The structure of obstructions to treewidth and pathwidth

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Abstract

It is known that the class of graphs with treewidth (resp. pathwidth) bounded by a constant $w$ can be characterized by a finite obstruction set $\text{obs}(\text{TW}(w))$ (resp. $\text{obs}(\text{PW}(w))$). These obstruction sets are known for $w \leq 3$ so far. In this paper we give a structural characterization of graphs from $\text{obs}(\text{TW}(w))$ (resp. $\text{obs}(\text{PW}(w))$) with a fixed number of vertices in terms of subgraphs of the complement. Our approach also essentially simplifies known characterization of graphs from $\text{obs}(\text{TW}(w))$ (resp. $\text{obs}(\text{PW}(w))$) with $(w + 3)$ vertices.

Also for any $w \geq 3$ a graph from $\text{obs}(\text{TW}(w)) \setminus \text{obs}(\text{PW}(w))$ is constructed, that solves an open problem. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

If $H$ are $G$ are graphs, then $H$ is a minor of $G$ if and only if $H$ can be obtained from a subgraph of $G$ by contracting edges. A class $\mathcal{F}$ of graphs is called minor-closed if for
every graph \( G \) in \( \mathcal{F} \), every minor of \( G \) is also a member of \( \mathcal{F} \). The obstruction set for a minor-closed class \( \mathcal{F} \), denoted by \( \text{obs}(\mathcal{F}) \), is the set of all graphs in the complement of \( \mathcal{F} \) that are minimal in the minor order. Robertson and Seymour [9] proved Wagner’s conjecture that every minor-closed class of graphs has a finite obstruction set. If we know all the graphs in \( \text{obs}(\mathcal{F}) \), then we can decide whether \( G \in \mathcal{F} \) in polynomial time using the fact, that for every fixed graph \( H \) there exists a polynomial-time algorithm that, when given an input graph \( G \), decided whether \( H \) is a minor of \( G \) [10].

Extensively studied are graphs of bounded treewidth. Note that for each \( k \), the class of graphs of treewidth at most \( k \) is minor closed.

The concept of treewidth seems to be interesting from the algorithmic point of view: many graph problems that are NP-complete in general can be polynomially solvable if graphs are constrained to have bounded treewidth.

A better comprehension of the obstructions for treewidth and pathwidth can help to design better algorithms for the graphs with treewidth (resp. pathwidth) bounded by a fixed constant.

**Definition 1.** A tree-decomposition of a graph \( G = (V,E) \) is a pair \((T,\mathcal{X})\), where \( T = (V(T),E(T)) \) is a tree and \( \mathcal{X} = (X_t, t \in V(T)) \) is a family of subsets of \( V \) with the following properties:

1. \( \bigcup (X_t, t \in V(T)) = V \);
2. for every edge \( e \in E \) there exists \( t \in V(T) \) such that \( e \) has both ends in \( X_t \);
3. for \( t, t', t'' \in V(T) \), if \( t' \) is on the path of \( T \) between \( t \) and \( t'' \) then
   \[ X_t \cap X_{t''} \subseteq X_{t'} \.
\]

The width of the tree-decomposition \((T,\mathcal{X})\) is
\[ \max_{t \in V(T)} (|X_t| - 1). \]

The treewidth of the graph \( G \), \( TW(G) \), is the smallest integer \( k \) such that \( G \) has a tree-decomposition of width \( k \).

A path-decomposition of the graph \( G \) is a tree-decomposition \((T,\mathcal{X})\) such that \( T \) is a path. The pathwidth of the graph \( G \), \( PW(G) \), is the smallest integer \( k \) such that \( G \) has a path-decomposition of width \( k \).

Several equivalent definitions of treewidth are extensively used, see e.g. [2] or [6]. Let us give one, which is frequently used in this paper.

