{\alpha}}_i} \bullet \dot{\tilde{\alpha}}_i = \sum_{i=1}^{N} \tilde{\chi}_i \bullet \dot{\tilde{\alpha}}_i \tag{5}$$

where $\tilde{\chi}_i = \partial d / \partial \dot{\tilde{\alpha}}_i$ is termed the dissipative generalized stress.

The formulation is completed by adopting Ziegler's orthogonality condition, which simply takes the form $\bar{\tilde{\chi}}_i = \tilde{\chi}_i$ for any $i \in [1, N]$. Although $\bar{\tilde{\chi}}_i = \tilde{\chi}_i$, for formal purposes it is convenient to keep $\tilde{\chi}_i$ and $\bar{\tilde{\chi}}_i$ as separate variables.

To develop rate independent thermomechanical models, Puzrin and Houlsby (2001) suggested a decoupled form of dissipation function that is appropriate for multiple surface kinematic hardening plasticity models. Section 2.2.1 presents some outcomes resulting from this form of dissipation. In the current work it is shown how this formulation could be used for developing plasticity models that also introduce damage. Adopting decoupled dissipation for plasticity-damage models may result in damage prior to plastic straining, or plastic straining prior to damage.

However, another form of the dissipation potential, this time coupled in its terms, is suggested in Section 2.2.2. This class of potentials defines models that introduce damage whenever plasticity occurs and vice versa. It is shown that the uncoupled theory of Section 2.2.1 corresponds to a singular case of the coupled theory. Moreover, while the coupled theory introduces a single yield surface in the higher-dimensional space $\mathscr{A}$, the uncoupled theory is linked to N individual yield surfaces.

2.2.1. Decoupled dissipation and discrete field of yield surfaces

Puzrin and Houlsby (2001) suggest the following decoupled form of dissipation function:

$$d = \sum_{i=1}^{N} d_i^f(\boldsymbol{\varepsilon}, \mathscr{A}, \dot{\tilde{\alpha}}_i) = \sum_{i=1}^{N} d_i^g(\boldsymbol{\sigma}, \mathscr{A}, \dot{\tilde{\alpha}}_i) \geqslant 0 \tag{6}$$

Here, the internal variables could be either the plastic strain or damage. With the condition that each of the dissipation terms must be non-negative

$$d_i^e = \frac{\partial d_i^e}{\partial \dot{\tilde{\alpha}}_i} \bullet \dot{\tilde{\alpha}}_i = \tilde{\chi}_i \bullet \dot{\tilde{\alpha}}_i \geqslant 0 \quad \forall i \in [1, N] \tag{7}$$

where the superscript e denotes either f or g.

In the formulation of multiple surface models, the incremental stress–strain response requires the functional form of the yield surfaces to be defined. The ith yield surface, associated with the evolution of the ith internal variable $\tilde{\alpha}_i$, is related to the ith component of dissipation function given in (6). The relationship is given by a degenerate special case of the Legendre transformation because the dissipation is homogeneous and first order in the rates (Collins and Houlsby, 1997)

$$\lambda_i y_i^e = \tilde{\chi}_i \bullet \dot{\tilde{\alpha}}_i - d_i^e = 0 \quad \forall i \in [1, N] \tag{8}$$

where $y_i^f = y_i^f(\boldsymbol{\varepsilon}, \mathscr{A}, \tilde{\chi}_i)$ and $y_i^g = y_i^g(\boldsymbol{\sigma}, \mathscr{A}, \tilde{\chi}_i)$ are the ith yield function in ith generalized stress space in f-form and g-form and λ_i is a non-negative multiplier in the same space. The properties of the degenerate Legendre transform require the flow rules

$$\dot{\tilde{\alpha}}_i = \lambda_i \frac{\partial y_i^e}{\partial \tilde{\chi}_i} \tag{9}$$

Since $\lambda_i \geqslant 0$ and $\lambda_i y_i^e = 0$ from (8), the Kuhn–Tucker complementary conditions are completed by requiring $y_i^e \leqslant 0$. The condition $y_i^e = 0$ represents the ith yield function. If this condition is met, and since y_i^e cannot be larger than zero, the consistency condition becomes

$$\dot{y}_i^f = \frac{\partial y_i^f}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \sum_{j=1}^{N} \frac{\partial y_i^f}{\partial \tilde{\alpha}_j} \bullet \dot{\tilde{\alpha}}_j + \frac{\partial y_i^f}{\partial \tilde{\chi}_i} \bullet \dot{\tilde{\chi}}_i = 0 \tag{10a}$$

$$\dot{y}_i^g = \frac{\partial y_i^g}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \sum_{j=1}^{N} \frac{\partial y_i^g}{\partial \tilde{\alpha}_j} \bullet \dot{\tilde{\alpha}}_j + \frac{\partial y_i^g}{\partial \tilde{\chi}_i} \bullet \dot{\tilde{\chi}}_i = 0 \tag{10b}$$

The summation of the $\partial y_i^e / \partial \tilde{\alpha}_j \bullet \dot{\tilde{\alpha}}_j$ terms represents a possible coupling of the evolution of the ith yield surface with the evolution of the jth internal variable. This coupling is weak since it happens only when both the ith and the jth yield surfaces are active. Each surface must be checked independently for loading or unloading.

2.2.2. Coupled dissipation and yield surface

Consider alternatively a dissipation function that cannot be decomposed into additive terms, as given by the summation operator in Eq. (6), so that we simply write

$$d = d^f(\boldsymbol{\varepsilon}, \mathscr{A}, \dot{\mathscr{A}}) = d^g(\boldsymbol{\sigma}, \mathscr{A}, \dot{\mathscr{A}}) \geqslant 0 \tag{11}$$

In that case, the dissipation potential is related to a single yield function by a single degenerate special case of the Legendre transformation in the form

$$\lambda y^e = \sum_{i=1}^{N} \tilde{\chi}_i \bullet \dot{\tilde{\alpha}}_i - d^e = 0 \tag{12}$$

where $y^f = y^f(\boldsymbol{\varepsilon}, \mathscr{A}, \mathscr{B})$, $y^g = y^g(\boldsymbol{\sigma}, \mathscr{A}, \mathscr{B})$ and $\mathscr{B}$ denotes the set of dissipative stresses $\mathscr{B} \equiv \mathscr{B}(\tilde{\chi}_1, \ldots, \tilde{\chi}_N)$, where the ith in the set is given by

$$\tilde{\chi}_i = \frac{\partial d^e}{\partial \dot{\tilde{\alpha}}_i} \tag{13}$$

Unlike the uncoupled theory, in this case the yield function must be expressed in N-dimensional generalized dissipative stress $\mathscr{B}$ space. However, this yield surface is linked to N flow rules, all containing a common non-negative multiplier λ, in the form

$$\dot{\tilde{\alpha}}_i = \lambda \frac{\partial y^e}{\partial \tilde{\chi}_i}, \quad \forall i \in [1, N] \tag{14}$$

Since $\lambda \geqslant 0$ and $\lambda y^e = 0$ from (8), the Kuhn–Tucker complementary conditions are completed by requiring $y^e \leqslant 0$, while $y^e = 0$ denotes a single yield surface. The above introduces a strong coupling between internal variables, as all of them evolve if yielding occurs. If $y^e = 0$ and $\lambda > 0$ only a single consistency condition is introduced by

$$\dot{y}^f = \frac{\partial y^f}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \sum_{i=1}^{N} \frac{\partial y^f}{\partial \tilde{\alpha}_i} \bullet \dot{\tilde{\alpha}}_i + \sum_{i=1}^{N} \frac{\partial y^f}{\partial \tilde{\chi}_i} \bullet \dot{\tilde{\chi}}_i = 0 \tag{15a}$$

$$\dot{y}^g = \frac{\partial y^g}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \sum_{i=1}^{N} \frac{\partial y^g}{\partial \tilde{\alpha}_i} \bullet \dot{\tilde{\alpha}}_i + \sum_{i=1}^{N} \frac{\partial y^g}{\partial \tilde{\chi}_i} \bullet \dot{\tilde{\chi}}_i = 0 \tag{15b}$$

The summation over the $\partial y^e / \partial \tilde{\alpha}_i \bullet \dot{\tilde{\alpha}}_i$ terms represents the same possible weak coupling between internal variables as in the uncoupled theory. This time, however, a stronger coupling is also introduced by the summation over the $\partial y^e / \partial \tilde{\chi}_i \bullet \dot{\tilde{\chi}}_i$ terms.

