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Approximating of Unstable Cycle in Nonlinear Autonomous Systems

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Abstract—Numerical procedures for approximate construction of cycles in nonlinear systems are studied. The procedures are based on functional parameter methods combined with mechanical quadratures, Newton's, and gradient methods. The convergence rate of the procedures is studied, as well as their range of applicability, and their stability with respect to small perturbations of the parameters. The results obtained can be applied to nonlinear problems described by ordinary differential equations, to the systems with delay, and to distributed systems.

Keywords—Limit cycle, Iteration scheme, Delay systems, Extension on parameter, Gradient method.

1. INTRODUCTION

The problem of approximate evaluating oscillatory conditions for nonlinear systems is studied in numerous papers (see [1-5] and their references). The most complete investigation for this problem was led in the case of the forced oscillations, when the period of the oscillatory mode is equal (or multiple) to the period of the external action. General theorems were proved for realizability and convergence of different types of iteration schemes: the harmonic balance method, the collocation method, the method of mechanical quadratures, methods of finite elements, and finite differences; ranges of validity for the above methods were studied. Applications of these methods for various mechanical, physical, and engineering problems are known [6-10].

The problem of approximate evaluating oscillatory conditions for autonomous systems as compared with the same problem for nonautonomous systems is more complicated. There are at least two reasons explaining specific features of autonomous problems. First, a period of an autonomous system is unknown *a priori*; second, a periodic solution of the autonomous system is nonisolated, i.e., the same cycle in the phase space is corresponding all phase lags of the periodic solution. Therefore, the special methods working in these conditions are necessary for the autonomous case.

The powerful method of investigating oscillatory conditions for autonomous systems was proposed in [11]. The parameter's functionalization method allows us to derive special integral equations which determine isolated periodic solutions for autonomous systems and their periods.

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In this paper, the parameter's functionalization method is applied to the problem of approximating cycles for autonomous systems. For this purpose the combination of this method with the mechanical quadrature's method and Newton method is used. The iteration scheme based on this combination is considered in Section 4.

Moreover, two more iteration schemes for approximating oscillatory modes of autonomous systems are analysed in the paper. One of them is based on the method of auxiliary relations [8] and the gradient method. The other solves the problem of approximating oscillatory modes for delay systems and is based on the gradient method of nonlinear minimization in Hilbert spaces (Section 3).

It should be noted that the gradient iteration scheme proposed in this paper has common features with the Sparrow algorithm [12]. However, the gradient algorithm differs in principle from the Sparrow algorithm. There are at least two main distinctions between these algorithms: first, the realization of the gradient method does not require solving of auxiliary equations which may have no solutions; second, the gradient algorithm is robust with respect to initial approximations (the Sparrow method requires more precise initial approximations). The lack of the gradient method is that its convergence is slower than the one for the Sparrow method, but often it is compensated by the above advantages.

2. AN ITERATION SCHEME BASED ON THE GRADIENT METHOD

Suppose the dynamical system

$$\frac{dx}{dt} = f(x), \qquad x \in \mathbb{R}^n \tag{2.1}$$

has a cycle $\Gamma \subset \mathbb{R}^n$. Consider a question on approximate evaluating the cycle Γ for system (2.1).

Assume that Γ is orbital asymptotically stable. If we compute the solution $p(t, x_0)$ of (2.1) with the initial condition x_0 from the attracting set of Γ, we obtain

$$\lim_{t\to\infty} \min_{y\in\Gamma} \|p(t, x_0) - y\| = 0.$$

So, we can get an arbitrarily exact approximation for the cycle Γ. In this case, we use *a priori* information about the location of the cycle Γ in the phase space of (2.1) to choose the initial condition x_0.

However, even if we have an exact localization for Γ, the above procedure is not effective when Γ is unstable.

We propose an iteration scheme for the approximate evaluating of an isolated unstable cycle of nonlinear autonomous systems.

2.1. Main Result

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^n$ be Euclidean real space with the scalar product $\langle\cdot,\cdot\rangle$; for any $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ by $\{x, t\}$, we denote the couples in $\mathbb{R}^{n+1}$; by $^\top$, we denote the operation of transposition.

Suppose the right-hand $f(x)$ of equation (2.1) is continuously differentiable; by $p(t, x)$ denote the solution of (2.1) with the initial condition $p(0, x) = x$.

Take any vector $a \in \mathbb{R}^n$ and number $b \in \mathbb{R}$; consider in the space of couples $\{x, t\}$ the system of equations

$$x = p(t, x), \tag{2.2}$$

$$\langle a, x\rangle = b. \tag{2.3}$$

It is evident that the solution $\{x^*, t^*\}$, $t^* \neq 0$ of system (2.2),(2.3) is a point of some t^*-periodic cycle of system (2.1). Therefore, the problem of evaluating cycles for system (2.1) is equivalent to the problem of computation of solutions for system (2.2),(2.3).

Let us use the following iteration scheme for computating solutions of (2.2),(2.3):

$$x_{k+1} = x_k - \gamma_k \cdot \left(\left(I - V^\top (t_k, x_k)\right) \cdot (x_k - p(t_k, x_k)) + (\langle a, x_k \rangle - b) \cdot a\right), \tag{2.4}$$

$$t_{k+1} = t_k + \mu_k \cdot \langle f(p(t_k, x_k)), x_k - p(t_k, x_k) \rangle, \qquad k = 0, 1, \dots, \tag{2.5}$$

where $V(t, x)$ is the solution of the linear matrix equation

$$\frac{dV}{dt} = f'_x(p(t, x)) \cdot V, \tag{2.6}$$

with the initial condition

$$V(0, x) \equiv I,$$

and γ_k, μ_k are control parameters of the iteration scheme (2.4),(2.5).

Assume that Γ is an isolated cycle of system (2.1) with period T^*, $x^* \in \Gamma$, and

$$\langle a, x^* \rangle = b. \tag{2.7}$$

Consider the hyperplane π, which is determined by equation (2.3). If π is transversal Γ at the point x^*, then the couple $\{x^*, T^*\}$ is an isolated solution of system (2.2),(2.3).

The main result is the following.

THEOREM 2.1. *Let the couple $\{x^*, T^*\}$ be an isolated solution for the system of equations*

$$\left(\left(I - V^\top (t, x)\right) \cdot (x - p(t, x)\right) - (\langle a, x \rangle - b) \cdot a) = 0, \tag{2.8}$$

$$\langle f(p(t, x)), x - p(t, x) \rangle = 0, \tag{2.9}$$

and control parameters γ_k, μ_k of iteration scheme (2.4),(2.5) be satisfied inequalities

$$0 < \alpha_0 \le \gamma_k \le \beta_0, \tag{2.10}$$

$$0 < \alpha_1 \le \mu_k \le \beta_1, \tag{2.11}$$

where numbers β_0, β_1 are fairly small.

Suppose the initial approximation $\{x_0, t_0\}$ is nearly of $\{x^*, T^*\}$, then the successive approximations $\{x_k, t_k\}$ of iteration scheme (2.4),(2.5) converge to $\{x^*, T^*\}$, i.e.,

$$\lim_{k \to \infty} (\|x_k - x^*\| + |t_k - T^*|) = 0. \tag{2.12}$$

PROOF. In a neighborhood of the point $\{x^*, T^*\} \in \mathbb{R}^{n+1}$, we consider the function

$$W(x, t) = \frac{1}{2}\left(\|x - p(t, x)\|^2 + (\langle a, x \rangle - b)^2\right); \tag{2.13}$$

here by $\|\cdot\|$, we denote the vector or matrix norm which corresponds to the scalar product.

Using a simple calculation, we get

$$\begin{aligned} W(x + h, t + \tau) - W(t, x) = {} & \langle h - p'_t(t, x)\tau - p'_x(t, x)h, x - p(t, x) \rangle \\ & + (\langle a, x \rangle - b) \cdot \langle a, h \rangle + o(\|h\| + |\tau|). \end{aligned} \tag{2.14}$$

Since

$$p'_t(t, x) = f(p(t, x)), \tag{2.15}$$

it follows that

$$\frac{dp'_x(t,x)}{dt} = f'_x(p(t,x)) \cdot p'_x(t,x). \tag{2.16}$$

Therefore the matrix $p'_x(t,x)$ satisfies the linear matrix equation (2.6) with initial condition (2.7). Hence, (2.14) may be written in the form

$$\begin{aligned} W(x+h,t+\tau) - W(t,x) = {} & \left\langle \left(I - V^\top(t,x)\right) \cdot \left(x - p(t,x)\right), h \right\rangle - \langle f(p(t,x)), x - p(t,x)\rangle \cdot \tau \\ & + (\langle a,x\rangle - b) \cdot \langle a,h\rangle + o(\|h\| + |\tau|). \end{aligned}$$

Let us introduce the following notation. By ∇, we denote a gradient along variables x,t; by ∇_x, we denote a gradient along x, and by ∇_t, we denote a gradient along t.

