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Relations between closed string amplitudes at higher-order tree level and open string amplitudes

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Abstract

KLT relations almost factorize closed string amplitudes on S_2 by two open string tree amplitudes which correspond to the left- and the right-moving sectors. In this paper, we investigate string amplitudes on D_2 and RP_2. We find that KLT factorization relations do not hold in these two cases. The relations between closed and open string amplitudes have new forms. On D_2 and RP_2, the left- and the right-moving sectors are connected into a single sector. Then an amplitude with closed strings on D_2 or RP_2 can be given by one open string tree amplitude except for a phase factor. The relations depends on the topologies of the world-sheets. Under T-duality, the relations on D_2 and RP_2 give the amplitudes between closed strings scattering from D-brane and O-plane respectively by open string partial amplitudes. In the low energy limits of these two cases, the factorization relations for graviton amplitudes do not hold. The amplitudes for gravitons must be given by the new relations instead.

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Keywords: Gauge-gravity correspondence; String amplitudes

1. Introduction

Superstring theories are theories without ultraviolet divergences. They contain both gravitational and gauge interactions as low energy limits [1,2]. Thus they offer a possible solution to the problem of unifying all of the fundamental interactions in a consistent quantum theory. In string

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theory, gravitons are massless states of closed strings and gauge particles are massless states of open strings. To study the relations between gravity and gauge field, we should explore the relations between closed and open strings. The duality between open and closed strings [3–8] also motivates us to explore the relations between closed and open strings.

The most simple relation is any excited mode of a free closed string $|N_L, N_R\rangle \otimes |p\rangle$ can be factorized by left- and right-moving open string excited modes:

$$|N_L\rangle \otimes |N_R\rangle \otimes |p\rangle. \tag{1}$$

However, when we consider the interactions among strings, there are nontrivial relations between closed and open string amplitudes. The first nontrivial relation was given by Kawai, Lewellen and Tye [9]. They express an amplitude for N closed strings on sphere (S_2) by the following equation[1]:

$$\mathcal{A}_{S_2}^{(N)} = \epsilon_{\alpha\beta}\mathcal{A}_{S_2}^{(N)\alpha\beta} = \left(\frac{i}{2}\right)^{N-3}\kappa^{N-2}\epsilon_{\alpha\beta}\sum_{P,P'}\mathcal{M}^{(N)\alpha}(P)\bar{\mathcal{M}}^{(N)\beta}(P')e^{i\pi F(P,P')}. \tag{2}$$

Here $\mathcal{A}_{S_2}^{N}$ is the amplitude for N closed strings on S_2 and $\mathcal{A}_{S_2}^{(N)\alpha\beta}$ is the closed string amplitude without polarization tensors. $\mathcal{M}^{(N)\alpha}(P)$ and $\bar{\mathcal{M}}^{(N)\beta}(P')$ are the open string partial amplitudes on D_2 corresponding to the left- and right-moving sectors respectively. They are dependent on the orderings of the external legs. If we sum over the orderings P and P', we get the total amplitudes $\sum_P \mathcal{M}^{(N)\alpha}(P)$ and $\sum_{P'} \bar{\mathcal{M}}^{(N)\beta}(P')$ for the left- and the right-moving open strings respectively. Then we can see, except for a phase factor, a closed string amplitude on S_2 can be factorized by two open string tree amplitudes corresponding to the left- and right-moving sectors (see Fig. 1(a)). There is no interaction between left- and right-moving open strings. Any closed string polarization tensor has left and right indices, they correspond to the left- and the right-moving modes respectively. The left and right indices of polarization tensors must contract with the indices in the amplitude for left- and right-moving open strings respectively. The phase factor is entirely independent of which open and closed string theories we are considering. It only depends on P and P'. Contour deformations can be used to reduce the number of the terms in Eq. (2). The number of the terms can be reduced to

$$(N-3)!\left(\frac{1}{2}(N-3)\right)!\left(\frac{1}{2}(N-3)\right)!, \quad N \text{ odd}, \tag{3}$$

and

$$(N-3)!\left(\frac{1}{2}(N-2)\right)!\left(\frac{1}{2}(N-4)\right)!, \quad N \text{ even}. \tag{4}$$

In the low energy limits, the massive modes decouple. Only massless states are left. Then KLT relations can be used to factorize the amplitudes for gravitons into products of two amplitudes for gauge particles [10]. Gauge theory has a better ultraviolet behavior than gravity. Then KLT relations can be used to investigate the ultraviolet properties of gravity. Researches with KLT relations support that $N = 8$ supergravity may be finite [11–16]. However, a question arises: Do KLT factorization relations hold for any gravity amplitude? In string theory, to calculate

[1] We use $\epsilon_{\alpha\beta}$ to denote all the polarization tensors for convenience. α correspond to the left indices and β correspond to the right indices. If there are open strings on the boundary of D_2, we use $\epsilon_{\alpha\beta\gamma}$ to denote all the polarization tensors for convenience. γ correspond to the indices of open strings.

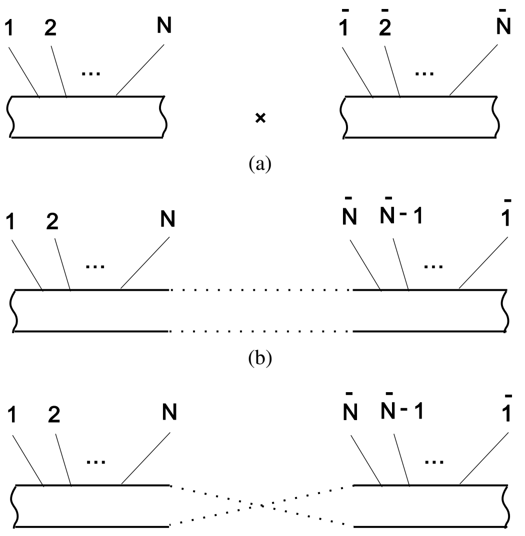

Fig. 1. (a) A closed string amplitude on S_2 can be factorize by two open string tree amplitudes corresponding to the left- and right-sectors. (b) A closed string amplitude on D_2 can be given by connecting the open string world-sheets for the two sectors with a time reverse in the right-moving sector. (c) A closed string amplitude on RP_2 can be given by connecting the open string world-sheets for the two sectors with a time reverse and a twist in the right-moving sector.

the S-matrix, we should sum over all the topologies of world-sheets. S_2 is just the most simple topology. If we consider other topologies, we should reconsider the relations between closed and open strings. Then the question becomes: Do the factorization relations hold for any topology?

Earlier works [17,25–27] have given some insights into the relations on Disk (D_2). In [17], some examples of the relations on D_2 are given. In [25–27], the most simple process of D-brane and closed string interactions are discussed. In the paper by Garousi and Myers [25], they found that the two-point scattering amplitudes of closed strings from a D-brane in Type II theory is identical with the four-point open string amplitudes upon a certain identification between the momenta and polarizations. In [26,27], the amplitude for one closed string and two open strings attached to a D-brane are calculated. They shown that this amplitude are also identical with the four-point open string amplitude. In these examples, the KLT factorization relations do not hold. Then they imply the KLT factorization relations may not hold for general amplitudes on D_2. The amplitudes on real projective plane (RP_2) have similar structures with open string amplitudes [30]. In fact, both D_2 and RP_2 can be obtained by a sphere with a $\mathbb{Z}_2$ identification. Then the KLT factorization relations may also not hold in the RP_2 case.

