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The structure of obstructions to treewidth and pathwidth

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Abstract

It is known that the class of graphs with treewidth (resp. pathwidth) bounded by a constant w can be characterized by a finite obstruction set $obs(TW(w))$ (resp. $obs(PW(w))$). These obstruction sets are known for $w \leqslant 3$ so far. In this paper we give a structural characterization of graphs from $obs(TW(w))$ (resp. $obs(PW(w))$) with a fixed number of vertices in terms of subgraphs of the complement. Our approach also essentially simplifies known characterization of graphs from $obs(TW(w))$ (resp. $obs(PW(w))$) with $(w+3)$ vertices.

Also for any $w \geqslant 3$ a graph from $obs(TW(w)) \setminus obs(PW(w))$ is constructed, that solves an open problem.

Keywords: Obstruction set; Treewidth; Pathwidth

All graphs in this paper are finite, undirected and without loops and parallel edges. The difference between isomorphism and equality of graphs is ignored. As a standard notation we use $G=(V,E)$, where G is a graph with the vertex set V (or $V(G)$) and the edge set E (or $E(G)$).

Let $G=(V,E)$ be a graph. If $v \in V(G)$ then $G \setminus v$ denotes the subgraph of G induced by $V \setminus \{v\}$. If $e \in E(G)$ then $G \setminus e$ denotes the subgraph $(V, E \setminus \{e\})$ of G. A discrete graph is a graph with empty edge set.

K_k stands for the complete graph with k vertices, $K_{r_1,\dots,r_k}$ for the complete k-partite graphs. The bipartite graph $K_{1,r}$ $(r \geqslant 1)$ is termed a *star* and the vertex of $K_{1,r}$ connected with r vertices is called the *central* vertex. $\overline{G}$ stands for the complement of a graph G and P_k is the path on $k+1$ vertices.

1. Introduction

If H are G are graphs, then H is a minor of G if and only if H can be obtained from a subgraph of G by contracting edges. A class $\mathscr{F}$ of graphs is called minor-closed if for

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every graph G in $\mathscr{F}$, every minor of G is also a member of $\mathscr{F}$. The *obstruction set* for a minor-closed class $\mathscr{F}$, denoted by $obs(\mathscr{F})$, is the set of all graphs in the complement of $\mathscr{F}$ that are minimal in the minor order. Robertson and Seymour [9] proved Wagner's conjecture that every minor-closed class of graphs has a finite obstruction set. If we know all the graphs in $obs(\mathscr{F})$, then we can decide whether $G \in \mathscr{F}$ in polynomial time using the fact, that for every fixed graph H there exists a polynomial-time algorithm that, when given an input graph G, decided whether H is a minor of G [10].

Extensively studied are graphs of bounded treewidth. Note that for each k, the class of graphs of treewidth at most k is minor closed.

The concept of treewidth seems to be interesting from the algorithmic point of view: many graph problems that are NP-complete in general can be polynomially solvable if graphs are constrained to have bounded treewidth.

A better comprehension of the obstructions for treewidth and pathwidth can help to design better algorithms for the graphs with treewidth (resp. pathwidth) bounded by a fixed constant.

Definition 1. A tree-decomposition of a graph $G=(V,E)$ is a pair $(T,\mathscr{X})$, where $T=(V(T),E(T))$ is a tree and $\mathscr{X}=(X_t,\ t\in V(T))$ is a family of subsets of V with the following properties:
(1) $\bigcup(X_t,\ t\in V(T))=V$;
(2) for every edge $e\in E$ there exists $t\in V(T)$ such that e has both ends in X_t;
(3) for $t,t',t''\in V(T)$, if t' is on the path of T between t and t'' then

$$X_t \cap X_{t''} \subseteq X_{t'}.$$

The width of the tree-decomposition $(T,\mathscr{X})$ is

$$\max_{t\in V(T)}(|X_t|-1).$$

The treewidth of the graph G, $TW(G)$, is the smallest integer k such that G has a tree-decomposition of width k.

