Orbits and co-orbits of ultrasymmetric spaces in weak interpolation

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Abstract

Ultrasymmetric spaces form a large class of rearrangement-invariant spaces which are not only intermediate but also interpolation between Lorentz and Marcinkiewicz spaces with the same fundamental function. They include Lebesgue, Lorentz, Lorentz–Zygmund and many other classical spaces. At the same time they have rather simple analytical description, making them suitable for stating various interpolation properties, especially in “extreme” cases of weak interpolation. In the present paper we consider ultrasymmetric spaces which are so “close” to the endpoint spaces that the ratio of their fundamental functions is a slowly varying function \(b(t) \sim b(t^2)\), and find for them explicitly the upper and the lower optimal interpolation spaces near the “right” and near the “left” endpoints. In result we obtain four new types of rearrangement-invariant spaces (not ultrasymmetric) and study some other properties of them.

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1. Introduction

One of the most difficult problems in interpolation theory of linear operators is description of optimal interpolation spaces, namely, the largest space \(A\) and the smallest space \(B\) such that, for any linear operator \(T : A_i \rightarrow B_i, i = 0, 1\), it follows that \(T : A \rightarrow B\). Existence and uniqueness of such spaces was shown by Aronszajn and Gagliardo still in 1965 in one of the first big papers of interpolation theory [1]. In fact, they divided this problem into two subproblems on one-sided optimality. For arbitrary given space \(A\) intermediate in the couple \((A_0, A_1)\), they constructed the optimal range space \(B\), using a special method of orbits, and thus the space \(B\) itself is usually named orbit space of the space \(A\). Correspondingly the optimal domain space \(A\) for a given space \(B\) intermediate in the couple \((B_0, B_1)\) got the name of co-orbit space.

Although the Aronszajn–Gagliardo’s method is very important for abstract theory of interpolation, it does not help to find explicit analytical description of optimal interpolation spaces in applications. That is why interpolation theory,
being now on a very high level, has only few results on optimal interpolation,\(^2\) which almost all concern the couples of rearrangement-invariant spaces.

In the present paper we consider only the so-called weak interpolation, that is, interpolation from a couple of Lorentz spaces \((L_{\phi_0}, L_{\phi_1})\) to a couple of Marcinkiewicz spaces \((M_{\psi_0}, M_{\psi_1})\) which takes its origin in the famous Marcinkiewicz theorem (see, e.g., [2] or [5]). This kind of interpolation is especially important for integral operators, which, as usual, satisfy only weak conditions on the endpoint spaces. That is why the weak interpolation is widely used for studying differential and integral equations, Fourier series and transforms, Sobolev embeddings, etc. Moreover, most results on optimal interpolation were stated just for weak interpolation and among them a very general theorem of Kreĭn and Semenov (see [5, Section II.6]).

Unfortunately, the optimality results of Kreĭn and Semenov and some other authors are applicable only to intermediate spaces which are not “too close” to the endpoint spaces of interpolation, i.e., satisfy some strong inequalities between the Boyd indices of these spaces. Such assertions turn out to be useless in the so-called “extreme” (or “limiting”) cases of various analytical problems, such as properties of integral transforms on Zygmund spaces \(L \log L\) and \(\exp L\), embeddings of Sobolev spaces \(W^{k, p}_0(\Omega)\), when \(pk = n\) (dimension of \(\Omega\)), etc. Generally speaking, we get here a problem of weak interpolation on intermediate spaces with the same Boyd indices as for the endpoint spaces, and even non-optimal interpolation in such cases is rather complicated or unknown at all.

The first optimal result for “extreme” cases of weak interpolation was obtained in [4] and applied to “limiting” Sobolev embeddings in [3]. It was then generalized in [6] to arbitrary spaces of Lorentz–Zygmund type, i.e., spaces with the norm \(\|t^{1/p}(\ln t)^{\alpha} f^*(t)\|_{L^E}\), where \(p > 1\), \(\alpha \in \mathbb{R}\) and \(E\) is arbitrary space, rearrangement-invariant on \((0, 1)\) with respect to the measure \(dt/t\). While the optimal interpolation for such spaces in “non-extreme” cases is attained on the same class of spaces, the “extreme” cases lead to spaces of new types with rather complicated norms, including additional integral and supremum operations.

Before speaking of intermediate spaces which are “too close” to the endpoint spaces of a given couple we should agree how to estimate this “closeness.” Since the Boyd indices of both spaces are equal, we need a more delicate measure of their difference. In the aforementioned examples this role was played by the exponent of logarithmic factor. Passing to more general classes of spaces, we should seek for a suitable replacement of this exponent. For instance, we may compare two rearrangement-invariant spaces \(F, G\) on \((0, 1)\) by the ratio of their fundamental functions \(b(t) = \varphi_F(t)/\varphi_G(t)\) and consider extension indices of the function \(B(t) = b(e^{1-1/t})\). This approach is still more productive if the fundamental functions enter explicitly into expression of norms in \(F\) and \(G\).

A large class of spaces with the last property was considered in [7]. A rearrangement-invariant space \(G\) is called ultrasymmetric if it is interpolation in the couple of Lorentz and Marcinkiewicz spaces with the same fundamental function \(\varphi(t)\). As shown in [7], the norm of \(G\) is equivalent to \(\|\varphi(t) f^*(t)\|_{L^E}\) for some space \(E\) described above. The class of ultrasymmetric spaces includes Lebesgue, Lorentz, Lorentz–Zygmund and many other classical spaces. It will be the subject of our consideration in the present paper.

The paper is organized as follows. In the next section we give some preliminary information about ultrasymmetric spaces and weak interpolation, discuss proximity of spaces to the endpoints of interpolation and principal difference between the “right” and the “left” endpoints. The main results of our paper are given in Sections 3 and 4, where we prove the theorems on optimal interpolation separately for each endpoint and give some properties of the optimal spaces thus obtained. It turns out that the technique of proofs is essentially different: while optimality on the right-hand side is proved by standard use of a specially constructed maximal operator, the left-hand spaces are constructed and studied via duality to the right-hand ones.

Throughout the paper we write \(X = Y\) for spaces with equivalent (quasi)norms and \(X \subset Y\) for continuous embedding, while \(T : X \rightarrow Y\) will stand for the continuous operator acting from \(X\) to \(Y\). We shall write \(f \lesssim g\) \((f \gtrsim g)\) instead of \(f \leq C g\) \((f \geq C g)\) with some constant \(C > 0\) and \(f \sim g\) for equivalent functions, i.e., if \(f \lesssim g\) and \(f \gtrsim g\) at the same time. Moreover, we do not distinguish the notions “increasing” and “non-decreasing” as well as “decreasing” and “non-increasing” and we say “a function \(f\) is almost increasing (decreasing)” if it is equivalent to an increasing (decreasing) function.

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\(^2\) Recall that we speak of absolute optimality, among all Banach spaces, and do not consider results on partial optimality among spaces from special classes like \(L_p, L_{pq}\), etc.
2. Preliminaries

For main definitions and properties concerning rearrangement-invariant spaces and interpolation theory, we refer the reader to the monographs [2] and [5]. Recall that a Banach function space $E$ is called rearrangement-invariant (r.i.) if, for any two measurable functions $f, g$, the conditions $g \in E$, $f^*(t) \leq g^*(t)$ for almost all $t > 0$ imply $f \in E$ and $\|f\|_E \leq \|g\|_E$, where $f^*$ and $g^*$ stand for the decreasing rearrangements of $f$ and $g$. As a consequence, all equimeasurable functions have the same norm in $\Lambda_{\varphi}$. Any r.i. space $E$ is intermediate in the Banach couple $(L_1, L_\infty)$; in addition we will always assume that $E$ is exact interpolation in this couple. As underlying measure spaces we will consider the intervals $(0, 1)$ or $(0, \infty)$ with usual Lebesgue measure $dt$ and $(0, 1)$ with homogeneous measure $\frac{dt}{t}$. In the last case we will use letters with the tilde, denoting by $\tilde{E}$ the space of all $f : (0, 1) \mapsto \mathbb{R}$ such that $g(u) = f(e^{-u}) \in E(0, \infty)$.

The spaces $\tilde{E}$ are used as parameters in definition of ultrasmooth symmetric spaces from [7], that is, spaces $L_{\varphi,E}$ of functions $f : (0, 1) \mapsto \mathbb{R}$ with the (quasi)norm $\|f^*\|_{\tilde{E}}$, where the second parameter $\varphi(t)$ is arbitrary positive, finite and almost increasing function on $(0, 1)$. As shown in [7], a r.i. space $G$ with fundamental function $\varphi(\lambda) = \|\chi(0,\lambda)\|_G$ is ultrasmooth if and only if it is interpolation between the corresponding Lorentz space $\Lambda_{\varphi}$ and Marcinkiewicz space $M_{\varphi}$, where

$$\|f\|_{\Lambda_{\varphi}} = \int_{0}^{1} f^*(u) d\varphi(u), \quad \|f\|_{M_{\varphi}} = \sup_{0 < u < \varphi(u) f^*(u)}. $$

Recall that any fundamental function of Banach r.i. spaces is quasiconcave, that is, both functions $\varphi(t)$ and $t/\varphi(t)$ are increasing. Therefore all parameter functions $\varphi(t)$ below will be supposed quasiconcave.