In this paper, we consider the general amplitudes on D_2 and RP_2. These two cases contribute to the higher-order tree amplitude [1] for closed strings. We find that the factorization relations (2) do not hold on D_2 and RP_2. The amplitudes with closed strings on D_2 and RP_2 cannot be factorized by the left- and the right-moving open string amplitudes. The amplitudes satisfy new relations. Particularly, an amplitude for N closed strings on D_2 can be given by an amplitude for $2N$ open strings:

$$\mathcal{A}^{(N)}_{D_2} = \epsilon_{\alpha\beta}\mathcal{A}^{(N)\alpha\beta}_{D_2} = \left(\frac{i}{4}\right)^{N-1}\kappa^{N-1}\epsilon_{\alpha\beta}\sum_{P}\mathcal{M}^{(2N)\alpha\beta}(P)e^{i\pi\Theta(P)}. \tag{5}$$

In this equation, $\mathcal{M}^{(2N)\alpha\beta}(P)$ is the tree amplitude for $2N$ open strings. N open strings come from the left-moving sector and the other N open strings come from the right-moving sector. The left- and the right-moving sectors are not independent of each other. The two sectors are connected into a single sector. Then the left indices contract with the right indices. The reason is that the left- (right-)moving waves must be reflected at the boundary of D_2 and then become right- (left-)moving waves. Then the interactions between the left- (right-)moving waves and their reflected waves become interactions between the two sectors. If there are open strings on the boundary of D_2, the left- and the right-moving sectors of closed strings also interact with the open strings, then an amplitude for N closed strings and M open strings on D_2 can be given by a tree amplitude for $2N+M$ open strings except for a phase factor:

$$\mathcal{A}_{D_2}^{(N,M)} = \epsilon_{\alpha\beta\gamma}\mathcal{A}_{D_2}^{(N,M)\alpha\beta\gamma} = \left(\frac{i}{4}\right)^{N-1}\kappa^{N-1}g^{M}\epsilon_{\alpha\beta\gamma}\sum_{P}\mathcal{M}^{(2N,M)\alpha\beta\gamma}(P)e^{i\pi\Theta'(P)}. \tag{6}$$

The amplitudes on RP_2 can also be factorized by one amplitude for open strings:

$$\mathcal{A}_{RP_2}^{(N)} = \epsilon_{\alpha\beta}\mathcal{A}_{RP_2}^{(N)\alpha\beta} = -\left(\frac{i}{4}\right)^{N-1}\kappa^{N-1}\epsilon_{\alpha\beta}\sum_{P}\mathcal{M}^{(2N)\alpha\beta}(P)e^{i\pi\Theta(P)}. \tag{7}$$

In this case, there is a crosscap but not a boundary here. However, the left- (right-)moving waves are also reflected at the crosscap and turn into the right- (left-)moving waves. Then there are also interactions between left- and right-moving sectors of closed strings. The two sectors are connected into one single sector again. The phase factors in (6) and (7) are complicated in concrete calculations. By considering the contour deformations, the number of the terms can be reduced [9,29]. It is noticed that the relations on D_2 are same with on RP_2 except for a minus. In a theory containing both D_2 and RP_2, the two amplitudes cancel out. Under a T-duality, the relation (6) gives the amplitude for N closed strings and M open strings attached to a D-brane by pure open string amplitudes while the relation (7) gives the amplitude for N closed strings scattering from an O-plane by pure open string amplitudes. In this case, the amplitudes on D_2 and RP_2 cannot cancel out.

An important fact will be used in our paper is that the amplitudes with closed strings are invariant under conformal transformations in each single sector. This allows us to transform the form of the interactions between left- and right-moving sectors. After some appropriate transformation in one sector, the interactions between left- and the right-moving sectors have the same form with interactions between open strings in a same sector. Then we can treat the two sectors of N closed strings as a single sector with $2N$ open strings.

In the low energy limit of an unoriented open string theory, the amplitudes for N closed strings on D_2, RP_2 and S_2 contribute to the tree amplitudes for N gravitons. In this case, we cannot only use KLT factorization relations on S_2 but also use the relations on D_2 and RP_2 to calculate the tree amplitudes for gravitons. The amplitudes for N closed strings and M open strings on D_2 become tree amplitudes for N gravitons and M gauge particles. Then the gauge-gravity interactions can be given by pure gauge interactions.

The structure of this paper is as follows. In Section 2 we will consider the correlation functions and the amplitudes on D_2. We will show KLT factorization relations do not hold on D_2. We will give the relations between closed string amplitudes on D_2 and open string tree amplitudes. We will also give the relations in the case of N closed strings and M open strings inserted on D_2. In Section 3 we will consider RP_2. We will show KLT factorization relations do not hold on

RP_2. The relations between amplitudes on RP_2 and open string amplitudes will be given in this section. Our conclusion will be given in Section 4.

2. Relations between amplitudes on D_2 and open string tree amplitudes

In this section, we will show the correlation functions on D_2 cannot be factorized into the left- and the right-moving sectors. The two sectors are connected together. Then we will give the relations between amplitudes on D_2 [1,2,18,19,22] and open string tree amplitudes [1,2,22–28].

In string theory, vertex operator for any closed string can be given as

$$\mathcal{V}(\omega,\bar{\omega}) = \mathcal{V}_L(\omega)\tilde{\mathcal{V}}_R(\bar{\omega})\mathcal{V}_0(\omega,\bar{\omega}), \tag{8}$$

where $\omega = \tau + i\sigma$. $\mathcal{V}_L$ and $\mathcal{V}_R$ are nonzero modes of open string vertex operators. They correspond to the left- and the right-moving sectors. $\mathcal{V}_0(\omega,\bar{\omega})$ correspond to the zero modes. Thus, the closed string vertex operators can be factorized by two open string vertex operators corresponding to the left- and the right-moving sectors (except for the zero modes).

Now we consider the correlation function of vertex operators. On S_2 the left- and the right-moving waves are independent of each other. Then a correlation function on S_2 can be factorized by the left- and the right-moving sectors [1,2]. However, when we add a boundary to S_2, we get D_2. The left- and the right-moving waves must be reflected at the boundary of D_2. The reflection waves of the left-moving waves are in the right-moving sector and the reflection waves of the right-moving waves are in the left-moving sector. Waves must interact with their reflection waves, then their must be interactions between the two sectors. To see this, we should use the boundary state [20,21] to give the correlation functions on D_2. The correlation function for N closed strings on D_2 is

$$\langle 0|\mathcal{V}_N(\omega,\tilde{\omega})\ldots\mathcal{V}_1(\omega,\tilde{\omega})|B\rangle, \tag{9}$$

where $|B\rangle \equiv B|0\rangle$ is the boundary state for D_2. In this paper, for convenience, we use the bosonized vertex operators[2]

$$\begin{aligned}\mathcal{V}(\omega,\bar{\omega}) = {}&:\!\exp(q\phi_6 + \tilde{q}\tilde{\phi}_6)\\ &\times \exp\left(i\lambda\circ\phi + i\sum_{i=1}^{m}\varepsilon^i\circ\partial\phi_i + i\tilde{\lambda}\circ\tilde{\phi} + i\sum_{i=1}^{\tilde{m}}\bar{\varepsilon}^i\circ\bar{\partial}\tilde{\phi}_i\right)\\ &\times \exp\left(ik\cdot X + i\sum_{i=1}^{n}\epsilon^i\cdot\partial X + i\sum_{j=1}^{\tilde{n}}\bar{\epsilon}^j\cdot\bar{\partial}X\right)(\omega,\bar{\omega})\!:\Big|_{\text{multilinear}}.\end{aligned} \tag{10}$$

[2] Here $\phi_i(z)$ $(i = 1\ldots5)$ and $\tilde{\phi}_i(\bar{z})$ $(i = 1\ldots5)$ are bosonic fields. They are used to bosonize holomorphic and antiholomorphic fermionic fields and spinor fields. $\phi_6(z)$ and $\tilde{\phi}_6(\bar{z})$ are used to bosonize the holomorphic and antiholomorphic superconformal ghost respectively. ϵ and $\bar{\epsilon}$ correspond to the components of polarization tensors contracting with bosonic fields ∂X and $\bar{\partial}X$ respectively. ε and $\bar{\varepsilon}$ correspond to the components contracting with $\partial\phi$ and $\bar{\partial}\tilde{\phi}$ respectively. We pick up the pieces multilinear in ϵ, $\bar{\epsilon}$ and ε, $\bar{\varepsilon}$, then replace these polarization vectors by the polarization tensor of the vertex operator. $\lambda_i' = i\lambda_i$ and $\tilde{\lambda}_i' = i\tilde{\lambda}_i$ $(i = 1\ldots5)$ are vectors in the weight lattice [18,19] of the left- and right-moving sectors respectively. q and $\tilde{q}$ are the γ ghost number in the left- and right-moving sectors respectively. We use $\circ$ to denote the inner product in the five-dimensional weight space and use $\cdot$ to denote the inner product in the space–time.