A path-decomposition of the graph G is a tree-decomposition $(T,\mathscr{X})$ such that T is a path. The pathwidth of the graph G, $PW(G)$, is the smallest integer k such that G has a path-decomposition of width k.

Several equivalent definitions of treewidth are extensively used, see e.g. [2] or [6]. Let us give one, which is frequently used in this paper.

Definition 2. k-trees are defined recursively as follows: a clique with $(k+1)$ vertices is a k-tree; given a k-tree G with n vertices, a k-tree with $(n+1)$ vertices is constructed by taking G and creating a new vertex v which is made adjacent to a k-clique of G and nonadjacent to the $(n-k)$ other vertices of G. A partial k-tree is any subgraph of a k-tree.

A k-path is a k-tree which is an interval graph. A partial k-path is a subgraph of a k-path.

It can be proved, that the treewidth of a graph G is k if and only if k is the minimum value for which G is a partial k-tree (see e.g. [2] or [6]). A similar statement can be formulated for pathwidth: the pathwidth of a graph G is k if and only if k is the minimum value for which G is a partial k-path (see [6]).

Let $TW(w)$ denote the class of graphs with treewidth at most w. For any fixed w, $TW(w)$ is minor-closed and consequently it can be characterized by a finite obstruction set $obs(TW(w))$. The same statement can be formulated also for the class $PW(w)$ of graphs with pathwidth at most w.

Only the obstruction sets for treewidth 1, 2, and 3 are known so far (see [1,13]). In [11] over 75 minimal forbidden minors for treewidth at most four of widely varying structures are presented. (The obstruction set for treewidth 4 could be probably determined using reductions given in [12]. To the best of author's knowledge the full list of graphs from $obs(TW(4))$ has not been given explicitly.) The obstruction sets for pathwidth 1 and 2 were described in [5]. In [8] a structural characterization of graphs from $obs(TW(w))$ (resp. $obs(PW(w))$) with $(w+3)$ vertices is given.

2. A characterization of graphs with bounded $|V(G)| - TW(G)$

In this section we give a characterization of graphs with bounded difference between the number of vertices and treewidth of the graph. This characterization is given in terms of forbidden subgraphs of the complement.

Definition 3. Let $\mathscr{T}^1 = \{K_2\}$. For $r \geqslant 2$, let $\mathscr{T}^{r+1}$ be the set of graphs G that can be constructed in the following way: take a graph H from $\mathscr{T}^r$ and an independent vertex set A (possibly empty) of H with at least $|V(H)| - (r+1)$ vertices. Denote $B = V(H) \setminus A$. Let $C \cup \{v_0\}$ be the set of new vertices such that $|C| = r + 1 - |B|$. Then G is the graph defined by

$$V(G) = V(H) \cup C \cup \{v_0\} \quad \text{and} \quad E(G) = E(H) \cup \{\{v_0, u\},\ u \in B \cup C\}.$$

Remark 4. Each graph from $\mathscr{T}^r$ is connected graph with $r(r+1)/2$ edges and $K_{r+1} \in \mathscr{T}^r$.

Obviously, $\mathscr{T}^2 = \{K_3, P_3\}$. Let us describe the set $\mathscr{T}^3$, which will be used later. The set $\mathscr{T}^3$ consists of five graphs, see Fig. 1.

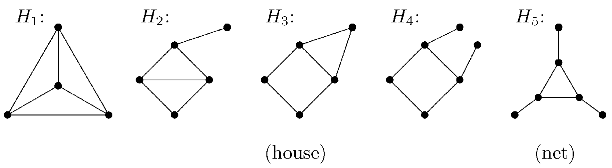

Fig. 1.

Rather than to test for each $H \in \mathscr{T}^3$ individually, we prefer the structurally-related test based on the following simple observation.

Observation 5. Let H be a graph. Then H contains a graph from $\mathscr{T}^3$ as a subgraph if and only if either
(a) H contains a triangle with each vertex of degree at least 3, or
(b) H contains a cycle of length 4 with two consecutive vertices of degree at least 3.