For example, consider a dissipation function

$$d^f = \sqrt[n]{\sum_{i=1}^{N} \left[c_i^f(\boldsymbol{\varepsilon}, \mathscr{A}) \Phi_i(\dot{\tilde{\alpha}}_i) \right]^n} \tag{16}$$

where $c_i^f(\boldsymbol{\varepsilon}, \mathscr{A})$ is a positive definite function and $\Phi_i(\dot{\tilde{\alpha}}_i)$ is a homogeneous first order function operator returning a positive scalar. For convenience we also require that the derivative $\Phi_i'(\dot{\tilde{\alpha}}_i) = \partial\Phi_i(\dot{\tilde{\alpha}}_i)/\partial\dot{\tilde{\alpha}}_i$ satisfies the condition $\Phi_i'(\dot{\tilde{\alpha}}_i) \bullet \Phi_i'(\dot{\tilde{\alpha}}_i) = 1$. For instance, if $\dot{\tilde{\alpha}}_i = \dot{\alpha}_i$ is a scalar, a possible operator could be the absolute function $\Phi(\dot{\alpha}_i) = |\dot{\alpha}_i|$, then the inner product sign '•' denotes simple multiplication between two scalars and the differential is just $\text{sgn}(\dot{\alpha}_i)$. If $\dot{\tilde{\alpha}}_i = \dot{\boldsymbol{\alpha}}_i$ is a second order tensor, a possible operator could be the norm of the tensor $\Phi(\dot{\boldsymbol{\alpha}}_i) = \sqrt{\dot{\boldsymbol{\alpha}}_i : \dot{\boldsymbol{\alpha}}_i}$. The inner product sign in this case corresponds to ':' between two second order tensors and the differential is $\dot{\boldsymbol{\alpha}}_i/\Phi(\dot{\boldsymbol{\alpha}}_i)$. Another possible operator on second order tensors could be related to the first invariant of the tensor $\dot{\boldsymbol{\alpha}}_i$ by $\Phi(\dot{\boldsymbol{\alpha}}_i) = |\dot{\boldsymbol{\alpha}}_i : \mathbf{1}/3|$. This time, the inner product sign denotes simple multiplication between two scalars and the differential is $\text{sgn}(\dot{\boldsymbol{\alpha}}_i : \mathbf{1})$.

From the definition of the ith dissipative generalized stress in (13), we have

$$\frac{\tilde{\chi}_i}{c_i^f(\boldsymbol{\varepsilon}, \mathscr{A})} = \frac{\left[c_i^f(\boldsymbol{\varepsilon}, \mathscr{A})\Phi_i(\dot{\tilde{\alpha}}_i)\right]^{n-1}\Phi_i'(\dot{\tilde{\alpha}}_i)}{\left(\sum_{i=1}^N\left[c_i^f(\boldsymbol{\varepsilon}, \mathscr{A})\Phi_i(\dot{\tilde{\alpha}}_i)\right]^n\right)^{\frac{n-1}{n}}} \tag{17}$$

The yield function can then be obtained, and is given by

$$y^f = \sum_{i=1}^N\left(\frac{\tilde{\chi}_i \bullet \tilde{\chi}_i}{c_i^f(\boldsymbol{\varepsilon}, \mathscr{A})^2}\right)^{\frac{n}{2(n-1)}} - 1 \leqslant 0 \tag{18}$$

In this theory, n could be identified as a parameter that controls the coupling intensity. The case $n = 1$ is a singular case, and the dissipation in Eq. (16) becomes uncoupled as described in Eq. (6).

Using the same arguments, the following dissipation and yield surface could be expressed for g-form potentials:

$$d^g = \sqrt[n]{\sum_{i=1}^N\left[c_i^g(\boldsymbol{\sigma}, \mathscr{A})\Phi_i(\dot{\tilde{\alpha}}_i)\right]^n} \tag{19}$$

$$y^g = \sum_{i=1}^N\left(\frac{\tilde{\chi}_i \bullet \tilde{\chi}_i}{c_i^g(\boldsymbol{\sigma}, \mathscr{A})^2}\right)^{\frac{n}{2(n-1)}} - 1 \leqslant 0 \tag{20}$$

The yield function in (20) is described in N-dimensional dissipative generalized stress space, but could be depicted in conventional Cauchy stress space by substituting orthogonality in the form $\bar{\tilde{\chi}}_i = \tilde{\chi}_i$ and Eq. (4b).

3. Single internal variable models

When the formulation employs only a single internal variable, hyperplasticity models (without damage), or damage hyperelastic models (without plasticity) can be developed. In these cases a single internal variable is used and the subscript 'i' omitted from $\tilde{\alpha}_i$. The dimension of $\tilde{\alpha}_i$ is defined for each specific theory and the tilde '$\sim$' removed; bold face indicates if the variable is a tensor.

3.1. Hyperplasticity theory

So far no particular emphasis has been placed on the interpretation of the internal variables. It is readily shown that their physical role is related to the functional form of the energy and dissipation expressions chosen. Collins and Houlsby (1997) showed for instance that if the Gibbs free energy can be decomposed in the following form:

$$g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_{\text{p}}) = g_1(\boldsymbol{\sigma}) - \boldsymbol{\sigma} : \boldsymbol{\alpha}_{\text{p}} + g_2(\boldsymbol{\alpha}_{\text{p}}) \tag{21}$$

then the strain is immediately derived as

$$\boldsymbol{\varepsilon} = -\frac{\partial g_1(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}} + \boldsymbol{\alpha}_{\text{p}} \tag{22}$$

So that it is seen that $\boldsymbol{\alpha}_{\mathrm{p}}$ plays the role of plastic strain. Furthermore, in such a model the elastic properties do not depend on plastic strains, so that the elasticity is said to be uncoupled. It is readily shown that the corresponding form of the Helmholtz free energy is

$$f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_{\mathrm{p}}) = f_1(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}_{\mathrm{p}}) + g_2(\boldsymbol{\alpha}_{\mathrm{p}}) = f_1(\boldsymbol{\varepsilon}_e) + g_2(\boldsymbol{\alpha}_{\mathrm{p}}) \tag{23}$$

From the definition of the generalized stress (4), in the case of potentials (21)

$$\bar{\boldsymbol{\chi}}_{\mathrm{p}} = \boldsymbol{\sigma} - \frac{\partial g_2(\boldsymbol{\alpha}_{\mathrm{p}})}{\partial \boldsymbol{\alpha}_{\mathrm{p}}} \tag{24}$$

where $\bar{\boldsymbol{\chi}}_{\mathrm{p}}$ differs from the true stress by the term $-\partial g_2/\partial \boldsymbol{\alpha}_{\mathrm{p}}$, which corresponds to the "back stress" in kinematic hardening plasticity. Moreover, it can also be observed that

$$\boldsymbol{\sigma} = \frac{\partial f_1(\boldsymbol{\varepsilon}_e)}{\partial \boldsymbol{\varepsilon}_e} \tag{25}$$

so that for energies in the form of (23), the stress definition is in the same form as for hyperelasticity. However, since in elasticity there are no plastic strains, in that case (23) degenerates to

$$f(\boldsymbol{\varepsilon}) = f_1(\boldsymbol{\varepsilon}) \tag{26}$$

Examples of this kind of model can be found in Houlsby and Puzrin (2000).

3.2. Damage hyperelasticity

We now demonstrate how, with a different choice of functional forms, the internal variable can instead play the role of damage variable.

3.2.1. Damage internal variable

The scalar damage concept was first introduced by Kachanov (1958) in the form of the phenomenologically based "effective stress" concept (see for example Lemaitre and Chaboche, 1978; Simo and Ju, 1987; Lemaitre, 1992). Alternatively, models can be based on the "effective strain" concept by Cordebois and Sidoroff (1982) and Simo and Ju (1987). In either case the damage variable is a scalar (starting from 0 and increasing to a maximum 1) is defined by

$$\alpha_{\mathrm{d}} = (A - A_{\mathrm{s}})/A \tag{27}$$

where A is the total cross-section area of a surface within the unit cell in one of the three perpendicular directions; A_{s} is the solid matrix area within A. We use the notation α_{d} rather than the common D in order to emphasize that this is an internal variable, and as such can used in the above formulation without conceptual changes.

Using the *hypothesis of strain equivalence* (Lemaitre, 1971), and definition (27), the relation between the macroscopic continuum mechanics stress $\boldsymbol{\sigma}$ and the corresponding "effective" stress could be found as

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1 - \alpha_{\mathrm{d}}) \tag{28}$$

If we consider a free energy in the following form:

$$f(\boldsymbol{\varepsilon}, \alpha_{\mathrm{d}}) = f_1(\boldsymbol{\varepsilon})(1 - \alpha_{\mathrm{d}}) \tag{29}$$

we can derive immediately

$$\boldsymbol{\sigma} = \frac{\partial f_1(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}(1 - \alpha_{\mathrm{d}}) \tag{30}$$

which can be seen to correspond to the formulation of Lemaitre (1971) if we identify the effective stress as

$$\bar{\boldsymbol{\sigma}} = \frac{\partial f_1(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \tag{31}$$

Using Legendre transformation (1) gives the Gibbs free energy

$$g(\boldsymbol{\sigma}, \alpha_d) = f_1(\boldsymbol{\varepsilon}(\bar{\boldsymbol{\sigma}}))(1 - \alpha_d) - \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\bar{\boldsymbol{\sigma}}) \tag{32}$$

When $\boldsymbol{\varepsilon}(\bar{\boldsymbol{\sigma}})$ is linear in $\bar{\boldsymbol{\sigma}}$, $f_1(\boldsymbol{\varepsilon})$ is quadratic in $\boldsymbol{\varepsilon}$ (and $\bar{\boldsymbol{\sigma}}$) and the Gibbs potential has the structure

$$g(\boldsymbol{\sigma}, \alpha_d) = \frac{g_1(\boldsymbol{\sigma})}{(1 - \alpha_d)} \tag{33}$$

while $g_1(\boldsymbol{\sigma})$ is the Gibbs free energy if no damage occurs.