**Definition 2.** \( k \)-trees are defined recursively as follows: a clique with \( (k+1) \) vertices is a \( k \)-tree; given a \( k \)-tree \( G \) with \( n \) vertices, a \( k \)-tree with \( (n+1) \) vertices is constructed by taking \( G \) and creating a new vertex \( v \) which is made adjacent to a \( k \)-clique of \( G \) and nonadjacent to the \( (n-k) \) other vertices of \( G \). A partial \( k \)-tree is any subgraph of a \( k \)-tree.

A \( k \)-path is a \( k \)-tree which is an interval graph. A partial \( k \)-path is a subgraph of a \( k \)-path.
It can be proved, that the treewidth of a graph \( G \) is \( k \) if and only if \( k \) is the minimum value for which \( G \) is a partial \( k \)-tree (see e.g. [2] or [6]). A similar statement can be formulated for pathwidth: the pathwidth of a graph \( G \) is \( k \) if and only if \( k \) is the minimum value for which \( G \) is a partial \( k \)-path (see [6]).

Let \( TW(w) \) denote the class of graphs with treewidth at most \( w \). For any fixed \( w \), \( TW(w) \) is minor-closed and consequently it can be characterized by a finite obstruction set \( obs(TW(w)) \). The same statement can be formulated also for the class \( PW(w) \) of graphs with pathwidth at most \( w \).

Only the obstruction sets for treewidth 1, 2, and 3 are known so far (see [1,13]). In [11] over 75 minimal forbidden minors for treewidth at most four of widely varying structures are presented. (The obstruction set for treewidth 4 could be probably determined using reductions given in [12]. To the best of author’s knowledge the full list of graphs from \( obs(TW(4)) \) has not been given explicitly.) The obstruction sets for pathwidth 1 and 2 were described in [5]. In [8] a structural characterization of graphs from \( obs(TW(w)) \) (resp. \( obs(PW(w)) \)) with \( (w+3) \) vertices is given.

2. A characterization of graphs with bounded \( |V(G)| - TW(G) \)

In this section we give a characterization of graphs with bounded difference between the number of vertices and treewidth of the graph. This characterization is given in terms of forbidden subgraphs of the complement.

**Definition 3.** Let \( \mathcal{F}^1 = \{K_2\} \). For \( r \geq 2 \), let \( \mathcal{F}^{r+1} \) be the set of graphs \( G \) that can be constructed in the following way: take a graph \( H \) from \( \mathcal{F}^r \) and an independent vertex set \( A \) (possibly empty) of \( H \) with at least \( |V(H)| - (r + 1) \) vertices. Denote \( B = V(H) \setminus A \). Let \( C \cup \{v_0\} \) be the set of new vertices such that \( |C| = r + 1 - |B| \). Then \( G \) is the graph defined by

\[
V(G) = V(H) \cup C \cup \{v_0\} \quad \text{and} \quad E(G) = E(H) \cup \{\{v_0, u\}, \ u \in B \cup C\}.
\]

**Remark 4.** Each graph from \( \mathcal{F}^r \) is connected graph with \( r(r+1)/2 \) edges and \( K_{r+1} \in \mathcal{F}^r \).

Obviously, \( \mathcal{F}^2 = \{K_3, P_3\} \). Let us describe the set \( \mathcal{F}^3 \), which will be used later. The set \( \mathcal{F}^3 \) consists of five graphs, see Fig. 1.
Rather than to test for each \( H \in \mathcal{F}^3 \) individually, we prefer the structurally-related test based on the following simple observation.

**Observation 5.** Let \( H \) be a graph. Then \( H \) contains a graph from \( \mathcal{F}^3 \) as a subgraph if and only if either

(a) \( H \) contains a triangle with each vertex of degree at least 3, or

(b) \( H \) contains a cycle of length 4 with two consecutive vertices of degree at least 3.

**Lemma 6.** Let \( r \geq 1 \) and \( G \) be a \( k \)-tree with \((k + r + 1)\) vertices. Then \( \overline{G} \) consists of a single nontrivial component from \( \mathcal{F}^r \) and zero or more isolated vertices.