Now, from the last expressions we get

$$\begin{aligned} \nabla W(x,t) &= \left\{ \left(I - V^\top(t,x)\right) \cdot (x - p(t,x)) + (\langle a,x\rangle - b) \cdot a, \right. \\ &\qquad \left. \langle -f(p(t,x)), x - p(t,x)\rangle \right\}, & (2.17) \\ \nabla_x W(x,t) &= \left(I - V^\top(t,x)\right) \cdot (x - p(t,x)) + (\langle a,x\rangle - b) \cdot a, & (2.18) \\ \nabla_t W(x,t) &= -\langle f(p(t,x)), x - p(t,x)\rangle. & (2.19) \end{aligned}$$

Now, by (2.7) and (2.13) it follows that $\{x^*, T^*\}$ is a minimum of the function $W(x,t)$. If we recall that $\{x^*, T^*\}$ is the isolated solution of system (2.8),(2.9) we can say, by virtue of (2.17)–(2.19), that $\{x^*, T^*\}$ is an isolated critical point of the function $W(x,t)$. Hence this point realizes a strong minimum of the function $W(x,t)$.

Therefore, there exists a spherical neighborhood $B \subset \mathbb{R}^{n+1}$ of the point $\{x^*, T^*\}$ containing a unique critical point of the function $W(x,t)$, and the point $\{x^*, T^*\}$ is an absolute strong minimum over the ball B.

Denote

$$\varepsilon = \min_{\{x,t\} \in \partial B} W(x,t). \tag{2.20}$$

Let $\delta < \varepsilon$ be a positive number such that

$$\max_{\{x,t\} \in L(\delta)} \|\nabla W(x,t)\| < \min_{\substack{\{x,t\} \in L(\delta), \\ \{y,\tau\} \in \partial B}} \|\{x,t\} - \{y,\tau\}\|, \tag{2.21}$$

where $L(\delta)$ is a Lebesgue set of the function $W(x,t)$

$$L(\delta) = \{\{x,t\} \in B : W(x,t) \le \delta\}. \tag{2.22}$$

Now, we shall prove that successive approximations $\{x_k, t_k\}$ of the iteration scheme (2.4),(2.5) converge to $\{x^*, T^*\}$ for any initial approximation $\{x_0, t_0\} \in L(\delta)$.

LEMMA 2.1. *If* $\{x_k, t_k\} \in L(\delta)$, *then*

$$\{x_{k+1}, t_{k+1}\} \in B, \tag{2.23}$$

$$W(x_{k+1}, t_{k+1}) \le W(x_k, t_k). \tag{2.24}$$

PROOF. Suppose the assumption of Lemma 2.1 is true with some k; then by virtue of (2.4),(2.5),

$$\|x_{k+1} - x_k\| \le \|\nabla_x W(x_k, t_k)\|, \tag{2.25}$$

$$|t_{k+1} - t_k| \le |\nabla_t W(x_k, t_k)|. \tag{2.26}$$

From estimations (2.25), (2.26), and inequality (2.21), we get

$$\{x_{k+1}, t_{k+1}\} \in B.$$

Let the numbers K_1, K_2, L_1, L_2 be satisfied inequalities

$$\|\nabla_x W(x_1,t_1) - \nabla_x W(x_2,t_2)\| \le K_1 \cdot \|x_1 - x_2\| + K_2 \cdot |t_1 - t_2|, \tag{2.27}$$

$$|\nabla_t W(x_1,t_1) - \nabla_t W(x_2,t_2)| \le L_1 \cdot \|x_1 - x_2\| + L_2 \cdot |t_1 - t_2|, \tag{2.28}$$

for any $\{x_1,t_1\}$, $\{x_2,t_2\} \in B$.

Suppose the numbers $\beta_0, \beta_1 < 1$ from (2.10),(2.11) satisfy

$$\frac{1}{2}K_1\beta_0 + \frac{1}{4}(K_2 + L_1)\cdot\beta_1 < 1; \tag{2.29}$$

$$\frac{1}{2}L_1\beta_1 + \frac{1}{4}(K_2 + L_1)\cdot\beta_0 < 1; \tag{2.30}$$

then

$$\begin{aligned}
&W(x_{k+1},t_{k+1}) - W(x_k,t_k)\\
&= \int_0^1 \langle \nabla_x W(x_k + \tau(x_{k+1}-x_k), t_k + \tau(t_{k+1}-t_k)), x_{k+1} - x_k\rangle\, d\tau\\
&\quad + \int_0^1 \nabla_t W(x_k + \tau(x_{k+1}-x_k), t_k + \tau(t_{k+1}-t_k)) \cdot (t_{k+1} - t_k)\, d\tau\\
&= -\gamma_k \cdot \int_0^1 \langle \nabla_x W(x_k - \tau\gamma_k\nabla_x W(x_k,t_k), t_k - \tau\mu_k\nabla_t W(x_k,t_K)), \nabla_x W(x_k,t_k)\rangle\, d\tau\\
&\quad - \mu_k \cdot \int_0^1 \nabla_t W(x_k - \tau\gamma_k\nabla_x W(x_k,t_k), t_k - \tau\mu_k\nabla_t W(x_k,t_k)) \cdot (\nabla_t W(x_k,t_k))\, d\tau\\
&= -\gamma_k \cdot \|\nabla_x W(x_k,t_k)\|^2 - \mu_k \cdot (\nabla_t W(x_k,t_k))^2,
\end{aligned}$$

and, finally,

$$\begin{aligned}
&-\gamma_k \cdot \int_0^1 \langle \nabla_x W(x_k - \tau\gamma_k\nabla_x W(x_k,t_k), t_k\\
&-\tau\mu_k\nabla_t W(x_k,t_k) - \nabla_x W(x_k,t_k)), \nabla_x W(x_k,t_k)\rangle\, d\tau\\
&-\mu_k \cdot \int_0^1 \nabla_t W(x_k - \tau\gamma_k\nabla_x W(x_k,t_k), t_k - \tau\mu_k\nabla_t W(x_k,t_k)\\
&-\nabla_t W(x_k,t_k)) \cdot \nabla_t W(x_k,t_k)\, d\tau\\
&\le -\gamma_k \cdot \|\nabla_x W(x_k,t_k)\|^2 - \mu_k \cdot (\nabla_t W(x_k,t_k))^2\\
&\quad + \frac{1}{2}\gamma_k \cdot (K_1\gamma_k \cdot \|\nabla_x W(x_k,t_k)\| + K_2\mu_k \cdot |\nabla_t W(x_k,t_k)|) \cdot \|\nabla_x W(x_k,t_k)\|\\
&\quad + \frac{1}{2}\mu_k \cdot (L_1\gamma_k \cdot \|\nabla_x W(x_k,t_k)\| + L_2\mu_k \cdot |\nabla_t W(x_k,t_k)|) \cdot \|\nabla_t W(x_k,t_k)\|\\
&\le -\gamma_k \cdot \left(1 - \frac{1}{2}K_1\gamma_k - \frac{1}{4}(K_2 + L_1)\cdot\mu_k\right) \cdot \|\nabla_x W(x_k,t_k)\|^2\\
&\quad - \mu_k \cdot \left(1 - \frac{1}{2}L_2\mu_k - \frac{1}{4}(K_2 + L_1)\cdot\gamma_k\right) \cdot |\nabla_t W(x_k,t_k)|^2\\
&\le -\alpha_0 \cdot \left(1 - \frac{1}{2}K_1\beta_0 - \frac{1}{4}(K_2 + L_1)\cdot\beta_1\right) \cdot \|\nabla_x W(x_k,t_k)\|^2\\
&\quad - \alpha_1 \cdot \left(1 - \frac{1}{2}L_2\beta_1 - \frac{1}{4}(K_2 + L_1)\cdot\beta_0\right) \cdot |\nabla_t W(x_k,t_k)|^2.
\end{aligned} \tag{2.31}$$

Using this estimation and inequalities (2.29),(2.30), we obtain inequality (2.24). This completes the proof of Lemma 2.1.

Now, by introduction on k we get that

$$\{x_k, t_k\} \in L(\delta), \qquad \text{for all } k = 1, 2, \ldots. \tag{2.32}$$

Suppose

$$\nu = \min\left\{\alpha_0 \cdot \left(1 - \frac{1}{2}K_1\beta_0 - \frac{1}{4}(K_2 + L_1)\cdot\beta_1\right),\right.$$
$$\left.\alpha_1 \cdot \left(1 - \frac{1}{2}L_2\beta_1 - \frac{1}{4}(K_2 + L_1)\cdot\beta_0\right)\right\}, \tag{2.33}$$

then for any k, the inequality is true

$$W(x_{k+1}, t_{k+1}) - W(x_k, t_k) \le -\nu \cdot \|\nabla W(x_k, t_k)\|^2. \tag{2.34}$$

Summing these inequalities over k from 0 to m, we get

$$W(x_{m+1}, t_{m+1}) - W(x_0, t_0) \le -\nu \cdot \sum_{k=0}^{m} \|\nabla W(x_k, t_k)\|^2. \tag{2.35}$$

Since (2.24) is true for all $k = 1, 2, \dots$, it follows that the series

$$\sum_{k=0}^{\infty} \|\nabla W(x_k, t_k)\|^2$$

is convergent. Therefore

$$\lim_{n\to\infty} \|\nabla W(x_k, t_k)\| = 0. \tag{2.36}$$

Finally there is a unique critical point $\{x^*, T^*\}$ of the function $W(x,t)$ in the ball B; hence, by (2.36), it follows the convergence (2.12). This completes the proof of Theorem 2.1.