The *hypothesis of stress equivalence* (Cordebois and Sidoroff, 1982; Simo and Ju, 1987) presents the inverse view to the former equivalence hypothesis. Although the hypothesis of strain equivalence is more widely used, the hypothesis of stress equivalence has the advantage of being related to stress space, thus more naturally combined with plasticity models. Using this hypothesis, and definition (27), the inverse stress space could be followed. In which case we start from the effective strain definition

$$\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(1 - \alpha_d) \tag{34}$$

relating between the macroscopic continuum mechanics strain $\boldsymbol{\varepsilon}$ and the corresponding effective strain $\bar{\boldsymbol{\varepsilon}}$. Differentiating (33) we obtain

$$\boldsymbol{\varepsilon} = -\frac{\partial g_1(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \frac{1}{(1 - \alpha_d)} \tag{35}$$

which is equivalent to the Simo and Ju (1987) formulation if we identify the effective strain as

$$\bar{\boldsymbol{\varepsilon}} = -\frac{\partial g_1(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \tag{36}$$

The transformed form of $f(\boldsymbol{\varepsilon}, \alpha_d) = f_1(\boldsymbol{\varepsilon})(1 - \alpha_d)$ again only applies if the material is linear elastic.

As observed by Simo and Ju (1987), and supported in the above, this means that the strain equivalence is naturally associated with strain-based formulation and the stress equivalence corresponds to a stress-spaced formulation. As shown here, they produce the same results only in linear elastic materials, in which case α_d has the same constitutive meaning in both cases. In the case of non-linear elasticity, each should be viewed according to the related space. However, in any case they rely on the same ideal microscopic idealisation. Li (2000) explores different microscopic expressions and other (energy based) equivalence principles, which result in, as before, different meanings for the internal damage variable α_d. In general, those cases introduce different factors instead of $(1 - \alpha_d)$ (say instead some monotonically decreasing function $M(\alpha_d)$ from 1 to 0). However, the energy potentials may be given by one of the forms

$$f(\boldsymbol{\varepsilon}, \alpha_d) = f_1(\boldsymbol{\varepsilon}) M(\alpha_d) \tag{37a}$$

$$g(\boldsymbol{\sigma}, \alpha_d) = \frac{g_1(\boldsymbol{\sigma})}{M(\alpha_d)} \tag{37b}$$

for the hypothesizes of strain and stress equivalence respectively, but with a possible change in meaning of the internal variables.

The use of $(1 - \alpha_d)$ or $M(\alpha_d)$ is of minor importance in this paper. The important consideration is that we have set a consistent way to develop damage elastic models, which can be completely derived from the explicit form of only two potential functions. In the same way, it could be demonstrated how anisotropic damage internal variables can be introduced to the formulation, simply by varying the structure of the energy potential functions. Ju (1990) highlighted the limitations and implications when the damage variable is taken as scalar and suggested a fourth order tensorial form of the damage internal variable if microcracks and/or microvoids are not spatially perfectly randomly distributed in all directions. Many authors prefer to develop anisotropic damage models using second order, or fourth order damage tensors (Yazdani and Schreyer, 1988; Kattan and Voyiadjis, 1990; Hansen and Schreyer, 1994; Li, 1999). If those measures agree with the laws of thermodynamics they could be easily incorporated here.

3.2.2. Damage generalized stress

Each of damage internal variable is associated with a stress dual, called the damage generalized stress. When the damage internal variable is scalar, then according to Eqs. (4a) and (37a), this is given by

$$\bar{\chi}_{\rm d} = -f_1(\boldsymbol{\varepsilon})M'(\alpha_{\rm d}) \tag{38}$$

Alternatively, according to Eqs. (4b) and (37b), the damage generalized stress is given by

$$\bar{\chi}_{\rm d} = g_1(\boldsymbol{\sigma})\frac{M'(\alpha_{\rm d})}{M(\alpha_{\rm d})^2} \tag{39}$$

Eqs. (38) and (39) show that the damage generalized "stress" in fact has the dimension of energy, as the damage internal variable is dimensionless. The two "stress" definitions are of course equivalent only when the meanings of the damage internal variables are the same.

For any monotonically decreasing function $M(\alpha_{\rm d})$ in $\alpha_{\rm d}$, $M'(\alpha_{\rm d})$ is always negative, and since $f_1(\boldsymbol{\varepsilon})$ in (38) corresponds to the undamaged elastic stored energy, which is by definition non-negative, $\bar{\chi}_{\rm d} \geqslant 0$ is also non-negative. For example, when $M(\alpha_{\rm d}) = (1 - \alpha_{\rm d})$, $M'(\alpha_{\rm d}) = -1$, thus giving $\bar{\chi}_{\rm d} = f_1(\boldsymbol{\sigma})$ as the undamaged elastic stored energy. The same result could be shown using Eq. (39).

3.2.3. Damage evolution

The model is completed by specifying either d^e (a function of the rate of the damage internal variable), or alternatively by specifying the damage yield surface (as a function of the damage generalised stress). If the latter is used the damage evolution equation is expressed using the flow rule (either Eqs. (9) or (14) for a single internal variable)

$$\dot{\alpha}_{\rm d} = \lambda\frac{\partial y^f(\boldsymbol{\varepsilon}, \chi_{\rm d}, \alpha_{\rm d})}{\partial \chi_{\rm d}} \tag{40a}$$

$$\dot{\alpha}_{\rm d} = \lambda\frac{\partial y^g(\boldsymbol{\sigma}, \chi_{\rm d}, \alpha_{\rm d})}{\partial \chi_{\rm d}} \tag{40b}$$

which are equivalent again only when the interpretation of the damage internal variable is the same.

Rewriting Eq. (5) in the notation of damage hyperelasticity

$$d^e = \chi_{\rm d}\dot{\alpha}_{\rm d} \tag{41}$$

Since $\chi_{\rm d} = \bar{\chi}_{\rm d}$ then for monotonically decreasing $M(\alpha_{\rm d})$ the damage internal variable rate would always be non-negative, i.e. $\dot{\alpha}_{\rm d} \geqslant 0$, provided that the proposed explicit dissipation is indeed non-negative $d^e = d^f(\boldsymbol{\varepsilon}, \alpha_{\rm d}, \dot{\alpha}_{\rm d}) = d^g(\boldsymbol{\sigma}, \alpha_{\rm d}, \dot{\alpha}_{\rm d}) \geqslant 0$. If $M(\alpha_{\rm d})$ is not monotonically decreasing it is possible that $\dot{\alpha}_{\rm d} < 0$. Thus, $M(\alpha_{\rm d})$ should be required to be a monotonically decreasing function if the damage internal variable $\alpha_{\rm d}$ is always to grow with time.

3.3. Example of one-dimensional damage hyperelastic model

Consider a model given by one of

$$f(\varepsilon, \alpha_{\rm d}) = \frac{E\varepsilon^2(1 - \alpha_{\rm d})}{2} \tag{42a}$$

$$g(\sigma, \alpha_{\rm d}) = -\frac{\sigma^2}{2E(1 - \alpha_{\rm d})} \tag{42b}$$

Together with the appropriate one of

$$d^f = \frac{k\Pi(\alpha_{\rm d})|\varepsilon||\dot{\alpha}_{\rm d}|}{2(1 - \alpha_{\rm d})} \geqslant 0 \tag{43a}$$

$$d^g = \frac{k\Pi(\alpha_{\rm d})|\sigma|}{2E(1 - \alpha_{\rm d})^2}|\dot{\alpha}_{\rm d}| \geqslant 0 \tag{43b}$$

$\Pi(\alpha_d)$ is a positive definite function describing the changes in dissipation as the material is damaged, satisfying $\Pi(\alpha_d = 0) \equiv 1$; we note that α_d in f-form and g-form has the same meaning as f is quadratic in ε; thus, using Eqs. (4) and (42a,b)

$$\bar{\chi}_d = \frac{E\varepsilon^2}{2} = \frac{\sigma^2}{2E(1-\alpha_d)^2} \tag{44a, b}$$

is always positive, and corresponds to the undamaged energy. The degenerate Legendre transformation of (43a,b) into the yield function gives

$$y^f = |\chi_d| - \frac{k\Pi(\alpha_d)|\varepsilon|}{2(1-\alpha_d)} \leqslant 0 \tag{45a}$$

$$y^g = |\chi_d| - \frac{k\Pi(\alpha_d)|\sigma|}{2E(1-\alpha_d)^2} \leqslant 0 \tag{45b}$$

Using the orthogonality condition in the form $\chi_d = \bar{\chi}_d$, and Eqs. (44a,b), the yield functions in strain and stress spaces are recovered

$$y^\varepsilon = E|\varepsilon|(1-\alpha_d) - k\Pi(\alpha_d) \leqslant 0 \tag{46a}$$

$$y^\sigma = |\sigma| - k\Pi(\alpha_d) \leqslant 0 \tag{46b}$$

During yielding, the condition $y^e = 0$, results in the pair of equations in which stress and strain are expressed parametrically in terms of α_d

$$\sigma_y(\alpha_d) = \pm k\Pi(\alpha_d) \tag{47a}$$

$$\varepsilon_y(\alpha_d) = \pm\frac{k\Pi(\alpha_d)}{E(1-\alpha_d)} \tag{47b}$$

where the subscript y was added to denote that these equations are followed only during yield. The determination of the function $\Pi(\alpha_d)$ should be based on these parametric equations.

