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Subgraphs and the Laplacian spectrum of a graph

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ABSTRACT

Let G be a graph and H a subgraph of G. In this paper, a set of pairwise independent subgraphs that are all isomorphic copies of H is called an *H-matching*. Denoting by $\nu(H, G)$ the cardinality of a maximum H-matching in G, we investigate some relations between $\nu(H, G)$ and the Laplacian spectrum of G.

1. Introduction

Let $G = (V, E)$ be a finite simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. Denoting by $d_G(i)$ the degree of vertex i, let

$$D(G) = \mathrm{diag}(d_G(1), d_G(2), \ldots, d_G(n))$$

be the diagonal matrix of vertex degrees. The Laplacian matrix $L(G)$ of G is defined by $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G. The matrix $L(G)$ is positive semi-definite. The spectrum of $L(G)$ is

$$Spec(G) = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}),$$

where $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ are eigenvalues of $L(G)$ arranged in nondecreasing order. We set $\mu_i = \lambda_{n-i}, \quad 1 \leq i \leq n;$

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0.$$

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When more than one graph is under discussion, we may write $\lambda_i(G)$(resp. $\mu_i(G)$) instead of λ_i (resp. μ_i).

We now fix some notation and terminology. For an eigenvalue $\lambda \in Spec(G)$, we denote its multiplicity by $m_G(\lambda)$.

We denote by K_n, $K_{1,n-1}$ and P_n, the complete graph, the star graph and the path graph of order n, respectively. A connected subgraph of a tree T will be called a *subtree* of T.

For graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ with disjoint vertex sets, the *union* of G_1 and G_2 is $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. If G_1 and G_2 are isomorphic, $G_1 + G_2$ may be written as $2G_1$. For graphs $G_1, G_2, \ldots, G_k$, we denote their union by $G_1 + G_2 + \cdots + G_k$.

A set of pairwise independent edges in a graph G is called a *matching* in G. The maximum cardinality of a matching is called the *matching number* of G and we denote it by $\nu(G)$. A perfect matching is a matching that satisfies $2\nu(G) = |V(G)|$.

Subgraphs $G_1, G_2, \ldots, G_k$ of a graph G are said to be *independent* if $V(G_i) \cap V(G_j) = \emptyset$ for all $1 \leq i < j \leq k$. We generalize the notion of matching as follows. For a subgraph H of a graph G, a set of pairwise independent subgraphs that are all isomorphic copies of H is called an *H-matching*. The maximum cardinality of an H-matching is called the *H-matching number* of G and we denote it by $\nu(H, G)$. A perfect H-matching is an H-matching that satisfies $|V(H)|\nu(H, G) = |V(G)|$.

The *deficiency* of a maximum H-matching of G is defined by $\delta(H, G) = |V(G)| - |V(H)|\nu(H, G)$. Evidently G has a perfect H-matching if and only if $\delta(H, G) = 0$.

It is proved in [3, Theorem 2.5] that, if a tree T has a perfect matching then 2 is a Laplacian eigenvalue of T. We shall prove in Theorem 2.4 that if a tree T has a perfect H-matching by a subtree H of T, then every Laplacian eigenvalue of H is that of G.

For an interval I in $\mathbb{R}$, $m_G(I)$ stands for the number of Laplacian eigenvalues of G, counting multiplicities, that belong to I.

The multiplicity of zero $m_G(0)$ is equal to the number of connected components of G. In particular, G is connected if and only if $\lambda_1 > 0$.

We recall some known results used in this paper. The following result is well known and follows from the Courant-Weyl inequalities.

