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Semilinear Integrodifferential Equations with Nonlocal Initial Conditions

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Abstract—Of concern is the Cauchy problem for semilinear integrodifferential equations with nonlocal initial conditions. Under general and natural hypotheses, we establish some new theorems about the existence and uniqueness of solutions for the Cauchy problem. As a consequence, we unify and extend the corresponding theorems given previously for the Cauchy problem for differential equations or integrodifferential equations with nonlocal initial conditions. Moreover, we present two examples, one of which comes from heat conduction in materials with memory, to show that the existing results are not applicable to them, in contrast with ours. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Cauchy problem, Nonlocal initial condition, Semilinear integrodifferential equation, Mild solution, Classical solution.

1. INTRODUCTION AND PRELIMINARY

Let X be a Banach space, $\mathbf{L}(X)$ the space of bounded linear operators from X to X, A the generator of a C_0 semigroup on X, $\mathcal{D}(A)$ the domain of A, and $[\mathcal{D}(A)]$ the space $\mathcal{D}(A)$ with the graph norm.

Of concern is the following Cauchy problem for a semilinear integrodifferential equation with a nonlocal initial condition:

$$\begin{aligned} u'(t) = A\left[u(t) + \int_{t_0}^{t} F(t-s)u(s)\,ds\right] + f(t,u(t)), \qquad t \in [t_0, t_0+T], \\ u(t_0) + g(t_1, \dots, t_p, u) = u_0, \end{aligned} \tag{1.1}$$

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where $\{F(t)\}_{t\in[0,T]} \subset \mathbf{L}(X)$ is a family of operators such that

$$\begin{aligned} F(t)([\mathcal{D}(A)]) &\subset [\mathcal{D}(A)], & t &\in [0,T], \\ AF(\cdot)u(\cdot) &\in L^1([0,T],X), & u(\cdot) &\in C([0,T],[\mathcal{D}(A)]), \\ F(\cdot)u &\in C^1([0,T],X), & u &\in X; \end{aligned} \tag{1.2}$$

$f(\cdot,\cdot) \in C([t_0, t_0+T] \times X, X)$ and

$$\|f(t,x) - f(t,y)\| \le L\|x-y\|, \qquad t \in [t_0, t_0+T], \quad x, y \in X, \tag{1.3}$$

for a constant $L > 0$ and $0 \le t_0 < t_1 < \cdots < t_p \le t_0 + T$. The X-valued function $g(t_1, \ldots, t_p, \cdot)$ on $C([t_0, t_0+T], X)$ satisfies

$$\begin{gathered} \|g(t_1,\ldots,t_p,\phi) - g(t_1,\ldots,t_p,\psi)\| \le K \max_{t\in[t_0,t_0+T]} \|\phi(t) - \psi(t)\|, \\ \phi, \psi \in C([t_0, t_0+T], X), \end{gathered} \tag{1.4}$$

for a constant $K > 0$.

Interest in the Cauchy problem for differential equations with nonlocal initial conditions stems mainly from the observation that nonlocal initial conditions are more realistic than the usual ones in treating physical problems. From [1] and the references given there, one can find more detailed information about the importance of nonlocal initial conditions in applications. There have been many papers concerning this topic (cf., e.g., [1–7] and references therein), especially since the work [1] in 1991.

However, much of the previous research was done under the condition "$M(K+TL) < 1$" or its analogues (cf., e.g., [1,3,6]). This condition turns out to be quite restrictive. In particular, limited by it, the results obtained for nonlocal problems cannot cover the classical results regarding the case when $F \equiv 0$ and $g \equiv 0$, i.e., the following differential equations with usual initial conditions:

$$u(t) = Au(t) + f(t, u(t)) \qquad (t_0 \le t \le t_0 + T), \qquad u(t_0) = u_0 \tag{1.5}$$

(cf. [8, Chapter 6]). Thus, there naturally arises a question: "Can the condition above be relaxed in such a way that the results for nonlocal problems cover the corresponding ones for (1.5)?" In this paper, among others we will give an affirmative answer to this question (see Corollary 2.2(1), Theorem 2.7, Remark 2.3(c), and Remark 2.9(a)).

In Section 2, we first study the existence and uniqueness of solutions for a general integral equation ((2.3) below), and then investigate the corresponding problems for (1.1). The theorems formulated are unifications and extensions of those given previously for the Cauchy problem for differential equations or integrodifferential equations with nonlocal initial conditions. As the reader will see, the hypotheses in our theorems are reasonably weak and the proofs provided are concise. Moreover, following every main result, we append a remark with a detailed analysis of how the result extends and improves the known ones. Finally, in Section 3, we apply our theorems to two concrete problems, one of which comes from heat conduction in materials with memory. It is shown that the existing results are not applicable to them, in contrast with ours.

To begin with, we recall that there is a strongly continuous $\{R(t)\}_{t\in[0,T]} \subset \mathbf{L}(X)$ such that $R(0) = I$, $R(\cdot)y \in C^1([0,T],X) \cap C([0,T],[\mathcal{D}(A)])$ $(y \in \mathcal{D}(A))$, and

$$\begin{aligned} \frac{d}{dt}R(t)y &= A\left[R(t)y + \int_0^t F(t-s)R(s)y\,ds\right] \\ &= R(t)Ay + \int_0^t R(t-s)AF(s)y\,ds, \qquad t \in [0,T], \end{aligned} \tag{1.6}$$

(cf. [6] or [9]).