**Proof.** We proceed by induction on \( r \). For \( r = 1 \) the complement of any \( k \)-tree with \((k + 2)\) vertices consists of \( K_2 \) (\( \in \mathcal{F}^1 \)) and isolated vertices.

Let \( r \geq 1 \). Let \( G \) be a \( k \)-tree with \((k + r + 2)\) vertices for some nonnegative integer \( k \). According to the recursive definition of \( k \)-trees there is a vertex \( v_0 \), which together with all \( k \) neighbors \( \{v_{r+2}, \ldots, v_{r+k+1}\} \) creates the clique \( K_{k+1} \) in \( G \).

Let \( G_0 = G \setminus v_0 \). Obviously, \( G_0 \) is a \( k \)-tree with \((k + r + 1)\) vertices \( \{v_1, \ldots, v_{k+r+1}\} \). By the induction hypothesis, \( \overline{G_0} \) consists of a single nontrivial component \( H_0 \in \mathcal{F}^r \) and zero or more isolated vertices.

Look closer how the component \( H_0 \) is modified when the graph \( \overline{G} \) from the graph \( \overline{G_0} \) is constructed:

- the new vertex \( v_0 \) is added,
- the vertex set \( V(H_0) \cap \{v_{r+2}, \ldots, v_{k+r+1}\} \) is an independent set in \( \overline{G} \), consequently in \( \overline{G_0} \) and also in \( H_0 \), with at least \( |V(H_0)| - (r + 1) \) vertices,
- new vertices \( \{v_1, \ldots, v_{r+1}\} \setminus V(H_0) \) are added, if \( V(H_0) \not\supseteq \{v_1, \ldots, v_{r+1}\} \),
- \( r + 1 \) new edges \( \{v_0, v_j\} \) \((j = 1, 2, \ldots, r + 1)\) are added.

As \( H_0 \) is not a discrete graph, necessarily \( V(H_0) \cap \{v_1, \ldots, v_{r+1}\} \neq \emptyset \). It implies that \( H \) consists of a single nontrivial component.

This construction follows the same steps as described in Definition 3. As a result we obtain that the component \( H \), constructed from \( H_0 \) as above, belongs to \( \mathcal{F}^{r+1} \).

**Lemma 7.** Let \( r \geq 1 \) and let \( H \) be a graph from \( \mathcal{F}^r \). Then a graph consisting of \( H \) and \( l \) (possibly \( l = 0 \)) isolated vertices is the complement of an \((l + |V(H)| - r - 1)\)-tree.

**Proof.** The proof uses induction on \( r \). For \( r = 1 \) we have \( H = K_2 \). Obviously, the graph consisting of \( K_2 \) and \( l \) isolated vertices is the complement of a \( l \)-tree.

Let \( r \geq 1 \). To prove the induction step, fix \( H \in \mathcal{F}^{r+1} \).

(a) First, we prove that \( H \) is a \((|V(H)| - r - 2)\)-tree.

The graph \( H \in \mathcal{F}^{r+1} \) was constructed from some graph \( H_0 \in \mathcal{F}^r \) following the construction described in Definition 3:

- for fixed independent vertex set \( A \) in \( H_0 \) and the set \( B = V(H_0) \setminus A \) with the property \( |B| \leq r + 1 \) we added a new vertex set \( C \) such that \( |B \cup C| = r + 1 \),
- we added a new vertex \( v_0 \) and \((r + 1)\) edges \( \{v_0, u\} \), \( u \in B \cup C \).
The graph $H \setminus v_0$ consists of $H_0$ and the set $C$ of isolated vertices. By the induction hypothesis, its complement is a $(|C| + |V(H_0)| - r - 1)$-tree. Obviously, $|C| + |V(H_0)| - r - 1 = |V(H)| - r - 2 = |A|$ and the induced subgraph with the vertex set $A$ is a clique in $H_0$. The graph $\overline{H}$ was created adding the new vertex $v_0$, which was made adjacent to a clique $A$. That means, $\overline{H}$ is a $(|V(H)| - r - 2)$-tree.