Theorem 1.1 [1, Theorem 2.1]**.** *For a given graph G let $G' = G + e$ be the graph obtained from G by inserting a new edge e into G. Then the eigenvalues of G and G' interlace:*

$$0 = \lambda_0(G) = \lambda_0(G') \leq \lambda_1(G) \leq \lambda_1(G') \leq \cdots \leq \lambda_{n-1}(G) \leq \lambda_{n-1}(G').$$

In particular,

$$\mu_i(G) \leq \mu_i(G'), \quad 1 \leq i \leq n, \tag{1.1}$$

and

$$\lambda_i(G') \leq \lambda_{i+1}(G), \quad 0 \leq i \leq n-2. \tag{1.2}$$

Theorem 1.2 [6, Theorem 2.2], [3, Theorem 2.1]**.** *Let λ be a Laplacian eigenvalue of a tree T. If λ is not a unit in the ring of algebraic integers, then it is a simple Laplacian eigenvalue of T.*

2. Matching by a subgraph and Laplacian spectrum

The following theorem is the key result in this paper.

Theorem 2.1

(a) *Let H be a connected subgraph of a connected graph G. Then*

$$m_G[\mu_i(H), \infty) \geq i\nu(H, G), \quad 1 \leq i \leq |V(H)|.$$

(b) *Let H be a subtree of a tree G. Then*

$$m_T(0, \lambda_i(H)] \geq i\nu(H, G), \quad 1 \leq i \leq |V(H)| - 1.$$

Proof. Set

$$F = \nu(H, G)H + (|V(G)| - \nu(H, G)|V(H)|)K_1,$$

where K_1 is the trivial graph.

We now prove (a). The graph G can be built up from the spanning subgraph F, by adding new edges. The Laplacian spectrum of F is

$$(\mu_1(H), \ldots, \mu_1(H), \mu_2(H), \ldots, \mu_2(H), \ldots, \mu_r(H), \ldots \mu_r(H), 0, \ldots, 0),$$

where $r = |V(H)|$ and $\mu_i(H)$ appears $\nu(H, G)$ times for each $1 \leq i \leq r$. Then by repeated applications of (1.1) in Theorem 1.1, we have

$$\mu_{i\nu(H,G)}(G) \geq \mu_i(H), \quad 1 \leq i \leq |V(H)|,$$

and this proves (a).

We shall prove (b). We consider the spanning forest F of the tree G. Set

$$s = |V(G)| - \nu(H, G)(|V(H)| - 1).$$

Then s is equal to the number of connected components of F and the Laplacian spectrum of F is

$$(0, \ldots, 0, \lambda_1(H), \ldots, \lambda_1(H), \lambda_2(H), \ldots, \lambda_2(H), \ldots, \lambda_{r-1}(H), \ldots \lambda_{r-1}(H)),$$

where 0 appears s times and $\lambda_i(H)$ appears $\nu(H, T)$ times for each $1 \leq i \leq r - 1$. Then the tree G can be built up from the spanning subgraph F by adding new $s - 1$ edges. Therefore by repeated applications of (1.2) in Theorem 1.1 $s - 1$ times, we find that

$$\lambda_{i\nu(H,G)}(G) \leq \lambda_i(H), \quad 1 \leq i \leq r - 1,$$

and this proves (b). □

Corollary 2.2

1. *Let H be a connected subgraph of a connected graph G. Then*

$$\mu_i(G) \geq \mu_i(H), \quad 1 \leq i \leq |V(H)|.$$

2. *Let H be a subtree of a tree G. Then*

$$\lambda_i(G) \leq \lambda_i(H), \quad 0 \leq i \leq |V(H)| - 1.$$

Example 2.3

1. If a graph G has a vertex of degree $d \geq 1$, then, since $\mu_1(K_{1,d}) = d + 1$, we have $\mu_1(G) \geq d + 1$ [2, Corollary 2].
2. If T is a tree with diameter d, then since $\lambda_1(P_{d+1}) = 2(1 - \cos(\pi/(d+1)))$, we have $\lambda_1(T) \leq 2(1 - \cos(\pi/(d+1))$ [3, Corollary 4.4].

Theorem 2.4 *Let μ be a Laplacian eigenvalue of a subtree H in a tree T. If*

$$\delta(H, T) \leq (m_H(\mu) - 1)\nu(H, T),$$

then μ is a Laplacian eigenvalue of T.