2.2. Stationary States of Dynamical Systems

Let us use the iteration scheme (2.4),(2.5) for detection of stationary states of system (2.1). In this case, the iteration scheme is very simple.

Let $x^* \in \Omega \subset \mathbb{R}^n$ be a unique stationary state of system (2.1).

Suppose x^* is asymptotically stable; then

$$\lim_{t\to\infty} |p(t, x_0) - x^*| = 0$$

for solution $p(t, x_0)$ of (1) with any initial condition x_0 from the attracting set of the point x^*. Therefore $p(t, x_0)$ is arbitrarily close to x^* as $t \to \infty$.

In general, this is not true if x^* is not asymptotically stable. In this case, using the idea of scheme (2.4),(2.5) we can reduce the problem to the one of a detection stationary states for the system

$$\frac{dx}{dt} = -\left(f_x'(x)\right)^{\mathsf{T}} \cdot f(x). \tag{2.37}$$

Suppose the equation

$$f_x'(x) \cdot f(x) = 0 \tag{2.38}$$

has a unique solution x^* in the domain Ω; then x^* is a unique stationary state of (2.37) in Ω. The following result is true.

THEOREM 2.2. *Suppose equation (2.38) has a unique solution $x^* \in \Omega$. Then x^* is an asymptotically stable stationary state of system (2.37).*

PROOF. The function

$$v(x) = \frac{1}{2} \cdot |f(x)|^2$$

is a Lyapunov function for system (2.37) in Ω. Theorem 2.2 is proved.

So, we see that a detection of unstable stationary states for system (2.1) may be reduced to a detection of asymptotically stable stationary states of system (2.37).

We can use the same approach for a detection of saddle points of smooth functions. Let $w(x)$ be a smooth function, x^* be a saddle point with a complex topological structure of the function $w(x)$.

Consider the differential equation

$$\frac{dx}{dt} = -\nabla^2 w(x) \cdot \nabla w(x). \tag{2.39}$$

Under natural conditions, the point x^* is an asymptotically stable stationary state of equation (2.39). By Theorem 2.2, it follows immediately.

THEOREM 2.3. *Suppose the equation*

$$\nabla^2 w(x) \cdot \nabla w(x) = 0 \tag{2.40}$$

has a unique solution $x^* \in \Omega$. *Then* x^* *is the asymptotically stable stationary state of system (2.39).*

2.3. Estimations for Iteration's Parameters

In this section, we give estimations for control parameters of the gradient algorithm (2.4),(2.5).

THEOREM 2.4. *Let a point* x^* *belong to cycle* Γ *and the next relationship*

$$\langle a, x^* \rangle = b$$

holds. Suppose that the pair $\{x^*, T^*\}$ *is an isolated solution of the following equations:*

$$\begin{aligned} \left((I - V^\top(t,x)) \cdot (x - p(t,x)) - (\langle a, x\rangle - b) \cdot a\right) &= 0, \\ \langle f(p(t,x)), x - p(t,x)\rangle &= 0. \end{aligned}$$

Let the controllable parameters γ_k, μ_k *of the iterative procedure (2.4),(2.5) satisfy the following inequalities:*

$$0 < \alpha_0 \le \gamma_k \le \beta_0, \tag{2.41}$$

$$0 < \alpha_1 \le \mu_k \le \beta_1, \tag{2.42}$$

and

$$\begin{aligned} \frac{1}{2} K_1 \beta_0 + \frac{1}{4}(K_2 + L_1)\beta_1 &< 1, \\ \frac{1}{2} L_1 \beta_1 + \frac{1}{4}(K_2 + L_1)\beta_0 &< 1, \end{aligned}$$

where K_1, K_2, L_1, L_2 *are Lipschitz constants so that*

$$\begin{aligned} \left\|\nabla_x W(x_1,t_1) - \nabla_x W(x_2,t_2)\right\| &\le K_1 \cdot \|x_1 - x_2\| + K_2 \cdot |t_1 - t_2|, \\ \left|\nabla_t W(x_1,t_1) - \nabla_t W(x_2,t_2)\right| &\le L_1 \cdot \|x_1 - x_2\| + L_2 \cdot |t_1 - t_2|. \end{aligned}$$

Above we used the next notations: ∇_x*—a gradient operator with respect to* x, ∇_t*—a differential operator with respect to* t, *and*

$$W(x,t) = \frac{1}{2}\left(\|x - p(t,x)\|^2 + \left(\langle a, x\rangle - b)^2\right)\right).$$

Let us suppose finally that an initial approximation $\{x_0, t_0\}$ *is rather close to* $\{x^*, T^*\}$. *Under these assumptions, an iterative process* $\{x_k, t_k\}$ *generated by the iterative procedure (2.4),(2.5) converges to* $\{x^*, T^*\}$, *or in other words, we have*

$$\lim_{k\to\infty} \left(\|x_k - x^*\| + |t_k - T^*|\right) = 0.$$

When we solve a tracing problem for concrete nonlinear systems, it is very important to know numbers β_0, β_1 in inequalities (2.41),(2.42), the vector $a \in \mathbb{R}^n$, the quantity $b \in \mathbb{R}$ in (2.4),(2.5), and also to have an initial pair $\{x_0, t_0\}$.

In this paper, we will suggest estimations for numbers β_0, β_1 and indicate a procedure for effective calculating parameters a, b and the initial pair $\{x_0, t_0\}$ for interactive procedure (2.4),(2.5).

2.4. Subsidiary Inequalities

Suppose R_0 is a ball's radius $\|x\| < R_0$ which has inside a cycle Γ; a quantity $T_0 > T^*$ is an arbitrary number. Moreover, later, the following inequalities are fulfilled for some collections of constants R_1, M_1, M_2, N_1, N_2:

$$\max_{\substack{\|x\| \le R_0, \\ 0 \le t \le T_0}} \|p(t,x)\| \le R_1, \tag{2.43}$$

$$\max_{\|x\| \le R_1} \|f(x)\| \le M_1, \tag{2.44}$$

$$\max_{\|x\| \le R_1} \|f'(x)\| \le M_2, \tag{2.45}$$

$$\|f(x_1) - f(x_2)\| \le N_1 \cdot \|x_1 - x_2\| \quad (\|x\| \le R_1), \tag{2.46}$$

$$\|f'(x_1) - f'(x_2)\| \le N_2 \cdot \|x_1 - x_2\| \quad (\|x\| \le R_1). \tag{2.47}$$

Under these assumptions, we will prove five lemmas.

LEMMA 2.2. *If inequalities $\|x_1\|, \|x_2\| \le R_0$, and $0 \le t \le T_0$ take place, then we have the following estimation:*

$$\|p(t,x_1) - p(t,x_2)\| \le \|x_1 - x_2\| e^{N_1 t}. \tag{2.48}$$

PROOF. It is evident that we have inequalities

$$p(t,x_1) = x_1 + \int_0^t f(p(\tau,x_1))\, d\tau,$$

$$p(t,x_2) = x_2 + \int_0^t f(p(\tau,x_2))\, d\tau.$$

From the above relationships and (2.43),(2.44), we have

$$\|p(t,x_1) - p(t,x_2)\| \le \|x_1 - x_2\| + N_1 \cdot \int_0^t \|p(\tau,x_1) - p(\tau,x_2)\|\, d\tau. \tag{2.49}$$

Using (2.49) and Gronwall-Bellman's lemma finally, we may get the estimation (2.48). This proves Lemma 2.2.

LEMMA 2.3. *If inequalities $\|x\| < R_0$ and $0 \le t_1, t_2 \le T_0$ hold, then we have an estimation*

$$\|p(t_1,x) - p(t_2,x)\| \le |t_1 - t_2| \cdot M_1. \tag{2.50}$$

PROOF. Directly from the differential equation, we have

$$p(t_1,x) = x + \int_0^{t_1} f(p(\tau,x))\, d\tau,$$

$$p(t_2,x) = x + \int_0^{t_2} f(p(\tau,x))\, d\tau.$$

Using that we may get

$$p(t_1,x) - p(t_2,x) = \int_{t_2}^{t_1} f(p(\tau,x))\, d\tau. \tag{2.51}$$

Finally from (2.51) and (2.44), we obtain the required estimation

$$\|p(t_1,x) - p(t_2,x)\| \le \max_{t_2 \le \tau \le t_1} \|f(p(\tau,x))\|\ |t_1 - t_2| \le |t_1 - t_2| \cdot M_2.$$

This proves Lemma 2.3.