Physical vertices containing higher derivatives can be transformed into the vertices with only first derivatives. In fact we can do partial integrals to reduce the order of the derivatives. After the integrals on the world-sheet, the surface terms turn to zero. Redefine the polarization tensor, the vertices then turn to those only contain first derivatives.

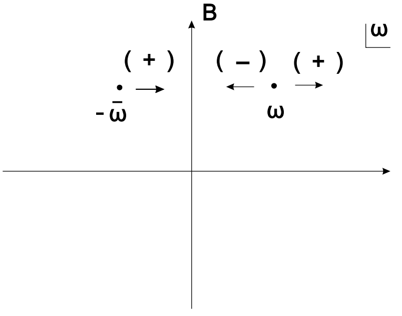

Fig. 2. Only the annihilation modes are reflected at the boundary of D_2.

With the definition of normal ordering, we have

$$\mathcal{V}(\omega,\bar{\omega}) = \mathcal{V}_L^{(+)}(\omega)\mathcal{V}_L^{(-)}(\omega)\tilde{\mathcal{V}}_L^{(+)}(\bar{\omega})\tilde{\mathcal{V}}_R^{(-)}(\bar{\omega})\mathcal{V}_0(\omega,\bar{\omega}), \tag{11}$$

where $(+)$ and $(-)$ correspond to the creation modes and the annihilation modes respectively. In $\mathcal{V}_0$, we consider x as creation operator and p as annihilation operator. Then in the normal ordering, x must on the left of p. The bosonized boundary operator is[3]

$$B = \exp\left(\sum_{n=1}^{\infty} a_n^\dagger \cdot \tilde{a}_n^\dagger\right) \otimes \exp\left(\sum_{n=1}^{\infty} b_n^\dagger \circ \tilde{b}_n^\dagger\right) \otimes \exp\left(\sum_{n=1}^{\infty} c_n^\dagger \tilde{c}_n^\dagger\right), \tag{12}$$

where $a^\dagger$ and $\tilde{a}^\dagger$ are creation modes of X, $b^\dagger$ and $\tilde{b}^\dagger$ are creation modes of ϕ_i and $\tilde{\phi}_i$ respectively, $c^\dagger$ and $\tilde{c}^\dagger$ are creation modes of ϕ_6 and $\tilde{\phi}_6$ respectively. To get the correlation function on D_2 we substitute the bosonized vertex operators and the bosonized boundary operators into (9). We can move the boundary operator B to the left of all the vertex operators. Then use the creation operators in B to annihilate the state $\langle 0|$. Because B is constructed by creation operators, it commutes with the creation modes and the zero modes of the vertex operators and does not commute with the annihilation modes of the vertex operators. It means only the annihilation modes are reflected at the boundary (see Fig. 2). When we move B to the left of the annihilation modes of the vertex operators $\mathcal{V}_L^{(-)}(\omega)$ and $\tilde{\mathcal{V}}_R^{(-)}(\bar{\omega})$, the "images" of the annihilation modes $\tilde{\mathcal{V}}_L^{(+)}(-\omega)$ and $\mathcal{V}_R^{(+)}(-\bar{\omega})$ are created respectively. Though $\tilde{\mathcal{V}}_L^{(+)}(-\omega)$ is depend on ω, it is constructed by $\tilde{a}^\dagger$, $\tilde{b}^\dagger$ and $\tilde{c}^\dagger$. It must interact with operators constructed by $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$. In a similar way, $\mathcal{V}_R^{(+)}(-\bar{\omega})$ must interact with operators constructed by a, b and c. Then the correlation function can be factorized as

$$\begin{aligned}
&\left\langle \mathcal{V}_L^{(+)}(\omega_N)\mathcal{V}_L^{(-)}(\omega_N)\mathcal{V}_R^{(+)}(-\bar{\omega}_N)\ldots\mathcal{V}_L^{(+)}(\omega_1)\mathcal{V}_L^{(-)}(\omega_1)\mathcal{V}_R^{(+)}(-\bar{\omega}_1)\right\rangle \\
&\quad\times\left\langle \tilde{\mathcal{V}}_R^{(+)}(\bar{\omega}_N)\tilde{\mathcal{V}}_R^{(-)}(\bar{\omega}_N)\tilde{\mathcal{V}}_L^{(+)}(-\omega_N)\ldots\tilde{\mathcal{V}}_R^{(+)}(\bar{\omega}_1)\tilde{\mathcal{V}}_R^{(-)}(\bar{\omega}_1)\tilde{\mathcal{V}}_L^{(+)}(-\omega_1)\right\rangle \\
&\quad\times\left\langle \mathcal{V}_0(\omega_N,\tilde{\omega}_N)\ldots\mathcal{V}_0(\omega_1,\tilde{\omega}_1)\right\rangle.
\end{aligned} \tag{13}$$

Here, the first correlation function only contain operators constructed by a, b, c and $a^\dagger$, $b^\dagger$, $c^\dagger$, the second correlation function only contain operators constructed by $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{a}^\dagger$, $\tilde{b}^\dagger$,

[3] Here, we only consider Neumann boundary condition for convenience. The case with Dirichlet boundary conditions has similar relations.

$\tilde{c}^\dagger$, the third correlation function only contain operators constructed by zero modes. Though the nonzero modes are factorized into two correlation functions, both of them contain the interactions between left- and the right-moving sectors. Actually, in the first correlation function, if we move $\mathcal{V}_L^{(-)}(\omega_i)$ to the right of $\mathcal{V}_L^{(+)}(\omega_j)$, we get the interaction in the left-moving sector and if we move $\mathcal{V}_L^{(-)}(\omega_i)$ to the right of $\mathcal{V}_R^{(+)}(-\bar{\omega}_j)$, we get the interactions between the two sectors. In the same way, the second correlation function contain interactions in the right-moving sector and the interactions between left- and right-moving sectors. The interactions between the two sectors are just the interactions between vertex operators and their images. Then the correlation function on D_2 cannot be factorized by the two sectors, interactions between the two sectors connect them together.