Criterion (45a,b) matches to the ideal elasto-plastic criterion if $\Pi(\alpha_d) = 1$. In that case, the energy threshold will be constant k upon yielding. If $\Pi(\alpha_d) > 1$ then during loading the stress will exceed the value k; If

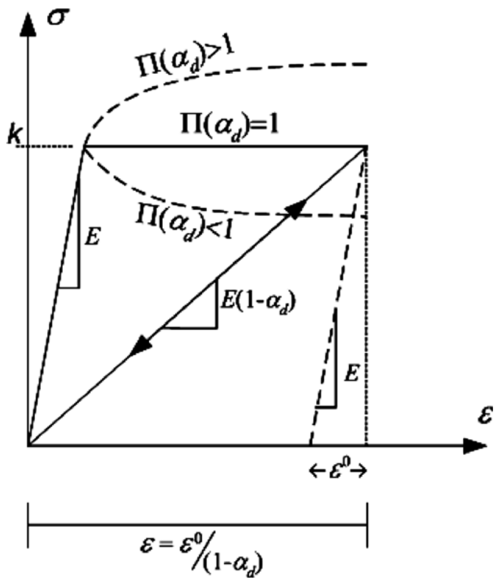

Fig. 1. Stress–strain curves of one-dimensional damage hyperelasticity model in ideal, hardening and softening conditions (reloading demonstrated in the ideal case).

$\Pi(\alpha_d) < 1$ the stress will go below the value k while yielding. These options are described by the three stress–strain curves given in Fig. 1. In this model, the reloading curve returns to initial conditions since no residual plastic strain is introduced. This fact is demonstrated in the figure only for the ideal case of $\Pi(\alpha_d) = 1$.

The instantaneous elastic modulus (damaged modulus) in this model is varying upon damage. Using Eqs. (2) and (42a) gives

$$\sigma = E(1 - \alpha_d)\varepsilon \tag{48}$$

4. Damage hyperplastic models

When the thermomechanical formulation, described in Section 2, depends on two internal variables, one identified as the plastic strain and the other as the damage variable, damage hyperplastic models can be developed. We present two options for modelling damage hyperplastic materials. The general structure of the energy potentials of the two theories is the same. Following Eqs. (23) and (29), the Helmholtz free energy potential is given by

$$f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_p) = f_1(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}_p)(1 - \alpha_d) + f_2(\boldsymbol{\alpha}_p) \tag{49}$$

Following Eqs. (21) and (33), the Gibbs free energy potential is given by:

$$g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d) = \frac{g_1(\boldsymbol{\sigma})}{(1 - \alpha_d)} - \boldsymbol{\sigma} : \boldsymbol{\alpha}_p + g_2(\boldsymbol{\alpha}_p) \tag{50}$$

As discussed before, the two approaches produce the same results in the case of linear elasticity (i.e. f_1 and g_1 are quadratic functions).

The difference between the two theories comes from the different structure of the dissipation potential. Corresponding to Section 3, the first option is defined using *decoupled dissipation* function either in *f*-form

$$d_d^f = d_d^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d, \dot{\alpha}_d) + d_p^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d, \dot{\boldsymbol{\alpha}}_p) \geqslant 0 \tag{51a}$$

or in *g*-form

$$d_d^g = d_d^g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d, \dot{\alpha}_d) + d_p^g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d, \dot{\boldsymbol{\alpha}}_p) \geqslant 0 \tag{51b}$$

Using the degenerate Legendre transformation gives yield functions in generalized stress space (damage yield function in damage generalized stress space χ_d and plastic yield function in plastic generalized stress space $\boldsymbol{\chi}_p$), either given in *f*-form

$$y_d^f = y_d^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d, \chi_d) \leqslant 0 \tag{52a}$$

$$y_p^f = y_p^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d, \boldsymbol{\chi}_p) \leqslant 0 \tag{53a}$$

or alternatively in *g*-form

$$y_d^g = y_d^g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d, \chi_d) \leqslant 0 \tag{52b}$$

$$y_p^g = y_p^g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d, \boldsymbol{\chi}_p) \leqslant 0 \tag{53b}$$

Corresponding to Section 3, the second option could be defined using *coupled dissipation* function, given respectively by the *f*-form and *g*-form functions

$$d^f = d^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d, \dot{\boldsymbol{\alpha}}_p, \dot{\alpha}_d) \geqslant 0 \tag{54a}$$

$$d^g = d^g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d, \dot{\boldsymbol{\alpha}}_p, \dot{\alpha}_d) \geqslant 0 \tag{54b}$$

Using the degenerate Legendre transformation gives, this time, a single yield function in combined damage-plastic generalized stress $\{\chi_d, \boldsymbol{\chi}_p\}$ space, given in one of the forms

$$y^f = y^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d, \boldsymbol{\chi}_p, \chi_d) \leqslant 0 \tag{55a}$$

$$y^g = y^g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d, \boldsymbol{\chi}_p, \chi_d) \leqslant 0 \tag{55b}$$

An example is given by applying Eq. (20)

$$y^f = \left(\frac{\boldsymbol{\chi}_p \bullet \boldsymbol{\chi}_p}{c_p^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d)^2}\right)^{\frac{n}{2(n-1)}} + \left(\frac{\chi_d}{c_d^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d)}\right)^{\frac{n}{n-1}} - 1 \leqslant 0 \tag{56a}$$

$$y^g = \left(\frac{\boldsymbol{\chi}_p \bullet \boldsymbol{\chi}_p}{c_p^g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d)^2}\right)^{\frac{n}{2(n-1)}} + \left(\frac{\chi_d}{c_d^g(\boldsymbol{\sigma}, \boldsymbol{\alpha}_p, \alpha_d)}\right)^{\frac{n}{n-1}} - 1 \leqslant 0 \tag{56b}$$

The following examples are given only for the second option because in the past the notion of separate yield surfaces for damage growth and plastic strain evolution has been explored, while the second option has largely been overlooked. The examples will be given in the stress-space form (i.e. in g-form); although we should mention that the f-form potentials could be recovered from the corresponding relations.

4.1. Example of one-dimensional coupled damage hyperplasticity model

Suppose a model which combines the hyperplasticity model of Eq. (21) (but with $g_2(\boldsymbol{\alpha}_p) = 0$) and the damage hyperelasticity of Eq. (42b) in the following way:

$$g(\sigma, \alpha_p, \alpha_d) = -\frac{\sigma^2}{2E(1-\alpha_d)} - \sigma\alpha_p \tag{57}$$

$$d^g(\sigma, \alpha_d, \dot{\alpha}_p, \dot{\alpha}_d) = k\Pi(\alpha_d)\sqrt{(r_p\dot{\alpha}_p)^2 + \left(r_d \frac{\sigma}{2E(1-\alpha_d)^2}\dot{\alpha}_d\right)^2} \geqslant 0 \tag{58}$$

where and r_d are factors relating to the ratio between the internal variable rates. Their interpretation will become clearer later on.

Following the g-form, Eq. (45b) gives a yield function in combined plastic-damage generalized stress space of the form:

$$y^g = \left(\frac{\chi_p}{kr_p\Pi(\alpha_d)}\right)^2 + \left(\frac{\chi_d \cdot 2E(1-\alpha_d)^2}{kr_d\Pi(\alpha_d)\sigma}\right)^2 - 1 \leqslant 0 \tag{59}$$

Using Eqs. (3) and (58) we find that

$$\varepsilon = \frac{\sigma}{E(1-\alpha_d)} + \alpha_p \tag{60}$$

such that the first component is recognized as the damage hyperelastic strain and the second as the plastic strain. Using Eqs. (4b) and (58) gives

$$\bar{\chi}_p = \sigma; \quad \bar{\chi}_d = \frac{\sigma^2}{2E(1-\alpha_d)^2} \tag{61a, b}$$

which upon substitution in the generalized stress space yield function (59) gives the stress space yield function

$$y^\sigma = \left(\frac{\sigma}{k}\right)^2 \left[\frac{r_p^2 + r_d^2}{r_p^2 r_d^2}\right] \frac{1}{\Pi^2(\alpha_d)} - 1 \leqslant 0 \tag{62}$$

This means that during yielding, the stress–strain curve is again a function of the damage internal variable, but now also of the plastic strain internal variable

$$\sigma_y(\alpha_d) = \pm \frac{r_p r_d}{\sqrt{r_p^2 + r_d^2}} k\Pi(\alpha_d) \tag{63a}$$

$$\varepsilon_y(\alpha_d, \alpha_p) = \frac{\sigma_y(\alpha_d)}{E(1-\alpha_d)} + \alpha_p \tag{63b}$$

For consistency with the two idealised models in Section 3, we set $\Pi(0) \equiv 1$, such that $\sigma_y(\alpha_d = 0) = \pm k$, thus according to (62) we can find that the factors r_p and r_d are related by the equation

$$\frac{1}{r_p^2} + \frac{1}{r_d^2} = 1 \tag{64}$$

from which it immediately follows that $r_p \geqslant 1$ and $r_d \geqslant 1$.

In the limiting case of $r_p \geqslant 1$, $r_d \to \infty$. In that case the yield function (59) is expressed only in plastic generalized stress space

$$y^g = \left(\frac{\chi_p}{k}\right)^2 - 1 \leqslant 0 \tag{65}$$

agreeing with the ideal hyperplastic model. In the same way, when $r_d = 1$, $r_p \to \infty$ and the yield function (59) can be expressed only in damage generalized stress

$$y^g = \left(\frac{\chi_d}{k\Pi(\alpha_d)}\right)^2 - 1 \leqslant 0 \tag{66}$$

agreeing with the ideal damage hyperelastic model.