LEMMA 2.4. *If inequalities $\|x\| < R_0$ and $0 \le t \le T_0$ hold, then the following estimation takes place:*

$$\|V(t,x)\| \le e^{M_2 t}. \tag{2.52}$$

PROOF. By virtue of (2.41),(2.42), we may write

$$V(t,x) = I + \int_0^t f'_x(p(\tau,x))V(\tau,x)\,d\tau.$$

Using (2.45), we obtain first an inequality

$$\|V(t,x)\| \le 1 + M_2 \cdot \int_0^t \|V(\tau,x)\|\,d\tau, \tag{2.53}$$

and finally, after using Gronwall-Bellman's lemma, we get the required the estimation (2.52). This proves Lemma 2.4.

LEMMA 2.5. *If inequalities $\|x\| < R_0$ and $0 \le t_0, t_1 \le T_0$ hold, then the following estimation takes place:*

$$\|V(t_1,x) - V(t_2,x)\| \le M_2 \cdot e^{M_2 T_0}\,|t_1 - t_2|. \tag{2.54}$$

PROOF. By virtue of (2.41),(2.42) we have

$$V(t_1,x) - V(t_2,x) = \int_{t_2}^{t_1} f'_x(p(\tau,x))V(\tau,x)\,d\tau.$$

Using this relationship (2.46) and (2.51), we finally obtain the required estimation

$$\begin{aligned}\|V(t_1,x) - V(t_2,x)\| &\le \left|\int_{t_2}^{t_1} M_2 \cdot e^{M_2\tau}\,d\tau\right| \\ &= \left|e^{M_2 t_1} - e^{M_2 t_2}\right| \le M_2 \cdot e^{M_2 T_0}\,|t_1 - t_2|.\end{aligned}$$

This proves Lemma 2.5.

LEMMA 2.6. *If inequalities $\|x_1\|, \|x_2\| \le R_0$, $0 \le t \le T_0$ hold, then the following estimation is valid:*

$$\|V(t,x_1) - V(t,x_2)\| \le \frac{N_2}{N_1 + M_2} \cdot \left(e^{(N_1+M_2)T_0} - 1\right) \cdot e^{M_2 t} \cdot \|x_1 - x_2\|. \tag{2.55}$$

PROOF. From the beginning we have the relationships

$$\begin{aligned}V(t,x_1) &= I + \int_0^t f'(p(\tau,x_1)) \cdot V(\tau,x_1)\,d\tau, \\ V(t,x_2) &= I + \int_0^t f'(p(\tau,x_2)) \cdot V(\tau,x_2)\,d\tau.\end{aligned}$$

By virtue of (2.45), (2.47), and (2.48), we may write the sequence of inequalities

$$\begin{aligned}
&\|V(t,x_1) - V(t,x_2)\| \\
&\quad \le \int_0^t \|f'_x(p(\tau,x_1))\cdot V(\tau,x_1) - f'_x(p(\tau,x_1))\cdot V(\tau,x_2)\|\,d\tau \\
&\qquad + \int_0^t \|f'_x(p(\tau,x_1))\cdot V(\tau,x_2) - f'_x(p(\tau,x_2))\cdot V(\tau,x_2)\|\,d\tau \\
&\quad \le M_2\cdot \int_0^t \|V(\tau,x_1) - V(\tau,x_2)\|\,d\tau \\
&\qquad + N_2\cdot \int_0^t \|p(\tau,x_1) - p(\tau,x_2)\|\cdot\|V(\tau,x_2)\|\,d\tau \\
&\quad \le M_2\cdot \int_0^t \|V(\tau,x_1) + V(\tau,x_2)\|\,d\tau + N_2\cdot\|x_1-x_2\|\cdot\int_0^t e^{N_1\tau}e^{M_2\tau}\,d\tau \\
&\quad \le M_2\cdot \int_0^t \|V(\tau,x_1) - V(\tau,x_2)\|\,d\tau \\
&\qquad + \frac{N_2}{N_1+M_2}\cdot\left(e^{(N_1+M_2)T_0}-1\right)\cdot\|x_1-x_2\|.
\end{aligned}$$

The last inequalities and Gronwall-Bellman's lemma give us the required relationship (2.55). This completes the proof of Lemma 2.6.

2.5. Estimation of Parameters

Direct calculations show us that

$$\nabla W(x,t) = \left(I - V^\top(t,x)\right)\cdot(x - p(t,x)) + (\langle a,x\rangle - b)\cdot a, \tag{2.56}$$

$$\nabla_t W(x,t) = -\langle f(p(t,x)), x - p(t,x)\rangle. \tag{2.57}$$

Using Lemmas 2.2–2.6 for the transformation's last relationships, we have the following sequence of estimations:

$$\begin{aligned}
&\|\nabla_x W(x_1,t_1) - \nabla_x W(x_2,t_2)\| \\
&\quad \le \left\|\left(I - V^\top(t_1,x_1)\right)\cdot(x_1 - p(t_1,x_1)) - \left(I - V^\top(t_2,x_2)\right)\cdot(x_2 - p(t_2,x_2))\right\| \\
&\qquad + \|a\|^2\cdot\|x_1-x_2\| \le \left(1+\|a\|^2\right)\cdot\|x_1-x_2\| \\
&\qquad + \|p(t_1,x_1) - p(t_1,x_2)\| + \|p(t_1,x_2) - p(t_2,x_2)\| \\
&\qquad + \left\|V^\top(t_1,x_1)\cdot x_1 - V^\top(t_1,x_1)\cdot x_2\right\| + \left\|V^\top(t_1,x_1)\cdot x_2 - V^\top(t_2,x_2)\cdot x_2\right\| \\
&\qquad + \left\|V^\top(t_1,x_1)\cdot p(t_1,x_1) - V^\top(t_1,x_1)\cdot p(t_2,x_2)\right\| \\
&\qquad + \left\|V^\top(t_1,x_1)\cdot p(t_2,x_2) - V^\top(t_2,x_2)\cdot p(t_2,x_2)\right\| \\
&\quad \le \left(1+\|a\|^2\right)\cdot\|x_1-x_2\| + \|x_1-x_2\|\cdot e^{N_1T_0} + |t_1-t_2|\cdot M_1 + \|x_1-x_2\|\cdot e^{M_2T_0} \\
&\qquad + \left\|V^\top(t_1,x_1)\cdot x_2 - V^\top(t_1,x_2)\cdot x_2\right\| + \left\|V^\top(t_1,x_2)\cdot x_2 - V^\top(t_2,x_2)\cdot x_2\right\| \\
&\qquad + \|p(t_1,x_1) - p(t_2,x_2)\|\cdot e^{M_2T_0} \\
&\qquad + \left\|V^\top(t_1,x_2)\cdot p(t_2,x_2) - V^\top(t_1,x_2)\cdot p(t_1,x_2)\right\| \\
&\qquad + \left\|V^\top(t_1,x_2)\cdot p(t_2,x_2) - V^\top(t_2,x_2)\cdot p(t_1,x_2)\right\| \\
&\quad \le \left(1+\|a\|^2 + e^{N_1T_0} + e^{M_2T_0}\right)\cdot\|x_1-x_2\| + |t_1-t_2|\cdot M_1 \\
&\qquad + \|x_2\|\cdot\frac{N_2}{N_1+M_2}\cdot\left(e^{(N_1+M_2)\cdot T_0}-1\right)\cdot e^{M_2T_0}\,\|x_1-x_2\| \\
&\qquad + \|x_2\|\cdot M_2\cdot e^{M_2T_0}\cdot|t_1-t_2| + |t_1-t_2|\cdot M_1\cdot e^{M_2T_0} + \|x_1-x_2\|\cdot e^{(M_2+N_1)\cdot T_0}
\end{aligned}$$

$$+ \|p(t_2, x_2)\| \cdot \frac{N_2}{N_1 + M_2} \cdot \left(e^{(N_1+M_2)\cdot T_2} - 1\right) \cdot e^{M_2 T_0} \cdot \|x_1 - x_2\|$$
$$+ \|p(t_2, x_2)\| \cdot M_2 \cdot e^{M_2 T_0} \cdot |t_1 - t_2|$$
$$\leq \left(1 + \|a\|^2 + e^{N_1 T_0} + e^{M_2 T_0} + \frac{N_2 \cdot (R_0 + R_1) \cdot \left(e^{(N_1+M_2)\cdot T_0} - 1\right) \cdot e^{M_2 T_0}}{N_1 + M_2}\right.$$
$$\left. + e^{(M_2+N_1)\cdot T_0}\right) \cdot \|x_1 - x_2\| + \left(M_1 + (R_0 M_2 + M_1 + R_1 M_2) \cdot e^{M_2 T_0}\right) \cdot |t_1 - t_2| .$$

Thus the gradient $\nabla_x W(x,t)$ satisfies to Lipschitz conditions with respect to variables x and t with approximate constants

$$K_1 = 1 + \|a\|^2 + e^{N_1 T_0} + e^{M_2 T_0} + e^{(M_2+N_1)\cdot T_0}$$
$$+ \frac{N_2 \cdot (R_0 + R_1) \cdot \left(e^{(N_1+M_2)\cdot T_0} - 1\right) \cdot e^{M_2 T_0}}{N_1 + M_2}, \tag{2.58}$$
$$K_2 = M_1 + ((R_0 + R_1) \cdot M_2 + M_1) \cdot e^{M_2 T_0}. \tag{2.59}$$