Lemma 6. *Let $r \geqslant 1$ and G be a k-tree with $(k+r+1)$ vertices. Then $\overline{G}$ consists of a single nontrivial component from $\mathscr{T}^r$ and zero or more isolated vertices.*

Proof. We proceed by induction on r. For $r=1$ the complement of any k-tree with $(k+2)$ vertices consists of K_2 ($\in \mathscr{T}^1$) and isolated vertices.

Let $r \geqslant 1$. Let G be a k-tree with $(k+r+2)$ vertices for some nonnegative integer k. According to the recursive definition of k-trees there is a vertex v_0, which together with all k neighbors $\{v_{r+2}, \dots, v_{r+k+1}\}$ creates the clique K_{k+1} in G.

Let $G_0 = G \setminus v_0$. Obviously, G_0 is a k-tree with $(k+r+1)$ vertices $\{v_1, \dots, v_{k+r+1}\}$. By the induction hypothesis, $\overline{G_0}$ consists of a single nontrivial component $H_0 \in \mathscr{T}^r$ and zero or more isolated vertices.

Look closer how the component H_0 is modified when the graph $\overline{G}$ from the graph $\overline{G_0}$ is constructed:

- the new vertex v_0 is added,
- the vertex set $V(H_0) \cap \{v_{r+2}, \dots, v_{k+r+1}\}$ is an independent set in $\overline{G}$, consequently in $\overline{G_0}$ and also in $\overline{H_0}$, with at least $|V(H_0)| - (r+1)$ vertices,
- new vertices $\{v_1, \dots, v_{r+1}\} \setminus V(H_0)$ are added, if $V(H_0) \not\supseteq \{v_1, \dots, v_{r+1}\}$,
- $r+1$ new edges $\{v_0, v_j\}$ $(j = 1, 2, \dots, r+1)$ are added.

As H_0 is not a discrete graph, necessarily $V(H_0) \cap \{v_1, \dots, v_{r+1}\} \neq \emptyset$. It implies that H consists of a single nontrivial component.

This construction follows the same steps as described in Definition 3. As a result we obtain that the component H, constructed from H_0 as above, belongs to $\mathscr{T}^{r+1}$. □

Lemma 7. *Let $r \geqslant 1$ and let H be a graph from $\mathscr{T}^r$. Then a graph consisting of H and l (possibly $l=0$) isolated vertices is the complement of an $(l+|V(H)|-r-1)$-tree.*

Proof. The proof uses induction on r. For $r=1$ we have $H = K_2$. Obviously, the graph consisting of K_2 and l isolated vertices is the complement of a l-tree.

Let $r \geqslant 1$. To prove the induction step, fix $H \in \mathscr{T}^{r+1}$.

(a) First, we prove that $\overline{H}$ is a $(|V(H)| - r - 2)$-tree.

The graph $H \in \mathscr{T}^{r+1}$ was constructed from some graph $H_0 \in \mathscr{T}^r$ following the construction described in Definition 3:

- for fixed independent vertex set A in H_0 and the set $B = V(H_0) \setminus A$ with the property $|B| \leqslant r+1$ we added a new vertex set C such that $|B \cup C| = r+1$,
- we added a new vertex v_0 and $(r+1)$ edges $\{v_0, u\}$, $u \in B \cup C$.

The graph $H \setminus v_0$ consists of H_0 and the set C of isolated vertices. By the induction hypothesis, its complement is a $(|C|+|V(H_0)|-r-1)$-tree. Obviously, $|C|+|V(H_0)|-r-1=|V(H)|-r-2=|A|$ and the induced subgraph with the vertex set A is a clique in $\overline{H_0}$. The graph $\overline{H}$ was created adding the new vertex v_0, which was made adjacent to a clique A. That means, $\overline{H}$ is a $(|V(H)|-r-2)$-tree.