The relation between the internal variable rates could be found by applying Eq. (14) to the yield function (59) and substitution of relations (61), (62) and (64)

$$\frac{\dot{\alpha}_d}{\dot{\alpha}_p} = \frac{\chi_p}{\chi_d}\left(\frac{r_p}{r_d}\right)^2 = \frac{2E(1-\alpha_d)^2}{k\Pi(\alpha_d)}\left(\frac{r_p}{r_d}\right)^2 \tag{67}$$

which shows that the ratio r_p/r_d determines the ratio of internal variable rates, and hence the balance between damage and plasticity.

Corresponding to Fig. 1, the stress–strain curve of the one-dimensional damage hyper-plastic model is given in Fig. 2. In this model, the reloading curve introduces residual plastic strain when returning to initial stress conditions. This is demonstrated in the figure only for the case $\Pi(\alpha_d) = 1$. The magnitude of the plastic strain is a function of the rate equation (67).

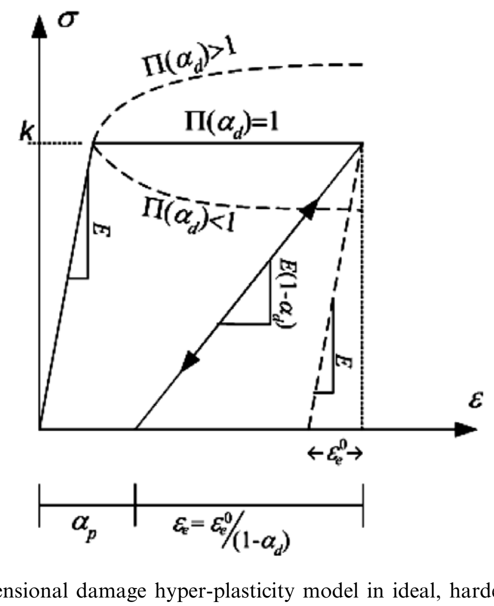

Fig. 2. Stress–strain curves of one-dimensional damage hyper-plasticity model in ideal, hardening and softening conditions (reloading demonstrated in the ideal case).

4.2. Example of coupled damage-plasticity von Mises model

Lemaitre (1985) presents a variant of coupled damage-plasticity von Mises model, often simply known as the Lemaitre model. In the following a different form of coupled damage-plasticity von Mises model is presented, and discussed in the context of the Lemaitre model. The energy potential is expressed similarly to the Lemaitre approach, but the dissipative mechanisms are expressed in quite a different way. While the plasticity yield surface and dissipation potential of the Lemaitre model were treated as quite independent, here they are interlinked using the Legendre transformation. This linkage is an important feature of our approach, and one which we consider lends it added consistency and rigour.

Lemaitre (1985) expresses the Helmholtz free energy potential for an isothermal linear elastic, isotropic coupled damage-plasticity model. With a slight change in notation, this potential could be written as

$$f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_{\mathrm{p}}, \alpha_{\mathrm{d}}) = G(1-\alpha_{\mathrm{d}})(\boldsymbol{\varepsilon}' - \boldsymbol{\alpha}'_{\mathrm{p}}) : (\boldsymbol{\varepsilon}' - \boldsymbol{\alpha}'_{\mathrm{p}}) + \frac{1}{2}K(1-\alpha_{\mathrm{d}})\,\mathrm{tr}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}_{\mathrm{p}})^2 \tag{68}$$

where G and K represent the shear and bulk moduli; $\boldsymbol{x}' = \boldsymbol{x} - \frac{1}{3}\mathrm{tr}(\boldsymbol{x})\mathbf{1}$ is the distortional (deviatoric) part of the second order tensor $\boldsymbol{x}$ and $\mathrm{tr}(\boldsymbol{x}) = x_{ii}$ is the trace of the tensor. The trace and deviator parts of the stress tensor are defined by applying Eq. (2)

$$\mathrm{tr}(\boldsymbol{\sigma}) = 3^* \frac{\partial f}{\partial \mathrm{tr}(\boldsymbol{\varepsilon})} = 3^* K^* (1-\alpha_{\mathrm{d}})^* \mathrm{tr}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}_{\mathrm{p}}) \tag{69}$$

$$\boldsymbol{\sigma}' = \frac{\partial f}{\partial \boldsymbol{\varepsilon}'} = 2G(1-\alpha_{\mathrm{d}})(\boldsymbol{\varepsilon}' - \boldsymbol{\alpha}'_{\mathrm{p}}) \tag{70}$$

and using Eqs. (4a,b) the trace and deviator parts of the plasticity generalised stress and damage scalar generalised stress are defined as

$$\mathrm{tr}(\bar{\boldsymbol{\chi}}_{\mathrm{p}}) = -3^* \frac{\partial f}{\partial\, \mathrm{tr}(\boldsymbol{\alpha}_{\mathrm{p}})} = \mathrm{tr}(\boldsymbol{\sigma}) \tag{71}$$

$$\bar{\boldsymbol{\chi}}'_{\mathrm{p}} = -\frac{\partial f}{\partial \boldsymbol{\alpha}'_{\mathrm{p}}} = \boldsymbol{\sigma}' \tag{72}$$

$$\bar{\chi}_{\mathrm{d}} = -\frac{\partial f}{\partial \alpha_{\mathrm{d}}} = G(\boldsymbol{\varepsilon}' - \boldsymbol{\alpha}'_{\mathrm{p}}) : (\boldsymbol{\varepsilon}' - \boldsymbol{\alpha}'_{\mathrm{p}}) + \frac{1}{2}K\,\mathrm{tr}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}_{\mathrm{p}}) = \frac{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'}{4G(1-\alpha_{\mathrm{d}})^2} R_v \tag{73}$$

where it is convenient to introduce

$$R_v = R_{v\sigma}(\boldsymbol{\sigma}) = 1 + \frac{G}{2K}\frac{\mathrm{tr}(\boldsymbol{\sigma})^2}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} \tag{74a}$$

$$R_v = R_{v\varepsilon}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}_{\mathrm{p}}) = 1 + \frac{K}{2G}\frac{\mathrm{tr}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}_{\mathrm{p}})^2}{(\boldsymbol{\varepsilon}' - \boldsymbol{\alpha}'_{\mathrm{p}}) : (\boldsymbol{\varepsilon}' - \boldsymbol{\alpha}'_{\mathrm{p}})} \tag{74b}$$

is defined as the "triaxiality function" (in two possible forms), and as Lemaitre noted is a function of the triaxiality ratio $\mathrm{tr}(\boldsymbol{\sigma})^2/\boldsymbol{\sigma}' : \boldsymbol{\sigma}'$. For zero triaxiality ratio $\mathrm{tr}(\boldsymbol{\sigma}) = 0$ and $R_v = 1$. Lemaitre (1985) postulated the following yield surface (without linking it to an explicit dissipation potential):

$$y^* = \bar{\chi}_{\mathrm{d}} - \frac{k^2}{G(1-\alpha_{\mathrm{d}})^2} R_v(\boldsymbol{\sigma}) = 0 \tag{75a}$$

where we use the asterisk on y^* to highlight that it was not derived from an explicit dissipation function. Lemaitre noticed that by applying Eq. (73), Eq. (75a) reverts to the classical elasto-plastic von Mises yield surface

$$y^* = \boldsymbol{\sigma}' : \boldsymbol{\sigma}' - 2k^2 = 0 \tag{75b}$$

where k is the strength parameter that corresponds to the simple shear test.

Lemaitre completes his formulation by postulating another function, this time the dissipation potential, but importantly *not* the one which is associated with the dissipative action of the yield surface in (75a). The relationship of the dissipation and the yield surface expressed through the Legendre transform was ignored, and the choice of the dissipation made apparently arbitrarily, allowing derivation of a convenient curve-fitting evolution equation for the damage internal variable. In the present formulation this is impossible, as yield and dissipation are explicitly interlinked. Furthermore, Lemaitre writes the dissipation potential, in an explicit form, only for rate dependent materials, giving the result $\dot{\alpha}_d = (\bar{\chi}_d/S_0)^{s_0}(1-\alpha_d)^{-1}$ and $\dot{\boldsymbol{\alpha}}_p = \frac{3}{2}\boldsymbol{\sigma}'\left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'\right)^{-1/2}(1-\alpha_d)^{-1}$ (where S_0 and s_0 are two material parameters which although not given an explicit physical meaning, clearly relate to the rate-dependent behavior of the material). For rate independent deformations Lemaitre suggested simply multiplying these expressions by λ, without any particular justification, and without writing the explicit form of the dissipation potential.