To get the relations between amplitudes, we should calculate the correlation function, then integral over the fundamental region and divide the integrals by the volume of conformal Killing group [1,2,22,23]. For convenience, we use the z coordinate instead of ω coordinate. They are connected by a conformal transformation:

$$z = e^{\omega}. \tag{14}$$

Then the amplitude for N closed strings on D_2 becomes

$$\begin{aligned}
\mathcal{A}_{D_2}^{(N,0)} = {} & \kappa^{N-1} \int\limits_{|z|<1} \prod_{i=1}^{N} d^2 z_i \frac{|1 - z_o \bar{z}_o|^2}{2\pi d^2 z_o} \\
& \times \prod_{s>r} (z_s - z_r)^{\frac{\alpha'}{2} k_r \cdot k_s + \lambda_r \circ \lambda_s - q_r q_s} (\bar{z}_r - \bar{z}_s)^{\frac{\alpha'}{2} k_r \cdot k_s + \tilde{\lambda}_r \circ \tilde{\lambda}_s - \tilde{q}_r \tilde{q}_s} \\
& \times \prod_{r,s} \bigl(1 - (z_r \bar{z}_s)^{-1}\bigr)^{\frac{\alpha'}{2} k_r \cdot k_s + \lambda_r \circ \tilde{\lambda}_s - q_r \tilde{q}_s} \\
& \times \exp \sum_{r=1}^{N} \left(\sum_{i=1}^{n_r} \sum_{j=1}^{\tilde{n}_s} \left(-\frac{\alpha'}{2}\right) \epsilon_r^{(i)} \cdot \bar{\epsilon}_r^{(j)} - \sum_{i=1}^{m_r} \sum_{j=1}^{\tilde{m}_s} \varepsilon_r^{(i)} \circ \bar{\varepsilon}_r^{(j)} \right) \bigl(1 - |z_r|^2\bigr)^{-2} \\
& \times \exp \sum_{s>r} \left[\left(\sum_{i=1}^{\tilde{n}_r} \sum_{j=1}^{n_s} \left(-\frac{\alpha'}{2}\right) \bar{\epsilon}_r^{(i)} \cdot \epsilon_s^{(j)} - \sum_{i=1}^{\tilde{m}_r} \sum_{j=1}^{m_s} \bar{\varepsilon}_r^{(i)} \circ \varepsilon_s^{(j)} \right) (1 - \bar{z}_r z_s)^{-2} + \text{c.c.} \right] \\
& \times \exp \left[-\sum_{s>r} \left(\sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \left(-\frac{\alpha'}{2}\right) \epsilon_r^{(i)} \cdot \epsilon_s^{(j)} - \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} \varepsilon_r^{(i)} \circ \varepsilon_s^{(j)} \right) (z_s - z_r)^{-2} + \text{c.c.} \right] \\
& \times \exp \sum_{r \neq s} \left[\left(\sum_{i=1}^{n_s} \left(-\frac{\alpha'}{2}\right) k_r \cdot \epsilon_s^{(i)} - \sum_{i=1}^{m_s} \lambda_r \circ \varepsilon_s^{(i)} \right) \bigl((z_r - z_s)^{-1} \right. \\
& \left. + \bigl(\bar{z}_r^{-1} - z_s\bigr)^{-1} \bigr) + \text{c.c.} \right] \\
& \times \exp \sum_{r=1}^{N} \left[\left(\left(-\frac{\alpha'}{2}\right) k_r \cdot \sum_{i=1}^{n_r} \epsilon_r^{(i)} - \lambda_r \circ \sum_{i=1}^{m_r} \varepsilon_r^{(i)} \right) \bigl(\bigl(\bar{z}_r^{-1} - z_r\bigr)^{-1} \right. \\
& \left. \left. + z_r^{-1} \bigr) + \text{c.c.} \right] \right|_{\text{multilinear}},
\end{aligned} \tag{15}$$

where we have

$$\sum_{r=1}^{N}\lambda_r=\sum_{r=1}^{N}\tilde{\lambda}_r=0,\qquad \sum_{r=1}^{N}k_r=0\quad\text{and}\quad\sum_{r=1}^{N}(q_r+\tilde{q}_r)=-2$$

correspond to the conservation of fermion number, the conservation of momentum and the fact that background superghost number is -2. $\frac{2\pi\, d^2 z_o}{|1-z_o\bar{z}_o|^2}$ is the volume element of the CKG,[4] it can be used to fix one complex coordinate.

An integral over the fundamental region $|z|<1$ is equal to an integral over the other fundamental region $|z|>1$. So we can use $(\frac{1}{2})^{N-1}\int_{\mathbb{C}}\prod_{i=1}^{N}d^2 z_i$ instead of the integrals over the unit disk. For any $z_r=x_r+iy_r$, the z_r integral can be given by $\int_{-\infty}^{\infty}dx_r\int_{-\infty}^{\infty}dy_r$. We then follow the same steps as in [9]. We rotate the contour of the y integrals along the real axis to pure imaginary axis. The fixed point should be transformed simultaneously to guarantee the conformal invariance. Define the new variables:

$$\begin{aligned}&\xi_1=\xi_o=x_o+iy_o,\qquad \eta_1=\eta_o=x_o-iy_o,\\ &\xi_r\equiv x_r+iy_r,\qquad \eta_r\equiv x_r-iy_r\quad(r>1).\end{aligned}\tag{16}$$

Then the integrals become real integrals:

$$\begin{aligned}\mathcal{A}_{D_2}^{(N,0)}={}&\kappa^{N-1}\left(\frac{1}{2}\right)^{N-1}\int\prod_{i=1}^{N}d\xi_i\,d\eta_i\,\frac{|1-\xi_o\eta_o|^2}{2\pi\,d\xi_o\,d\eta_o}\\ &\times\prod_{s>r}(\xi_s-\xi_r)^{\frac{\alpha'}{2}k_r\cdot k_s+\lambda_r\circ\lambda_s-q_rq_s}(\eta_r-\eta_s)^{\frac{\alpha'}{2}k_r\cdot k_s+\tilde{\lambda}_r\circ\tilde{\lambda}_s-\tilde{q}_r\tilde{q}_s}\\ &\times\prod_{r,s}\left(1-(\xi_r\eta_s)^{-1}\right)^{\frac{\alpha'}{2}k_r\cdot k_s+\lambda_r\circ\tilde{\lambda}_s-q_r\tilde{q}_s}\\ &\times\exp\sum_{r=1}^{N}\left(\sum_{i=1}^{n_r}\sum_{j=1}^{\tilde{n}_s}\left(-\frac{\alpha'}{2}\right)\epsilon_r^{(i)}\cdot\bar{\epsilon}_r^{(j)}-\sum_{i=1}^{m_r}\sum_{j=1}^{\tilde{m}_s}\varepsilon_r^{(i)}\circ\bar{\varepsilon}_r^{(j)}\right)(1-\xi_r\eta_r)^{-2}\\ &\times\exp\sum_{s>r}\left[\left(\sum_{i=1}^{\tilde{n}_r}\sum_{j=1}^{n_s}\left(-\frac{\alpha'}{2}\right)\bar{\epsilon}_r^{(i)}\cdot\epsilon_s^{(j)}-\sum_{i=1}^{\tilde{m}_r}\sum_{j=1}^{m_s}\bar{\varepsilon}_r^{(i)}\circ\varepsilon_s^{(j)}\right)(1-\eta_r\xi_s)^{-2}+\text{c.c.}\right]\\ &\times\exp\left[-\sum_{s>r}\left(\sum_{i=1}^{n_r}\sum_{j=1}^{n_s}\left(-\frac{\alpha'}{2}\right)\epsilon_r^{(i)}\cdot\epsilon_s^{(j)}-\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}\varepsilon_r^{(i)}\circ\varepsilon_s^{(j)}\right)(\xi_s-\xi_r)^{-2}+\text{c.c.}\right]\\ &\times\exp\sum_{r\neq s}\left[\left(\sum_{i=1}^{n_s}\left(-\frac{\alpha'}{2}\right)k_r\cdot\epsilon_s^{(i)}-\sum_{i=1}^{m_s}\lambda_r\circ\varepsilon_s^{(i)}\right)\right.\\ &\left.\times\left((\xi_r-\xi_s)^{-1}+\left(\eta_r^{-1}-\xi_s\right)^{-1}\right)+\text{c.c.}\right]\end{aligned}$$