DEFINITION 1.1. *A mild solution of (1.1) is a function* $u \in C([t_0, t_0 + T], X)$ *satisfying*

$$u(t) = R(t - t_0)[u_0 - g(t_1, \ldots, t_p, u)] + \int_{t_0}^{t} R(t - s) f(s, u(s))\, ds, \quad t \in [t_0, t_0 + T]. \tag{1.7}$$

A classical solution of (1.1) is a function

$$u \in C^1([t_0, t_0 + T], X) \cap C([t_0, t_0 + T], [\mathcal{D}(A)])$$

satisfying (1.7).

2. RESULTS AND PROOFS

Assume that

(H1) $\{S(t)\}_{t\in[0,T]} \subset \mathbf{L}(X)$ is a strongly continuous family, and $\|S(t)\| \le Me^{-\omega t}$ $(t \in [0, T])$, where M and $\omega \ge 0$ are constants;

(H2) $h : C([t_0, t_0 + T], X) \to X$ and there exists a nonnegative function Φ on $C([t_0, t_0 + T], [0, \infty))$ satisfying

$$\begin{aligned} &\Phi(k\mu) \le k\Phi(\mu), && \forall k > 0, \quad \mu \in C([t_0, t_0 + T], [0, \infty)), \\ &\Phi(\mu_1) \le \Phi(\mu_2), && \forall \begin{cases} \mu_1, \mu_2 \in C([t_0, t_0 + T], [0, \infty)), \\ \text{with } \mu_1(t) \le \mu_2(t) \ (t \in [t_0, t_0 + T]), \end{cases} \end{aligned} \tag{2.1}$$

such that

$$\|h(\phi) - h(\psi)\| \le \Phi(\|\phi - \psi\|), \qquad \phi, \psi \in C([t_0, t_0 + T], X). \tag{2.2}$$

We first look at a general integral equation

$$v(t) = S(t - t_0)(u_0 - h(v)) + \int_{t_0}^{t} S(t - s) f(s, v(s))\, ds, \qquad t \in [t_0, t_0 + T]. \tag{2.3}$$

THEOREM 2.1. *Let (1.3), (H1), and (H2) hold and* $M\Phi(e^{(ML-\omega)(\cdot - t_0)}) < 1$. *Then, for all* $u_0 \in X$, *(2.3) has a unique solution* $v \in C([t_0, t_0 + T], X)$.

PROOF. Let $u_1 \in C([t_0, t_0 + T], X)$ be fixed, and $u_{1,0} := u_0 - h(u_1)$. Define an operator $\mathcal{F}$ on $C([t_0, t_0 + T], X)$ by

$$(\mathcal{F}u)(t) = S(t - t_0)u_{1,0} + \int_{t_0}^{t} S(t - s) f(s, u(s))\, ds, \qquad t \in [t_0, t_0 + T]. \tag{2.4}$$

Clearly, $\mathcal{F}(C([t_0, t_0 + T], X)) \subset C([t_0, t_0 + T], X)$. By a standard argument, we see that $\mathcal{F}$ has a unique fixed point $u_2 \in C([t_0, t_0 + T], X)$. Using mathematical induction, we infer that there exists a sequence $\{u_n\}_{n=2}^{\infty} \subset C([t_0, t_0 + T], X)$ such that

$$u_n(t) = S(t - t_0)u_{n-1,0} + \int_{t_0}^{t} S(t - s) f(s, u_n(s))\, ds, \qquad t \in [t_0, t_0 + T], \quad n \ge 2, \tag{2.5}$$

where

$$u_{n-1,0} = u_0 - h(u_{n-1}). \tag{2.6}$$

A combination of (1.3), (H1), (2.2), (2.5), and (2.6) shows

$$e^{\omega t}\|u_3(t) - u_2(t)\| \le e^{\omega t} M\Phi(\|u_2(\cdot) - u_1(\cdot)\|) + ML \int_{t_0}^{t} e^{\omega s}\|u_3(s) - u_2(s)\|\, ds, \qquad t \in [t_0, t_0 + T].$$

By Bellman-Gronwall's inequality,

$$\|u_3(t) - u_2(t)\| \le Me^{(ML-\omega)(t-t_0)}\Phi(\|u_2(\cdot) - u_1(\cdot)\|), \qquad t \in [t_0, t_0 + T].$$

Using (2.1) and mathematical induction, we have for each $t \in [t_0, t_0 + T]$,

$$\|u_n(t) - u_{n-1}(t)\| \le Me^{(ML-\omega)(t-t_0)} \left(M\Phi\left(e^{(ML-\omega)(\cdot - t_0)}\right)\right)^{n-3} \Phi(\|u_2(\cdot) - u_1(\cdot)\|), \qquad n \ge 3.$$

According to the assumption, we obtain for any $m > n \ge 3$

$$\begin{aligned} &\max_{t\in[t_0,t_0+T]} \|u_m(t) - u_n(t)\| \\ &\le \sum_{i=n}^{m-1} \max_{t\in[t_0,t_0+T]} \|u_{i+1}(t) - u_i(t)\| \\ &\le \max\left\{M, Me^{(ML-\omega)(T-t_0)}\right\} \Phi(\|u_2(\cdot) - u_1(\cdot)\|) \sum_{i=n}^{m-1} \left(M\Phi\left(e^{(ML-\omega)(\cdot-t_0)}\right)\right)^{i-2} \longrightarrow 0, \\ &\quad \text{as } n \to \infty; \end{aligned}$$

that is, $\{u_n\}_{n=2}^{\infty}$ is a Cauchy sequence in $C([t_0, t_0 + T], X)$. Therefore, there is a $u \in C([t_0, t_0 + T], X)$ such that

$$\lim_{n\to\infty} u_n(t) = u(t), \qquad \text{uniformly for } t \in [t_0, t_0 + T].$$

This together with (2.4)–(2.6) implies that $u(t)$ is a continuous solution of (2.3).