In the following, the model is modified into a more consistent form, without introducing ad hoc assumptions about the evolution equations. The yield surface will be associated directly to the dissipation. We shall assume the following dissipation:

$$d^f(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}'_p, \alpha_d, \dot{\boldsymbol{\alpha}}'_p, \dot{\alpha}_d) = k\sqrt{2r_p^2\dot{\boldsymbol{\alpha}}'_p : \dot{\boldsymbol{\alpha}}'_p + \frac{1}{2}R_{v\varepsilon}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha}_p)\left(r_d\frac{1}{G(1-\alpha_d)^2}\dot{\alpha}_d\right)^2} \geqslant 0 \tag{76}$$

Using Eq. (56a) the yield function in combined plastic-damage generalized stress space takes the following form:

$$y^f = y^f(\boldsymbol{\chi}_p, \chi_d, \boldsymbol{\varepsilon}, \boldsymbol{\alpha}_p, \alpha_d) = \frac{\boldsymbol{\chi}'_p : \boldsymbol{\chi}'_p}{2(kr_p)^2} + \frac{\chi_d \cdot 2G(1-\alpha_d)^2}{(kr_d)^2 R_{v\varepsilon}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha}_p)} - 1 \leqslant 0 \tag{77}$$

On the use of Eqs. (72)–(74b), $\bar{\boldsymbol{\chi}}'_p = \boldsymbol{\chi}'_p$, and $\bar{\chi}_d = \chi_d$ the stress space yield function is derived

$$y^\sigma = \boldsymbol{\sigma}' : \boldsymbol{\sigma}'\left[\frac{r_p^2 + r_d^2}{r_p^2 r_d^2}\right] - 2k^2 \leqslant 0 \tag{78a}$$

If we set the coupling damage-plasticity parameters according to Eq. (64), this function reduces to the von Mises yield function

$$y^\sigma = \boldsymbol{\sigma}' : \boldsymbol{\sigma}' - 2k^2 \leqslant 0 \tag{78b}$$

such that $y^\sigma = y^*$ is the same as in the Lemaitre model, but this time derived from the dissipation. However, now it is possible to derive the evolution equation of the damage and plasticity internal variables directly from the flow rule

$$\dot{\alpha}_d = \lambda\frac{2G(1-\alpha_d)^2}{(kr_d)^2 R_v}; \quad \dot{\boldsymbol{\alpha}}'_p = \lambda\frac{\boldsymbol{\chi}'_p}{(kr_p)^2} \tag{79a, b}$$

giving

$$\frac{\dot{\alpha}_d}{\sqrt{\dot{\boldsymbol{\alpha}}'_p : \dot{\boldsymbol{\alpha}}'_p}} = \frac{\sqrt{2}G(1-\alpha_d)^2}{R_v k}\left(\frac{r_p}{r_d}\right)^2 \tag{80}$$

showing once more that the ratio r_p/r_d determines the balance between damage and plasticity, though in this case the balance also depends on the degree of triaxiality. For zero triaxiality ratio, Eq. (80) becomes

$$\frac{\dot{\alpha}_d}{\sqrt{\dot{\boldsymbol{\alpha}}'_p : \dot{\boldsymbol{\alpha}}'_p}} = \frac{G(1-\alpha_d)^2}{\sqrt{2}k}\left(\frac{r_p}{r_d}\right)^2 \quad \text{when } \operatorname{tr}(\boldsymbol{\sigma}) = 0 \tag{81}$$

which upon integration gives (provided zero triaxaility is maintained)

$$\alpha_d = \frac{\left(\frac{r_d}{r_p}\right)^2\xi^*}{1+\left(\frac{r_d}{r_p}\right)^2\xi^*}; \quad \xi^* = \frac{\sqrt{2}k}{2G}\xi \tag{82}$$

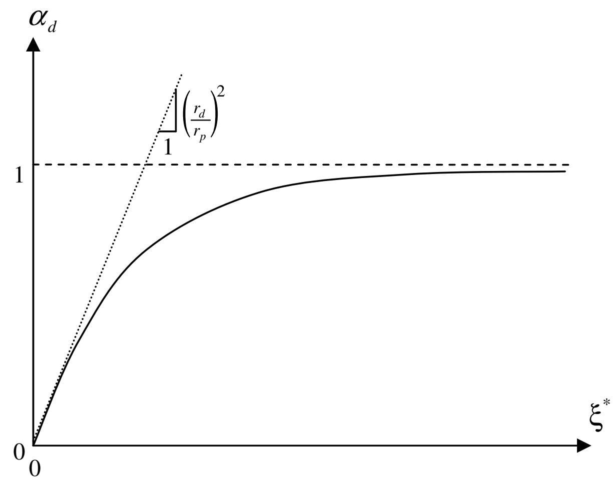

Fig. 3. Relation between damage and cumulative plastic shear strain in coupled damage-plasticity von Mises model.

such that in this loading case, damage is purely a function of the cumulative plastic shear strain ξ (with ξ^* being a normalised form of ξ)

$$\xi = \int_t \sqrt{\dot{\boldsymbol{\alpha}}'_{\mathrm{p}} : \dot{\boldsymbol{\alpha}}'_{\mathrm{p}}}\,\mathrm{d}t = \int_0^{\boldsymbol{\alpha}'_{\mathrm{p}}} \sqrt{\mathrm{d}\boldsymbol{\alpha}'_{\mathrm{p}} : \mathrm{d}\boldsymbol{\alpha}'_{\mathrm{p}}} \tag{83}$$

As is seen from Eq. (82) the relation between the damage and cumulative plastic shear strain is given by a hyperbola, with the ratio $r_{\mathrm{p}}/r_{\mathrm{d}}$ determining the initial slope in $\alpha_{\mathrm{d}} - \xi^*$ space (see Fig. 3).

4.3. Example of coupled damage-plasticity Modified Cam Clay model

Modern trends in the offshore industry are towards deeper water developments, where the seabed soils typically comprise soft sediments, many of which are highly sensitive clays, i.e. with strength loss by factors of 3–10, when they are strained. The term "sensitive clay" was introduced by Mitchell (1976) to describe this characteristic of the material constitutive behaviour. Alternatively, the term "structured clay" is used to describe the microscopic-scale chemical, fabric and stability features (Mitchell, 1976; Burland, 1990) of the same materials. The most distinctive characteristic of the stress–strain compression curves of sensitive clays, when compared with remoulded clays, is that the isotropic pre-consolidation pressure property undergoes degradation (Burland, 1990; Liu and Carter, 1999). This behaviour of sensitive clays have been described using a variety of approaches, focusing on the strength loss. However, experiments suggest that during compression, the strength loss of sensitive clays is additional to further degradation in the elastic moduli (Holtz et al., 1986). Existing constitutive models fail to represent this strength-stiffness degradation coupling via damage. This is the motive behind the following section, were we adopt the damage hyperplasticity approach to extend the Modified Cam Clay model. This is facilitated by following the ideas presented in Example 4.1 and employing the "non-damage" Hyperplastic Modified Cam Clay formulation by Houlsby (1981) and Collins and Houlsby (1997).

Many authors (e.g. Ladeveze and Le-Dantec, 1992; Lemaitre, 1992; Fish and Yu, 2001) have introduced two scalar damage internal variables (or a two-dimensional vectorial damage internal variable) to model the damage evolution in two separate modes of deformations. This method is popular for materials such as laminated composites, in which the mechanism of damage may be different in tension and shear. Our model for sensitive clays will also incorporate two measures of scalar damage, one for volumetric and another for shear deformation modes.

According to Collins and Houlsby an entire Modified Cam Clay model could be encapsulated (in terms of the so-called "triaxial" variables commonly used in soil mechanics) through the following two potentials:

$$g = -\kappa^* p \left(\log\left(\frac{p}{p_0}\right) - 1 \right) - \frac{q^2}{6G} - p\alpha_{\mathrm{p}}^{\mathrm{v}} - q\alpha_{\mathrm{p}}^{\mathrm{s}} \tag{84a}$$

$$d^g = \frac{p_y}{2} \left(\dot{\alpha}_{\mathrm{p}}^{\mathrm{v}} + \sqrt{(\dot{\alpha}_{\mathrm{p}}^{\mathrm{v}})^2 + (M\dot{\alpha}_{\mathrm{p}}^{\mathrm{s}})^2} \right) \geqslant 0 \tag{84b}$$

where

$$\Pi\left(\alpha_{\mathrm{p}}^{\mathrm{v}}\right) = \exp\left(\alpha_{\mathrm{p}}^{\mathrm{v}}/(\lambda^* - \kappa^*)\right) \tag{85a}$$

$$p_y\left(\alpha_{\mathrm{p}}^{\mathrm{v}}\right) = p_{y0}\Pi\left(\alpha_{\mathrm{p}}^{\mathrm{v}}\right) \tag{85b}$$

is the "preconsolidation pressure", which together with the volumetric stress–strain relation $\varepsilon_v = -\partial g/\partial p = \kappa^* \log(p/p_0) + \alpha_{\mathrm{p}}^{\mathrm{v}}$, gives the conventional MCC compression curve in Fig. 4a.

Consider a similar model, but this time a model that includes damage, which could be completely derived from the following two potentials:

$$g = -\frac{\kappa^* p}{(1-\alpha_{\mathrm{d}}^{\mathrm{v}})} \left(\log\left(\frac{p}{p_0}\right) - 1 \right) - \frac{q^2}{6G(1-\alpha_{\mathrm{d}}^{\mathrm{s}})} - p\alpha_{\mathrm{p}}^{\mathrm{v}} - q\alpha_{\mathrm{p}}^{\mathrm{s}} \tag{86a}$$

$$d^g = \frac{p_y}{2} \left(\dot{\alpha}_{\mathrm{p}}^{\mathrm{v}} + R_{\mathrm{d}}^{\mathrm{v}}\dot{\alpha}_{\mathrm{d}}^{\mathrm{v}} + \sqrt{\left(r_{\mathrm{p}}\dot{\alpha}_{\mathrm{p}}^{\mathrm{v}}\right)^2 + \left(r_{\mathrm{d}}R_{\mathrm{d}}^{\mathrm{v}}\dot{\alpha}_{\mathrm{d}}^{\mathrm{v}}\right)^2 + \left(r_{\mathrm{p}}M\dot{\alpha}_{\mathrm{p}}^{\mathrm{s}}\right)^2 + \left(r_{\mathrm{d}}MR_{\mathrm{d}}^{\mathrm{s}}\dot{\alpha}_{\mathrm{d}}^{\mathrm{s}}\right)^2} \right) \geqslant 0 \tag{86b}$$

where $R_{\mathrm{d}}^{\mathrm{v}}(p,\alpha_{\mathrm{d}}^{\mathrm{v}}) = \kappa^*\left(\log\left(\frac{p}{p_0}\right) - 1\right)/(1-\alpha_{\mathrm{d}}^{\mathrm{v}})^2$, $R_{\mathrm{d}}^{\mathrm{s}}(q,\alpha_{\mathrm{d}}^{\mathrm{s}}) = q/6G(1-\alpha_{\mathrm{d}}^{\mathrm{s}})^2$ and we require as before that $1/r_{\mathrm{p}}^2 + 1/r_{\mathrm{d}}^2 = 1$.