[4] To divide the amplitude by the volume of CKG, we can fix three real coordinate. We can also fix two real coordinate or one complex coordinate, then divide the amplitude by volume of the one-parameter subgroup left. The two method are equivalence. Here, we use the second method to fix $z_1=z_o$.

$$\times \exp \sum_{r=1}^{N} \left[\left(\left(-\frac{\alpha'}{2} \right) k_r \cdot \sum_{i=1}^{n_r} \epsilon_r^{(i)} - \lambda_r \circ \sum_{i=1}^{m_r} \varepsilon_r^{(i)} \right) \times \left(\left(\eta_r^{-1} - \xi_r \right)^{-1} + \xi_r{}^{-1} \right) + \text{c.c.} \right] \Bigg|_{\text{multilinear}}. \tag{17}$$

The real variables ξ_r correspond to the left-moving sector and η_r correspond to the left-moving sector. An open string tree amplitude for M bosonized vertices has the form

$$\begin{aligned}
\mathcal{M}_{D_2}^{(N)} = (g)^{M-2} \int \prod_{i=1}^{M} dx_i \frac{|x_a - x_b||x_b - x_c||x_c - x_a|}{dx_a\, dx_b\, dx_c} \\
\times \prod_{s>r} |x_s - x_r|^{2\alpha' k_r \cdot k_s} (x_s - x_r)^{\lambda_r \circ \lambda_s - q_r q_s} \\
\times \exp \left[\sum_{s>r} \left(\sum_{i=1}^{n_r} \sum_{j=1}^{n_s} (2\alpha') \epsilon_r^{(i)} \cdot \epsilon_s^{(j)} + \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} \varepsilon_r^{(i)} \circ \varepsilon_s^{(j)} \right) (x_s - x_r)^{-2} \right] \\
\times \exp \left[\sum_{r \neq s} \left(\sum_{i=1}^{n_s} (-2\alpha') k_r \cdot \epsilon_s^{(i)} - \sum_{i=1}^{m_s} \lambda_r \circ \varepsilon_s^{(i)} \right) (x_r - x_s)^{-1} \right] \Bigg|_{\text{multilinear}},
\end{aligned} \tag{18}$$

where g is the coupling constant for open strings, it can be related with closed string coupling constant by $\kappa \sim g^2$. By comparing (17) with the open string amplitude (18), we can see, the interactions in one sector can be considered as interactions between open strings. The interactions between left- and right-moving sectors look like those between open strings inserted at ξ_r and $(\eta_s)^{-1}$. η_s can be considered as the coordinates of the right-moving open string. Then in the (τ, σ) coordinate, $\eta_s{}^{-1} = e^{-\tau}$ can be considered as a time reverse in the right-moving sector. Thus the interactions between the two sectors can be regarded as interactions between left- and right-moving open strings with a time reverse in the right moving sector (see Fig. 1(b)). The amplitude (17) then can be considered as an amplitude for $2N$ open strings. N of them correspond to the left-moving sector and the other N of them correspond to the right-moving sector. In the amplitude we have a time reverse in the right-moving sector.

From Fig. 1(b), we can see, if we reverse the time in the right-moving sector, we will get an open string tree amplitude. In fact, we can replace all the $\eta_r{}^{-1}$ by η_r. By using the mass-shell condition [18,19] which is determined by the conformal invariance in one sector, the interactions between the two sectors as well as the interactions in one sector become those between open strings. Define

$$\begin{aligned}
&\xi_{r+N} \equiv \eta_r, \qquad k_{r+N} \equiv k_r, \qquad \tilde{\lambda}_{r+N} \equiv \lambda_r, \\
&\bar{\epsilon}_{r+N} \equiv \epsilon_r, \qquad \bar{\varepsilon}_{r+N} \equiv \varepsilon_r.
\end{aligned} \tag{19}$$

After the simultaneous transformations, the volume of CKG becomes $\frac{1}{2\pi} \int \frac{d\xi_o\, d\eta_o}{|\xi_o - \eta_o|^2}$. The fixed points become $\xi_1 = \xi_o$ and $\xi_{1+N} = \xi_o$. The conformal Killing volume has another form $\int \frac{dx_a\, dx_b\, dx_c}{|x_a - x_b||x_b - x_c||x_c - x_a|}$, it can be used to fix three real variables. We reset the fixed points at:

$$\xi_1 = x_a = 0, \qquad \xi_2 = x_b = 1, \qquad \xi_{2N} = x_c = \infty. \tag{20}$$

The amplitude for N closed strings on D_2 then becomes

$$\mathcal{A}_{D_2}^{(N,0)} = \kappa^{N-1}\left(\frac{i}{4}\right)^{N-1}\int\prod_{i=1}^{2N} d\xi_i \frac{|\xi_a-\xi_b||\xi_b-\xi_c||\xi_c-\xi_a|}{d\xi_a\,d\xi_b\,d\xi_c}$$
$$\times\prod_{s>r}(\xi_s-\xi_r)^{\frac{\alpha'}{2}k_r\cdot k_s}(\xi_s-\xi_r)^{\lambda_r\circ\lambda_s-q_rq_s}$$
$$\times\exp\left[\sum_{s>r}\left(\sum_{i=1}^{n_r}\sum_{j=1}^{n_s}(2\alpha')\epsilon_r^{(i)}\cdot\epsilon_s^{(j)}+\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}\varepsilon_r^{(i)}\circ\varepsilon_s^{(j)}\right)(\xi_s-\xi_r)^{-2}\right]$$
$$\times\exp\left[\sum_{r\neq s}\left(\sum_{i=1}^{n_s}(-2\alpha')k_r\cdot\epsilon_s^{(i)}-\sum_{i=1}^{m_s}\lambda_r\circ\varepsilon_s^{(i)}\right)(\xi_r-\xi_s)^{-1}\right]\Bigg|_{\text{multilinear}} e^{i\pi\Theta(P)}. \tag{21}$$

After taking an appropriate phase factor out, we get

$$\mathcal{A}_{D_2}^{(N,0)} = \kappa^{N-1}\left(\frac{i}{4}\right)^{N-1}\int\prod_{i=1}^{2N} d\xi_i \frac{|\xi_a-\xi_b||\xi_b-\xi_c||\xi_c-\xi_a|}{d\xi_a\,d\xi_b\,d\xi_c}$$
$$\times\prod_{s>r}|\xi_s-\xi_r|^{\frac{\alpha'}{2}k_r\cdot k_s}(\xi_s-\xi_r)^{\lambda_r\circ\lambda_s-q_rq_s}$$
$$\times\exp\left[\sum_{s>r}\left(\sum_{i=1}^{n_r}\sum_{j=1}^{n_s}(2\alpha')\epsilon_r^{(i)}\cdot\epsilon_s^{(j)}+\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}\varepsilon_r^{(i)}\circ\varepsilon_s^{(j)}\right)(\xi_s-\xi_r)^{-2}\right]$$
$$\times\exp\left[\sum_{r\neq s}\left(\sum_{i=1}^{n_s}(-2\alpha')k_r\cdot\epsilon_s^{(i)}-\sum_{i=1}^{m_s}\lambda_r\circ\varepsilon_s^{(i)}\right)(\xi_r-\xi_s)^{-1}\right]\Bigg|_{\text{multilinear}} e^{i\pi\Theta(P)}, \tag{22}$$

where we have absorbed a factor $\frac{1}{2}$ into each ϵ. $\Theta(P)$ is defined as

$$\Theta(P)=\sum_{s>r}2\alpha' k'_s\cdot k'_r\theta(\xi_s-\xi_r), \tag{23}$$

where $k_r'^{\mu}=\frac{1}{2}k_r^{\mu}$ is the momentum of the open string and

$$\theta(\xi_s-\xi_r)=\begin{cases}0 & (\xi_s>\xi_r),\\ 1 & (\xi_s<\xi_r).\end{cases} \tag{24}$$