Let $u(t)$ and $v(t)$ be the solution of (2.3). Then, by (1.3), (H1), (2.2), and (2.3),

$$e^{\omega t}\|u(t) - v(t)\| \le e^{\omega t} M\Phi(\|u(\cdot) - v(\cdot)\|) + ML \int_{t_0}^{t} e^{\omega s}\|u(s) - v(s)\|\, ds, \qquad t \in [t_0, t_0 + T],$$

which implies, by Bellman-Gronwall's inequality and (2.1), that

$$\|u(t) - v(t)\| \le Me^{(ML-\omega)(t-t_0)} \left(M\Phi\left(e^{(ML-\omega)(\cdot-t_0)}\right)\right)^{n-1} \Phi(\|u(\cdot) - v(\cdot)\|), \quad t \in [t_0, t_0 + T], \quad n \ge 1.$$

Letting $n \to \infty$, we have $u(t) \equiv v(t)$ on $[t_0, t_0 + T]$. This means that the solution of (2.3) is unique.

COROLLARY 2.2. *Let (1.3), (H1), and one of the following assumptions hold.*

(1) *There is a constant $K > 0$ such that*

$$\|h(\phi) - h(\psi)\| \le K \max_{s\in[t_0,t_0+T]} \|\phi(s) - \psi(s)\|, \qquad (\phi, \psi \in C([t_0, t_0 + T], X)),$$

and $KMe^{T\max\{ML-\omega,0\}} < 1$.

(2) *There are constants $K > 0$, $t_0 \le q < r \le t_0 + T$ such that*

$$\|h(\phi) - h(\psi)\| \le K \int_q^r \|\phi(s) - \psi(s)\|\, ds, \qquad (\phi, \psi \in C([t_0, t_0 + T], X)),$$

and

$$\begin{aligned} KM(r - q) &< 1, \qquad \text{if } ML = \omega, \\ \frac{KM}{ML - \omega}\left(e^{(ML-\omega)(r-t_0)} - e^{(ML-\omega)(q-t_0)}\right) &< 1, \qquad \text{if } ML \ne \omega. \end{aligned}$$

(3) *There are $c_1, \ldots, c_p \in \mathbf{C}$ (the set of complex numbers) such that*

$$\|h(\phi) - h(\psi)\| \le \sum_{i=1}^{p} |c_i|\|\phi(t_i) - \psi(t_i)\|, \qquad (\phi, \psi \in C([t_0, t_0 + T], X)),$$

and $M\sum_{i=1}^{p} |c_i|e^{(ML-\omega)(t_i-t_0)} < 1$.

Then, for all $u_0 \in X$, equation (2.3) has a unique solution $v \in C([t_0, t_0 + T], X)$.

PROOF. Applying Theorem 2.1 to the functions

$$\Phi(\mu) = K \max_{s\in[t_0,t_0+T]} \mu(s), \qquad \Phi(\mu) = K \int_q^r \mu(s)\,ds, \qquad \Phi(\mu) = \sum_{i=1}^{p} |c_i|\mu(t_i),$$

respectively, we get the desired conclusions.

REMARK 2.3.

(a) The proof of Theorem 2.1 shows a way to compute the continuous solution of (2.3).
(b) Corollary 2.2(1) gives a generalization of [6, Theorem 3.2], because
   (1) the operator family $\{S(\cdot)\}$ and the mapping $h(u)$ in Corollary 2.2(1) are more general than the operator family $\{R(\cdot)\}$ and the mapping $g(t_1,\dots,t_p,u(t_1),\dots,u(t_p))$, respectively;
   (2) if we let

$$t_0 = 0, \qquad \omega = 0, \qquad S(\cdot) = R(\cdot), \qquad h(u) = g(t_1,\dots,t_p,u(t_1),\dots,u(t_p)),$$

   then, Corollary 2.2 says that (1.10),(1.11) in [6] has a unique mild solution for any $u_0 \in X$ provided $MK < e^{-MTL}$. But, Theorem 3.2 in [6] is not applicable for any $K \geq 0$ when $MTL \geq 1$, since

$$MK + MTL \geq 1, \qquad \text{for any } K \geq 0;$$

   (3) for $M,K,T,L \geq 0$, the inequality $MKe^{MTL} < 1$ does not imply $M(K+TL) < 1$ even if $MTL < 1$ (for example, let $MK = 3/4$ and $MTL = 1/4$, then $MTL < 1$ and $MKe^{MTL} < 1$, but $M(K+TL) = 1$), however, the converse holds, in fact, for $M,K,T,L \geq 0$, the inequality $M(K+TL) < 1$ implies

$$MKe^{MTL} < MKe^{1-MK} < 1,$$

   by noting that the function $\xi \mapsto \xi e^{1-\xi}$ is increasing on $[0,1]$.
(c) Corollary 2.2(1) covers naturally and directly the "existence and uniqueness" part of [8, p. 184, Theorem 6.1.2], because if $h \equiv 0$, then $K \equiv 0$ which means that the assumption $KMe^{T\max\{ML-\omega,0\}} < 1$ always holds.