(b) It easily follows from (a) that the graph consisting of $H \in \mathscr{T}^{r+1}$ and l isolated vertices is the complement of an $(l+|V(H)|-r-2)$-tree. □

Lemma 8. *Let $r \geqslant 1$ and G be a graph with property $TW(G) \leqslant |V(G)|-r-1$. Then $\overline{G}$ contains some graph from $\mathscr{T}^r$ as a subgraph.*

Proof. Fix G as above. $TW(G) \leqslant |V(G)|-r-1$ implies G is a partial $(|V(G)|-r-1)$-tree. It is well known that there exists a supergraph G_0 of G with the same vertex set $V(G)$ such that G_0 is a $(|V(G)|-r-1)$-tree (see e.g. [6, Lemma 2.1.13]).

Due to Lemma 6, $\overline{G_0}$ contains some graph from $\mathscr{T}^r$ as a subgraph. Obviously, $\overline{G_0}$ is a subgraph of $\overline{G}$ and the proof is finished. □

Lemma 9. *Let $r \geqslant 1$ and $\overline{G}$ contains a graph H from $\mathscr{T}^r$ as a subgraph. Then $TW(G) \leqslant |V(G)|-r-1$.*

Proof. Let $H \in \mathscr{T}^r$ be a subgraph of $\overline{G}$. Let G_0 be the graph, whose complement consists of H and isolated vertices of $V(G)\setminus V(H)$ (if $V(H) \neq V(G)$). Due to Lemma 7 the graph G_0 is a $(|V(G)|-r-1)$-tree. Obviously, G_0 is a supergraph of G, which implies $TW(G) \leqslant |V(G)|-r-1$. □

Theorem 10. *Let G be a graph and $r \geqslant 1$. Then the following conditions are equivalent*:
(1) *$TW(G) \geqslant |V(G)|-r$;*
(2) *$\overline{G}$ contains no graph from $\mathscr{T}^r$ as a subgraph.*

Proof. According to Lemmas 8 and 9 the following statement is true: $\overline{G}$ contains some graph from $\mathscr{T}^r$ as a subgraph if and only if $TW(G) \leqslant |V(G)|-r-1$. □

Definition 11. An undirected graph $G=(V,E)$ is called a comparability graph, or a transitively orientable graph, if there exists an orientation of the edges such that the resulting oriented graph (V,F) satisfies the following conditions:

$$F \cap F^{-1} = \emptyset \quad \text{and} \quad F + F^{-1} = E \quad \text{and} \quad F^2 \subseteq F$$

where $F^2 = \{(u,w) \mid \exists v \in V\ (u,v) \in F\ \&\ (v,w) \in F\}$.

For a k-tree G the following properties are equivalent:
(C_1) G is a k-path;
(C_2) $\overline{G}$ is a comparability graph;
(C_3) G does not contain a triple of vertices with the property that any two of them are connected by a path which avoids the neighborhood of the third.

The equivalence $(C_1) \Leftrightarrow (C_2)$ easily follows from the characterization of interval graphs found by Gilmore and Hoffman [3].

The equivalence $(C_1) \Leftrightarrow (C_3)$ is a consequence of another important characterization of interval graphs proved by Lekkerkerker and Boland [7]. Hence analogous results as we obtained in this section for the treewidth can be proved for the pathwidth.

Definition 12. Define $\mathscr{P}^r = \{H \in \mathscr{T}^r,\ H$ is a comparability graph$\}$.

Remark 13. It is easy to see that $\mathscr{P}^1 = \mathscr{T}^1$ and $\mathscr{P}^2 = \mathscr{T}^2$. Further, $\mathscr{T}^3 \setminus \mathscr{P}^3 = \{H_5\}$ (net), as $\overline{H_5}$ (3-sun) is a 2-tree, which does not satisfy (C_3), hence H_5 is not a comparability graph (see Fig. 1 for the set $\mathscr{T}^3$). Moreover, for every $r > 3$ there exists $H \in \mathscr{T}^r$ such that H_5 is an induced subgraph of H and hence H is not a comparability graph. It implies that $\mathscr{P}^r$ is a proper subset of $\mathscr{T}^r$ for every $r \geqslant 3$.