As an important characteristic of parameter-function $\varphi$ we will take its lower and upper extension indices

$$\pi_{\varphi} = \lim_{t \to 0} \frac{\ln m_{\varphi}(t)}{\ln t}, \quad \rho_{\varphi} = \lim_{t \to \infty} \frac{\ln m_{\varphi}(t)}{\ln t},$$

where $m_{\varphi}(t) = \sup_{0 < s < 1} \frac{\varphi(ts)}{\varphi(s)}$.

(Since the function $\varphi(t)$ is defined only on $(0, 1)$, we agree to set $\varphi(t) = \varphi(1) := \varphi(0)$ for any $t > 1$ whenever such values of $t$ are needed.)

In general $0 \leq \pi_{\varphi} \leq \rho_{\varphi} \leq \infty$, but $0 \leq \pi_{\varphi} \leq \rho_{\varphi} \leq 1$ if the function $\varphi$ is quasiconcave. In a similar way, the Boyd indices of a r.i. space $G$ are defined by

$$\pi_{G} = \lim_{t \to 0} \frac{\ln d_{G}(t)}{\ln t}, \quad \rho_{G} = \lim_{t \to \infty} \frac{\ln d_{G}(t)}{\ln t}, \quad d_{G}(t) = \sup_{f \in G} \frac{\|f(s/t)\|_G}{\|f(s)\|_G},$$

Notice that, for any ultrasmooth symmetric space, its Boyd indices coincide with the extension indices of parameter-function $\varphi$.

The problem of optimal interpolation for r.i. spaces which are “too close” to the spaces $L_1$ and $L_\infty$ (that is, have the Boyd indices 0 or 1) is rather special and will not be considered in this paper. In order to exclude such spaces, we shall work below mainly with quasi-power parameter functions, i.e., such quasiconcave functions $\varphi$ that

$$\varphi(t) \sim \int_{0}^{1} \min \left(1, \frac{t}{s}\right) \varphi(s) \frac{ds}{s} \quad \text{for all } t \in (0, \infty).$$

It can be proved that $\varphi$ is a quasi-power function if and only if $0 < \pi_{\varphi} \leq \rho_{\varphi} < 1$ (see [5, p. 57]).

Recall that we do not differentiate spaces with equivalent norms and thus we may replace the given parameter function by equivalent one with more convenient properties. In result we may always assume that the parameter function $\varphi(t)$ is strictly increasing, continuous and even smooth. For quasi-power functions, we get a stronger property as stated in the following lemma:

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3 The conditions for this are very mild, and in some books (e.g., in [2]) this property is included into definition of r.i. spaces.
Lemma 2.1. Let \( \varphi \) be a quasi-power function, then, for any \( \epsilon > 0 \) such that \( \epsilon < \min(\pi_\varphi, 1 - \rho_\varphi) \), there exists a smooth function \( \psi(t) \sim \varphi(t) \) such that

\[
0 < \pi_\varphi - \epsilon \lesssim \frac{t \psi'(t)}{\psi(t)} \lesssim \rho_\varphi + \epsilon < 1 \quad \text{for all} \; t \in (0, 1).
\]

Proof. Let \( \epsilon > 0 \) satisfy the required inequalities. By properties of extension indices (see, e.g., [5, Section II.1.2]), the function \( \varphi(t) / t^{\pi_\varphi - \epsilon} \) is almost increasing and the function \( \varphi(t) / t^{\rho_\varphi + \epsilon} \) is almost decreasing. This implies existence of a function \( \psi(t) \sim \varphi(t) \) such that \( \psi(t) / t^{\pi_\varphi - \epsilon} \) is strictly increasing while \( \psi(t) / t^{\rho_\varphi + \epsilon} \) is strictly decreasing. For example, we may take

\[
\psi(t) = \sup_{s > 0} \min\left( (t/s)^{\pi_\varphi - \epsilon}, (t/s)^{\rho_\varphi + \epsilon} \right) \varphi(s).
\]

Moreover, within the framework of equivalence, we can make \( \psi(t) \) smooth everywhere. Then \( (\psi(t) / t^{\pi_\varphi - \epsilon})' \geq 0 \), that is, \( \psi'(t) / (\pi_\varphi - \epsilon) \psi(t) t^{\pi_\varphi - \epsilon - 1} \geq 0 \) and thus \( \pi_\varphi - \epsilon \lesssim t \psi'(t) / \psi(t) \). Similarly, \( t \psi'(t) / \psi(t) \lesssim \rho_\varphi + \epsilon \).

In what follows we denote \( \tilde{\pi}_\varphi = \pi_\varphi - \epsilon \) and \( \tilde{\rho}_\varphi = \rho_\varphi + \epsilon \) and keep the same notation \( \varphi \) for equivalent parameter-functions. In result, we may always assume that if \( \varphi \) is a quasi-power function then there exist two numbers \( \tilde{\pi}_\varphi \leq \pi_\varphi \) and \( \tilde{\rho}_\varphi \geq \rho_\varphi \) such that

\[
0 < \tilde{\pi}_\varphi \lesssim \frac{t \varphi'(t)}{\varphi(t)} \lesssim \tilde{\rho}_\varphi < 1. \tag{2.1}
\]

The same reason of norm equivalence allows us to replace (if needed) the parameter-space \( E \) by \( E \cap L_\infty \), because always \( L_{\psi,E} \subset M_{\varphi} = L_{\psi,L_\infty} \). Let us collect some other useful properties of parameter-spaces \( \tilde{E} \) (the proofs see, e.g., in [4] or [6]).

Lemma 2.2.

1. Any space \( \tilde{E} \) is an exact interpolation space in the couple \( \tilde{L}_1, L_\infty \), where

\[
\| f \|_{\tilde{L}_1} = \int_0^1 |f(t)| \frac{dt}{t}.
\]

2. For any numbers \( 0 < a \leq 1, b > 0 \) and each function \( g \in \tilde{E} \), the following inequality is valid

\[
\| g(at^b) \|_{\tilde{E}} \leq \max\left( 1, \frac{1}{b} \right) \| g(t) \|_{\tilde{E}}.
\]

3. If \( \rho_\varphi < 1 \) then \( \| \varphi f^* \|_{\tilde{E}} \sim \| \varphi f^{**} \|_{\tilde{E}} \) where, as usual, \( f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds \). As a consequence we obtain that the ultrasymmetric space \( L_{\psi,E} \) in the case of \( \rho_\varphi < 1 \) is a Banach space with the norm \( \| \varphi f^{**} \|_{\tilde{E}} \).

Let two quasiconcave functions \( \phi(t), \psi(t) \) on \((0, 1)\) be given. A linear operator \( T \) is said to be of weak type \((\phi, \psi)\) if \( T: L_\phi \to M_{\psi} \). We denote by \( W(\phi_0, \psi_0; \phi_1, \psi_1) \) the set of all linear operators which are at the same time of weak type \((\phi_0, \psi_0)\) and of weak type \((\phi_1, \psi_1)\). Some general properties of such operators can be found in [2, Section 4.7]. For obtaining more informative results, we assume additionally that the ratios \( \phi_0/\phi_1 \) and \( \psi_0/\psi_1 \) are also monotone with sufficiently rapid growth.

Weak type interpolation for ultrasymmetric spaces which are not “too close” to the endpoint spaces was studied in [7, Section 5] with complete description of orbit and co-orbit spaces. It turned out that both spaces of optimal interpolation belong to the same class of ultrasymmetric spaces. Another situation occurs when the space \( L_{\psi,E} \) is “too close” to one of endpoints. Even non-optimal interpolation in this case changes essentially, replacing the range space \( L_{\psi,E} \) by \( L_\kappa \psi,E \) with additional “decay factor” \( \kappa(t) = o(1) \) for \( t \to 0 \) (see [9]), whereas the optimal interpolation spaces will be shown below as not ultrasymmetric at all.

An important feature of the above mentioned results from [7] and [9] is their symmetry with respect to both endpoints of interpolation. As we shall see below, the situation in optimal interpolation “near” these endpoints is
rather different. Let us, for definiteness, refer to \( \phi_0 \) as the left and to \( \phi_1 \) as the right endpoint of interpolation. The right endpoint space \( A_{\phi_1} \) is the smallest among all r.i. spaces with the same fundamental function \( \phi_1 \), hence we can describe analytically any degree of proximity to it, taking, e.g., the space \( L_{\phi_1,E} \) with \( \tilde{E} \) approaching \( \tilde{L}_1 \). On the contrary, any intermediate space near the left endpoint should be smaller than \( A_{\phi_0} \), and this kind of proximity is very difficult for analytical description. That is why interpolation for the spaces near the right endpoint is investigated now much better (see, e.g., [6,10]).

Let us describe the kind of proximity between spaces which will be considered in this paper. For arbitrary ultrasymmetric space \( L_{\phi,E} \), we define a function \( b(t) = \phi(t)/\phi_1(t) \) and suppose that \( b \) is bounded, increasing and such that \( b(t^2) \sim b(t) \) (evidently we get also that \( b(t^{2\alpha}) \sim b(t) \) for any \( \alpha > 0 \)). This condition implies that \( \pi_b = \rho_b = 0 \) and thus \( \pi_f = \pi_{\phi_1}, \rho_f = \rho_{\phi_1} \), i.e., \( L_{\phi,E} \) is really “too close” to the right endpoint. Note also that \( b(t)/t^\alpha \) is (almost) decreasing for any \( \alpha > 0 \).