In the above p and q are the mean effective and shear stresses and p_0 is a reference pressure; $\alpha_{\mathrm{d}}^{\mathrm{v}}$ and $\alpha_{\mathrm{d}}^{\mathrm{s}}$ are two damage internal variables, associated with volumetric and shear deformations respectively; p_y is the preconsolidation pressure; $\alpha_{\mathrm{p}}^{\mathrm{v}}$ and $\alpha_{\mathrm{p}}^{\mathrm{s}}$ are the plastic strain internal variables associated with the volumetric and shear deformations respectively; G is the shear modulus; κ^* and λ^* are the elastic compressibility index related to the bulk modulus and the slope of the virgin compression line respectively. The yield function may be obtained as

$$y^g = \left(\frac{\chi_{\mathrm{p}}^{\mathrm{v}} - p_y/2}{r_{\mathrm{p}}}\right)^2 + \left(\frac{\chi_{\mathrm{d}}^{\mathrm{v}} - R_{\mathrm{d}}^{\mathrm{v}}p_y/2}{r_{\mathrm{d}}R_{\mathrm{d}}^{\mathrm{v}}}\right)^2 + \left(\frac{\chi_{\mathrm{p}}^{\mathrm{s}}}{r_{\mathrm{p}}M}\right)^2 + \left(\frac{\chi_{\mathrm{d}}^{\mathrm{s}}}{r_{\mathrm{d}}MR_{\mathrm{d}}^{\mathrm{s}}}\right)^2 - \left(\frac{p_y}{2}\right)^2 = 0 \tag{87}$$

The yield function in stress space can be extracted from the two potentials, after some manipulation, by adopting Eq. (4b) in the form $\bar{\chi}_{\mathrm{p}}^{\mathrm{v}} = -\partial g/\partial\alpha_{\mathrm{p}}^{\mathrm{v}}$, $\bar{\chi}_{\mathrm{p}}^{\mathrm{s}} = -\partial g/\partial\alpha_{\mathrm{p}}^{\mathrm{s}}$, $\bar{\chi}_{\mathrm{d}}^{\mathrm{v}} = -\partial g/\partial\alpha_{\mathrm{d}}^{\mathrm{v}}$, $\bar{\chi}_{\mathrm{d}}^{\mathrm{s}} = -\partial g/\partial\alpha_{\mathrm{d}}^{\mathrm{s}}$, Eq. (13) in the form $\chi_{\mathrm{p}}^{\mathrm{v}} = -\partial d/\partial\dot{\alpha}_{\mathrm{p}}^{\mathrm{v}}$, $\chi_{\mathrm{p}}^{\mathrm{s}} = -\partial d/\partial\dot{\alpha}_{\mathrm{p}}^{\mathrm{s}}$, $\chi_{\mathrm{d}}^{\mathrm{v}} = -\partial d/\partial\dot{\alpha}_{\mathrm{d}}^{\mathrm{v}}$, $\chi_{\mathrm{d}}^{\mathrm{s}} = -\partial d/\partial\dot{\alpha}_{\mathrm{d}}^{\mathrm{s}}$, and (87)

$$y = \left(p - \frac{p_y}{2}\right)^2 + \left(\frac{q}{M}\right)^2 - \left(\frac{p_y}{2}\right)^2 \leqslant 0 \tag{88}$$

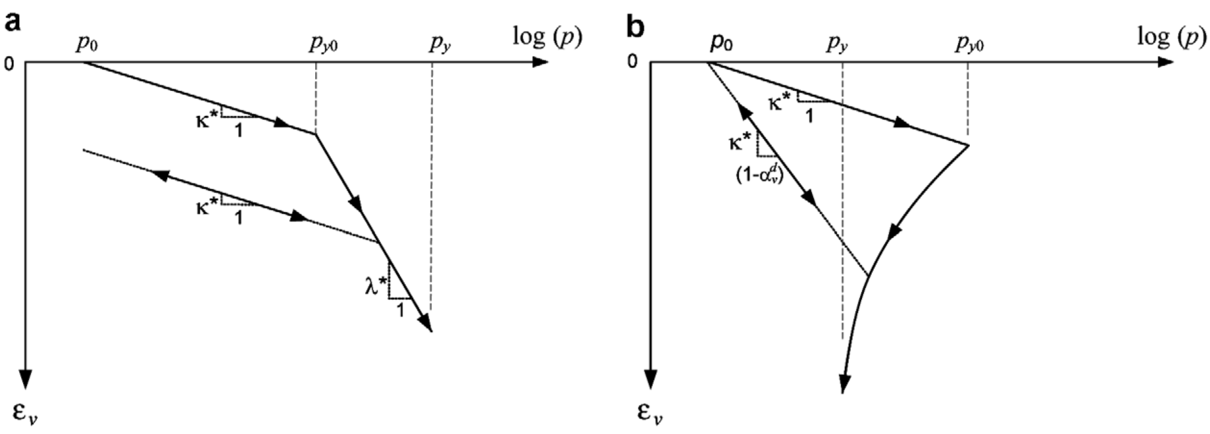

Fig. 4. Compression models. (a) Conventional and (b) damage hyperelastic.

where $p_y = p_y(\alpha_p^v, \alpha_d^v, \alpha_d^s)$. The specific form of this function will determine the role of the damage parameters, as we explore later. If this expression reduces to $p_y = p_y(\alpha_p^v) = p_{y0}\Pi(\alpha_p^v)$ the yield function of the damage Modified Cam Clay model becomes that of the conventional Modified Cam Clay model.

The volumetric and shear strains are defined from Eq. (3)

$$\varepsilon_v = -\frac{\partial g}{\partial p} = \bar{\kappa}^* \log\left(\frac{p}{p_0}\right) + \alpha_p^v = \frac{\kappa^*}{(1-\alpha_d^v)} \log\left(\frac{p}{p_0}\right) + \alpha_p^v \tag{89a}$$

$$\varepsilon_s = -\frac{\partial g}{\partial q} = \frac{q}{3\bar{G}} + \alpha_p^s = \frac{q}{3G\left(1-\alpha_d^s\right)} + \alpha_p^s \tag{89b}$$

which agrees again with Collins and Houlsby's hyperplastic version of Modified Cam Clay, but here the model includes the two damage internal variables. We identify the effective shear modulus and effective compressibility index by

$$\overline{G} = G(1-\alpha_d^s) \tag{90a}$$

$$\bar{\kappa}^* = \frac{\kappa^*}{(1-\alpha_d^v)} \tag{90b}$$

We now turn attention to the normal compression behaviour of the model. First, let us postulate a hypothetical damage hyperelastic model (i.e., by imposing $r_d = 1$ and $r_p \to \infty$ in the above model, and removing the dependence on α_p). Further assume the following expression to describe the degradation of the strength parameter due to damage, by replacing Eq. (85)

$$\Gamma(\alpha_d^v) = \left(\delta_{rem} + (1-\delta_{rem})\exp\left(-3\alpha_d^v(1-D_{95})/D_{95}\left(1-\alpha_d^v\right)\right)\right) \tag{91a}$$

$$p_y\left(\alpha_d^v\right) = p_{y0}\Gamma\left(\alpha_d^v\right) \tag{91b}$$

where δ_{rem} is the fully remoulded strength ratio, and D_{95} is the amount of damage required to cause 95% reduction (from peak to remoulded). (Note that $\exp(-3) \approx 0.05$.) This formula and Eq. (89a), gives the damage hyperelastic MCC compression curve in Fig. 4b.

However, it is well established that clays (whether sensitive or not), are strongly dependent on the plastic strain. Therefore, let us now remove the imposition of damage hyperelastic models ($r_p = 1$, $r_d \to \infty$) and go back to the full version of the damage hyperplastic model by combining Eqs. (85) and (91)

$$p_y\left(\alpha_p^v, \alpha_d^v\right) = p_{y0}\Pi\left(\alpha_p^v\right)\Gamma\left(\alpha_d^v\right) \tag{92}$$

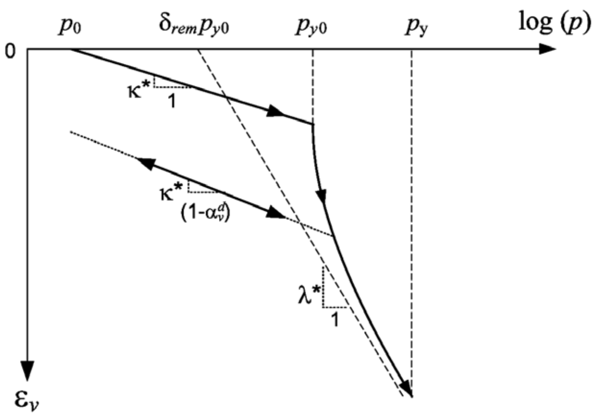

Fig. 5. The compression stress–strain behaviour of the damage hyperplastic MCC model.