Replacing the set $\mathscr{T}^r$ by the set $\mathscr{P}^r$, Lemmas 6–9 can be formulated and proved in the same way also for pathwidth. Consequently, the analogous result of Theorem 10 holds also for pathwidth:

Theorem 14. *Let G be a graph and* $r \geqslant 1$. *Then the following conditions are equivalent*:
(1) $PW(G) \geqslant |V(G)| - r$,
(2) $\overline{G}$ *contains no graph from* $\mathscr{P}^r$ *as a subgraph.*

In the following, Theorems 10 and 14 will be used to give a structural characterization of graphs from $obs(TW(w))$, resp. $obs(PW(w))$ with $(w + r + 1)$ vertices in terms of subgraphs of the complement (for any $r \geqslant 1$). This description can be used to construct explicitly some graphs from $obs(TW(w))$, resp. $obs(PW(w))$.

3. Graphs from $obs(TW(w))$ (resp. $obs(PW(w))$) with $(w + 3)$ vertices

Ramachandramurthi [8] has found a structural characterization of graphs from $obs(TW(w))$ (resp. $obs(PW(w))$) with $(w + 3)$ vertices.

We give the method how to construct graphs from $obs(TW(w))$ (resp. $obs(PW(w))$) with $(w + r + 1)$ vertices for any $r \geqslant 2$. Our approach also essentially simplifies the results of [8].

Theorem 15. *For every* w, *a graph* G *with* $(w + 3)$ *vertices is in* $obs(TW(w))$ (*equivalently in* $obs(PW(w))$) *if and only if all components of* $\overline{G}$ *are stars and the number of them is at least* 3.

Proof. We will prove the theorem for treewidth. As $\mathscr{P}^2 = \mathscr{T}^2$, one can obtain the proof for pathwidth replacing TW by PW in our proof.

($\Rightarrow$) Let G be a graph from $obs(TW(w))$ with $(w + 3)$ vertices. Then $TW(G) = w + 1 = |V(G)| - 2$. Due to Theorem 10 neither K_3 nor P_3 are subgraphs of $\overline{G}$.

Consequently, the components of $\overline{G}$ are stars and isolated vertices. Moreover, $\overline{G}$ is not a discrete graph.

If $\overline{G}$ contains an isolated vertex v, then we delete an edge e of G connecting v and the central vertex of a star. In virtue of Theorem 10 $TW(G \setminus e) = TW(G)$, a contradiction. Hence components of $\overline{G}$ are just stars.

Suppose that $\overline{G}$ consists of one star (resp. two stars). Then deleting the central vertex (resp. contracting an edge connecting two central vertices) we obtain K_{w+2} as a proper minor of G. A contradiction with $G \in obs(TW(w))$.

($\Leftarrow$) Let G be a graph with $(w+3)$ vertices of all required properties. $\overline{G}$ contains no graph from $\mathcal{T}^2$ as a subgraph, that implies $TW(G) \geqslant w+1$. Our aim is to prove that $TW(H) \leqslant w$ for any proper minor H of G. It is clear, if $|V(H)| \leqslant w+1$.

(i) The number of stars implies that the complement of any minor H of G with $w+2$ vertices contains an edge, which implies $TW(H) \leqslant w$.

(ii) Because $\overline{G}$ contains no isolated vertices, deleting any edge e the graph $\overline{G \setminus e}$ contains K_3 or P_3. Due to Theorem 10 it follows $TW(G \setminus e) \leqslant w$.

This concludes the proof. □

Remark 16. Due to the previous theorem, the number of graphs from $obs(TW(n-3))$ with n vertices is equal to the number of partitions of number n into at least three parts of size at least 2.

4. A structure of graphs from $obs(TW(w))$ (resp. $obs(PW(w))$)

The following theorem gives a structural characterization of graphs from $obs(TW(w))$ with a fixed number of vertices in terms of subgraphs of the complement.

Theorem 17. *Given $r \geqslant 2$. A graph G with $(w+r+1)$ vertices is in $obs(TW(w))$ if and only if G satisfies the following three conditions*:

$T_1(r)$: *$\overline{G}$ contains no graph from $\mathcal{T}^r$ as a subgraph.*

$T_2(r)$: *If H is a minor of G with $|V(G)| - 1$ vertices, then $\overline{H}$ contains some graph from $\mathcal{T}^{r-1}$ as a subgraph.*

$T_3(r)$: *For every $e \in E(G)$, $\overline{G \setminus e}$ contains some graph from $\mathcal{T}^r$ as a subgraph.*

Proof. ($\Rightarrow$) This part follows from the definition of the obstruction set and Theorem 10.