For every function \( b \) as above, we define a new function \( B \) by the formula \( B(u) = b(e^{1/u}) \). In result, \( B(u) \) is also positive and increasing on \((0,1)\). Moreover, the condition \( b(t^2) \sim b(t) \) implies that \( B(2t^2) \sim B(t^2) \) and thus \( 0 \leq \pi_B \leq \rho_B < \infty \). The case of \( \pi_B = 0 \) requires special consideration, because the function \( b(t) \) becomes too slow (like \( \ln|\ln|t|| \)); that is why we assume\(^4\) in what follows that \( \pi_B > 0 \).

In order to show explicitly which parameter-functions define the space \( L_{\phi,E} \), we denote it further by \( L_{\phi_1,b,E} \). Analogous notation will be used for other kinds of spaces considered below.

### 3. Optimal interpolation near the right endpoint

In this section we shall prove that optimal interpolation “near the right endpoint” can be described, involving the r.i. spaces of \( A \) and \( B \) types that were already considered in [10] as giving better interpolation results than those from [9]. The spaces of type \( A \) were used there for improving the domain spaces of interpolation. For given functions \( \phi_1 \) and \( b \) with the aforementioned properties, the space \( A_{\phi_1,b,E} \) consists of all measurable functions \( f : (0,1) \mapsto \mathbb{R} \) with finite norm

\[
\| f \|_{A_{\phi_1,b,E}} = \left\| \frac{1}{t} \int_0^{1/t} \phi_1(s) f^{**}(s) \frac{ds}{s} \right\|_{\tilde{E}}.
\]

This space is non-trivial if and only if \( (e^{-u}) \in E \), hence this condition should be mentioned in any use of \( A_{\phi_1,b,E} \) (or replaced by a stronger condition like \( \rho_E < \pi_B \)). For quasi-power function \( \phi_1 \), the norm in \( A \) space is equivalent to the quasinorm, obtained via replacing \( f^{**} \) by \( f^* \).

The fact that the space \( A_{\phi_1,b,E} \) gives better result than the corresponding ultrasymmetric space from [9] follows from the direct comparison of two kinds of spaces.

**Proposition 3.1.** For any r.i. space \( E \) such that \( b(e^{-u}) \in E \) if \( \pi_B > 1 \) or \( \rho_E < \pi_B \) if \( 0 < \pi_B \leq 1 \), the \( A \)-type and the ultrasymmetric spaces are connected by the relation

\[
L_{\phi_1,b_1,E} \subset A_{\phi_1,b,E} \subset L_{\phi_1,b,E}.
\]

Moreover, these embeddings are strict whenever \( \lim_{t \to \infty} \varphi_E(t)/t = 0 \), and \( A_{\phi_1,b,L_1} = L_{\phi_1,b_1,L_1} \) if \( \pi_B > 1 \).

The spaces of type \( B \) were used for improving the range spaces of interpolation. For functions \( \psi_1 \) and \( b \) as above, the space \( B_{\psi_1,b,E} \) consists of all measurable functions \( f : (0,1) \mapsto \mathbb{R} \) with finite norm

\[
\| f \|_{B_{\psi_1,b,E}} = \sup_{0 < s < t} \psi_1(s) \| b(s)f^{**}(s) \|_{\tilde{E}},
\]

where again \( f^{**} \) may be replaced by \( f^* \) in the case of quasi-power function \( \psi_1 \). The condition \( b(e^{-u}) \in E \) is also important here, although the space \( B_{\psi_1,b,E} \) exists (is non-trivial) even without it. It can be shown that this inclusion is necessary and sufficient for the space \( B_{\psi_1,b,E} \) to be intermediate in the couple \((M_{\phi_0}, M_{\psi_1})\). Evidently \( B_{\psi_1,b,E} \subset L_{\psi_1,b,E} \), and this embedding, in general, is strict (the exceptional case is \( E \supseteq L_{\infty} \) when both spaces coincide).

The main statement of this section is as follows:

\(^4\) This assumption is important only for proving optimality of interpolation; other interpolation results do not need it.
**Theorem 3.2.** Let \( \phi_0, \psi_0 \) be two quasiconcave functions with \( \pi_{\phi_0}, \pi_{\psi_0} > 0 \). Let \( \phi_1 \) and \( \psi_1 \) be two quasi-power functions such that \( \pi_{\phi_0}/\phi_1 > 0 \) and \( \pi_{\psi_0}/\psi_1 > 0 \). Let \( E \) be any r.i. space such that \( b(e^{-m}) \in E \) and let \( T \in W(\phi_0, \psi_0; \phi_1, \psi_1) \). Then \( T : A_{\phi_1, b, E} \rightarrow B_{\psi_1, b, E} \) and, consequently, \( T : A_{\phi_1, b, E} \rightarrow L_{\psi_1, b, E} \).

If, in addition, \( \rho_E < \pi_B \) when \( 0 < \pi_B \leq 1 \), then

\[
\begin{align*}
& (i) \quad T : L_{\phi_1, b, E} \rightarrow B_{\psi_1, b, E}, \\
& (ii) \quad T : L_{\phi_1, b, E} \rightarrow L_{\psi_1, b, E}.
\end{align*}
\]

Moreover, \( A_{\phi_1, b, E} \) is the upper optimal (co-orbit) space for \( L_{\psi_1, b, E} \) and \( B_{\psi_1, b, E} \) is the lower optimal (orbit) space for \( L_{\phi_1, b, E} \) in this interpolation.

Note that the additional condition on the space \( E \) in the second part of this theorem is caused by Proposition 3.1, since we need the embedding \( L_{\phi_1, b, E} \subset A_{\phi_1, b, E} \).

**Proof.** As was mentioned before, the interpolation properties of spaces of \( A \) and \( B \) types were stated in [10] and the proof of them can be found there. So we need to show only optimality of interpolation that will be proved with the help of a particular linear operator \( T \), defined in the following lemma:

**Lemma 3.3.** Let the functions \( \phi_0, \psi_0, \phi_1, \psi_1 \) satisfy all conditions of Theorem 3.2 and let \( T \) be an operator, defined for each \( f \in A_{\phi_0} + A_{\phi_1} \) by

\[
T f(t) = \frac{1}{\psi_1(t)} \int_{m(t)}^{1} \phi_1(s) f(s) \frac{ds}{s} \quad \text{for all } t \in (0, 1),
\]

(3.1)

where \( m(t) = \Phi^{-1}(\Psi(t)) \) for \( \Phi(t) = \frac{\phi_0(t)}{\phi_1(t)} \) and \( \Psi(t) = \frac{\psi_0(t)}{\psi_1(t)} \).

Then \( T \in W(\phi_0, \psi_0; \phi_1, \psi_1) \).

**Proof.** Passing (if necessary) to equivalent functions, we may assume that the functions \( \Phi \) and \( \Psi \) are strictly increasing and \( \Phi(1) = \Psi(1) = 1 \). Hence the function \( m(t) \) and the operator \( T \) are well-defined.

As known (see, e.g., [5, Section II.5.2]), \( (A_{\phi_1})^* = M_{\tilde{\phi}_1} \) with \( \tilde{\phi}_1(t) = t/\phi_1(t) \). Moreover, \( 1/\tilde{\phi}_1 \in M_{\tilde{\phi}_1} \). In result, for any \( t \in (0, 1) \), we obtain that

\[
(T f(t))^* = |T f(t)| \leq \frac{1}{\psi_1(t)} \int_{0}^{1} \phi_1(s) f(s) \frac{ds}{s} \leq \frac{1}{\psi_1(t)} \|f\|_{A_{\phi_1}} \|1/\tilde{\phi}_1\|_{M_{\tilde{\phi}_1}},
\]

hence \( |T f| \leq \|f\|_{A_{\phi_0}} \).

On the other hand, exploiting again the fact that the function \( \Phi(t) \) is increasing, we obtain that

\[
|T f(t)| \leq \frac{1}{\psi_1(t)} \int_{m(t)}^{1} \phi_1(s) \phi_0(s) f(s) \frac{ds}{s} \leq \frac{1}{\psi_1(t)} \frac{1}{\Phi(m(t))} \int_{0}^{1} \phi_0(s) f(s) \frac{ds}{s}.
\]

As for any fundamental function, the function \( \phi_0(s)/s \) is decreasing, thus

\[
\int_{0}^{1} \phi_0(s) f(s) \frac{ds}{s} \leq \int_{0}^{1} \phi_0(s) f^*(s) \frac{ds}{s}.
\]

Moreover, the inequality (2.1) implies that \( \phi_0(s)/s \leq (1/\tilde{\pi}_{\phi_0}) \phi_0(s) \), hence

\[
|T f(t)| \leq \frac{1}{\psi_1(t) \Psi(t)} \|f\|_{A_{\phi_0}} = \frac{1}{\psi_0(t)} \|f\|_{A_{\phi_0}}.
\]

Thus \( \|T f\|_{M_{\phi_0}} \leq \|f\|_{A_{\phi_0}} \) and the lemma is proved. \( \square \)
Lemma 3.4. Using the properties of function $A$ provides all needed relations between interpolation spaces. We start with consideration of spaces of the type $A_{\psi_1, b, E}$. From [1] it follows that such a space necessarily exists and must be interpolation in the couple $(A_{\phi_0}, A_{\phi_1})$; from [5, Theorem II.4.2] we then obtain that this space is r.i. Let us show that the space $D$ cannot be larger than $A_{\phi_0, b, E}$ even as a domain space for the single operator (3.1).