Analogously we may obtain

$$|\nabla_t W(x_1, t_1) - \nabla_t W(x_2, t_2)|$$
$$= |\langle f(p(t_1, x_1)), x_1 - p(t_1, x_1)\rangle - \langle f(p(t_2, x_2)), x_2 - p(t_2, x_2)\rangle|$$
$$\leq |\langle f(p(t_1, x_1)), x_1\rangle - \langle f(p(t_2, x_2)), x_2\rangle|$$
$$+ |\langle f(p(t_1, x_1)), p(t_1, x_1)\rangle - \langle f(p(t_2, x_2)), p(t_2, x_2)\rangle|$$
$$\leq |\langle f(p(t_1, x_1)) - f(p(t_2, x_2)), x_1\rangle| + |\langle f(p(t_2, x_2)), x_1\rangle - \langle f(p(t_2, x_2)), x_2\rangle|$$
$$+ |\langle f(p(t_1, x_1)), p(t_1, x_1)\rangle - \langle f(p(t_2, x_2)), p(t_1, x_1)\rangle|$$
$$+ |\langle f(p(t_2, x_2)), p(t_1, x_1)\rangle - \langle f(p(t_2, x_2)), p(t_2, x_2)\rangle|$$
$$\leq \|x_1\| \cdot N_1 \cdot \|p(t_1, x_1) - p(t_2, x_2)\| + \|f(p(t_2, x_2))\| \cdot \|x_1 - x_2\|$$
$$+ \|p(t_1, x_1)\| \cdot N_1 \cdot \|p(t_1, x_1) - p(t_2, x_2)\| + \|f(p(t_2, x_2))\| \cdot \|p(t_1, x_1) + p(t_2, x_2)\|$$
$$\leq \left(R_0 \cdot e^{N_1 T_0} + M_1 + R_1 \cdot N_1 \cdot e^{N_1 T_0} + M_1 \cdot e^{N_1 T_0}\right) \cdot \|x_1 - x_2\|$$
$$+ \left(R_0 \cdot N_1 \cdot M_1 + R_1 \cdot N_1 \cdot M_1 + M_1^2\right) \cdot |t_1 - t_2| .$$

The last relationship shows us that the derivative $\nabla_t W(x,t)$ satisfies to Lipschitz condition with the following constants:

$$L_1 = (R_0 + R_1) \cdot e^{N_1 T_0} + M_1 \cdot \left(1 + e^{N_1 T_0}\right), \tag{2.60}$$
$$L_2 = M_1 \cdot (R_0 \cdot N_1 + R_1 \cdot N_1 + M_1) . \tag{2.61}$$

Thus we proved that if constants β_0, β_1 for the iterative procedure (2.4),(2.5) satisfy the following inequalities:

$$\frac{1}{2} \cdot K_1 \cdot \beta_0 + \frac{1}{4} (K_2 + L_1) \cdot \beta_1 < 1,$$
$$\frac{1}{2} \cdot L_1 \cdot \beta_1 + \frac{1}{4} (K_2 + L_1) \cdot \beta_0 < 1,$$

where constants K_1, K_2, L_1, L_2 were defined in (2.58)–(2.61), that for the appropriate initial approximation pair $\{x_0, t_0\}$, successive approximations $\{x_k, t_k\}$ converge.

2.6. Determination of the Required Initial Approximation

The reasonings given below have a heuristic nature and may be used only as a recommendation. Consider one parameter family of autonomous systems

$$\frac{dx}{dt} = h(x;\lambda), \qquad (x \in \mathbb{R}^n,\ 0 \le \lambda \le 1) \tag{2.62}$$

having the following properties:

(1) $h(x;1) = f(x)$,

(2) for every value of the parameter λ, the system (2.62) has an isolated cycle Γ_λ, continuously depending on λ,

(3) the cycle Γ_0 for the system

$$\frac{dx}{dt} = h(x;0) \tag{2.63}$$

is known.

Under these assumptions for finding the parameters a, b and the initial approximation $\{x_0, t_0\}$ for the iterative procedure (2.4),(2.5), we may be use the following consideration.

Divide segment $[0, 1]$ with the help of points $\lambda_1, \dots, \lambda_k$ so that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k < 1.$$

As far as the cycle Γ_0 for system (2.63) is known, we may find a vector a_0 and a number b_0 such that the hyperplane $\langle a_0, x\rangle = b_0$ intersects T_0 transversally. Then, for arbitrary λ_1 from the small vicinity of zero, this hyperplane will transversally cut the cycle Γ_{λ_1} for the system

$$\frac{dx}{dt} = h\left(x;\lambda_1\right). \tag{2.64}$$

For tracing the cycle Γ_{λ_1} for system (2.64), we suggest using an analog of the iterative procedure (2.4),(2.5) with the initial pair $x_0 = x^*$, $t_0 = T_0$, where x^* is an arbitrary point of Γ_0, and T_0 is the period of Γ_0. Moreover, we recommend choosing the parameters a_0, b_0, as was done in the previous section.

After obtaining a good approximation of the cycle Γ_{λ_1} we have to repeat the procedure for tracing a cycle Γ_{λ_2} for the system

$$\frac{dx}{dt} = h\left(x;\lambda_2\right). \tag{2.65}$$

After m iterations, we may obtain good approximations of parameters a, b and an initial pair $\{x_0, t_0\}$ which are appropriate for tracing the required cycle Γ in system (2.1).

3. APPROXIMATING UNSTABLE CYCLES IN NONLINEAR DELAY SYSTEMS

The problem of evaluating periodic solutions for ordinary differential equations is well known. There are a lot of different methods for approximating periodic solutions: the harmonic balance method, the method of finite differences, the method of mechanical quadratures, and the collocation method. The rates of convergence for these methods and the ranges of their validities were studied in various papers [2–4,8–11,13,14]. The problem of evaluating periodic solutions in delay systems is more complicated than the similar problem for ordinary differential equations, and has specific features.

3.1. Abstract Theorems for Convergence

Let H be a real Hilbert space with a scalar product $\langle u, v\rangle$. Suppose $f(u)$ is a Frechet differentiable functional, the gradient $\nabla f(u)$ satisfies the Lipschitz conditions and Browder condition (S).

(S)-condition: if the sequence u_n weakly converges to the point u_0 and

$$\overline{\lim_{n\to\infty}} \left\langle \nabla f\left(u_n\right), u_n - u_0\right\rangle \le 0, \tag{3.1}$$

then u_n strongly converges to u_0 and

$$\lim_{n\to\infty} \|u_n - u_0\| = 0. \tag{3.2}$$

Suppose that the point $u^* \in H$ is a minimum of the functional f. Consider the following procedure for approximating the point u^*:

$$u_{n+1} = u_n - \gamma_n \nabla f(u_n), \qquad n = 0, 1, \dots. \tag{3.3}$$

Let u^* be a unique critical point of functional $f(u)$ in the ball $B(r) = \{u \in H : |u\| \le r\}$. By L denote the Lipschitz constant for the gradient $\nabla f(u)$ of the functional $f(u)$ on the ball $B(r)$. In Section 2, the following theorem was proved.

THEOREM 3.1. *Suppose the control parameters γ_n in procedure (3.3) satisfy inequalities*

$$0 \le a \le \gamma_n \le \frac{2}{L},$$

and the initial approximation u_0 in (3.3) is sufficiently close to u^*. Then, iteration scheme (3.3) converges to u_0, i.e.,

$$\lim_{n\to\infty} \|u_n - u^*\| = 0. \tag{3.4}$$

Theorem 3.1 will be used in the future for the proof of iteration scheme's convergence. In the space H, we consider the equation

$$u = A(u), \tag{3.5}$$

where the operator $A : H \to H$ is completely continuous and differentiable; its Frechet derivative $A'(\cdot) : H \to H$ satisfies the Lipschitz condition

$$\|A(u_1) - A(u_2)\|_{\mathcal{L}(H,H)} \le L \cdot \|u_1 - u_2\|_H. \tag{3.6}$$

Now, solving equation (3.5), we find the minimum points of functional

$$f(u) = \frac{1}{2} \cdot \|u - A(u)\|^2. \tag{3.7}$$

LEMMA 3.1. *Let the operator $(I - A'(u_1))^*$ be a positive definite at the point $u_1 \in H$, i.e.,*

$$\left\langle (I - A'(u_1))^* h, h \right\rangle > a \cdot \langle h, h\rangle, \qquad a > 0,\ h \in H. \tag{3.8}$$

Then the gradient $\nabla f(u)$ of the functional $f(u)$ satisfies condition (3.5) in a ball

$$B(r, u_1) = \{u \in H : \|u_1 - u_2\| \le r\}.$$

PROOF. Suppose

$$K = \sup_{u \in B(1,u_1)} \|A(u)\|, \tag{3.9}$$

$$r = \min\left(1, \frac{a}{2L(1 + \|u_1\| + K)}\right). \tag{3.10}$$

Now, we prove that the gradient $\nabla f(u)$ of the functional $f(u)$ satisfies the Lipschitz condition in the ball $B(r, u_1)$. The gradient $\nabla f(u)$ of the functional $f(u)$ is $\nabla f(u) = (I - A'(u))^*(u - A(u))$.