Using the idea in the proof of Theorem 2.1, we can also get the following theorem.

THEOREM 2.4. *Let A generate a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Write $\Omega_r := \{u;\ u \in X$ and $\|u\| \leq r\}$ $(r > 0)$. Assume that*

(i) *there exists a constant $L_0 > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq L_0\|x-y\|, \qquad t \in [t_0, t_0+T], \quad x,y \in \Omega_r;$$

(ii) *there exists a constant $K_0 > 0$ such that*

$$\|g(t_1,\dots,t_p,\phi) - g(t_1,\dots,t_p,\psi)\| \leq K_0 \max_{t\in[t_0,t_0+T]} \|\phi(t)-\psi(t)\|, \qquad \phi,\psi \in C([t_0,t_0+T],\Omega_r);$$

(iii) *the inequality $M_0(\|u_0\| + G + T(rL_0 + F)) \leq r$ holds with*

$$M_0 := \max_{s\in[t_0,t_0+T]} \|T(s)\|, \qquad F := \max_{s\in[t_0,t_0+T]} \|f(s,0)\|,$$

   and $G := \sup_{\phi\in C([t_0,t_0+T],\Omega_r)} \|g(t_1,\dots,t_p,\phi)\|$;
(iv) *$M_0K_0e^{M_0TL_0} < 1$.*

Then,

$$u'(t) = Au(t) + f(t, u(t)), \qquad t_0 \le t \le t_0 + T, \\ u(t_0) + g(t_1, \dots, t_p, u) = u_0, \tag{2.7}$$

has a unique mild solution $u \in C([t_0, t_0 + T], \Omega_r)$.

REMARK 2.5.

(a) Theorem 2.4 is an extension of [1, Theorem 3.1] for the same reasons in (1) and (3) of Remark 2.3.

(b) The conclusion of Theorem 2.4 is also true replacing Assumption (iii) with the following weaker one.

(iii′) The inequality $M_0(\|u_0\| + G + TF_0) \le r$ holds with

$$M_0 := \max_{s \in [t_0, t_0+T]} \|T(s)\|, \qquad F_0 := \sup_{s \in [t_0, t_0+T], \phi \in C([t_0, t_0+T], \Omega_r)} \|f(s, \phi(s))\|,$$

and $G := \sup_{\phi \in C([t_0, t_0+T], \Omega_r)} \|g(t_1, \dots, t_p, \phi)\|$.

For the case of $h(\cdot)$ taking the form $h(\phi) = \sum_{i=1}^p c_i \phi(t_i)$ for every $\phi \in C([t_0, t_0 + T], X)$, $c_1, \dots, c_p \in \mathbf{C}$, we present the following Theorem 2.6 which is sharper than Corollary 2.2(3). Furthermore, this result unifies and extends both of [6, Theorem 4.3] and [3, Theorem 3.1] (see Remark 2.9 below).

THEOREM 2.6. *Let (1.3) and (H1) hold and for some* $c_1, \dots, c_p \in \mathbf{C}$,

$$h(\phi) = \sum_{i=1}^p c_i \phi(t_i) \qquad (\phi \in C([t_0, t_0 + T], X)).$$

Assume that $B := (I + \sum_{i=1}^p c_i S(t_i - t_0))^{-1} \in \mathbf{L}(X)$ *and*

$$\|B\| M \sum_{i=1}^p |c_i| e^{-\omega(t_i - t_0)} \left(e^{ML(t_i - t_0)} - 1 \right) < 1.$$

Then, for all $u_0 \in X$, *equation (2.3) has a unique solution* $v \in C([t_0, t_0 + T], X)$.

PROOF. By the standard arguments, we see that for every $x \in X$, there is a unique $v_x(\cdot) \in C([t_0, t_0 + T], X)$ satisfying

$$v_x(t) = S(t - t_0)x + \int_{t_0}^t S(t - s) f(s, v_x(s))\, ds, \qquad t \in [t_0, t_0 + T]. \tag{2.8}$$

Hence,

$$v_x(t_i) = S(t_i - t_0)x + \int_{t_0}^{t_i} S(t_i - s) f(s, v_x(s))\, ds, \qquad i = 1, \dots, p, \tag{2.9}$$

and (1.3) implies that for every x_1, $x_2 \in X$,

$$e^{\omega t} \|v_{x_1}(t) - v_{x_2}(t)\| \le e^{\omega t_0} M \|x_1 - x_2\| + ML \int_{t_0}^t e^{\omega s} \|v_{x_1}(s) - v_{x_2}(s)\|\, ds.$$

Thus, Gronwall-Bellman's inequality shows that

$$\|v_{x_1}(t) - v_{x_2}(t)\| \le M e^{(ML - \omega)(t - t_0)} \|x_1 - x_2\|, \qquad x_1, x_2 \in X. \tag{2.10}$$

Fix $u_0 \in X$ and define an operator $\mathcal{G} : X \to X$ by

$$\mathcal{G}x = Bu_0 - B \sum_{i=1}^p c_i \int_{t_0}^{t_i} S(t_i - s) f(s, v_x(s))\, ds, \qquad x \in X. \tag{2.11}$$