By Lemma 3.3 we have that $T : D \rightarrow L_{\psi_1, b, E}$. Let $f \in D$, then also $h = f^* \in D$ and $Th \in L_{\psi_1, b, E}$, hence

$$\|Th\|_{L_{\psi_1, b, E}} = \left\|\psi_1(t) b(t)(Th)^*(t)\right\|_E = \|b(t) \int_{m(t)}^{1} \phi_1(s) f^*(s) \frac{ds}{s}\|_E < \infty$$

(as before, we may consider the function $m(t)$ as strictly increasing).

On the other hand, the conditions on functions $\phi_0$ and $\psi_0$ ensure that the functions $\Phi$ and $\Psi$ are quasi-power, hence by Lemma 2.1 there exist four numbers $\tilde{\pi}_\Phi$ and $\tilde{\rho}_\Phi$ such that

$$0 < \tilde{\pi}_\Phi \leq \frac{t\Phi'(t)}{\Phi(t)} \leq \tilde{\rho}_\Phi < 1, \quad 0 < \tilde{\pi}_\Psi \leq \frac{t\Psi'(t)}{\Psi(t)} \leq \tilde{\rho}_\Psi < 1.$$ 

Therefore

$$\Psi'(t) = \Phi'(m(t)) m'(t) \leq \tilde{\rho}_\Phi \frac{\Phi(m(t))}{m(t)} m'(t) = \tilde{\rho}_\Phi \phi(t) m'(t).$$

hence

$$\frac{tm'(t)}{m(t)} \geq \frac{\tilde{\pi}_\Psi}{\tilde{\rho}_\Phi} > 0 \quad \text{for all } t \in (0, 1).$$

In result $m(t) \leq t^{\tilde{\pi}_\Psi / \tilde{\rho}_\Phi}$ for all $t \in (0, 1)$ and, using the equivalence $b(t) \sim b(t^{\tilde{\pi}_\Psi / \tilde{\rho}_\Phi})$, we obtain that

$$\|Th\|_{L_{\psi_1, b, E}} \gtrsim \left\|b(t^{\tilde{\pi}_\Psi / \tilde{\rho}_\Psi}) \int_{t^{\tilde{\pi}_\Psi / \tilde{\rho}_\Phi}}^{1} \phi_1(s) f^*(s) \frac{ds}{s}\right\|_E.$$

Now, applying Lemma 2.2 (part 2) with $a = 1$ and $b = \tilde{\rho}_\Phi / \tilde{\pi}_\Psi > 0$, we get

$$\|Th\|_{L_{\psi_1, b, E}} \gtrsim \left\|b(t) \int_{t}^{1} \phi_1(s) f^*(s) \frac{ds}{s}\right\|_E \sim \|f\|_{A_{\phi_1, b, E}},$$

which gives that $f \in A_{\phi_1, b, E}$. Since $f$ is arbitrary, this means that $D \subseteq A_{\phi_1, b, E}$ and the proof of upper optimality of the space $A_{\phi_1, b, E}$ is complete.

For proceeding to spaces of the type $B$, we have to prove an additional property of functions $b$.

Lemma 3.4. For any $\alpha > 0$ and $t \in (0, e^{-1})$,  

$$\int_{t^\alpha}^{1} \frac{\ln s}{b(s) s^{\frac{\alpha}{\ln 2}}} \frac{ds}{s} \gtrsim \frac{1}{b(t)} \cdot \frac{\alpha}{1 + \alpha}.$$

Proof. Using the properties of function $b$, it is easy to observe that

$$\int_{t^\alpha}^{1} \frac{\ln s}{b(s) s^{\frac{\alpha}{\ln 2}}} \frac{ds}{s} \gtrsim \frac{1}{b(t^{\alpha/2})} \cdot \frac{1}{\ln s} \int_{t^\alpha}^{1} \frac{ds}{s} \sim \frac{1}{b(t)} \left( \frac{\alpha \ln \frac{1}{t}}{1 + \alpha \ln \frac{1}{t}} \right).$$

Therefore, for all $t \leq e^{-1}$, we get (3.2). \qed
Denote now by $G$ the smallest Banach function space such that any linear operator from $W(\psi_0, \psi_1; \phi_1, \psi_1)$ acts boundedly from $L_{\phi_1, \ln \xi, E}$ to $G$. Referring again to [1] and [5], we obtain that such a lower optimal space exists and is rearrangement invariant. Hence we have only to prove that $B \subseteq G$, that is, taking arbitrary $g \in B_{\psi_1, b, E}$, to show that $g^* \in G$. We shall do this, constructing a function $f \in L_{\phi_1, \ln \xi, E}$ such that $Tf \gtrsim g^*$.

On the first stage we define a function

$$\xi(t) = \sup_{0 < s < t} \psi_1(s) b(s) g^*(s).$$

Since $\|\xi(t)\|_{E} \gtrsim g\|_{B_{\psi_1, b, E}} < \infty$, we obtain that $\xi \in \tilde{E}$. Evidently $\xi(t)$ is positive and increasing on the interval $(0, 1)$. Moreover,

$$\xi(1) = \sup_{0 < s < 1} \psi_1(s) b(s) g^*(s) = \|g\|_{M_{\phi_1, b}} < \infty,$$

because $B_{\psi_1, b, E} \subseteq M_{\phi_1, b}$, and we may assume, without loss of generality, that $\xi(1) = 1$. At the second endpoint we have two possibilities: $\lim_{t \to 0} \xi(t) > 0$ or $\lim_{t \to 0} \xi(t) = 0$.

If $\lim_{t \to 0} \xi(t) > 0$ then $\chi_{(0, 1)}(t) \in \tilde{E}$, i.e., $\chi_{(0, \infty)}(u) \in E$, which means that $E$ contains all bounded functions on $(0, \infty)$. As was mentioned above (before Lemma 2.2), any parameter space $E$ (accurate to equivalence of norms) can be replaced by $E \cap L_{\infty}$, so in our case we may replace $E$ by $L_{\infty}$. Taking a function $f(t) = \frac{1}{\phi_1(t) b(t) \ln \xi}$, we obtain that it is almost decreasing, i.e., $f \sim f^*$ and

$$\|f\|_{L_{\phi_1, \ln \xi, E}} = \sup_{0 < s < 1} \frac{\phi_1(s) b(s) \ln s}{s} f^*(s) \sim 1.$$

Thus $f \in L_{\phi_1, \ln \xi, E}$ and $Tf \in G$. Moreover, we know that $m(t) \lesssim \tilde{T} \psi / \tilde{\rho}$ for all $t \in (0, 1)$, hence, for $\alpha = \tilde{T} \psi / \tilde{\rho}$,

$$Tf(t) \gtrsim \frac{1}{\psi_1(t)} \int_{a}^{b} \frac{1}{b(s) \ln \xi} ds \gtrsim \frac{1}{\psi_1(t) b(t)}$$

for all $t \in (0, \varepsilon^{-1})$ due to Lemma 3.4. Since for $t > \varepsilon^{-1}$ the function $\frac{1}{\psi_1(t) b(t)}$ is bounded, it belongs to $G$ on the whole interval $(0, 1)$. On the other hand, in the considered case $g \in B_{\psi_1, b, L_{\infty}}$, so

$$g^*(t) \lesssim \frac{1}{\psi_1(t) b(t)} \in G,$$

and the proof in the case when $\lim_{t \to 0} \xi(t) > 0$ is finished.

Next we consider only the case when $\lim_{t \to 0} \xi(t) = 0$ and define a sequence

$$a_0 = 1, \quad a_n = \max \{u : \xi(e^{-u}) \geq 2^{-n}\}, \quad n = 1, 2, \ldots.$$

The numbers $a_n$ are well-defined and even different for different $n$; moreover, they are increasing and tend to infinity. At last, $\xi(e^{-a_n}) = 2^{-n}$ since the function $\xi(t)$ is continuous.

Basing on the sequence $\{a_n\}$, we define the second function

$$\xi(u) = \begin{cases} 1, & \text{for } 0 < u < 1, \\ 2^{-n}, & \text{for } a_n \leq u \leq a_{n+1}, \quad n = 0, 1, \ldots. \end{cases}$$

It is easy to see that $\xi(e^{-u}) \leq \xi(u) \leq 2 \xi(e^{-u})$ for all $u \geq 1$, thus $\xi \in E$ and $\xi(\ln \xi) \gtrsim \xi(t)$ for all $t \in (0, 1)$. Observe also that $\xi(u)$ is decreasing and $\lim_{u \to \infty} \xi(u) = 0$.

Let now $\varepsilon = \tilde{T}_{\phi_1} / (\tilde{T} \phi_1 + 1)$. We set for every $n = 1, 2, \ldots$

$$\tau_n(u) = \begin{cases} 0, & \text{for } 0 < u \leq a_n, \\ 2^{-n+1} e^{-\varepsilon(a_n-a_u)}, & \text{for } u > a_n. \end{cases}$$

Evidently each $\tau_n(u)$ is continuous when $u \neq a_n$, and if $\tau_n(u_0) \geq \tau_m(u_0) > 0$ at some point $u_0$, then $\tau_n(u) \geq \tau_m(u)$ for all $u > u_0$. Moreover, for any $n = 1, 2, \ldots$

$$\|e^{-\varepsilon(a_n-a_u)} \chi_{(a_n, \infty)}(u)\|_E \leq \phi E(1) \cdot \max \{\|e^{-\varepsilon u}\|_{L_1}, \|e^{-\varepsilon u}\|_{L_{\infty}}\} = \frac{1}{\varepsilon} \phi E(1),$$

thus $\|\tau_n\| \leq 2^{-n+1} \phi E(1) / \varepsilon$. 