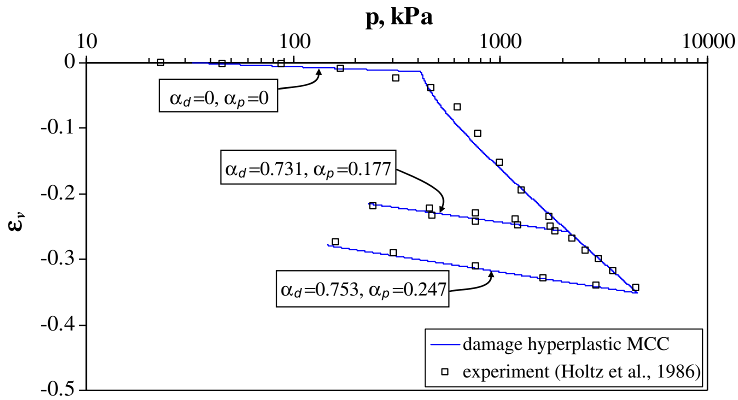

Fig. 6. The damage hyperplastic model predictions of experimental results from Oedometer tests (Holtz et al., 1986) of the volumetric stress–strain response.

This time the plastic strain is not eliminated from Eq. (89). Prior to yielding, the elastic behaviour is given by the effective compressibility index $\bar{\kappa}^* = \kappa^*/(1 - \alpha_d^v)$, denoting the damaged linear compressibility slope in the $\varepsilon_v - \log(p)$ space. Upon yielding, the normal compression curve ($q = 0$) satisfies $p = p_y$, and we get the combine effects of the hardening plasticity and softening damage. When $\alpha_d^v = 1$ and the material is fully remoulded, the pressure p and preconsolidation pressure reduce to $p = p_y(\alpha_p^v, \alpha_d^v) = \delta_{rem} p_y(\alpha_p^v) = \delta_{rem} \exp(\alpha_p^v/(\lambda^* - \kappa^*))$, and this is represented in Fig. 5.

Let us compare the above with the experimental data. For example, Holtz et al. (1986) presented results from carefully executed oedometer tests on undisturbed samples of a natural, sensitive clay deposit. During the test, both the vertical and lateral stresses were measured, allowing to recover the stress–strain normal compression in Fig. 6. The same test was repeated using the damage hyperplastic MCC model, and the predictions are plotted in the same figure. The results agree well with the fact that both the strength and compressibility modulus κ^* are being degraded during loading.

In this figure, we added the amount of damage and plastic strain that correspond to the reloading stages. As noted, the damage grows from being zero ($\alpha_d = 0$) at the beginning, to $\alpha_d = 0.731$ and $\alpha_d = 0.753$ during the first and second reloading stages. This entails an increase in the effective compressibility index from $\bar{\kappa}^* = \kappa^* = 0.005$ to $\bar{\kappa}^* = 0.018$ and $\bar{\kappa}^* = 0.02$, i.e., a reduction factor of about 4 in the bulk modulus. Obviously, this aspect of stiffness reduction has a great influence on engineering problems. The capability of the damage hyperplasticity approach to predict this characteristic of sensitive clays, which is frequently ignored by other models, motivates further work in this area.

The model is completed, by updating Eq. (92) to represent the effect of damage on the pre-consolidation pressure via shear deformations

$$p_y\left(\alpha_p^v, \alpha_d^v\right) = p_{y0}\Pi\left(\alpha_p^v\right)\sqrt{\Gamma\left(\alpha_d^v\right)\Gamma\left(\alpha_d^s\right)} \tag{93}$$

where the square root term was chosen such that in normal compression the model will behave exactly as in Figs. 5 and 6, and that in shear deformations the reduction factor is symmetric. Fig. 7 presents an example of how the model behaves under undrained shear test conditions (i.e., $\varepsilon_v = 0$), of normally consolidated sensitive clay, where we examine the effect of the damage-plasticity coupling parameter r_p. The rest of the parameters remains as those used for Fig. 6 ($\kappa^* = 0.005$, $\lambda^* = 0.09$, $p_{y0} = 410$ kPa, $M = 1.2$, $D_{95} = 0.93$, $\delta_{rem} = 0.3$, $G = 20{,}000$ kPa). When $r_p = 1$, the model performs exactly as the conventional MCC model, as the damage mechanism is deactivated. However, when $r_p > 1$ the model is allowed to undergo damage and therefore softening. When $r_p \to \infty$ ($r_d = 0$), the model is purely a damage hyperelastic, without any plastic straining. In this model, the stress paths always end along the failure line $q = Mp$ (i.e. the friction angle is constant) but the final stresses (q and p) are smaller than in the original MCC, reflecting the sensitivity of the material in shear. Fig. 7b presents how the shear mode damage scalar evolves during the test, reaching a critical asymptotic damage value that increases with increase in coupling via r_p.

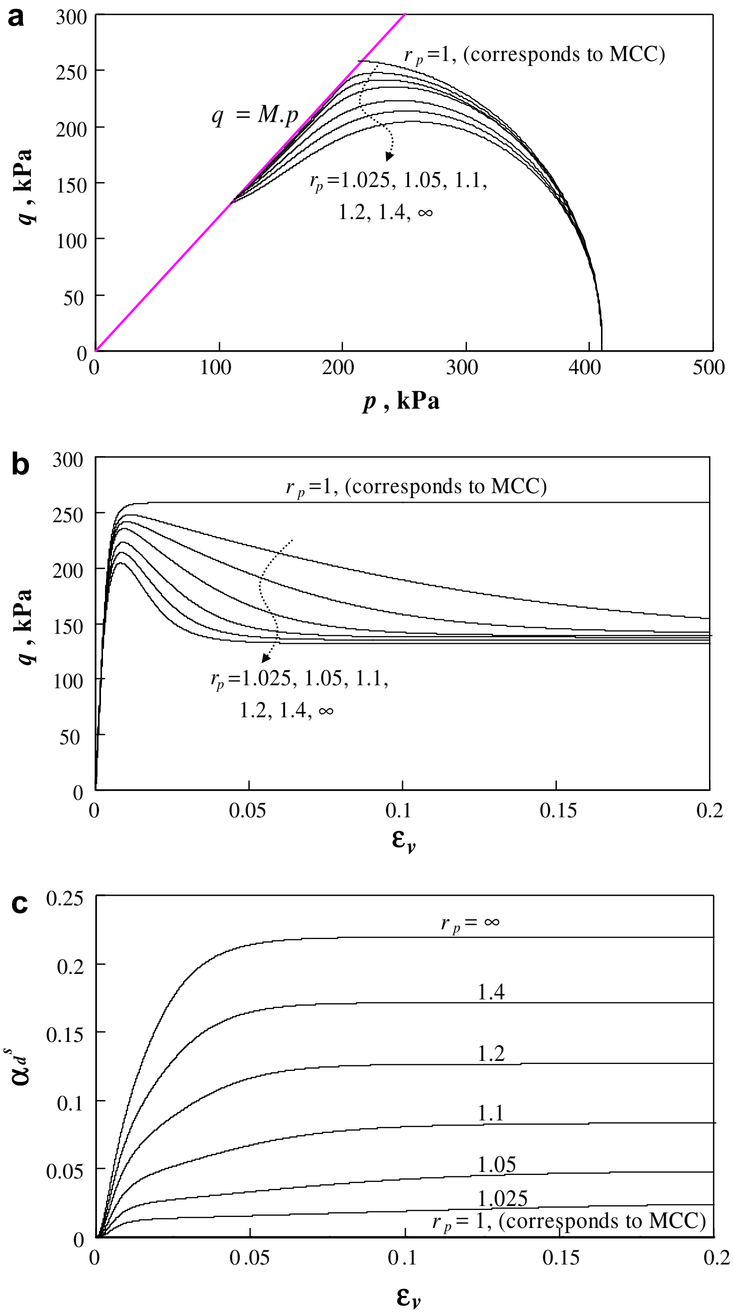

Fig. 7. The effect of the damage-plasticity coupling parameter r_p in undrained shear test: (a) stress paths, (b) shear stress–strain response and (c) evolution of shear-mode damage.

5. Conclusions

We demonstrate how the same constitutive framework can be adopted to describe both the growth of damage and the evolution of plastic strains, solely from two energy potentials. Both damage and plastic strain are identified as internal variables. Their physical interpretation comes from their particular role in the potential functions.

The most important advantage of the present work is that the energy potentials, in the form of dissipation and Helmholtz or Gibbs free energies, are given explicitly. The entire constitutive behaviour is encapsulated in these two functions. This guarantees a consistent and coherent formulation. Features such as the yield surface, flow rule and damage evolution law are the outcome of the explicit form of the potential functions. No additional *ad hoc* assumptions, such as a damage evolution law are needed.

An important distinction is made, identifying two families of combined damage-plasticity models. The first encompasses models with decoupled plasticity and damage yield surfaces. In this case, pure damage can occur without plastic straining, or vice versa, pure plastic straining can happen without damage. The second family describes models with a coupled damage-plasticity yield surface, such that any plastic straining is accompanied to damage.

A discussion was made regarding the interpretation of damage internal variable, in accordance with the effective stress and strain concepts. These two concepts give identical meaning to the damage internal variable only when the hyperelastic component of the potential is quadratic in the strain/stress variables, and produce linear damage hyperelasticity.

Finally, examples of constitutive models were given and showed the versatility of the framework and the possible applications to development of geomaterials undergoing damage.

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