Then, by virtue of (1.3) and (2.10), we obtain for every $x_1, x_2 \in X$,

$$\|\mathcal{G}x_1 - \mathcal{G}x_2\| \le \|B\| \sum_{i=1}^{p} |c_i| \int_{t_0}^{t_i} M e^{-\omega(t_i-s)} L \|v_{x_1}(s) - v_{x_2}(s)\| \, ds$$
$$= \|B\| M \sum_{i=1}^{p} |c_i| e^{-\omega(t_i-t_0)} \left(e^{ML(t_i-t_0)} - 1\right) \|x_1 - x_2\|.$$

This means that $\mathcal{G}$ is a contractive operator on X. Therefore, $\mathcal{G}$ has a unique fixed point $x_* \in X$. Thus, from (2.11) and (2.9), it follows that

$$x_* = u_0 - \sum_{i=1}^{p} c_i S(t_i - t_0) x_* - \sum_{i=1}^{p} c_i \int_{t_0}^{t_i} S(t_i - s) f(s, v_{x_*}(s)) \, ds$$
$$= u_0 - \sum_{i=1}^{p} c_i v_{x_*}(t_i).$$

This together with (2.8) shows that $v_{x_*}(t)$ is the solution of (2.3) as desired.

We now return to the nonlocal Cauchy problem (1.1).

THEOREM 2.7. *Let (1.2)–(1.4) hold. Suppose that M and ω are constants such that $\|R(t)\| \le Me^{-\omega t}$ $(t \in [0,T])$ and $\lambda := MKe^{T \max\{ML-\omega, 0\}} < 1$. Then, for every $u_0 \in X$, (1.1) has a unique mild solution u.*

Moreover, (1.1) has a unique classical solution provided

$$u_0 - g(t_1, \ldots, t_p, u) \in \mathcal{D}(A), \qquad f \in C^1([t_0, t_0+T] \times X, X). \tag{2.12}$$

PROOF. From Corollary 2.2(1) and the plain fact that a classical solution of (1.1) is also a mild solution of (1.1), we deduce that (1.1) has at most one classical solution.

On the other hand, Corollary 2.2(1) says that for every $u_0 \in X$, (1.1) has a mild solution $u(t)$. Next, we show that $u(t)$ is continuously differentiable on $[t_0, t_0+T]$. The proof of this fact is almost standard (cf. [8]). We give it here for completeness.

For $s \in [t_0, t_0+T]$ and $x \in X$, denote

$$y_1(s,x) = \frac{\partial}{\partial s} f(s,x), \qquad y_2(s,x) = \frac{\partial}{\partial x} f(s,x). \tag{2.13}$$

By (1.3), we have

$$\max_{s \in [t_0, t_0+T]} \|y_2(s, u(s))\| < \infty \tag{2.14}$$

and

$$\begin{aligned} f(s, u(s+\sigma)) - f(s, u(s)) &= y_2(s, u(s))(u(s+\sigma) - u(s)) + \omega_1(s,\sigma), \\ f(s+\sigma, u(s+\sigma)) - f(s, u(s+\sigma)) &= y_1(s, u(s+\sigma))\sigma + \omega_2(s,\sigma), \end{aligned} \tag{2.15}$$

where $\lim_{\sigma\to 0}(\|\omega_i(s,\sigma)\|/\sigma) = 0$ uniformly on $[t_0, t_0+T]$ for $i = 1, 2$.

Let (2.12) hold. Then,

$$\frac{d}{dt}(R(t-t_0)(u_0 - g(t_1, \ldots, t_p, u))) \in C([0,T], X).$$

Thus, by the standard arguments we deduce that the integral equation

$$x(t) = \left\{ \frac{d}{dt}(R(t-t_0)(u_0 - g(t_1, \ldots, t_p, u))) + R(t-t_0) f(t_0, u(t_0)) + \int_0^t R(t-s) y_1(s, u(s)) \, ds \right\} + \int_0^t R(t-s) y_2(s, u(s)) x(s) \, ds,$$
$$t \in [t_0, t_0+T],$$

has a unique solution $x(t) \in C([t_0, t_0+T], X)$.

Making use of (1.7), (2.13)–(2.15), we obtain

$$\begin{aligned}\frac{u(t+\sigma)-u(t)}{\sigma}-x(t)&=\frac{1}{\sigma}[R(t+\sigma-t_0)-R(t-t_0)][u_0-g(t_1,\ldots,t_p,u)]\\&\quad+\frac{1}{\sigma}\int_{t_0}^{t}R(t-s)[\omega_1(s,\sigma)+\omega_2(s,\sigma)]\,ds\\&\quad+\int_{t_0}^{t}R(t-s)[y_1(s,u(s+\sigma))-y_1(s,u(s))]\,ds\\&\quad+\frac{1}{\sigma}\int_{t_0}^{t_0+\sigma}R(t+\sigma-s)f(s,u(s))\,ds-R(t-t_0)f(t_0,u(t_0))\\&\quad+\int_{t_0}^{t}R(t-s)y_2(s,u(s))\left[\frac{u(s+\sigma)-u(s)}{\sigma}-x(s)\right]ds.\end{aligned}\tag{2.16}$$

By virtue of the fact that the norm of each of the four terms on the right-hand side of (2.16) tends to 0 as $\sigma\to 0$, in conjunction with the Gronwall-Bellman inequality, we see that $u(t)$ is continuously differentiable on $[t_0,t_0+T]$, and its derivative is $x(t)$. This implies that $f(t,u(t))\in C^1([t_0,t_0+T],X)$. Thus, by (1.6) and (1.7), we conclude that $u(\cdot)$ satisfies

$$u'(t)=A\left[u(t)+\int_{t_0}^{t}F(t-s)u(s)\,ds\right]+f(t,u(t)),\qquad t\in[t_0,t_0+T],$$

i.e., $u(\cdot)$ is the unique classical solution of (1.1).