Because the Frechet derivative $A'(u)$ of operator $A(u)$ is continuous, we can always decrease the radius r of the ball $B(r, u_1)$, and without loss of generality, we can suppose that

$$\sup_{u \in B(r,u_1)} \left\langle \left(I - A'(u_1)\right)^* h, h \right\rangle \geq a \cdot \langle h, h \rangle, \qquad h \in H. \tag{3.11}$$

Suppose the sequence $u_n \in B(r, u_1)$ weakly converges to some point u_0. The sequence $A(u_n)$ is compact by boundedness of the sequence u_n and complete continuity of the operator $A(u)$. So, without loss of generality, we suppose that the sequence $A(u_n)$ converges to some point $v \in H$:

$$\lim_{n \to \infty} \|A(u_n) - v\| = 0. \tag{3.12}$$

By virtue of (3.9),

$$\|v\| \leq K. \tag{3.13}$$

So, we get

$$\begin{aligned} \langle \nabla f(u_n), u_n - u_0 \rangle &= \left\langle (I - A'(u_n))^* (u_n - A(u_n)), u_n - u_0 \right\rangle \\ &= \left\langle (I - A'(u_n))^* (u_n - u_0), u_n - u_0 \right\rangle \\ &\quad + \left\langle (I - A'(u_n))^* (u_0 - A(u_n)), u_n - u_0 \right\rangle \\ &= \left\langle (I - A'(u_n))^* (u_n - u_0), u_n - u_0 \right\rangle \\ &\quad + \left\langle (I - A'(u_n))^* (u_0 - v), u_n - u_0 \right\rangle \\ &\quad + \left\langle (I - A'(u_n))^* (v - A(u_n)), u_n - u_0 \right\rangle. \end{aligned}$$

Estimating each of the terms in the right-hand side of the latter expression, by (3.11) we get

$$\left\langle (I - A'(u_1))^* u_n - u_0, u_n - u_0 \right\rangle \leq a \cdot \|u_1 - u_2\|^2. \tag{3.14}$$

Further, by (3.10)

$$\begin{aligned} \left| \left\langle (I - A'(u_n))^* (u_0 - v), u_n - u_0 \right\rangle \right| &= \left| \left\langle (I - A'(u_0))^* (u_0 - v), u_n - u_0 \right\rangle \right| \\ &\quad + \left| \left\langle (A'(u_0) - A'(u_n))^* (u_0 - v), u_n - u_0 \right\rangle \right| \\ &\leq \varepsilon_n + \left\| (A'(u_0) - A'(u_n))^* \right\| \cdot \|u_0 - v\| \cdot \|u_n - u_0\| \\ &\leq \varepsilon_n + L \cdot \|u_n - u_0\|^2 \cdot (\|u_0\| + \|v\|) \\ &\leq \varepsilon_n + L \cdot \|u_n - u_0\|^2 \cdot (1 + \|u_1\| + K) \\ &\leq \varepsilon_n + \frac{a}{2} \cdot \|u_n - u_0\|^2, \end{aligned} \tag{3.15}$$

where

$$\varepsilon_n = \left| \left\langle (I - A'(u_0))^* (u_0 - v), u_n - u_0 \right\rangle \right|,$$

and by weakly convergence of u_n to u_0,

$$\lim_{n \to \infty} \varepsilon_n = 0. \tag{3.16}$$

Finally, by (3.12)

$$\begin{aligned} &\lim_{n \to \infty} \left| \left\langle (I - A'(u_n))^* (v - A(u_n)), u_n - u_0 \right\rangle \right| \\ &\qquad \leq \lim_{n \to \infty} \sup_{u \in B(r,u^*)} \left\| (I - A'(u_n))^* \right\| \cdot \|v - A(u_n)\| \cdot r = 0. \end{aligned} \tag{3.17}$$

From (3.14)–(3.17) we get

$$\langle \nabla f(u_n), u_n - u_0 \rangle \geq \delta_n + \frac{a}{2} \cdot \|u_n - u_0\|, \tag{3.18}$$

where

$$\lim_{n\to\infty} \delta_n \to 0. \tag{3.19}$$

Therefore, if

$$\overline{\lim_{n\to\infty}} \langle \nabla f(u_n), u_n - u_0 \rangle \le 0, \tag{3.20}$$

then, by (3.18) and (3.19)

$$\lim_{n\to\infty} \|u_n - u_0\| = 0. \tag{3.21}$$

This completes the proof of Lemma 3.1.

Let u^* be an isolated extremal of functional (3.7) realizing its local minimum. Let L be the Lipschitz constant for the gradient $\nabla f(u)$ of the functional on a ball $B(r, u^*)$. Consider a gradient procedure for approximating the minimum u^* of the functional $f(u)$:

$$u_{n+1} = u_n - \gamma_n \cdot (I - A'(u_n))^* (u_n - A(u_n)), \qquad n = 0, 1, \ldots. \tag{3.22}$$

From Theorem 3.1 and Lemma 3.1, Theorem 3.2 follows.

THEOREM 3.2. *Suppose the control parameters γ_n of gradient procedure (3.22) satisfy inequalities*

$$0 \le a \le \gamma_n \le \frac{2}{L}, \tag{3.23}$$

and the initial approximation u_0 of algorithm (3.22) is sufficiently close to u^*.

Then, the convergence

$$\lim_{n\to\infty} \|u_n - u^*\| = 0 \tag{3.24}$$

is valid.

3.2. Periodic Solutions of Delay Systems

In this section, Theorem 3.2 is applied to the proof of a convergence for the iteration scheme of approximating periodic solutions for delay systems.

Consider the problem of finding periodic solutions for the equation

$$\begin{aligned} \frac{dx(t)}{dt} &= ax(t) + f(x(t-h)), \\ x(s) &= \vartheta(s), \qquad -h \le s \le 0. \end{aligned} \tag{3.25}$$

By $p(t; \vartheta(s))$ we denote the solution of equation (3.25) with initial condition $\vartheta(s)$, $-h \le s \le 0$, and by E, we denote the space of initial conditions for (3.25). The operator

$$U(t; \vartheta(\cdot)) = p(t + s; \vartheta(s)), \qquad -h \le s \le 0, \tag{3.26}$$

is called the translation operator along solutions of equation (3.25). The translation operator is a mapping $E \times R_+ \to E$. The problem of finding periodic solutions of equation (3.25) is equivalent to the problem of finding initial conditions $\vartheta(s)$ and a time t such that the operator $U(t; \vartheta(s))$ has a fixed point.

Let us choose some function $\varphi(s)$, $-h \le s \le 0$, and consider the functional

$$\Phi(t; \vartheta(s)) = \frac{1}{2} \int_{-h}^{0} (\vartheta(s) - U(t; \vartheta(s)))^2 \, ds + \frac{1}{2} \cdot \left(\int_{-h}^{0} \varphi(s)\vartheta(s) \, ds - t \right)^2. \tag{3.27}$$

Obviously, the points of global minimum of the functional are the approximate solutions for the periodic problem. In this case the first component T^* of the minimum point $\{T^*, \vartheta^*\}$ is a value of the period, and ϑ^* is its initial condition.

Let us use a gradient method for solving a minimum problem with functional (3.27). By direct calculations, we get

$$\nabla_\vartheta \Phi(t;\vartheta(s)) = (I-U'_\vartheta(t;\vartheta(s)))^* (\vartheta(s)-U(t;\vartheta(s)))+\varphi(s)\left(\int_{-h}^{0} \varphi(s)\vartheta(s)\,ds - t\right), \tag{3.28}$$

$$\nabla_t \Phi(t;\vartheta(s)) = -\int_{-h}^{0} (\vartheta(s) - U(t;\vartheta(s)))U'_t(t;\vartheta(s))\,ds - \left(\int_{-h}^{0} \varphi(s)\vartheta(s)\,ds - t\right), \tag{3.29}$$

where

$$U'_\vartheta(t;\vartheta(\cdot))\tau(s) = e^{a(t+s)}\tau(0) + \int_{-h}^{s} e^{a(t+s-\mu)} f'(p(t+\mu-h;\vartheta(s)))\tau(\mu)\,d\mu, \tag{3.30}$$

$$\nabla_t \Phi(t;\vartheta(s)) = -\int_{-h}^{0} (\vartheta(s) - U(t;\vartheta(s)))U'_t(t;\vartheta(s))\,ds - \left(\int_{-h}^{0} \varphi(s)\vartheta(s)\,ds - t\right). \tag{3.31}$$

The conjugation operation in (3.28) demands an extension onto $L_2[-h,0]$ space. This extension is connected with an introduction of distributions. Therefore, at first, we make a regularization of operator's derivative to simplify the numerical realization of the gradient procedure, then we use a conjugation operation for regularized operator in (3.28). Suppose

$$\begin{aligned} A_\varepsilon(t;\vartheta(s)) &= \vartheta(s) - p(t+s;\vartheta(s)) - B_\varepsilon[\vartheta(s) - p(t+s;\vartheta(s))] \\ &\quad - C[\vartheta(s) - p(t+s;\vartheta(s))] + \varphi(s)\left(\int_{-h}^{0} \varphi(s)\vartheta(s)\,ds - t\right), \end{aligned} \tag{3.32}$$

where the linear operators B_ε and C are defined by formulae

$$B[g(s)] = \begin{cases} \dfrac{1}{\varepsilon} e^{at} \displaystyle\int_{-h}^{0} e^{a\mu} g(\mu)\,d\mu, & \text{for } -\varepsilon \le s \le 0, \\ 0, & \text{for } -h \le s \le -\varepsilon, \end{cases} \tag{3.33}$$

and

$$C[g(s)] = \int_{s}^{0} e^{a(t+s-\mu)} f'(p(t+\mu-h;\vartheta(s)))g(\mu)\,d\mu, \tag{3.34}$$

where $\varepsilon > 0$ is a parameter of regularization.