($\Leftarrow$) Due to Theorem 10 the property $T_1(r)$ implies $TW(G) \geqslant w+1$. If H is a minor of G with at most $|V(G)| - 1$ vertices, then $TW(H) \leqslant w$ due to the property $T_2(r)$.

Let H be a proper minor (equivalently, subgraph) of G with $|V(G)|$ vertices. Then the property $T_3(r)$ implies $TW(H) \leqslant w$. □

Lemma 18. *Given $r \geqslant 2$ and a graph G with the property $T_2(r)$. Let F be a graph such that $\overline{F}$ contains $\overline{G}$ as a subgraph. Then F has the property $T_2(r)$.*

Proof. For fixed r we denote

$$\mathscr{M}^r := \{H: \overline{H} \text{ is a graph satisfying } T_2(r)\}.$$

Now lemma can be formulated in the following way: if $H \in \mathscr{M}^r$ and G is a supergraph of H, then also $G \in \mathscr{M}^r$.

To prove this statement it is enough to show that if $H \in \mathscr{M}^r$, then the graph H_e (resp. H_v) obtained from H adding a new edge e (resp. a new isolated vertex v) has to belong to $\mathscr{M}^r$.

As $\overline{H_e}$ is a subgraph of $\overline{H}$, then each minor of $\overline{H_e}$ is the minor of $\overline{H}$ and the conclusion is obvious.

Now let M be a minor of $\overline{H_v}$ with $|V(H)|$ vertices.

(i) If M is a subgraph of $\overline{H}$, the conclusion is trivial.

(ii) If M is a subgraph of $\overline{H_v}$ ($v \in V(M)$) or a graph obtained from $\overline{H_v}$ by contracting some edge from $E(\overline{H})$, then $M \setminus v$ is a minor of $\overline{H}$ with $|V(H)| - 1$ vertices. Hence $\overline{H \setminus v}$ contains a graph from $\mathscr{T}^{r-1}$ as a subgraph.

(iii) Finally, let us suppose, that M is a subgraph of the graph obtained from $\overline{H_v}$ by contracting some edge $\{u, v\} \in E(\overline{H_v})$. (A new vertex which is result of contracting $\{u, v\}$ is denoted by w.) Then $M \setminus w$ is a minor of $\overline{H}$ with $|V(H)| - 1$ vertices, which completes the proof. □

Remark 19. According to Lemma 18 the class of graphs $\mathscr{M}^r$ (for fixed r) is supergraph-closed. It easily follows that $\mathscr{M}^r$ can be characterized in terms of subgraphs of the complement. A graph G possesses $T_2(r)$ if and only if $\overline{G}$ contains some graph from $\min \mathscr{M}^r$ as a subgraph ($\min \mathscr{M}^r$ stands for the set of minimal elements of $\mathscr{M}^r$ with respect to subgraph relation).

The proofs of the following Theorems 20 and 22 give a general method, how to find graphs from the obstruction set, $obs(TW(w))$.

Theorem 20. *Let $r \geqslant 2$ be given and F be a graph with property $T_1(r)$ and $T_2(r)$. Then for every w, $w \geqslant |V(F)| - (r+1)$ there exists a graph $G \in obs(TW(w))$ with $(w+r+1)$ vertices such that some subgraph of F with vertex set $V(F)$ is an induced subgraph of G.*

Proof. Let F be a graph with property $T_1(r)$ and $T_2(r)$ for given $r \geqslant 2$ and $w \geqslant |V(F)| - (r+1)$ be fixed. Define $\overline{G}$ with $(w+r+1)$ vertices in the following way: $\overline{G}$ consists of $\overline{F}$ and $(w+r+1-|V(F)|)$ new isolated vertices. The graph G satisfies the condition $T_1(r)$ and also $T_2(r)$ (by Lemma 18).