Evidently each $\tau_n(u)$ is continuous when $u \neq a_n$, and if $\tau_n(u_0) \geq \tau_m(u_0) > 0$ at some point $u_0$, then $\tau_n(u) \geq \tau_m(u)$ for all $u > u_0$. Moreover, for any $n = 1, 2, \ldots$
After all preliminaries, we proceed to definition of the main function being used in this proof:

\[ z(u) = \max\{\xi(u), \tau_1(u), \tau_2(u), \ldots\}. \]

For any fixed point \( u \), we have only a finite number of functions \( \tau_n(u) \neq 0 \), hence the maximum here always exists. Moreover, on any interval \((a_n, a_{n+1})\) only one of the following cases occurs:

(a) \( z(u) = \xi(u) \) for all \( u \in (a_n, a_{n+1}) \);
(b) \( z(u) = \tau_k(u) \) for some \( k \leq n \) and all \( u \in (a_n, a_{n+1}) \);
(c) there exists \( \epsilon_n \in (a_n, a_{n+1}) \) such that \( z(u) = \tau_k(u) \) for some \( k \leq n \) when \( u \in (a_n, \epsilon_n) \) and \( z(u) = \xi(u) \) for all \( u \in (\epsilon_n, a_{n+1}) \);

at the point \( \epsilon_n \) itself the “left” and the “right” functions always are equal.

The examination of all possible transitions at the boundary points \( a_n, n = 1, 2, \ldots \), shows that the function \( z(u) \) is continuous and decreasing. Moreover, it is differentiable at any point \( u \) different from \( a_n, \epsilon_n \), i.e. almost everywhere on \((0, \infty)\). Furthermore,

\[
\|z\|_E \leq \|\xi\|_E + \sum_{n=1}^{\infty} \|\tau_n\|_E \leq \|\xi\|_E + \frac{2}{\epsilon} \varphi_E(1) < \infty,
\]

that is, \( z \in E \). Notice also that \( z(\ln \frac{1}{t}) \geq z(\ln \frac{\epsilon}{t}) \geq \xi(t) \) for all \( t \in (0, 1) \).

The function \( z(u) \) enables construction of a function \( f(t) \) suitable for applying the operator \( T \):

\[
f(t) = \frac{1}{\phi_1(t) b(t) \ln \frac{\xi}{t}} z \left( \ln \frac{1}{t} \right), \quad 0 < t < 1.
\]

Let us show that \( f \in L_{\phi_1, b \ln \frac{\xi}{t}, E} \).

Since \( \pi_B > 0 \), arguing as in Lemma 2.1 and passing (if necessary) to an equivalent function, we can assure existence of a number \( \tilde{\pi}_B \leq \pi_B \) such that

\[
0 < \tilde{\pi}_B \leq \frac{\mu B'(u)}{B(u)}, \quad \text{for all } u > 0.
\]

Then we take \( c = \min(e^{-1}, e^{(\tilde{\pi}_B - 1)/\tilde{\pi}_B}) \) and decompose the function \( f(t) \) into the sum \( f(t) = f_1(t) + f_2(t) \), where

\[
f_1(t) = f(t) \chi_{(0, c)}(t) \quad \text{and} \quad f_2(t) = f(t) \chi_{(c, 1)}(t).
\]

It is easy to see that \( f_2(t) \) is bounded on \((0, 1)\) and thus belongs to any r.i. space on this interval. In order to prove that the function \( f_1 \) belongs to \( L_{\phi_1, b \ln \frac{\xi}{t}, E} \), let us show that it is decreasing, using its differentiability a.e. on \((0, 1)\).

For every \( t < c \) where \( z'(\ln \frac{1}{t}) \) exists, we have that

\[
f'_1(t) = \frac{\phi'_1(t)}{\phi_1(t)} \cdot \frac{1}{\phi_1(t) b(t) \ln \frac{\xi}{t}} z \left( \ln \frac{1}{t} \right) - \frac{b'(t)}{b(t)} \cdot \frac{1}{\phi_1(t) b(t) \ln \frac{\xi}{t}} z \left( \ln \frac{1}{t} \right) + \frac{1}{t \ln \frac{\xi}{t}} \cdot \frac{1}{\phi_1(t) b(t) \ln \frac{\xi}{t}} z \left( \ln \frac{1}{t} \right)
\]

\[
= -\frac{1}{t \phi_1(t) b(t) \ln \frac{\xi}{t}} \left[ \phi'_1(t) \ln \frac{1}{t} + t b'(t) - \frac{1}{\ln \frac{\xi}{t}} \right] z \left( \ln \frac{1}{t} \right) + z' \left( \ln \frac{1}{t} \right).
\]

Since \( b(t) = B(1/\ln \frac{\xi}{t}) \), we obtain that

\[
\frac{t b'(t)}{b(t)} = \frac{1}{\ln \frac{\xi}{t}} \cdot \frac{1/\ln \frac{\xi}{t} \cdot B'(1/\ln \frac{\xi}{t})}{B(1/\ln \frac{\xi}{t})} \geq \frac{\tilde{\pi}_B}{\ln \frac{\xi}{t}},
\]

and hence

\[
f'_1(t) \leq -\frac{1}{t \phi_1(t) b(t) \ln \frac{\xi}{t}} \left[ \left( \tilde{\pi}_B + \frac{\tilde{\pi}_B - 1}{\ln \frac{\xi}{t}} \right) z \left( \ln \frac{1}{t} \right) + z' \left( \ln \frac{1}{t} \right) \right].
\]
Let us define the function $R(u) = (\pi_1 + \frac{\pi_B - 1}{1 + u})z(u) + z'(u)$ for $u > \ln \frac{1}{e} = \max(1, (1 - \pi_B)/\pi_1)$. Then in order to prove that $f_1$ is decreasing, we get for all $t \in (0, 1)$ at each point $u \in (\ln \frac{1}{e}, \infty)$ where $z'(u)$ exists.

In some neighborhood of each point of its differentiability, the function $z(u)$ either equals $\xi(u)$ and thus is constant or has a form $z(u) = ke^{-e^u}$ for some $k > 0$. In the first occurrence $z'(u) = 0$, hence if $\pi_B \geq 1$

$$R(u) = \left(\pi_1 + \frac{\pi_B - 1}{1 + u}\right)z(u) > 0.$$ 

If $0 < \pi_B < 1$, we use the condition $u > (1 - \pi_B)/\pi_1$, which gives that

$$R(u) > \left(\pi_1 + \frac{(1 - \pi_B)\pi_1}{\pi_1 + 1 - \pi_B}\right)z(u) = \frac{\pi_1^2}{\pi_1 + 1 - \pi_B}z(u) > 0.$$ 

In the second occurrence $z'(u) = -e^u$, so, for $\pi_B \geq 1$,

$$R(u) = \left(\pi_1 + \frac{\pi_B - 1}{1 + u} - e\right)z(u) = \left(\pi_1 + \frac{\pi_B - 1}{1 + u} - \frac{\pi_1^2}{\pi_1 + 1}\right)z(u) \geq \left(\pi_1 - \frac{\pi_1^2}{\pi_1 + 1}\right)z(u) = \frac{\pi_1}{\pi_1 + 1}z(u) > 0,$$

while for $0 < \pi_B < 1$

$$R(u) = \left(\pi_1 + \frac{\pi_B - 1}{1 + u} - \frac{\pi_1^2}{\pi_1 + 1}\right)z(u) > \left(\frac{\pi_1^2}{\pi_1 + 1 - \pi_B} - \frac{\pi_1^2}{\pi_1 + 1}\right)z(u) > 0.$$ 

So we have proved the monotonicity of the function $f_1(t)$, which gives that $f_1''(t) = f_1(t)$ a.e. Hence

$$\|f_1\|_{L_{\psi_1,\ln \xi,E}} = \left\|\phi_1(t)b(t)\ln \frac{e}{t}f_1''(t)\right\|_E = \left\|z\left(\ln \frac{1}{t}\right)\chi_{(0,e^{-1})}(t)\right\|_E \leq \|z(u)\|_E < \infty,$$

and we get that $f \in L_{\psi_1,\ln \xi,E}$. Consequently, $Tf \in G$.

Next we are going to show that $Tf(t) \geq \frac{1}{\psi_1(t)b(t)}\xi(t)$ for every $t \in (0, e^{-1})$. We assume first that $\psi_1/\rho_\Phi \geq 1$, then

$$m(t) \leq t\psi_1/\rho_\Phi \leq t < t$$

and thus

$$Tf(t) \geq \frac{1}{\psi_1(t)b(t)}z\left(\ln \frac{1}{t}\right)\int t \frac{1}{b(s)\ln \frac{e}{s}}z\left(\ln \frac{1}{s}\right)ds.$$ 

Applying Lemma 3.4 with $\alpha = 1$ and the fact that the function $z(\ln \frac{1}{s})$ is increasing, we obtain that

$$Tf(t) \geq \frac{1}{\psi_1(t)b(t)}z\left(\ln \frac{1}{t}\right)$$

for all $t \in (0, e^{-1})$. Furthermore, $z(\ln \frac{1}{t}) \geq \xi(t)$, hence $Tf(t) \geq \frac{1}{\psi_1(t)b(t)}\xi(t)$ for all $t \in (0, e^{-1})$.