(b) It easily follows from (a) that the graph consisting of $H \in \mathcal{F}_{r+1}$ and $l$ isolated vertices is the complement of an $(l + |V(H)| - r - 2)$-tree. □

Lemma 8. Let $r \geq 1$ and $G$ be a graph with property $TW(G) \leq |V(G)| - r - 1$. Then $\overline{G}$ contains some graph from $\mathcal{F}_r$ as a subgraph.

Proof. Fix $G$ as above. $TW(G) \leq |V(G)| - r - 1$ implies $G$ is a partial $(|V(G)| - r - 1)$-tree. It is well known that there exists a supergraph $G_0$ of $G$ with the same vertex set $V(G)$ such that $G_0$ is a $(|V(G)| - r - 1)$-tree (see e.g. [6, Lemma 2.13]).

Due to Lemma 6, $\overline{G}_0$ contains some graph from $\mathcal{F}_r$ as a subgraph. Obviously, $\overline{G}_0$ is a subgraph of $\overline{G}$ and the proof is finished. □

Lemma 9. Let $r \geq 1$ and $\overline{G}$ contains a graph $H$ from $\mathcal{F}_r$ as a subgraph. Then $TW(G) \leq |V(G)| - r - 1$.

Proof. Let $H \in \mathcal{F}_r$ be a subgraph of $\overline{G}$. Let $G_0$ be the graph, whose complement consists of $H$ and isolated vertices of $V(G) \setminus V(H)$ (if $V(H) \neq V(G)$). Due to Lemma 7 the graph $G_0$ is a $(|V(G)| - r - 1)$-tree. Obviously, $G_0$ is a supergraph of $G$, which implies $TW(G) \leq |V(G)| - r - 1$. □

Theorem 10. Let $G$ be a graph and $r \geq 1$. Then the following conditions are equivalent:

(1) $TW(G) \geq |V(G)| - r$;

(2) $\overline{G}$ contains no graph from $\mathcal{F}_r$ as a subgraph.

Proof. According to Lemmas 8 and 9 the following statement is true: $\overline{G}$ contains some graph from $\mathcal{F}_r$ as a subgraph if and only if $TW(G) \leq |V(G)| - r - 1$. □

Definition 11. An undirected graph $G = (V, E)$ is called a comparability graph, or a transitively orientable graph, if there exists an orientation of the edges such that the resulting oriented graph $(V, F)$ satisfies the following conditions:

$$F \cap F^{-1} = \emptyset \quad \text{and} \quad F + F^{-1} = E \quad \text{and} \quad F^2 \subseteq F$$

where $F^2 = \{(u, w) \mid \exists v \in V \ (u, v) \in F \ & \ (v, w) \in F\}$.

For a $k$-tree $G$ the following properties are equivalent:

$(C_1)$ $G$ is a $k$-path;

$(C_2)$ $\overline{G}$ is a comparability graph;

$(C_3)$ $G$ does not contain a triple of vertices with the property that any two of them are connected by a path which avoids the neighborhood of the third.
The equivalence \( (C_1) \iff (C_2) \) easily follows from the characterization of interval graphs found by Gilmore and Hoffman [3].

The equivalence \( (C_1) \iff (C_3) \) is a consequence of another important characterization of interval graphs proved by Lekkerkerker and Boland [7]. Hence analogous results as we obtained in this section for the treewidth can be proved for the pathwidth.

**Definition 12.** Define \( \mathcal{P}^r = \{ H \in \mathcal{T}^r, H \text{ is a comparability graph} \} \).

**Remark 13.** It is easy to see that \( \mathcal{P}^1 = \mathcal{T}^1 \) and \( \mathcal{P}^2 = \mathcal{T}^2 \). Further, \( \mathcal{T}^3 \setminus \mathcal{P}^3 = \{ H_5 \} \) (net), as \( H_5 \) (3-sun) is a 2-tree, which does not satisfy \( (C_3) \), hence \( H_5 \) is not a comparability graph (see Fig. 1 for the set \( \mathcal{T}^3 \)). Moreover, for every \( r \geq 3 \) there exists \( H \in \mathcal{T}^r \) such that \( H_5 \) is an induced subgraph of \( H \) and hence \( H \) is not a comparability graph. It implies that \( \mathcal{P}^r \) is a proper subset of \( \mathcal{T}^r \) for every \( r \geq 3 \).