Deleting a maximal set of edges from G for which the resulting graph has the property $T_3(r)$, we obtain a graph satisfying $T_1(r)$, $T_2(r)$ and $T_3(r)$, hence a graph from $obs(TW(w))$ with $(w+r+1)$ vertices. □

Remark 21. Let $r \geqslant 2$ be fixed. Obviously, if $\overline{G}$ consists of three disjoint copies of K_r, then G has the property $T_1(r)$ and $T_2(r)$. Theorem 20 implies the existence of

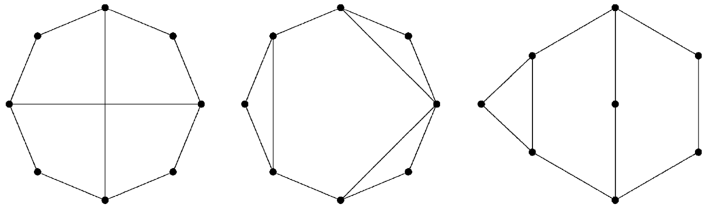

Fig. 2.

graphs from $obs(TW(w))$ for each $w \geqslant (2r-1)$. But this bound is not always optimal. For example, for $r=3$ we have found three graphs from $obs(TW(4))$ with 8 vertices. Their complements are in Fig. 2.

Theorem 22. *Let H be a graph from* $obs(TW(w))$ *for some* $w \geqslant 0$. *Then for every* $k > w$ *there exists a graph G from* $obs(TW(k))$ *such that*

$$|V(G)| - k = |V(H)| - w$$

and H is an induced subgraph of G.

Proof. It easily follows from Theorem 20. □

Remark 23. The results analogous to Theorems 17, 20 and 22, where treewidth is replaced by pathwidth and $\mathscr{T}^r$ by $\mathscr{P}^r$, can be proved in the same way.

5. Relation between $obs(TW(w))$ and $obs(PW(w))$

It is well known [4] that for each $w > 0$ there are trees which are obstructions to $PW(w)$. However, since the treewidth of a tree is 1, no tree can be in $obs(TW(w))$ for any $w > 0$. Therefore, $obs(PW(w)) \nsubseteq obs(TW(w))$.

For $w=1$ and $w=2$ it holds $obs(TW(w)) = \{K_{w+2}\} \subset obs(PW(w))$.

Ramachandramurthi [8] formulated a conjecture that for any $w \geqslant 3$ there exists a graph $H \in obs(TW(w)) \setminus obs(PW(w))$. He gave an example of such a graph for $w=3$ only. It can be seen (from Theorem 15) that any such a graph has at least $(w+4)$ vertices.

In the next we will construct for every $w \geqslant 3$ a graph which belongs to $obs(TW(w)) \setminus obs(PW(w))$.

Construction. Let $w > 5$ be fixed and k, l be integers such that $w + 4 = l + 3k$, $l \in \{1,2,3\}$. Let $K_{l,3,\dots,3}$ be the complete $(k+1)$-partite graphs with the corresponding partition of the vertices $\{V_i\}_{i=1}^{k+1}$, where $V_i = \{v_1^i, v_2^i, v_3^i\}$ for $2 \leqslant i \leqslant k+1$ and

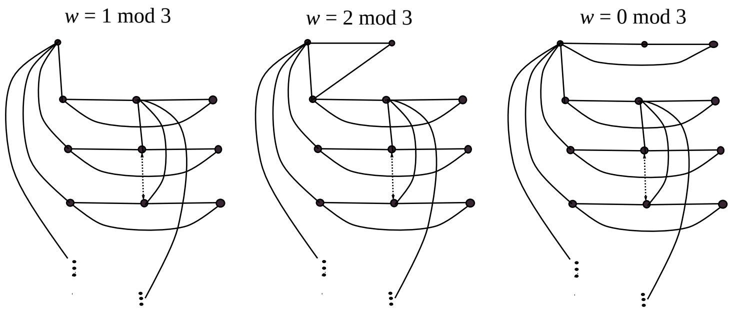

Fig. 3. The complements of graphs G_w.