From (18) and (22) we can see amplitudes for N closed strings on D_2 can be given by one open string tree amplitude for $2N$ open strings except for a phase factor. The phase factor is caused by taking absolute number of $(\xi_s-\xi_r)$ in $(\xi_s-\xi_r)^{\frac{\alpha'}{2}k_r\cdot k_s}$. It is used to guarantee the integrals in the right branch cut. It only depend on the orderings of the open strings. For a certain order P, the phase factor decouple from the integrals. So we can break the integrals into pieces, take the multilinear terms in ϵ, $\bar{\epsilon}$, ε and $\bar{\varepsilon}$, replace the polarization vectors by the polarization tensors of closed strings. Then we get the relation between closed string amplitudes and partial amplitudes for open strings on D_2:

$$\mathcal{A}_{D_2}^{(N,0)}=\kappa^{N-1}\epsilon_{\alpha\beta}\mathcal{A}_{D_2}^{(N)\alpha\beta}=\left(\frac{i}{4}\right)^{N-1}\kappa^{N-1}\epsilon_{\alpha\beta}\sum_P\mathcal{M}^{(2N)\alpha\beta}(P)e^{i\pi\Theta(P)}, \tag{25}$$

where $\mathcal{M}$ is the open string amplitude without the coupling constant g, and we sum over all the orderings P of the open strings.

If there are open strings on the boundary of D_2, we can insert the open string vertices into the amplitude (22). Because (22) is already an amplitude for open strings on the real axis except for a phase factor, we just increase the number of the open strings on the boundary of D_2 and adjust the phase factor to make the integrals in the right branch cuts. The phase factor should be adjusted because we must consider the interactions between closed and open strings. Then we have

$$\begin{aligned}\mathcal{A}_{D_2}^{(N,M)} &= \epsilon_{\alpha\beta\gamma}\mathcal{A}_{D_2}^{(N,M)\alpha\beta\gamma} \\ &= \left(\frac{i}{4}\right)^{N-1}\kappa^{N-1}g^M\epsilon_{\alpha\beta\gamma}\sum_P \mathcal{M}^{(2N,M)\alpha\beta\gamma}(P)e^{i\pi\Theta'(P)},\end{aligned} \tag{26}$$

where we have defined the coordinates of the left-moving open strings are $\xi_1,\dots,\xi_N$, those of right-moving open strings are $\xi_{1+N+M},\dots,\xi_{2N+M}$ and the coordinates of other open strings are $\xi_{1+N},\dots,\xi_{M+N}$.

$$\Theta'(P) = \sum_{s>r} 2\alpha' k'_s \cdot k'_r \theta'(\xi_s - \xi_r), \tag{27}$$

where k'_r are the momentums of the open strings. If $\xi_s > \xi_r$, $\theta'(\xi_s - \xi_r) = 0$, else if $\xi_s < \xi_r$ but $N < s$, $r < N + M + 1$, $\theta'(\xi_s - \xi_r) = 0$, otherwise $\theta'(\xi_s - \xi_r) = 1$. This relation can also be derived by choosing the fundamental region as the upper half-plane, then repeat the similar steps in the case of N closed strings on D_2. We can see if $M = 0$, (6) gives the relation for N closed strings on D_2 (25) and if $N = 0$ it gives the open string tree amplitude (18).

By comparing the relations (25) with KLT factorization relations (2), we can see, in (2), the left- and the right-moving sectors are independent of each other. In (25), they are not independent of each other. The interactions connect the two open string amplitudes into a single one. Because the interactions between the two sectors are just the open string interactions, the amplitudes for N closed strings then can be given by tree amplitudes for $2N$ open strings.

We can consider the relations on D_2 as any closed strings can be splitted into two open strings. Each open string catch half of the momentum of the closed string. Move the open strings corresponding to the two sectors of closed strings onto the boundary of D_2. Then an amplitude for N closed strings and $2M$ open strings on D_2 is given by an amplitude for $N + 2M$ open strings.

In (26), the D_2 amplitudes have been given by the sum of open string partial amplitudes with $2N + M$ external legs correspondingly. We have to sum over all the orderings of the open strings in this relation. However, as in the case of S_2 [9], the contour treatment [29] can reduce the number of the terms in this relation. The main points of the treatment of [29] is there are relations among open string partial amplitudes. Then any open string partial amplitudes can be expressed in terms of a minimal basis. All the M-point open string partial amplitude can be expressed in terms of the minimal basis of $(M - 3)!$ independent partial amplitudes. Then for the (N, M) case, the amplitude can be given by $(2N + M - 3)!$ open string partial amplitudes.

If we consider the interactions between open strings attached to a Dp-brane and closed strings, we should do appropriate replacements in the right-moving sector. For example, if the external legs are gravitons, we just need to replace the momenta $\frac{1}{2}k_r^\mu$ corresponding to the right-moving sector by $\frac{1}{2}D^\mu_\nu \cdot k_r^\nu$ and replace the polarization tensor $\epsilon_{\mu\nu}$ by $\epsilon_{\mu\lambda}D^\lambda_\nu$ in the relation (26) [25–27], where D^μ_ν is defined as

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}. \tag{28}$$

Then this relation reveals the amplitudes between N closed string and M open strings on a Dp-brane can be given by $2N + M$-point open string partial amplitudes. Though in [25–27], $(2, 0)$ amplitude and $(1, 2)$ amplitude are four-point open string amplitudes upon a certain identification between the momenta and polarizations, in general case, there is a phase factor in the relations.

In the low energy limit of an open string theory, gravitons are closed string states and gauge particles are open string states. Then in this case, the KLT factorization relations do not hold. We should use one amplitude for $2N$ gauge particles instead of the product of two amplitudes for N gauge particles to give an amplitude for N gravitons.

3. Relations between amplitudes on RP_2 and open string tree amplitudes

In this section, we will explore the amplitudes on RP_2 [1,2,28]. We first show the correlation functions on RP_2 cannot be factorized by the left- and the right-moving sectors. The two sectors are connected together. Then we will give the relations between amplitude on RP_2 and tree amplitude for open strings.

RP_2 is an unoriented surface, it can be derived by identifying the diametrically opposite points on S_2. It can be considered as a sphere with a crosscap. With this equivalence, the waves must be reflected at the crosscap. The reflection waves of the left-moving waves are in the right-moving sector and the reflection waves of the right-moving waves are in the left-moving sector. The waves must interact with their reflection waves, thus the two sectors must interact with each other. This is similar with the case of D_2.

Particularly, the correlation function on RP_2 is given as

$$\langle 0|\mathcal{V}_N(\omega, \tilde{\omega}) \ldots \mathcal{V}_1(\omega, \tilde{\omega})|C\rangle, \tag{29}$$

where $|C\rangle = C|0\rangle$ is the boundary sate for RP_2 [20,21]. The bosonized boundary operator C is

$$\begin{aligned} C = {} & \exp\left(\sum_{n=1}^{\infty}(-1)^n a_n^\dagger \cdot \tilde{a}_n^\dagger\right)|0\rangle_X \otimes \exp\left(\sum_{n=1}^{\infty}(-1)^n b_n^\dagger \circ \tilde{b}_n^\dagger\right)|0\rangle_\phi \\ & \otimes \exp\left(\sum_{n=1}^{\infty}(-1)^n c_n^\dagger \tilde{c}_n^\dagger\right)|0\rangle_{\phi_6}. \end{aligned} \tag{30}$$