Likewise, by Corollary 2.2(2),(3) and Theorem 2.6, we get the following result.

THEOREM 2.8. *Let M and ω be constants such that $\|R(t)\|\le Me^{-\omega t}$ ($t\in[0,T]$), and let one of the following assumptions hold.*

(1) *There are constants $K>0$, q and r with $t_0\le q<r\le t_0+T$ such that*

$$\|g(t_1,\ldots,t_p,\phi)-g(t_1,\ldots,t_p,\psi)\|\le K\int_q^r\|\phi(s)-\psi(s)\|\,ds\qquad(\phi,\psi\in C([t_0,t_0+T],X))$$

and

$$\begin{aligned}KM(r-q)&<1,&&\text{if } ML=\omega,\\\frac{KM}{ML-\omega}\left(e^{(ML-\omega)(r-t_0)}-e^{(ML-\omega)(q-t_0)}\right)&<1,&&\text{if } ML\ne\omega.\end{aligned}$$

(2) *For some $c_1,\ldots,c_p\in\mathbf{C}$,*

$$g(t_1,\ldots,t_p,\phi)=\sum_{i=1}^{p}c_i\phi(t_i)\qquad(\phi\in C([t_0,t_0+T],X)).$$

Suppose that $B:=(I+\sum_{i=1}^{p}c_iR(t_i-t_0))^{-1}\in\mathbf{L}(X)$ and

$$\|B\|M\sum_{i=1}^{p}|c_i|e^{-\omega(t_i-t_0)}\left(e^{ML(t_i-t_0)}-1\right)<1.\tag{2.17}$$

Then, the conclusions of Theorem 2.7 hold.

REMARK 2.9.

(a) Theorem 2.7 covers naturally and directly [8, p. 187, Theorem 6.1.5].

(b) Theorem 2.8 unifies and generalizes [3, Theorems 3.1 and 4.3] and [6, Theorems 4.3 and 4.4]. Let us illustrate this point in detail.

(i) Specialized to the case of $F \equiv 0$ and $\omega = 0$, Theorem 2.8(2) extends [3, Theorems 3.1 and 4.3]. Actually, in this case, inequality (2.17) becomes

$$\|B\| M \sum_{i=1}^{p} |c_i| \left(e^{MLt_i} - 1\right) < 1. \tag{2.18}$$

Suppose that the hypotheses in [3, Theorems 3.1 and 4.3] hold. Then,

$$MLT \left(1 + \|B\| M \sum_{i=1}^{p} |c_i|\right) < 1. \tag{2.19}$$

So $MLT < 1$, $\|B\| M \sum_{i=1}^{p} |c_i| < (MLT)^{-1} - 1$ (if $MLT \neq 0$), and hence,

$$\begin{aligned}\|B\| M \sum_{i=1}^{p} |c_i| \left(e^{MLt_i} - 1\right) &\le \|B\| M \sum_{i=1}^{p} |c_i| \left(e^{MLT} - 1\right) \\ &< \left((MLT)^{-1} - 1\right)\left(e^{MLT} - 1\right) < 1.\end{aligned}$$

Thus, (2.19) implies (2.18).

Clearly, the converse is not true.

Moreover, we mention that the assumption on initial data in [3, Theorem 4.3] was

$$Bu_0 \in \mathcal{D}(A), \quad B \int_{t_0}^{t_i} R(t_i - s) f(s, u(s))\, ds \in \mathcal{D}(A), \qquad i = 1, 2, \dots, p. \tag{2.20}$$

Write $w_1 := u_0 - \sum_{i=1}^{p} c_i u(t_i)$. Then, by

$$u(t) = R(t - t_0) w_1 + \int_{t_0}^{t} R(t - s) f(s, u(s))\, ds \qquad (t \in [t_0, t_0 + T])$$

and (2.20), we have

$$w_1 = Bu_0 - \sum_{i=1}^{p} c_i B \int_{t_0}^{t_i} R(t_i - s) f(s, u(s))\, ds \in \mathcal{D}(A).$$

(ii) Taking $t_0 = 0$ in Theorem 2.8(2), we get

$$\|B\| M \sum_{i=1}^{p} |c_i| e^{-\omega t_i} \left(e^{MLt_i} - 1\right) < 1. \tag{2.21}$$

We say that (2.21) is implied by the hypotheses

$$\omega - ML > 0, \qquad M \sum_{i=1}^{p} |c_i| e^{(ML - \omega) t_i} < 1, \tag{2.22}$$

given in [6, Theorems 4.3 and 4.4], and (2.21) is indeed much weaker than (2.22).

In fact, if $\alpha := M \sum_{i=1}^{p} |c_i| e^{(ML-\omega)t_i} < 1$, then $\beta := M \sum_{i=1}^{p} |c_i| e^{-\omega t_i} < 1$. This implies $\| \sum_{i=1}^{p} c_i R(t_i) \| < 1$, so that $B \in \mathbf{L}(X)$ and $\|B\| \le 1/(1 - \beta)$. Therefore,

$$\|B\| M \sum_{i=1}^{p} |c_i| e^{-\omega t_i} \left(e^{MLt_i} - 1\right) \le \frac{1}{1 - \beta}(\alpha - \beta) < \frac{1}{1 - \beta}(\alpha - \alpha\beta) = \alpha < 1.$$

This shows that (2.22) implies (2.21). On the other hand, for

$$\gamma := \left\| \sum_{i=1}^{p} c_i R(t_i) \right\| < \beta < 1, \qquad 1 \le \alpha < 1 + \beta - \gamma,$$

we have

$$\|B\| M \sum_{i=1}^{p} |c_i| e^{-\omega t_i} \left(e^{MLt_i} - 1\right) \le \frac{1}{1 - \gamma}(\alpha - \beta) < 1;$$

i.e., (2.21) holds but not (2.22).