$V_1 = \{v_j^1\}_{j=1}^l$. Define a graph G_w in the following way:

$$V(G_w) = V(K_{l,3,3,\dots,3}),$$

$$E(G_w) = E(K_{l,3,3,\dots,3}) \setminus \{\{\{v_1^1, v_1^i\},\ 2 \leqslant i \leqslant k+1\}$$

$$\cup \{\{v_2^2, v_2^i\},\ 3 \leqslant i \leqslant k+1\} \cup E_l\}.$$

where $E_l = \{\{v_2^1, v_1^2\}\}$ if $l = 2$; $E_l = \emptyset$ if $l = 1$ or $l = 3$ (see Fig. 3).

Theorem 24. *For every $w \geqslant 3$ the set $obs(TW(w)) \setminus obs(PW(w))$ is nonempty.*

Proof. Let $w > 5$ be fixed. We show that $G_w \in obs(TW(w)) \setminus obs(PW(w))$.

Applying Theorem 17 to prove $G_w \in obs(TW(w))$ it is enough to verify properties $T_1(r)$–$T_3(r)$ for G_w and $r = 3$.

$T_1(3)$: The graph $\overline{G_w}$ has not the property (a) from Observation 5 and contains no cycle of length 4, which concludes the proof of this part.

$T_2(3)$: Obviously, the complement of each minor of G_w with $|V(G_w)| - 1$ vertices contains K_3 as a subgraph.

$T_3(3)$: Let $e \in E(G_w)$ be fixed. It is enough to prove that $\overline{G_w \setminus e}$ has property (a) or (b) from Observation 5.

If $v_3^i \in e$ $(2 \leqslant i \leqslant k+1)$, then property (a) holds. Suppose $v_i^2 \in e$ $(3 \leqslant i \leqslant k+1)$. If the second vertex of e is v_1^1 or v_2^j $(3 \leqslant j \leqslant k+1)$, then (a) holds. If $e = \{v_2^i, v_1^j\}$ $(2 \leqslant j \leqslant k+1)$, property (b) holds. The same argument is true, whenever the second vertex of e is v_2^1 (if $l \geqslant 2$) or v_3^1 (if $l = 3$).

Now assume $v_1^i \in e$ $(2 \leqslant i \leqslant k+1)$. If any of the vertices v_1^j $(2 \leqslant j \leqslant k+1)$, v_2^1 (if $l \geqslant 2$) or v_3^1 (if $l = 3$) is the second vertex of e, then property (a) trivially holds. Finally, if $e = \{v_1^i, v_2^2\}$ $(3 \leqslant i \leqslant k+1)$, then property (b) holds. The other cases for $e = \{v_i^1, v_2^2\}$ $(1 \leqslant i \leqslant l)$ can be simply discussed.

Due to Remark 23 to prove $G_w \notin obs(PW(w))$ it is enough to find $e \in E(G_w)$ such that $\overline{G_w \setminus e}$ contains no graph from $\mathscr{P}^3$ as a subgraph. Let $e = \{v_2^3, v_2^4\}$ (dashed edge in Fig. 3). One can easily check that $\overline{G_w \setminus \{v_2^3, v_2^4\}}$ contains no cycle of length 4. Consequently no graph from $\mathscr{P}^3$.

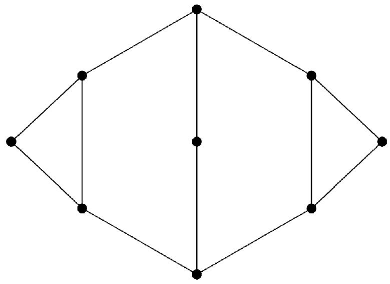

Fig. 4.

In the case $w=4$ (resp. $w=5$) an example of a graph from $obs(TW(w))\setminus obs(PW(w))$ is the graph whose complement is the third graph in Fig. 2 (resp. graph whose complement is in Fig. 4). This fact can be easily verified using Theorem 17 and Remark 23. □

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