In this case we can see, the image point of ω is $-\bar{\omega} + i\pi$. When we move C to the left of a vertex operator it commute with the creation modes and the zero modes of the vertex operator. It does not commute with the annihilation modes $\mathcal{V}_L^{(-)}(\omega)$ and $\tilde{\mathcal{V}}_R^{(-)}(\bar{\omega})$. This means only the annihilation modes can be reflected at the crosscap (see Fig. 3). After moving the boundary operator to the left of the annihilation modes $\mathcal{V}_L^{(-)}(\omega)$ and $\tilde{\mathcal{V}}_R^{(-)}(\bar{\omega})$, the images $\tilde{\mathcal{V}}_L^{(+)}(-\omega - i\pi)$ and $\mathcal{V}_R^{(+)}(-\bar{\omega} + i\pi)$ are created respectively. We move the boundary operator to the left of all the vertex operators. Then use the creation operators in the boundary operator to annihilate the state $\langle 0|$. The correlation becomes

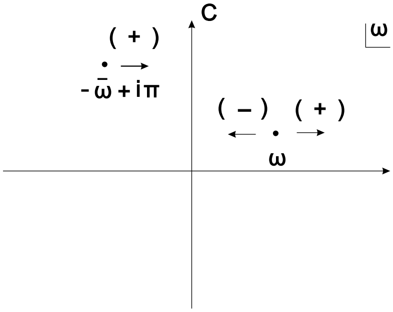

Fig. 3. Only the annihilation modes are reflected at the crosscap of RP_2.

$$\begin{aligned}&\big\langle \mathcal{V}_L^{(+)}(\omega_N)\mathcal{V}_L^{(-)}(\omega_N)\mathcal{V}_R^{(+)}(-\bar{\omega}_N+i\pi)\dots\mathcal{V}_L^{(+)}(\omega_1)\mathcal{V}_L^{(-)}(\omega_1)\mathcal{V}_R^{(+)}(-\bar{\omega}_1+i\pi)\big\rangle\\&\quad\times\big\langle \tilde{\mathcal{V}}_R^{(+)}(\bar{\omega}_N)\tilde{\mathcal{V}}_R^{(-)}(\bar{\omega}_N)\tilde{\mathcal{V}}_L^{(+)}(-\omega_N-i\pi)\dots\tilde{\mathcal{V}}_R^{(+)}(\bar{\omega}_1)\tilde{\mathcal{V}}_R^{(-)}(\bar{\omega}_1)\tilde{\mathcal{V}}_L^{(+)}(-\omega_1-i\pi)\big\rangle\\&\quad\times\big\langle \mathcal{V}_0(\omega_N,\tilde{\omega}_N)\dots\mathcal{V}_0(\omega_1,\tilde{\omega}_1)\big\rangle.\end{aligned}\tag{31}$$

As in the case of D_2, the first correlation function in (31) only contain a, b, c and $a^\dagger$, $b^\dagger$, $c^\dagger$. When we move the left-moving modes of a vertex operator $\mathcal{V}_L^{(-)}(\omega_r)$ to the right of the operator $\mathcal{V}_L^{(+)}(\omega_s)$, we get the interaction in the left-moving sector. When we move $\mathcal{V}_L^{(-)}(\omega_r)$ to the right of $\mathcal{V}_R^{(+)}(-\bar{\omega}_s+i\pi)$, we get the interaction between the left- and the right-moving sectors. In the same way, the second correlation function in (31) gives the interactions in the right-moving sector and those between the two sectors. Thus the correlation function cannot be factorized by the two sectors. Interactions connect the two sectors together.

Now we consider the amplitude for N closed strings on RP_2. We calculate the correlation function, integral over the fundamental region and divide the integrals by the volume of the CKG [1,2,22,23] on RP_2. As we have done in the case of D_2, we also extend the integral region to the complex pane, rotate the y integrals to the real axis and redefine the integral variables. The amplitude for N closed strings on RP_2 can be given as

$$\begin{aligned}\mathcal{A}_{RP_2}^{(N)}&=\kappa^{N-1}\left(\frac{1}{2}\right)^{N-1}\int\prod_{i=1}^{N}d\xi_i\,d\eta_i\,\frac{|1+\xi_o\eta_o|^2}{2\pi\,d\xi_o\,\eta_o}\\&\quad\times\prod_{s>r}(\xi_s-\xi_r)^{\frac{\alpha'}{2}k_r\cdot k_s+\lambda_r\circ\lambda_s-q_rq_s}(\eta_r-\eta_s)^{\frac{\alpha'}{2}k_r\cdot k_s+\tilde{\lambda}_r\circ\tilde{\lambda}_s-\tilde{q}_r\tilde{q}_s}\\&\quad\times\prod_{r,s}\big(1+(\xi_r\eta_s)^{-1}\big)^{\frac{\alpha'}{2}k_r\cdot k_s+\lambda_r\circ\tilde{\lambda}_s-q_r\tilde{q}_s}\\&\quad\times\exp\sum_{r=1}^{N}\left(\sum_{i=1}^{n_r}\sum_{j=1}^{\tilde{n}_s}\left(-\frac{\alpha'}{2}\right)\epsilon_r^i\cdot\bar{\epsilon}_r^j-\sum_{i=1}^{m_r}\sum_{j=1}^{\tilde{m}_s}\varepsilon_r^i\circ\bar{\varepsilon}_r^j\right)(1+\xi_r\eta_r)^{-2}\\&\quad\times\exp\sum_{s>r}\left[\left(\sum_{i=1}^{n_r}\sum_{j=1}^{n_s}\left(-\frac{\alpha'}{2}\right)\bar{\epsilon}_r^{(i)}\cdot\epsilon_s^{(j)}-\sum_{i=1}^{\tilde{m}_r}\sum_{j=1}^{m_s}\bar{\varepsilon}_r^{(i)}\circ\varepsilon_s^{(j)}\right)(1+\eta_r\xi_s)^{-2}+\text{c.c.}\right]\\&\quad\times\exp\left[-\sum_{s>r}\left(\sum_{i=1}^{n_r}\sum_{j=1}^{n_s}\left(-\frac{\alpha'}{2}\right)\epsilon_r^{(i)}\cdot\epsilon_s^{(j)}-\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}\varepsilon_r^{(i)}\circ\varepsilon_s^{(j)}\right)(\xi_s-\xi_r)^{-2}+\text{c.c.}\right]\end{aligned}$$

$$\times \exp \sum_{r \neq s} \Bigg[\Bigg(\sum_{i=1}^{n_s} \bigg(-\frac{\alpha'}{2}\bigg) k_r \cdot \epsilon_s^{(i)} - \sum_{i=1}^{m_s} \lambda_r \circ \varepsilon_s^{(i)} \Bigg)$$
$$\times \big((\xi_r - \xi_s)^{-1} + \big(-\eta_r^{-1} - \xi_s\big)^{-1} \big) + \text{c.c.} \Bigg]$$
$$\times \exp \sum_{r=1}^{N} \Bigg[\Bigg(\bigg(-\frac{\alpha'}{2}\bigg) k_r \cdot \sum_{i=1}^{n_r} \epsilon_r^{(i)} - \lambda_r \circ \sum_{i=1}^{m_r} \varepsilon_r^{(i)} \Bigg)$$
$$\times \big(\big(-\eta_r^{-1} - \xi_r\big)^{-1} + \xi_r{}^{-1} \big) + \text{c.c.} \Bigg] \Bigg|_{\text{multilinear}}. \tag{32}$$

Then the amplitude has been given by real integrals. The interactions in one sector are the open string interactions. The interaction between left- and right-moving sectors can be considered as interactions between open strings inserted at ξ_r and $(-\eta_s)^{-1}$. $\frac{1}{-\bar{\eta}_s}$ can be considered as a time reverse and a twist in the right-moving sector. Then the interactions between left- and right-moving sectors can be regarded as interactions between left- and right-moving open strings with a time reverse and a twist in the right-moving sector (see Fig. 1(c)).