In addition, similar to Theorem 2.4, we have the following extension of [1, Theorem 5.1].

THEOREM 2.10. *Let A generate a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Assume the following.*

(i) *The function $f : [t_0, t_0 + T] \times X \to X$ is continuously differentiable and there exists a constant $L_0 > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq L_0\|x - y\|, \qquad t \in [t_0, t_0 + T], \quad x, y \in \Omega_r,$$

where Ω_r is as in Theorem 2.4.

(ii) *The function $g : [t_0, t_0 + T]^p \times C([t_0, t_0 + T], X) \to \mathcal{D}(A)$ and there exists a constant $K_0 > 0$ such that*

$$\|g(t_1, \dots, t_p, \phi) - g(t_1, \dots, t_p, \psi)\| \leq K_0 \max_{t\in[t_0,t_0+T]} \|\phi(s) - \psi(s)\|, \qquad \phi, \psi \in C([t_0, t_0 + T], \Omega_r).$$

(iii) *The vector $u_0 \in \mathcal{D}(A)$ and the inequality $M_0(\|u_0\| + G + TF_0) \leq r$ is true for*

$$M_0 := \max_{s\in[t_0,t_0+T]} \|T(s)\|, \qquad F_0 := \sup_{s\in[t_0,t_0+T],\phi\in C([t_0,t_0+T],\Omega_r)} \|f(s, \phi(s)\|,$$

and $G := \sup_{\phi\in C([t_0,t_0+T],\Omega_r)} \|g(t_1, \dots, t_p, \phi)\|$.

(iv) $M_0K_0e^{M_0TL_0} < 1$.

Then, (2.7) has a unique classical solution.

3. APPLICATIONS

EXAMPLE 3.1. Let us consider an operator A on a Banach space X, which generates an analytic semigroup $\{R(t)\}_{t\geq 0}$ on X such that

$$\|R(t)\| \leq e^{-t/3}, \quad \|AR(t)\| \leq \frac{1}{t}e^{-t/3} \qquad (t \geq 0).$$

Clearly, the operator $A = \Delta - (1/3)I$ in the Banach space $X = L^2(R^n)$ with $\mathcal{D}(A) = H^2(R^n)$ is an example for such A and X. From [8,10–14], one can find many other examples.

Suppose that $f : [0,3] \times C([0,3], X) \to C([0,3], X)$ is continuous with

$$\|f(t,x) - f(t,y)\| \leq \frac{1}{3}\|x - y\|, \qquad t \in [0,3], \quad x, y \in X,$$

and

$$g(1,2,\phi) = \frac{1}{2}\phi(1) - \frac{1}{2}\phi(2) \qquad (\phi \in C([0,3], X)).$$

Set $t_0 = 0$, $T = 3$, $L = \omega = 1/3$, $M = 1$, $p = 2$, $c_1 = 1/2$, $c_2 = -1/2$, $t_1 = 1$, and $t_2 = 2$. Then,

$$\alpha = M\sum_{i=1}^{p} |c_i|e^{(ML-\omega)t_i} = \frac{1}{2}\left(e^{L-\omega} + e^{2(L-\omega)}\right) = 1,$$
$$\beta = M\sum_{i=1}^{p} |c_i|e^{-\omega t_i} = \frac{1}{2}\left(e^{-1/3} + e^{-2/3}\right) < 1,$$

and

$$\gamma = \left\|\sum_{i=1}^{p} c_iR(t_i)\right\| = \frac{1}{2}\|R(2) - R(1)\| = \frac{1}{2}\left\|\int_1^2 AR(s)\, ds\right\|$$
$$\leq \frac{1}{2}\int_1^2 \frac{e^{-s/3}}{s}\, ds \leq \frac{1}{2}e^{-1/3}\ln 2.$$

Hence,

$$\beta - \gamma \geq \frac{1}{2}e^{-1/3}\left[\left(1 + e^{-1/3}\right) - \ln 2\right] > 0,$$

and $1 = \alpha < 1 + \beta - \gamma$. By Remark 2.9, (2.17) holds. So the nonlocal Cauchy problem

$$\begin{gathered} u'(t) = Au(t) + f(t, u(t)) \qquad (0 \leq t \leq 3), \\ u(0) + g(1, 2, u) = u_0 \end{gathered}$$

has a unique mild solution $u \in C([0,3], X)$ by Theorem 2.6. But [6, Theorem 4.3] is not applicable since $\alpha = 1$; neither is [3, Theorem 3.1] since

$$MTL\left(I + M\|B\|\sum_{i=1}^{p}|c_i|\right) = 1 + \|B\| \geq 1.$$

EXAMPLE 3.2. Let Ω be a bounded open connected subset of R^3 with C^∞ boundary, and let α and β be in $C^2([0,\infty), R)$ with $\alpha(0)$ and $\beta(0)$ positive. We consider an equation arising in the study of heat conduction in materials with memory

$$\begin{aligned} \begin{pmatrix} \theta'(t) \\ \eta'(t) \end{pmatrix} &= \begin{pmatrix} 0 & I \\ \alpha(0)\Delta & -\beta(0)I \end{pmatrix} \begin{pmatrix} \theta(t) \\ \eta(t) \end{pmatrix} \\ &+ \int_0^t \begin{pmatrix} 0 & I \\ \alpha'(t-s)\Delta & -\beta'(t-s)I \end{pmatrix} \begin{pmatrix} \theta(s) \\ \eta(s) \end{pmatrix} ds + \begin{pmatrix} 0 \\ a(t, \theta(t)) \end{pmatrix}. \end{aligned} \tag{3.1}$$