The iteration scheme of approximating the minimum points for functional (3.27) is the following:

$$\begin{aligned} \vartheta_{n+1}(s) &= \vartheta_n(s) - \gamma_n \cdot A_{\varepsilon_n}(t;\vartheta(s)), \\ t_{n+1} &= t_n - \mu_n \nabla_t \Phi(t;\vartheta(s)). \end{aligned} \tag{3.35}$$

The next theorem follows from Theorem 3.2.

THEOREM 3.3. *Let $x^*(t)$ be a periodic solution of equation (3.25) with period T^*. Suppose the pair $\{T^*, \vartheta^*(s)\}$ is the isolated critical point for functional (3.27) over the solution $x^*(t)$, and with some α and β, the inequalities*

$$\int_{-h}^{0} \langle \varphi(s), x^*(s+\alpha)\rangle\, ds < T^*, \tag{3.36}$$

$$\int_{-h}^{0} \langle \varphi(s), x^*(s+\beta)\rangle\, ds < T^*, \tag{3.37}$$

hold.

Suppose, in addition, that control parameters γ_n and μ_n in (3.35) satisfy inequalities

$$0 < a_1 \le \gamma_n \le b_1, \tag{3.38}$$

$$0 < a_2 \le \mu_n \le b_2; \tag{3.39}$$

the initial condition $\{t_0, \vartheta_0(s)\}$ satisfies inequalities

$$|t_0 - T^*| \le \delta_1, \tag{3.40}$$

$$\max_{-h\le s\le 0} |\vartheta^*(s) - \vartheta_0(s)| \le \delta_2, \tag{3.41}$$

and finally, numbers $b_1, b_2, \beta_1, \beta_2$ are sufficiently small, and

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{3.42}$$

Then

$$\lim_{n\to\infty} |t_n - T^*| = 0, \tag{3.43}$$

$$\lim_{n\to\infty} \max_{-h\le s\le 0} |\vartheta^*(s) - \vartheta_n(s)| = 0. \tag{3.44}$$

4. PARAMETER'S FUNCTIONALIZATION METHOD FOR APPROXIMATING CYCLES IN AUTONOMOUS SYSTEMS

The problem of evaluating periodic solutions for autonomous systems as compared with the same problem for nonautonomous systems has some specific features.

First, a period of an autonomous system is unknown *a priori*. Second, the periodic solution for an autonomous system may be nonisolated; namely, if $x_0(t)$ is a T-periodic solution of the autonomous differential equation

$$\frac{dx}{dt} = f(x), \tag{4.1}$$

then all functions $x_h(t) = x_0(t+h)$, $0 \le t \le T$, are also T-periodic solutions for equation (4.1). These two arguments essentially complicate the problem of evaluating periodic solutions for autonomous systems.

The parameter's functionalization method was proposed in [11] for a qualitative analysis of an autonomous system.

Let us recall a procedure of this method. Consider an equation

$$\frac{dx}{dt} = \lambda \cdot f(x). \tag{4.2}$$

Obviously, the problem of searching T-periodic solution of equation (4.1) is equivalent to the problem of finding values of λ such that equation (4.2) has a periodic solution with fixed period 1. The latter problem, in turn, is equivalent to locating the values of parameter λ such that an integral equation

$$x(t) = x(1) + \lambda \cdot \int_0^t f(x(s))\,ds, \tag{4.3}$$

has a solution.

The parameter's functionalization method is based on substituting integral equation (4.3) by equivalent integro-functional equation

$$x(t) = x(1) + \lambda(x(t)) \cdot \int_0^t f(x(s))\,ds, \tag{4.4}$$

where $\lambda(x) = \lambda(x(t))$ is a functional. Equation (4.4) does not include a parameter.

Moreover, if function $x_0(t)$, $0 \le t \le 1$, is a solution of (4.4), then function $x_0(T_0 t)$ is a T_0-periodic solution of (4.1), where

$$T_0 = \lambda\left(x_0\left(t\right)\right).$$

Note that solutions of (4.4) are in most cases isolated.

Thus, the parameter's functionalization method reduces the periodic solutions problem for autonomous systems to the problem of evaluating isolated solutions for an equation which does not include any parameter.

In this paper, the parameter's functionalization method is applied to the problem of the approximating of cycles for autonomous systems. For this purpose the combination of the mechanical quadrature's method and the Newton method is used.

4.1. Description of Algorithm

Suppose equation (4.1) has a periodic solution $x_0(t)$ with period T_0. Then the function $x_1(t) = x_0(T_0^{-1}t)$ is a solution of integral equation (4.3) with $\lambda = T_0$. Note that, for this λ, all functions $x_h(t) = x_1(t+h)$, $0 \le h \le 1$ are solutions of equation (4.3).

Choose the functional $\lambda(x)$ in (4.4) in the form

$$\lambda(x) = \int_0^1 \langle a(s), x(s)\rangle\, ds, \tag{4.5}$$

where $a(t)$, $0 \le t \le 1$, is a vector-function; and $\langle\cdot,\cdot\rangle$ denotes a scalar product. The equation

$$x(t) = x(1) + \int_0^1 \langle a(s), x(s)\rangle\, ds \cdot \int_0^1 f(x(s))\, ds, \tag{4.6}$$

is solvable, if the function $a(t)$ is such that the scalar function

$$\alpha(h) = \int_0^1 \langle a(s), x_1(s+h)\rangle \; ds - T_0, \qquad 0 \le h \le 1, \tag{4.7}$$

is alternating in sign over the interval $[0,1]$.

In this case, the function

$$x^*(t) = x_1\left(t + h^*\right), \tag{4.8}$$

where h^* is a zero of scalar function $\alpha(h)$ over $[0,1]$, is a solution of (4.6).

In practice, in most cases, the vector-function $a(t)$ determining functional (4.5) may be a constant vector $a \in \mathbb{R}^N$.

Choose a discretization for equation (4.6) by the mechanical quadrature's method. Let α_{1m}, $\dots$, α_{mm} be positive real numbers and

$$0 \le t_{1m} < \dots < t_{mm} = 1,$$

be points of the interval $[0,1]$.

Define a quadrature process by the sequence of quadrature formulae

$$\int_0^1 x(s)\, ds = \sum_{j=1}^{m} \alpha_{jm} x\left(t_{jm}\right) + R_m(x), \qquad m = 1, 2, \dots. \tag{4.9}$$

Here the numbers $\alpha_{1m}, \dots, \alpha_{mm}$ are the coefficients and points $t_{1m}, \dots, t_{mm}$ are the nodes of quadrature formula (4.9). The difference

$$R_m(x) = \int_0^1 x(s)\, ds - \sum_{j=1}^{m} \alpha_{jm} x\left(t_{jm}\right), \tag{4.10}$$

is a remainder term of quadrature formula (4.9).

DEFINITION 4.1. *Quadrature process (4.9) is called convergent if for each continuous function $x(t)$*

$$\lim_{m\to\infty} R_m(x) = 0. \tag{4.11}$$

For example, the quadrature process defined by rectangular formula with uniformly distributed nodes, i.e.,

$$t_{km} = \frac{k}{m}, \qquad k = 1, \dots, m, \tag{4.12}$$

$$\alpha_{km} = \frac{1}{m}, \qquad k = 1, \dots, m, \tag{4.13}$$

is convergent.

Let us fix a positive integer m and denote by $\xi_k^* \in \mathbb{R}^N$ successive approximations for periodic solution $x^*(t)$ in the nodes t_{km}

$$\xi_k^* \approx x^*\left(t_{km}\right). \tag{4.14}$$

Substituting integral (4.6) by finite sums (4.9) without the remainder term, we get a system of equations

$$\begin{aligned}
\xi_1 &= \xi_m + \sum_{k=1}^{m} \alpha_{km} \cdot \langle a_k^m, \xi_k \rangle \cdot \alpha_{1m} f\left(\xi_1\right), \\
\xi_2 &= \xi_m + \sum_{k=1}^{m} \alpha_{km} \cdot \langle a_k^m, \xi_k \rangle \cdot \left(\alpha_{1m} f\left(\xi_1\right) + \alpha_{2m} f\left(\xi_2\right)\right), \\
&\dots \\
\xi_m &= \xi_m + \sum_{k=1}^{m} \alpha_{km} \cdot \langle \alpha_k^m, \xi_k \rangle \cdot \sum_{k=1}^{m} \alpha_{km} f\left(\xi_k\right),
\end{aligned} \tag{4.15}$$

where a_k^m are the values of vector-function $a(t)$ in the nodes t_{km}.