From Fig. 1(c), we can see, if we twist the right-moving sector and reverse the time in the right-moving sector, we will get an open string tree amplitude. In fact, we can replace all the η_r by $-\frac{1}{\eta_r}$. Then by using the mass-shell condition, the interactions between the two different sectors as well as in one sector become the interactions between open strings. Redefine the variables in the right-moving sector by Eq. (19). The amplitude on RP_2 then becomes

$$\mathcal{A}_{RP_2}^{(N)} = -\bigg(\frac{i}{4}\bigg)^{N-1} \kappa^{N-1} \int \prod_{i=1}^{2N} d\xi_i \frac{|\xi_a - \xi_b||\xi_b - \xi_c||\xi_c - \xi_a|}{d\xi_a\, d\xi_b\, d\xi_c}$$
$$\times \prod_{s>r} |\xi_s - \xi_r|^{\frac{\alpha'}{2} k_r \cdot k_s} (\xi_s - \xi_r)^{\lambda_r \circ \lambda_s - q_r q_s}$$
$$\times \exp \Bigg[\sum_{s>r} \Bigg(\sum_{i=1}^{n_r} \sum_{j=1}^{n_s} (2\alpha') \epsilon_r^{(i)} \cdot \epsilon_s^{(j)} + \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} \varepsilon_r^{(i)} \circ \varepsilon_s^{(j)} \Bigg) (\xi_s - \xi_r)^{-2} \Bigg]$$
$$\times \exp \Bigg[\sum_{r \neq s} \Bigg(\sum_{i=1}^{n_s} (-2\alpha') k_r \cdot \epsilon_s^{(i)} - \sum_{i=1}^{m_s} \lambda_r \circ \varepsilon_s^{(i)} \Bigg) (\xi_r - \xi_s)^{-1} \Bigg] \Bigg|_{\text{multilinear}} e^{i\pi\Theta(P)}. \tag{33}$$

This amplitude is different from D_2 amplitude by a factor -1. It is caused by the difference between the measure of the CKG on RP_2 and D_2. When we change the topology, this -1 appears. The phase factor only depends on the ordering of the open strings. We can break the integrals into pieces as in the case of D_2 and keep the multilinear terms of the polarization tensors. Then Eq. (33) becomes

$$\mathcal{A}_{RP_2}^{(N)} = \epsilon_{\alpha\beta} \mathcal{A}_{RP_2}^{(N)\alpha\beta} = -\bigg(\frac{i}{4}\bigg)^{N-1} \kappa^{N-1} \epsilon_{\alpha\beta} \sum_P \mathcal{M}^{(2N)\alpha\beta}(P) e^{i\pi\Theta(P)}. \tag{34}$$

As in the case of D_2, KLT factorization relations (2) do not hold on RP_2. The left- and the right-moving sectors are not independent of each other again. The interactions between the two sectors connect them into a single sector. Since the interactions between the two sectors are just the those between open strings, the two open string tree amplitude in the case of S_2 are connected into one amplitude for open strings. In the relation (34), we also sum over all the orderings of the external legs of the open strings. By using the same method in [29], the relations on RP_2 for N closed strings can be reduced to $(2N-3)!$ terms.

From the relations (25) and (34) we can see, the amplitudes on D_2 and RP_2 with same external closed string states are equal except for a factor -1. In fact, after we transform the complex variables into real ones, the image of a point ξ_r in the left-moving sector becomes $\frac{1}{\eta_r}$ on D_2 and $-\frac{1}{\eta_r}$ on RP_2. The minus means a twist in the right-moving sector. After this twist, the amplitude on RP_2 becomes that on D_2 except for a factor -1. Then if we consider a theory containing both D_2 and RP_2, the amplitudes with same external states cancel out. However, if we consider T-duality, the interactions on D_2 becomes interactions between closed strings and D-brane, while the interactions on RP_2 becomes interactions between closed strings and O-plane. As we have seen in Section 2, we should make appropriate replacement on the momenta and polarizations in the right-moving sector to give the relations in D_2 case. Under the T-duality, we also need to replace the vertex operators on RP_2 by new ones [30]. We take the massless NS–NS vertex operator as an example. The vertex operator after T-duality becomes

$$\begin{aligned}\mathcal{V}^{RP_2}(\epsilon,k,z,\bar{z}) = \frac{1}{2}\big(&\epsilon_{\mu\nu}{:}\mathcal{V}^{\mu}_{\alpha}(k,z){:}{:}\tilde{\mathcal{V}}^{\mu}_{\beta}(k,\bar{z}){:}\\ &+\big(D\cdot\epsilon^{T}\cdot D\big)_{\mu\nu}{:}\mathcal{V}^{\mu}_{\alpha}(k\cdot D,z){:}{:}\tilde{\mathcal{V}}^{\nu}_{\beta}(k\cdot D,\bar{z}){:}\big).\end{aligned} \tag{35}$$

Here $\mathcal{V}^{\mu}_{\alpha}(p,\epsilon)$ in 0 and -1 picture are

$$\begin{aligned}&\mathcal{V}^{\mu}_{-1}(k,z) = e^{-\phi(z)}\psi^{\mu}(z)e^{ik\cdot X(z)},\\ &\mathcal{V}^{\mu}_{0}(k,z) = \big(\partial X^{\mu}(z) + ik\cdot\psi(z)\psi^{\mu}(z)\big)e^{ik\cdot X}.\end{aligned} \tag{36}$$

The second term in (35) can also be given by replacing $\epsilon_{\mu\nu}$ and k^{μ} in the original vertex by $(D\cdot\epsilon^{T}\cdot D)_{\mu\nu}$ and $(k\cdot D)^{\mu}$ respectively. Now we consider the amplitudes for N NS–NS strings. Each vertex operator (35) have two terms, each term can be considered as a vertex operator on RP_2 under appropriate replacement. The amplitude then is given by 2^N terms, each term can be obtained from the amplitude before T-duality by appropriate replacements. Then each term can be given by partial amplitudes for $2N$ open strings again. So the RP_2 relation gives the amplitudes for closed strings scattering from an O-plane by open string amplitudes. Generally, under the T-duality, the D_2 amplitudes cannot be canceled by the RP_2 amplitudes [30].

In the low energy limit of an unoriented string theory, the amplitudes for closed strings on RP_2 contribute to the amplitudes for gravitons. Then the KLT factorization relations do not hold in this case as in the case of D_2. The amplitudes for N gravitons cannot be factorized by two amplitudes for N gauge particles. They can be given by an amplitude for $2N$ gauge particles.

4. Conclusion

In this paper, we investigated the relations between closed and open strings on D_2 and RP_2. We have shown that the KLT factorization relations do not hold for these two topologies. The closed string amplitudes cannot be factorized by tree amplitudes for left- and right-moving open strings. However, the two sectors are connected into a single sector. We can give the amplitudes with closed strings in these two cases by amplitudes in this single sector. The terms in the relations on D_2 and RP_2 can be reduced by contour deformations.

Under the T-duality, the relations on D_2 and RP_2 give the amplitudes between closed strings scattering from D-brane and O-plane respectively by open string partial amplitudes.

In the low energy limits of these two cases, we cannot use KLT relations to factorize amplitudes for gravitons into products of two amplitudes for gauge particles. Interactions between the "left-" and the "right-"moving gauge fields connect the two amplitudes into one. Then an graviton amplitude in these two cases can be given by one amplitude for both left- and right-moving gauge particles.

The relations for other topologies have not been given. However, we expect there are also some relations between closed and open string amplitudes. If there are more boundaries and crosscaps on the world-sheet, the boundaries and the crosscaps also connect left- and the right-moving sectors, then in these cases, KLT factorization relations do not hold.

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