Replacing the set \( \mathcal{T}^r \) by the set \( \mathcal{P}^r \), Lemmas 6–9 can be formulated and proved in the same way also for pathwidth. Consequently, the analogous result of Theorem 10 holds also for pathwidth:

**Theorem 14.** Let \( G \) be a graph and \( r \geq 1 \). Then the following conditions are equivalent:

1. \( PW(G) \geq |V(G)| - r \).
2. \( \overline{G} \) contains no graph from \( \mathcal{P}^r \) as a subgraph.

In the following, Theorems 10 and 14 will be used to give a structural characterization of graphs from \( obs(TW(w)) \) (resp. \( obs(PW(w)) \)) with \( (w + r + 1) \) vertices in terms of subgraphs of the complement (for any \( r \geq 1 \)). This description can be used to construct explicitly some graphs from \( obs(TW(w)) \), resp. \( obs(PW(w)) \).

### 3. Graphs from \( obs(TW(w)) \) (resp. \( obs(PW(w)) \)) with \( (w + 3) \) vertices

Ramachandramurthi [8] has found a structural characterization of graphs from \( obs(TW(w)) \) (resp. \( obs(PW(w)) \)) with \( (w + 3) \) vertices.

We give the method how to construct graphs from \( obs(TW(w)) \) (resp. \( obs(PW(w)) \)) with \( (w + r + 1) \) vertices for any \( r \geq 2 \). Our approach also essentially simplifies the results of [8].

**Theorem 15.** For every \( w \), a graph \( G \) with \( (w + 3) \) vertices is in \( obs(TW(w)) \) (equivalently in \( obs(PW(w)) \)) if and only if all components of \( \overline{G} \) are stars and the number of them is at least 3.

**Proof.** We will prove the theorem for treewidth. As \( \mathcal{P}^2 = \mathcal{T}^2 \), one can obtain the proof for pathwidth replacing \( TW \) by \( PW \) in our proof.

\( \Rightarrow \) Let \( G \) be a graph from \( obs(TW(w)) \) with \( (w + 3) \) vertices. Then \( TW(G) = w + 1 = |V(G)| - 2 \). Due to Theorem 10 neither \( K_3 \) nor \( P_3 \) are subgraphs of \( \overline{G} \).
Consequently, the components of $G$ are stars and isolated vertices. Moreover, $G$ is not a discrete graph.

If $G$ contains an isolated vertex $v$, then we delete an edge $e$ of $G$ connecting $v$ and the central vertex of a star. In virtue of Theorem 10 $TW(G \setminus e) = TW(G)$, a contradiction. Hence components of $G$ are just stars.

Suppose that $G$ consists of one star (resp. two stars). Then deleting the central vertex (resp. contracting an edge connecting two central vertices) we obtain $K_{w+2}$ as a proper minor of $G$. A contradiction with $G \in \text{obs}(TW(w))$.

$(\Leftarrow)$ Let $G$ be a graph with $(w+3)$ vertices of all required properties. $G$ contains no graph from $T^2$ as a subgraph, that implies $TW(G) \geq w+1$. Our aim is to prove that $TW(H) \leq w$ for any proper minor $H$ of $G$. It is clear, if $|V(H)| \leq w+1$.

(i) The number of stars implies that the complement of any minor $H$ of $G$ with $w+2$ vertices contains an edge, which implies $TW(H) \leq w$.

(ii) Because $G$ contains no isolated vertices, deleting any edge $e$ the graph $\overline{G \setminus e}$ contains $K_3$ or $P_3$. Due to Theorem 10 it follows $TW(G \setminus e) \leq w$.