Alternatively, let $\alpha := \psi_1/\rho_\Phi < 1$. Since $m(t) \leq t^\alpha$, using again Lemma 3.4 and taking in account that $z(u)$ is a decreasing function, we get

$$Tf(t) \geq \frac{1}{\psi_1(t)b(t)}z\left(\ln \frac{1}{t}\right)\int t^\alpha \frac{1}{b(s)\ln \frac{e}{s}}z\left(\ln \frac{1}{s}\right)ds \geq \frac{1}{\psi_1(t)b(t)}z\left(\ln \frac{1}{t}\right)\int t^\alpha \frac{1}{b(s)\ln \frac{e}{s}}ds \geq \frac{1}{\psi_1(t)b(t)}z\left(\ln \frac{1}{t}\right)\xi(t)$$

for all $t \in (0, e^{-1})$ as before. Hence $\frac{1}{\psi_1(t)b(t)}\xi(t)\chi_{(0,e^{-1})} \in G$ for any value of $\psi_1/\rho_\Phi$. 

On the other hand, for all \( t \in (e^{-1}, 1) \), the function \( \frac{1}{\phi(t)} \zeta(t) \) is bounded, thus it belongs to \( G \) on the whole interval \((0, 1)\). But the definition of function \( \zeta(t) \) implies that
\[
\frac{1}{\psi_1(t)} \zeta(t) \geq g^*(t),
\]
thus \( g^* \in G \) and the proof of the theorem is complete. \( \square \)

Optimality of \( A \) and \( B \) type spaces with respect to ultrasymmetric spaces can be extended to their mutual optimality if to take in account the embeddings
\[
L_{\phi_1,b} \subseteq A_{\phi_1,b,E}, \quad B_{\psi_1,b} \subseteq L_{\psi_1,b,E}.
\]

**Corollary 3.5.** Under conditions of Theorem 3.2 the space \( B_{\psi_1,b,E} \) is the orbit space of \( A_{\phi_1,b,E} \) and the space \( A_{\phi_1,b,E} \) is the co-orbit space of \( B_{\psi_1,b,E} \), i.e., the interpolation triples \((A_{\phi_0}, A_{\phi_1}; A_{\phi_1,b,E})\) and \((M_{\psi_0}, M_{\psi_1}; B_{\psi_1,b,E})\) are optimal for each other.

**Remark.** Recall that in theory of real interpolation, for a given Banach couple \((A_0, A_1)\), the notation \((A_0, A_1)^E_E\) with some Banach function space \( E \) means the space of all \( f \in A_0 + A_1 \) such that
\[
K(u) = K(u, f, A_0, A_1) = \| f \|_{A_0 + u A_1} \in E.
\]
In the course of studying \( A \) and \( B \) type spaces in [10] we have proved two embeddings
\[
A_{\phi_1,b,E} \subseteq (A_{\phi_1}, A_{\phi_1,b,L_{\infty}})^K_{D_{\psi_1}} \quad (M_{\psi_1}, M_{\psi_1,b})^K_{D_{\psi_1}} \subseteq B_{\psi_1,b,E}.
\]
principal for stating all interpolation properties of spaces \( A_{\phi_1,b,E} \) and \( B_{\psi_1,b,E} \). Here \( D'' \) is a function space, constructed by a special manner for a given space \( E \), that we do not specify here. Using Theorem 3.2, we may now assert that both these embeddings in fact are equalities
\[
A_{\phi_1,b,E} = (A_{\phi_1}, A_{\phi_1,b,L_{\infty}})^K_{D_{\psi_1}} \quad (M_{\psi_1}, M_{\psi_1,b})^K_{D_{\psi_1}} = B_{\psi_1,b,E}.
\]
Indeed, any strict embedding above contradicts to optimality of \( A \) and \( B \) spaces in corresponding interpolation.

4. Optimal interpolation near the left endpoint

As was already mentioned above, a direct analytical description of weak interpolation near the left endpoint is rather difficult. Fortunately, we may transform to the left-hand side of interpolation all result already stated for the right-hand spaces, since weak interpolation is, in some sense, “self-dual” due to mutual duality of Lorentz and Marcinkiewicz spaces.

Recall that, for any Banach function space \( G \), there exists an associate (Köthe dual) space \( G' \) with the norm
\[
\| f \|_{G'} = \sup \left\{ \langle f, g \rangle \mid \int_0^1 f(t)g(t) \, dt: \| g \|_G \leq 1 \right\}.
\]
This space coincides with the conjugate space \( G^* \) if and only if the space \( G \) is separable, which is equivalent to density of the set of bounded measurable functions in \( G \). This property justifies the preference of associate spaces to conjugate ones, since the last can be not function spaces. We have also that \( G'' = G \) if and only if the space \( G \) has the so-called Fatou property, that is, for any sequence \( f_n \in E, n = 1, 2, \ldots \), which converges almost everywhere to a measurable function \( f \) and is bounded in \( E \), one has that \( f \in E \) and \( \| f \|_E \leq \lim \inf \| f_n \|_E \). Notice that the Fatou property is included into the definition of r.i. spaces given in [2].

The relations \((A_0') = M_\phi^* \) and \((M_\psi') = A_\phi^* \) with \( \phi(t) = t/\phi(t) \) allow us to consider the set of operators \( W(\psi_1, \phi_1; \psi_0, \phi_0) \) as a dual to the set \( W(\phi_0, \psi_0; \phi_1, \psi_1) \) and to state the following assertion.

**Lemma 4.1.** Let \( \phi_0, \psi_0, \phi_1, \psi_1 \) be quasi-power functions with \( \pi_{\phi_0/\phi_1}, \pi_{\psi_0/\psi_1} > 0 \). Let \( B \) be a r.i. space with the Fatou property. Then the space \( A \) is the upper optimal interpolation space of \( B \) and/or the space \( B \) is the lower optimal
interpolation space of $A$ for all operators from $W(\phi_0, \psi_0; \phi_1, \psi_1)$ if and only if the space $A'$ is the lower optimal interpolation space of $B'$ and/or the space $B'$ is the upper optimal interpolation space of $A'$ for all operators from $W(\tilde{\psi}_1, \tilde{\phi}_1; \tilde{\psi}_0, \tilde{\phi}_0)$.

**Proof.** As follows from some results of [2, Section 4.7] (see also [7]), an operator $T \in W(\phi_0, \psi_0; \phi_1, \psi_1)$ if and only if $(Tf^*)'(t) \lesssim S(f^*)'(t)$ for all $t \in (0, 1)$ and any $f \in A_{\phi_0} + A_{\phi_1}$, where the operator

$$Sf(t) = \frac{1}{\psi_0(t)} \int_0^{m(t)} \phi_0(s) f(s) \frac{ds}{s} + \frac{1}{\psi_1(t)} \int_{m(t)}^1 \phi_1(s) f(s) \frac{ds}{s},$$

is usually called [Calderón maximal operator](here $m(t)$ is the same function as in Lemma 3.3). Defining its dual operator $S'$ by the relation $(Sf, g) = (f, S'g)$, it is easy to find that

$$S'g(t) = \frac{1}{\phi_1(t)} \int_0^{\delta(t)} \tilde{\psi}_1(s) f(s) \frac{ds}{s} + \frac{1}{\phi_0(t)} \int_{\delta(t)}^1 \tilde{\psi}_0(s) f(s) \frac{ds}{s},$$

where $\delta(t)$ is the inverse function for $m(t)$. We see that $S'$ is exactly the Calderón maximal operator for $W(\tilde{\psi}_1, \tilde{\phi}_1; \tilde{\psi}_0, \tilde{\phi}_0)$, so any its action implies analogous action for all linear operators from this set. But duality between the operators $S$ and $S'$ and the Fatou property of the space $B$ imply (see, e.g., [8]) that $S$ is bounded from $A$ into $B$ if and only if $S'$ is bounded from $B'$ into $A'$, and this finishes the proof. \[\Box\]

Comparing operator $S$ with operator (3.1), we reveal that $T$ coincides with the second summand of $S$, that is, for interpolation near the right endpoint, only this summand should be considered as maximal operator. By duality we obtain that, for interpolation near the left endpoint, it is enough to consider the first summand of the operator $S'$, namely, the operator

$$T'f(t) = \frac{1}{\phi_1(t)} \int_0^{\delta(t)} \tilde{\psi}_1(s) f(s) \frac{ds}{s},$$

which in fact is dual to $T$. Let us formulate the optimal properties of $T'$ following from Theorem 3.2.

**Lemma 4.2.** Let $\phi_0, \psi_0, \phi_1, \psi_1$ be quasi-power functions with $\pi_{\phi_0/\phi_1}, \pi_{\psi_0/\psi_1} > 0$. Let $E$ be any r.i. space such that $b(e^{-u}) \in E$ if $\pi_E > 1$ or $\rho_E < \pi_E$ if $0 < \pi_E \leq 1$ and let $T'$ be the operator (4.1). Then the following assertions hold:

1. If $T': G \to L_{\phi_1, \psi_1}^{\frac{1}{\ln \gamma}} E'$ for some r.i. space $G$, then $G \subset (B_{\psi_1, b}, E)'$.
2. If $T': L_{\tilde{\psi}_1, \tilde{b}, E'} \to D$ for some r.i. space $D$ with Fatou property, then $D \supset (A_{\phi_1, b}, E')'$.