System (4.15) is a system of mN scalar linear equations with mN variables ξ_{jk}, $j = 1, \dots, N$; $k = 1, \dots, m$. The variables ξ_{jk} are the components of vectors

$$\xi_k = \begin{bmatrix} \xi_{1k} \\ \vdots \\ \xi_{Nk} \end{bmatrix}, \qquad k = 1, \dots, m. \tag{4.16}$$

Let us use a Newton method to solve system (4.15). For this system, a procedure of the Newton method is the following.

Successive approximations ξ_k^n, $n = 0, 1, \dots$, to the solution ξ_k^* of system (4.15) are

$$\xi_k^{n+1} = \xi_k^n - y_k^n, \qquad k = 1, \dots, m; \qquad n = 1, 2, \dots, \tag{4.17}$$

where y_k^n are solutions of linear system

$$\begin{aligned}
&y_1^n - y_m^n - \alpha_{1m} f\left(\xi_1^n\right) \cdot \sum_{k=1}^{m} \alpha_{km} \langle a_k^m, y_k^n \rangle - \alpha_{1m} \frac{\partial f\left(\xi_1^n\right)}{\partial x} y_1^n \cdot \sum_{k=1}^{m} \alpha_{km} \langle a_k^m, \xi_k^n \rangle \\
&\quad = \xi_1^n - \xi_m^n - f\left(\xi_1^n\right) \cdot \sum_{k=1}^{m} \alpha_{km} \langle a_k^m, \xi_k^n \rangle, y_2^n - y_m^n - \left(\alpha_{1m} f\left(\xi_1^n\right) + \alpha_{2m} f\left(\xi_2^n\right)\right) \cdot \sum_{k=1}^{m} \alpha_{km} \langle a_k^m, y_k^n \rangle \\
&\qquad - \left(\alpha_{1m} \frac{\partial f\left(\xi_1^n\right)}{\partial x} y_1^n + \alpha_{2m} \frac{\partial f\left(\xi_2^n\right)}{\partial x} y_2^n\right) \cdot \sum_{k=1}^{m} \alpha_{km} \langle a_k^m, \xi_k^n \rangle \\
&\quad = \xi_2^n - \xi_m^n - \left(\alpha_{1m} f\left(\xi_1^n\right) + \alpha_{2m} f\left(\xi_2^n\right)\right) \cdot \sum_{k=1}^{m} \alpha_{km} \langle a_k^m, \xi_k^n \rangle,
\end{aligned} \tag{4.18}$$

$\dots$

$$y_m^n - y_m^n - \left(\sum_{k=1}^{m} \alpha_{km} f\left(\xi_k^n\right)\right) \cdot \left(\sum_{k=1}^{m} \alpha_{km} \left\langle a_k^m, y_k^n \right\rangle\right) - \left(\sum_{k=1}^{m} \alpha_{km} \frac{\partial f\left(\xi_k^n\right)}{\partial x} y_k^n\right) \cdot \left(\sum_{k=1}^{m} \alpha_{km} \left\langle a_k^m, \xi_k^n \right\rangle\right)$$
$$= \xi_m^n - \xi_m^n - \left(\sum_{k=1}^{m} \alpha_{km} f\left(\xi_k^n\right)\right) \cdot \left(\sum_{k=1}^{m} \alpha_{km} \left\langle a_k^m, \xi_k^n \right\rangle\right).$$

In (4.18) by $\frac{\partial f(\xi_k^n)}{\partial x}$, we denote the Jacobian of the map

$$f(x) = \begin{bmatrix} f_1\left(x_1, \dots, x_N\right) \\ \vdots \\ f_N\left(x_1, \dots, x_N\right) \end{bmatrix}, \tag{4.19}$$

at the point

$$\xi_k^n = \begin{bmatrix} \xi_{1k}^n \\ \vdots \\ \xi_{Nk}^n \end{bmatrix}, \tag{4.20}$$

i.e.,

$$\frac{\partial f(\xi_k^n)}{\partial x} = \begin{bmatrix} \frac{\partial f_1\left(\xi_{1k}^n, \dots, \xi_{Nk}^n\right)}{\partial x_1}, & \dots, & \frac{\partial f_1\left(\xi_{1k}^n, \dots, \xi_{Nk}^n\right)}{\partial x_N} \\ \dots & \dots & \dots \\ \frac{\partial f_N\left(\xi_{1k}^n, \dots, \xi_{Nk}^n\right)}{\partial x_1}, & \dots, & \frac{\partial f_N\left(\xi_{1k}^n, \dots, \xi_{Nk}^n\right)}{\partial x_N} \end{bmatrix}. \tag{4.21}$$

4.2. Convergence's Proof

The proof of convergence for the above algorithm is based on the following statements.

Let the right-hand side of equation (4.1) be continuously differentiable function of its arguments. Suppose equation (4.1) has a T_0-periodic solution $x_0(t)$ and the corresponding orbit is an isolated cycle in the phase space $\mathbb{R}^N$. Suppose

$$x_1(t) = x_0\left(T^{-1}t\right), \tag{4.22}$$

and by h^* denote a zero of function (4.7) over the interval $[0, 1]$.

PROPOSITION 4.1. *Suppose*

$$\int_0^1 \left\langle a(s), f\left(x_1\left(s + h^*\right)\right)\right\rangle ds \neq 0. \tag{4.23}$$

Then the function

$$x^*(t) = x_1\left(t + h^*\right), \tag{4.24}$$

is an isolated solution of equation (4.9) in $C([0, 1])$.

Consider the linearization of equation (4.1) along the solution $x_0(t)$:

$$\frac{dy}{dt} = f'\left(x_0(t)\right) y. \tag{4.25}$$

By $\mu_1, \dots, \mu_N$, denote the multiplicators of equation (4.25). Because equation (4.1) is autonomous, there is the multiplicator $\mu = 1$ among them.

PROPOSITION 4.2. *Let* $\mu = 1$ *be the simple multiplicator of equation (4.25) and the other multiplicators do not lie on the unit circle. There exists* $\varepsilon > 0$ *such that for a sufficiently large* m*, there is a unique solution*

$$\xi^* = \left\{\xi_1^*, \dots, \xi_m^*\right\}$$

of system (4.15) in the ball

$$B(\varepsilon, \eta^*) = \{\xi \in \mathbb{R}^{Nm} : |\xi - \eta^*| \le \varepsilon\}, \tag{4.26}$$

and

$$\lim_{m\to\infty} \max_{1\le k\le m} |x^*(t_{km}) - \xi_k^*| = 0. \tag{4.27}$$

Here

$$\eta^* = \{x^*(t_{1m}), \dots, x^*(t_{mm})\}. \tag{4.28}$$

In addition, if m is sufficiently large and the initial approximation $\xi^0 = \{\xi_1^0, \dots, \xi_m^0\}$ of method (4.17) is fairly close to the solution $\xi^* = \{\xi_1^*, \dots, \xi_m^*\}$ of (4.15), then the linear system (4.16) is solvable and the convergence

$$\lim_{n\to\infty} \|\xi^n - \xi^*\| = 0$$

is valid.

Proofs of Propositions 4.1 and 4.2 are based on the global theorems from monograph [8].

5. BIBLIOGRAPHICAL COMMENTARY

A large portion of the bibliography is devoted to iteration schemes for approximate computation of solutions of nonlinear systems. There is the literature on basic directions of this theory in the references of monograph [14]. Papers [11,15–17] are devoted to some results for the approximate evaluation of oscillating regimes in the autonomous systems. One may see theorems on convergence of projective methods for approximate evaluation of auto-oscillations in [8].

A large portion of the bibliography is devoted to the analysis of approximate procedures as the gradient descent method (see, for example, [18–21] and the references therein). The proof of convergence for the simple iteration methods in the case of regular critical points was given in [22]. In the paper [23], the convergence for the method of steepest descent under (S)-condition was proved. (S)-condition was introduced by Browder [24]. In [25], Skrypnik independently has formulated (S)-condition for solvable boundary-value elliptic problems. Classes of operators with properties like (S)-condition were considered in [26,27].

The algorithm of approximating cycles for nonlinear systems is based on three methods: the parameter's functionalization method, the mechanical quadrature's method, and the Newton methods.

The parameter's functionalization method was proposed in [11]. The detailed description of this method and of its applications for qualitative and approximate analysis of nonlinear systems, one may see in [8].

The mechanical quadrature's method was studied in [28–35]. In [13,14], this method was applied to approximating periodic solutions of ordinary differential equations. There are applications of the mechanical quadrature's method for approximate solving general integral equations in [6,36,37].

The Newton method is well known and was described in several monograph and handbooks.

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