This concludes the proof.  

**Remark 16.** Due to the previous theorem, the number of graphs from $\text{obs}(TW(n-3))$ with $n$ vertices is equal to the number of partitions of number $n$ into at least three parts of size at least 2.

4. **A structure of graphs from $\text{obs}(TW(w))$ (resp. $\text{obs}(PW(w))$)**

The following theorem gives a structural characterization of graphs from $\text{obs}(TW(w))$ with a fixed number of vertices in terms of subgraphs of the complement.

**Theorem 17.** Given $r \geq 2$. A graph $G$ with $(w+r+1)$ vertices is in $\text{obs}(TW(w))$ if and only if $G$ satisfies the following three conditions:

- $T_1(r)$: $\overline{G}$ contains no graph from $T^r$ as a subgraph.
- $T_2(r)$: If $H$ is a minor of $G$ with $|V(G)| - 1$ vertices, then $\overline{H}$ contains some graph from $T^{r-1}$ as a subgraph.
- $T_3(r)$: For every $e \in E(G)$, $\overline{G \setminus e}$ contains some graph from $T^r$ as a subgraph.

**Proof.** ($\Rightarrow$) This part follows from the definition of the obstruction set and Theorem 10.

($\Leftarrow$) Due to Theorem 10 the property $T_1(r)$ implies $TW(G) \geq w+1$. If $H$ is a minor of $G$ with at most $|V(G)| - 1$ vertices, then $TW(H) \leq w$ due to the property $T_2(r)$.

Let $H$ be a proper minor (equivalently, subgraph) of $G$ with $|V(G)|$ vertices. Then the property $T_3(r)$ implies $TW(H) \leq w$.  

**Lemma 18.** Given $r \geq 2$ and a graph $G$ with the property $T_2(r)$. Let $F$ be a graph such that $\overline{F}$ contains $\overline{G}$ as a subgraph. Then $F$ has the property $T_2(r)$.  

Proof. For fixed \( r \) we denote 
\[
\mathcal{M}^r := \{ H : \overline{H} \text{ is a graph satisfying } T_2(r) \}.
\]

Now lemma can be formulated in the following way: if \( H \in \mathcal{M}^r \) and \( G \) is a supergraph of \( H \), then also \( G \in \mathcal{M}^r \).

To prove this statement it is enough to show that if \( H \in \mathcal{M}^r \), then the graph \( H_e \) (resp. \( H_v \)) obtained from \( H \) adding a new edge \( e \) (resp. a new isolated vertex \( v \)) has to belong to \( \mathcal{M}^r \).

As \( H_e \) is a subgraph of \( H \), then each minor of \( H_e \) is the minor of \( H \) and the conclusion is obvious.

Now let \( M \) be a minor of \( H_v \) with \( |V(H)| \) vertices.

(i) If \( M \) is a subgraph of \( H \), the conclusion is trivial.

(ii) If \( M \) is a subgraph of \( H_v \) (\( v \in V(M) \)) or a graph obtained from \( H_v \) by contracting some edge from \( E(H) \), then \( M \setminus v \) is a minor of \( H \) with \( |V(H)| - 1 \) vertices.

Hence \( \overline{H} \setminus v \) contains a graph from \( \mathcal{F}^{r-1} \) as a subgraph.

(iii) Finally, let us suppose, that \( M \) is a subgraph of the graph obtained from \( H_v \) by contracting some edge \( \{u, v\} \in E(H_v) \). (A new vertex which is result of contracting \( \{u, v\} \) is denoted by \( w \).) Then \( M \setminus w \) is a minor of \( \overline{H} \) with \( |V(H)| - 1 \) vertices, which completes the proof.