Set $X = H_0^1(\Omega) \times L^2(\Omega)$,

$$A = \begin{pmatrix} 0 & I \\ \alpha(0)\Delta & -\beta(0)I \end{pmatrix}, \qquad \mathcal{D} = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega).$$

From [15], we know that A generates a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on X with

$$\|T(t)\| \leq Me^{-\gamma t}, \qquad t \geq 0,$$

for constants $M, \gamma > 0$. For any given $l > 0$ and each $t \in [0, 4l]$ set

$$F(t) = (F_{ij}(t)),$$

where

$$\begin{gathered} F_{11}(t) \equiv F_{12}(t) = 0, \qquad F_{22}(t) = \frac{\alpha'(t)}{\alpha(0)}I, \\ F_{21}(t) = -\beta'(t)I + \beta(0)F_{22}(t). \end{gathered}$$

Assume that

$$\begin{aligned} \|F_{22}(t)\|, \|F_{21}(t)\| &\leq \frac{\gamma}{2M}e^{-\gamma t}, \qquad t \in [0, 4l], \\ \|F'_{22}(t)\|, \|F'_{21}(t)\| &\leq \frac{\gamma^2}{4M^2}e^{-\gamma t}, \qquad t \in [0, 4l]. \end{aligned}$$

Then, from [9, p. 344], it follows that the resolvent operator $R(t)$ for (3.1) satisfies

$$\|R(t)\| \leq Me^{-\gamma t/2}, \qquad t \in [0, 4l].$$

Suppose that $a(t,\theta) : [0,\infty) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ satisfies

$$\|a(t,x) - a(t,y)\|_{L^2(\Omega)} \le \frac{\gamma}{2M}\|x-y\|_{H_0^1(\Omega)}, \qquad x,y \in H_0^1(\Omega), \quad t \in [0,4l], \tag{3.2}$$

and define $b(\theta) : C([0,4l], H_0^1(\Omega)) \longrightarrow L^2(\Omega)$ by

$$b(\theta) = (Ml)^{-1}\left(\int_{(2-\varepsilon)l}^{2l} (\operatorname{grad}\theta)(s)\,ds + \int_{(4-\varepsilon)l}^{4l} (\operatorname{grad}\theta)(s)\,ds\right), \tag{3.3}$$

where $\varepsilon < 1/2$. Then, by virtue of Theorem 2.7, we infer that for each $\theta_0 \in H_0^1(\Omega)$, $\eta_0 \in L^2(\Omega)$, equation (3.1) (for $t \in [0,4l]$) together with the nonlocal initial data

$$\begin{pmatrix}\theta(0)\\ \eta(0)\end{pmatrix} + \begin{pmatrix} 0 \\ (Ml)^{-1}\left(\int_{(2-\varepsilon)l}^{2l} (\operatorname{grad}\theta)(s)\,ds + \int_{(4-\varepsilon)l}^{4l} (\operatorname{grad}\theta)(s)\,ds\right)\end{pmatrix} = \begin{pmatrix}\theta_0\\ \eta_0\end{pmatrix} \tag{3.4}$$

has a unique mild solution

$$\begin{pmatrix}\theta(\cdot)\\ \eta(\cdot)\end{pmatrix} \in C\left([0,4l], H_0^1(\Omega)\times L^2(\Omega)\right).$$

In fact, if we write

$$\begin{aligned} f(t,u) &= \begin{pmatrix} 0 \\ a(t,\theta)\end{pmatrix}, &&\text{for } t\in[0,4l], \quad u = \begin{pmatrix}\theta\\ \eta\end{pmatrix} \in X,\\ g(2l,4l,\phi) &= \begin{pmatrix} 0 \\ b(\theta)\end{pmatrix}, &&\text{for } \phi = \begin{pmatrix}\theta\\ \eta\end{pmatrix} \in C([0,4l],X), \end{aligned}$$

then by (3.2) and (3.3),

$$\begin{aligned} \|f(t,u) - f(t,v)\| &\le \frac{\gamma}{2M}\|u-v\|, && u,v\in X, \quad t\in[0,4l],\\ \|g(2l,4l,\phi) - g(2l,4l,\psi)\| &\le 2\varepsilon M^{-1}\max_{t\in[0,l]}\|\phi(t)-\psi(t)\|, && \phi,\psi \in C([0,4l],X). \end{aligned} \tag{3.5}$$

Clearly, λ (in Theorem 2.7) $= 2\varepsilon < 1$. Therefore, by using Theorem 2.7, we get immediately the desired conclusion for any γ and $l > 0$. Nevertheless, Theorem 3.2 in [6] is not applicable to the nonlocal Cauchy problem (3.1) and (3.4) if $\gamma l \ge (1/2)(1-\varepsilon) > 1/4$. From (3.5), it is plain that the larger γ is, the larger the set of admissible f.

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