Proof. By Theorem 2.1, we have

$$\mu_{\delta(H,T)+i\nu(H,G)}(T) \leq \mu_i(H) \leq \mu_{i\nu(H,T)}(T), \quad 1 \leq i \leq r. \tag{2.3}$$

Set $k = m_H(\mu) - 1$ and $\mu = \mu_i(H) = \mu_{i+1}(H) = \cdots = \mu_{i+k}(H)$. Then since $\delta(H,T) \leq k\nu(H,T)$, it follows from (2.3) that

$$\mu_{(i+k)\nu(H,T)}(T) \leq \mu_{\delta(H,T)+i\nu(H,T)}(T) \leq \mu_i(H) = \mu_{i+k}(H) \leq \mu_{(i+k)\nu(H,T)}(T),$$

and we have

$$\mu_{(i+k)\nu(H,T)}(T) = \mu_{\delta(H,T)+i\nu(H,T)}(T) = \mu_i(H).$$

This proves the desired result. □

From Theorem 2.4 together with (2.3), we obtain the following:

Theorem 2.5 *Let H be a subtree of order r of a tree T.*

(a) *If T has a perfect H-matching, then*

$$\mu_i(H) = \mu_{i\nu(H,T)}(T), \quad 1 \leq i \leq r.$$

In particular, if T has a perfect H-matching, every Laplacian eigenvalue of H is that of T.

(b) *If $\delta(H,T) \leq \nu(H,T)$, then every multiple Laplacian eigenvalue of H is a Laplacian eigenvalue of T.*

Example 2.6 Let T be a tree. Since $\mu_1(P_2) = 2$, by Theorem 2.4, we recover a result of Guo and Tan [5, Theorem 2] that if T has a perfect matching, then $\mu_{\nu(T)}(T) = 2$.

Example 2.7

(1) If a tree T has a perfect P_3-matching, then since $\mu_1(P_3) = 3$ and $\mu_2(P_3) = 1$,

$$\mu_{\nu(P_3,T)}(T) = 3, \quad \mu_{2\nu(P_3,T)}(T) = 1.$$

(2) Let m and n be positive integers with $m|n$. Then each Laplacian eigenvalue of P_m is a Laplacian eigenvalue of P_n. This also follows from the explicit formula for the Laplacian eigenvalues of P_n and P_m.

Corollary 2.8 *Let T be a tree with a perfect H-matching. If $\mu_i(H)$ is not a unit in the ring of algebraic integers, then*

$$m_T[\mu_i(H), \infty) = i\nu(H,T).$$

Proof. By Theorem 1.2, $\mu_i(H)$ is a simple Laplacian eigenvalue of T and the result follows from Theorem 2.5. □

3. Code and Laplacian spectrum

By a code in a graph G, we mean a subset of $V(G)$. The *minimum distance* $\delta(C)$ of a code C is the minimum distance between distinct vertices in C. For a vertex v in G we denote by $B_e(v)$ the set of all vertices at distance at most e from v. The *packing radius* of C is the maximum integer e such that $B_e(v)$, $v \in C$ are pairwise disjoint. A code C is called an *$e - code$* if the packing radius of C is at least e. The following proposition gives a relation between 1-codes and Laplacian spectrum of a graph.

Proposition 3.1

(a) *Let C be a 1-code of a connected graph G. Then*

$$\mu_{|C|}(G) \geq \min_{v \in C} d_G(v) + 1.$$

(b) *Let C be a 1-code of a tree T. If* $\min_{v \in C} d_T(v) \geq 2$ *then*

$$\lambda_{|C|}(T) \leq 1 \quad \lambda_{2|C|}(T) \leq 3$$

Proof. Set $d = \min_{v \in C} d_G(v)$. Then, since $\nu(K_{1,d}, G) \geq |C|$, $\lambda_1(K_{1,d}) = 1$ and $\mu_1(K_{1,d}) = d + 1$, the result follows from the Theorem 2.1. □

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