Remark 19. According to Lemma 18 the class of graphs \( \mathcal{M}^r \) (for fixed \( r \)) is supergraph-closed. It easily follows that \( \mathcal{M}^r \) can be characterized in terms of subgraphs of the complement. A graph \( G \) possesses \( T_1(r) \) and \( T_2(r) \) if and only if \( \overline{G} \) contains some graph from \( \min \mathcal{M}^r \) as a subgraph (\( \min \mathcal{M}^r \) stands for the set of minimal elements of \( \mathcal{M}^r \) with respect to subgraph relation).

The proofs of the following Theorems 20 and 22 give a general method, how to find graphs from the obstruction set, \( obs(TW(w)) \).

Theorem 20. Let \( r \geq 2 \) be given and \( F \) be a graph with property \( T_1(r) \) and \( T_2(r) \). Then for every \( w, w \geq |V(F)| - (r + 1) \) there exists a graph \( G \in obs(TW(w)) \) with \( (w + r + 1) \) vertices such that some subgraph of \( F \) with vertex set \( V(F) \) is an induced subgraph of \( G \).

Proof. Let \( F \) be a graph with property \( T_1(r) \) and \( T_2(r) \) for given \( r \geq 2 \) and \( w \geq |V(F)| - (r + 1) \) be fixed. Define \( \overline{G} \) with \( (w + r + 1) \) vertices in the following way: \( \overline{G} \) consists of \( \mathcal{F} \) and \( (w + r + 1 - |V(F)|) \) new isolated vertices. The graph \( G \) satisfies the condition \( T_1(r) \) and also \( T_2(r) \) (by Lemma 18).

Deleting a maximal set of edges from \( G \) for which the resulting graph has the property \( T_2(r) \), we obtain a graph satisfying \( T_1(r) \), \( T_2(r) \) and \( T_3(r) \), hence a graph from \( obs(TW(w)) \) with \( (w + r + 1) \) vertices.

Remark 21. Let \( r \geq 2 \) be fixed. Obviously, if \( \overline{G} \) consists of three disjoint copies of \( K_r \), then \( G \) has the property \( T_1(r) \) and \( T_2(r) \). Theorem 20 implies the existence of
graphs from $\text{obs}(TW(w))$ for each $w \geq (2r - 1)$. But this bound is not always optimal. For example, for $r = 3$ we have found three graphs from $\text{obs}(TW(4))$ with 8 vertices. Their complements are in Fig. 2.

**Theorem 22.** Let $H$ be a graph from $\text{obs}(TW(w))$ for some $w \geq 0$. Then for every $k > w$ there exists a graph $G$ from $\text{obs}(TW(k))$ such that

$$|V(G)| - k = |V(H)| - w$$

and $H$ is an induced subgraph of $G$.

**Proof.** It easily follows from Theorem 20. □

**Remark 23.** The results analogous to Theorems 17, 20 and 22, where treewidth is replaced by pathwidth and $\mathcal{F}^r$ by $\mathcal{P}^r$, can be proved in the same way.

5. Relation between $\text{obs}(TW(w))$ and $\text{obs}(PW(w))$

It is well known [4] that for each $w > 0$ there are trees which are obstructions to $PW(w)$. However, since the treewidth of a tree is 1, no tree can be in $\text{obs}(TW(w))$ for any $w > 0$. Therefore, $\text{obs}(PW(w)) \not\subseteq \text{obs}(TW(w))$.

For $w = 1$ and $w = 2$ it holds $\text{obs}(TW(w)) = \{K_{w+2}\} \subseteq \text{obs}(PW(w))$.

Ramachandramurthi [8] formulated a conjecture that for any $w \geq 3$ there exists a graph $H \in \text{obs}(TW(w)) \setminus \text{obs}(PW(w))$. He gave an example of such a graph for $w = 3$ only. It can be seen (from Theorem 15) that any such a graph has at least $(w + 4)$ vertices.

In the next we will construct for every $w \geq 3$ a graph which belongs to $\text{obs}(TW(w)) \setminus \text{obs}(PW(w))$.