**Proof.** We need the main result on duality of ultrasymmetric spaces from [7], namely, that $(L_{\phi, E})' = L_{\phi, E}$.* In particular, $(L_{\phi_1, \bar{\psi}_1}^{\frac{1}{\ln \gamma}} E') = L_{\phi_1, \bar{\psi}_1}^{\frac{1}{\ln \gamma}} E'$. Thus if $T': G \to L_{\phi_1, \bar{\psi}_1}^{\frac{1}{\ln \gamma}} E'$, then by duality we obtain that $T: L_{\phi_1, \bar{\psi}_1}^{\frac{1}{\ln \gamma}} E' \to G'$ (since $E \subset E'$). By Theorem 3.2 this implies that $G' \supset B_{\psi_1, b, E}$ and the next duality gives that $G'' \subset (B_{\psi_1, b, E})'$. Since always $G \subset G''$, we get the desired embedding. Analogously in the second case, applying Theorem 3.2 and duality, we get that $D'' \supset (A_{\phi_1, b, E'})'$, and we again may pass to $D$, since now $D = D''$. \[\Box\]

Having so powerful tool as Lemma 4.1, we can now proceed to studying optimal weak interpolation near the left endpoint. The one thing we need for this is to find analytical description of spaces dual to the spaces of $A$ and $B$ types. We shall do this, using the operator (4.1) and his properties stated in Lemma 4.2. In this connection it is important to observe that the definition of operator (4.1) includes all parameter-functions $\phi_0, \psi_0, \phi_1, \psi_1$, while the norm of spaces $A_{\phi_1, b, E}$ or $B_{\psi_1, b, E}$ contains only one of these functions. This means that we are free in choice of other three functions and may exploit this opportunity for obtaining additional properties of operator $T'$, needed for the proof.
**Theorem 4.3.** Let $\psi_1$ be a quasi-power function and let $E$ be any r.i. space such that $b(e^{-u}) \in E$ if $\pi_B > 1$ or $\rho_E < \pi_B$ if $0 < \pi_B \leq 1$. Then $(B_{\psi_1, b, E})^\prime = G$, where $G$ is a r.i. space with the norm

$$
\| f \|_G \sim \left\| \frac{1}{b(t) \ln \frac{\varepsilon}{t}} \int_0^t \widetilde{\psi}_1 (s) f^*(s) \frac{ds}{s} \right\|_{E^\prime}.
$$

**Proof.** Let us take two positive numbers $a, b$ such that $\frac{1}{a} < \frac{1}{b} < 1 - \rho_{\psi_1}$ and define the functions

$$
\psi_0(t) = t^{1 - \frac{1}{b}}, \quad \phi_0(t) = t^{1 - \frac{1}{a}}, \quad \phi_1(t) = t^{1 - \frac{1}{a}} \psi_1 \left( t^{\frac{b}{a}} \right).
$$

By elementary calculations it is easy to verify that all these functions are quasi-power and $\pi_{\phi_0}/\phi_1, \pi_{\psi_0}/\psi_1 > 0$. In addition, $\Phi(t) = \Psi \left( t^{b/a} \right)$, giving $\delta(t) = t^{b/a}$ and $m(t) = t^{a/b}$. Using all these data, we can construct by the formula (4.1) the operator $T'$ which will be used in the following proof.

Let us prove first that $T' : G \to L_{\phi_1, \frac{1}{b^{1/a}}, E}$. The well-known properties of function rearrangements (see, e.g., [5]) imply that $|T' f(t)| \leq T' f^*(t)$ for any $f \in G$. Moreover, the function $T' f^*(t)$ is almost decreasing. Indeed, by conditions of the theorem, the function $\psi_1$ is quasi-power and thus

$$
\widetilde{\phi}_1(t) = \widetilde{\psi}_1 \left( t^{\frac{b}{a}} \right) = \int_0^{t^{\frac{b}{a}}} \widetilde{\psi}_1(s) \frac{ds}{s}.
$$

Therefore

$$
T' f^*(t) \sim \left\| \frac{1}{b(t) \ln \frac{\varepsilon}{t}} \int_0^{t^{\frac{b}{a}}} \widetilde{\psi}_1(s) f^*(s) \frac{ds}{s} \right\|_{E^\prime},
$$

and the right-hand side of this expression decreases as an integral mean of the decreasing function $f^*$ with respect to a positive measure $d\mu(s) = \widetilde{\psi}_1(s) \frac{ds}{s}$ on the increasing interval $(0, t^{b/a})$. In result, $(T' f^*)^\sim \sim T' f^*$ and

$$
\| T' f^* \|_{L_{\phi_1, \frac{1}{b^{1/a}}, E}} \sim \left\| \frac{1}{b(t) \ln \frac{\varepsilon}{t}} \int_0^{t^{\frac{b}{a}}} \widetilde{\psi}_1(s) f^*(s) \frac{ds}{s} \right\|_{E^\prime} \sim \left\| \frac{1}{b(t) \ln \frac{\varepsilon}{t}} \int_0^t \widetilde{\psi}_1(s) f^*(s) \frac{ds}{s} \right\|_{E^\prime} = \| f \|_G,
$$

(4.2)
giving $\| T' f \|_{L_{\phi_1, \frac{1}{b^{1/a}}, E}} \leq \| f \|_G$, as desired. Using Lemma 4.2, we get from this the embedding $G \subset (B_{\psi_1, b, E})^\prime$.

On the other hand, by Theorem 3.2(i) we have that $T : L_{\phi_1, \frac{1}{b^{1/a}}, E} \to B_{\psi_1, b, E}$, which implies that $T' : (B_{\psi_1, b, E})^\prime \to L_{\phi_1, \frac{1}{b^{1/a}}, E^\prime}$. Now, using (4.2), we obtain that

$$
\| f \|_G \sim \| T' f^* \|_{L_{\phi_1, \frac{1}{b^{1/a}}, E}} \sim \| f^* \|_{(B_{\psi_1, b, E})^\prime} = \| f \|_{(B_{\psi_1, b, E})^\prime}
$$

and the inverse embedding $G \supset (B_{\psi_1, b, E})^\prime$ is also proved. \(\square\)

Next we shall study duality for a given space $A_{\phi_1, b, E}$. As before we may choose other parameter functions $\phi_0, \psi_0, \psi_1$ arbitrarily.

**Theorem 4.4.** Let $\phi_1$ be a quasi-power function and let $E$ be any r.i. space with the Fatou property and such that $b(e^{-u}) \in E$ if $\pi_B > 1$ or $\rho_E < \pi_B$ if $0 < \pi_B \leq 1$. Then $(A_{\phi_1, b, E})^\prime = D$, where $D$ is a r.i. space with the quasinorm

$$
\| f \|_D \sim \left\| \frac{1}{b(t) \ln \frac{\varepsilon}{t}} \sup_{0 < s < t} \widetilde{\phi}_1(s) f^*(s) \right\|_{E^\prime}.
$$

**Proof.** Let us prove first the embedding $D \supset (A_{\phi_1, b, E})^\prime$. We take two numbers $a, b$ such that

$$
a > b > 1, \quad 1 - \frac{b}{a} < \pi_{\phi_1} \leq \rho_{\phi_1} < 1 - \frac{1}{a}
$$

Then
and define the functions
\[ \phi_0(t) = t^{1-1/\alpha}, \quad \psi_0(t) = t^{1-1/b}, \quad \psi_1(t) = t^{1-\alpha/b} \phi_1(t^{\alpha/b}). \]

As in the previous theorem, we can verify that these functions are quasi-power and \( \pi \phi_0/\phi_1, \pi \psi_0/\psi_1 > 0 \). We also obtain once again that \( \delta(t) = t^{b/\alpha} \) and \( m(t) = t^{\mu/b} \).

Taking all aforementioned functions, we define the operator (4.1) (which turns out to be exactly the same as in Theorem 4.3) and show that
\[ T' : L_{\tilde{\psi},1,\frac{1}{b},E} \to D. \]
Indeed, using as before the relations \(|T' f(t)| \leq T' f^*(t) \sim (T' f^*(t))^*\), we obtain that for any admissible \( f \)
\[
\|T' f\|_D \leq \|T' f^*\|_D \sim \frac{1}{b(t) \ln \frac{\zeta(t)}{\tilde{\psi}_1(t)}} \sup_{0 < s \leq t} \int_0^{b/a} \tilde{\psi}_1(u) f^*(u) \frac{du}{u} = \frac{1}{b(t) \ln \frac{\zeta(t)}{\tilde{\psi}_1(t)}} \int_0^{b/a} \tilde{\psi}_1(u) f^*(u) \frac{du}{u} = \|T' f^*\|_{L_{\tilde{\psi},1,\frac{1}{b},E}}.
\]

Moreover, from Theorem 3.2(ii) we have that \( T : L_{\pi \phi_1,\frac{1}{b},\ln \frac{\zeta}{\tilde{\psi}_1},E} \to \psi_1, b, E \) and thus \( T' : L_{\tilde{\psi},1,\frac{1}{b},E} \to L_{\tilde{\psi},1,\frac{1}{b},\ln \frac{\zeta}{\tilde{\psi}_1},E} \). Consequentially,
\[ \|T' f\|_D \leq \|T' f^*\|_{L_{\tilde{\psi},1,\frac{1}{b},\ln \frac{\zeta}{\tilde{\psi}_1},E}} \lesssim \|f\|_{L_{\tilde{\psi},1,\frac{1}{b},E}} , \]
which means that \( T' : L_{\tilde{\psi},1,\frac{1}{b},E} \to D. \) But the space \( D \) always has the Fatou property that follows evidently from the analogous embedding of the parameter space \( E' \), thus by Lemma 4.2 we obtain the embedding \( D \supset (A_{\phi, b, E'})' = (A_{\phi, b, E})' \).