**Construction.** Let $w > 5$ be fixed and $k, l$ be integers such that $w + 4 = l + 3k$, $l \in \{1, 2, 3\}$. Let $K_{l,3,\ldots,3}$ be the complete $(k + 1)$-partite graphs with the corresponding partition of the vertices $\{V_i\}_{i=1}^{k+1}$, where $V_i = \{v'_i, v'_i, v'_i\}$ for $2 \leq i \leq k + 1$ and
Fig. 3. The complements of graphs $G_w$.

$V_1 = \{v^1_j\}_{j=1}^l$. Define a graph $G_w$ in the following way:

$V(G_w) = V(K_{l,3,3,...,3})$

$E(G_w) = E(K_{l,3,3,...,3}) \setminus \{\{v^1_i, v^1_j\}, 2 \leq i \leq k + 1\}$

$\cup \{\{v^2_i, v^2_j\}, 3 \leq i \leq k + 1\} \cup E_l\}$.

where $E_l = \{\{v^2_i, v^2_j\}\}$ if $l = 2$; $E_l = \emptyset$ if $l = 1$ or $l = 3$ (see Fig. 3).

**Theorem 24.** For every $w \geq 3$ the set $\text{obs}(TW(w)) \setminus \text{obs}(PW(w))$ is nonempty.

**Proof.** Let $w > 5$ be fixed. We show that $G_w \notin \text{obs}(TW(w)) \setminus \text{obs}(PW(w))$.

Applying Theorem 17 to prove $G_w \notin \text{obs}(TW(w))$ it is enough to verify properties $T_1(r)$–$T_3(r)$ for $G_w$ and $r = 3$.

$T_1(3)$: The graph $\overline{G_w}$ has not the property (a) from Observation 5 and contains no cycle of length 4, which concludes the proof of this part.

$T_2(3)$: Obviously, the complement of each minor of $G_w$ with $|V(G_w)| - 1$ vertices contains $K_3$ as a subgraph.

$T_3(3)$: Let $e \in E(G_w)$ be fixed. It is enough to prove that $\overline{G_w \setminus e}$ has property (a) or (b) from Observation 5.

If $v^1_i \in e$ $(2 \leq i \leq k + 1)$, then property (a) holds. Suppose $v^1_i \in e$ $(3 \leq i \leq k + 1)$. If the second vertex of $e$ is $v^1_j$ or $v^1_j$ $(3 \leq j \leq k + 1)$, then (a) holds. If $e = \{v^1_i, v^1_i\}$ $(2 \leq j \leq k + 1)$, property (b) holds. The same argument is true, whenever the second vertex of $e$ is $v^1_j$ $(if \ l \geq 2)$ or $v^1_1$ $(if \ l = 3)$.

Now assume $v^1_i \in e$ $(2 \leq i \leq k + 1)$. If any of the vertices $v^1_j$ $(2 \leq j \leq k + 1)$, $v^1_1$ $(if \ l \geq 2)$ or $v^1_1$ $(if \ l = 3)$ is the second vertex of $e$, then property (a) trivially holds. Finally, if $e = \{v^1_i, v^1_j\}$ $(3 \leq i \leq k + 1)$, then property (b) holds. The other cases for $e = \{v^1_i, v^1_j\}$ $(1 \leq i \leq l)$ can be simply discussed.

Due to Remark 23 to prove $G_w \notin \text{obs}(PW(w))$ it is enough to find $e \in E(G_w)$ such that $\overline{G_w \setminus e}$ contains no graph from $\mathcal{P}^3$ as a subgraph. Let $e = \{v^1_3, v^1_2\}$ (dashed edge in Fig. 3). One can easily check that $\overline{G_w \setminus \{v^1_i, v^1_j\}}$ contains no cycle of length 4. Consequently no graph from $\mathcal{P}^3$. 
In the case $w = 4$ (resp. $w = 5$) an example of a graph from $\text{obs}(\text{TW}(w)) \setminus \text{obs}(\text{PW}(w))$ is the graph whose complement is the third graph in Fig. 2 (resp. graph whose complement is in Fig. 4). This fact can be easily verified using Theorem 17 and Remark 23.

References