Let us proceed to proving the inverse embedding \( D \subset (A_{\phi, b, E})' \), constructing another operator (4.1). We take two numbers \( \sigma, \tau \in (0, 1) \) such that \( \sigma < \pi \phi_1, \sigma \tau < 1 - \rho \phi_1 \), and define the functions
\[ \phi_0(t) = \phi_1(t) t^{\sigma \tau}, \quad \psi_0(t) = \phi_1(t), \quad \psi_1(t) = \phi_1(t) t^{-\sigma} \]
which again are quasi-power with \( \pi \phi_0/\phi_1, \pi \psi_0/\psi_1 > 0 \). Observing that \( \phi_0(t)/\phi_1(t) = \psi_0(t^\tau)/\psi_1(t^\tau) \), we get immediately that \( \delta(t) = t^\tau \) and \( m(t) = t^{1-\tau} \).

For an arbitrary given function \( g \in D \), we define the function
\[ \zeta(t) = \sup_{0 < s < t} \tilde{\phi}_1(s) g^*(s), \quad 0 < t < 1. \]

This function is increasing and \( \frac{1}{b(t) \ln \zeta(t)} \zeta(t) \in \tilde{E}' \) since \( \|g\|_D = \|\frac{1}{b(t) \ln \zeta(t)} \zeta(t)\|_{\tilde{E}} < \infty \). Let us consider the function
\[ f(t) = \frac{1}{\tilde{\psi}_1(t) \ln \zeta(t)} (t^{\zeta(t)} - \tilde{\phi}_1(t)) \] and prove that \( f \in L_{\tilde{\psi}_1,1,\frac{1}{b},E} \).

In order to do this we define a number \( c = e^{1-1/\alpha} \) and represent \( f \) in the form \( f(t) = f_1(t) + f_2(t) \), where \( f_1(t) = f(t) \chi_{(0,c)}(t) \) and \( f_2(t) = f(t) \chi_{(c,1)}(t) \). The function \( f_2 \) is bounded on \((0, 1)\) and thus belongs to any r.i. space. So we may investigate only the summand \( f_1(t) \), showing at first that it is decreasing.

Decomposing \( f_1 \) as a product
\[ f_1(t) = \frac{1}{t^{\sigma} \ln \zeta(t)} \chi_{(0,c)}(t) \cdot \frac{\phi_1(t) \zeta(t)}{t}, \]
we can check each factor separately. The behaviour of the first factor can be easily verified via its derivative. A little more complicated is to check the second factor which is convenient to write as \( \zeta(t)/\tilde{\phi}_1(t) \).

Note that both functions \( \zeta(t) \) and \( \tilde{\phi}_1(t) \) are increasing, so we have to compare the rates of their growth. Let us take two numbers \( t_1 < t_2 \) from the interval \((0, 1)\) and show that \( \zeta(t_2)/\zeta(t_1) \leq \tilde{\phi}_1(t_2)/\tilde{\phi}_1(t_1) \). If \( \zeta(t_1) = \zeta(t_2) \), this inequality is obvious. So let \( \zeta(t_2) > \zeta(t_1) \). Note that the function \( \tilde{\phi}_1(t) \) may be supposed continuous and \( g^*(t) \) is left-continuous, thus the supremum in definition of \( \zeta(t) \) is always attained at some point, e.g., \( \zeta(t_2) = \tilde{\phi}_1(t_0) g^*(t_0) \), where \( t_1 < t_0 \leq t_2 \).

This implies that \( \zeta(t_2) \leq \tilde{\phi}_1(t_2) g^*(t_1) \). At the same time, \( \zeta(t_1) \geq \tilde{\phi}_1(t_1) g^*(t_1) \), and we are done.

In result \( f_1(t) \) is a decreasing function, i.e., \( f_1^* = f_1 \), and we can calculate its norm
\[ \|f_1\|_{L_{\tilde{\psi}_1,1,\frac{1}{b},E}} = \|\tilde{\psi}_1(t) \frac{1}{b(t)} f_1(t)\|_{\tilde{E}} = \left\| \frac{1}{b(t) \ln \zeta(t)} \chi_{(0,c,1-\sigma)}(t) \right\|_{\tilde{E}} < \infty. \]
Hence \( f = f_1 + f_2 \in L^\infty_{\psi_1, b, E'} \). Moreover, since \( 0 < \tau < 1 \) and \( \zeta(t) \) is an increasing function, we have that

\[
T' f(t) = \frac{1}{\phi_1(t)} \int_0^t \psi_1(s) f(s) \frac{ds}{s} \geq \frac{1}{\phi_1(t)} \int_0^t \frac{1}{\ln e} \zeta(s) \frac{ds}{s} \geq \frac{\zeta(t)}{\phi_1(t)} \int_0^t \frac{1}{\ln e} \frac{ds}{s} = \frac{\zeta(t)}{\phi_1(t)} \ln \left( \frac{1 + \ln \frac{1}{\tau}}{1 + \tau \ln \frac{1}{\tau}} \right)
\]

and thus

\[
T' f(t) \geq k \frac{\zeta(t)}{\phi_1(t)} \quad \text{for all } t \leq e^{-1/\tau} \quad \text{and } k = \ln \left( \frac{1 + 1/\tau}{2} \right) > 0.
\] (4.3)

We know from Theorem 3.2 that \( T : A_{\phi_1, b, E} \to L_{\psi_1, b, E} \), hence by duality \( T' : L_{\psi_1, b, E'} \to (A_{\phi_1, b, E})' \). Since \( f \in L_{\psi_1, b, E'} \), this implies that \( T' f \in (A_{\phi_1, b, E})' \) and by (4.3) we obtain that also \( \frac{\zeta(t)}{\phi_1(t)} \in (A_{\phi_1, b, E})' \). On the other hand, for \( t > e^{-1/\tau} \), the function \( \frac{\zeta(t)}{\phi_1(t)} \) is bounded, thus it belongs to \( (A_{\phi_1, b, E})' \) for all \( t \in (0, 1) \). But the definition of function \( \zeta(t) \) implies that \( g^*(t) \leq \frac{\zeta(t)}{\phi_1(t)} \) and thus \( g \in (A_{\phi_1, b, E})' \). This gives the desired inverse embedding \( D \subset (A_{\phi_1, b, E})' \) and finishes the proof. \( \square \)

Theorem 4.3 and 4.4 define two new types of r.i. spaces that will be called spaces of \( G \) and \( D \) types, respectively. Due to Lemma 4.1 they play the same role for the left endpoint of weak interpolation as the spaces of \( A \) and \( B \) types play for the right endpoint. In order to make the definition of new spaces independent of duality, we introduce new functions, namely, \( g(t) = \frac{1}{b(t) \ln t} \) and \( G(u) = \frac{u}{\Phi(u)} \), obtaining \( \rho_G = 1 - \pi_B \). In fact, \( g(t) \) is an arbitrary positive function such that \( g(t^2) \sim g(t) \); this function may increase but becomes always decreasing after multiplication by \( \ln \frac{e}{t} \). Also the function \( G(u) \) may increase but slower than \( u \).

The main properties of \( D \) and \( G \) spaces are similar to those of spaces \( A \) and \( B \) and can be obtained via duality. Let us collect them in two propositions, assuming always that the parameter space \( E \) has the Fatou property.

**Proposition 4.5.** For a given quasi-power function \( \phi_0 \), let \( G = G_{\phi_0, b, E} \) be the set of all measurable functions \( f : (0, 1) \mapsto \mathbb{R} \) with the finite quasinorm

\[
\| f \|_{G_{\phi_0, b, E}} = \left\| g(t) \int_0^t \phi_0(s) f^*(s) \frac{ds}{s} \right\|_E
\] (4.4)

where \( \pi_E > \rho_G \) if \( 0 \leq \rho_G < 1 \) or \( \frac{1}{(u+1)^{\rho_G}} \in E' \) if \( \rho_G < 0 \). Then

(i) \( G_{\phi_0, b, E} \) is a r.i. space with the norm equivalent to the quasinorm (4.4); this norm can be obtained via replacement of \( f^* \) in (4.4) by \( f^{**} \).

(ii) This space has the Boyd indices \( \pi_G = \pi_{\phi_0}, \rho_G = \rho_{\phi_0} \) and the fundamental function

\[
\varphi_G(\lambda) \sim \phi_0(\lambda) g(\lambda) \varphi_E \left( \ln \frac{e}{\lambda} \right).
\]

(iii) The embedding \( L_{\phi_0, b, \ln \frac{e}{t}, E} \subset G_{\phi_0, b, E} \) holds for any \( E \) and is strict whenever \( \lim_{t \to \infty} \varphi(t)/t = 0 \).

**Proposition 4.6.** For a given quasi-power function \( \psi_0 \), let \( D = D_{\psi_0, b, E} \) be the set of all measurable functions \( f : (0, 1) \mapsto \mathbb{R} \) with the finite quasinorm

\[
\| f \|_{D_{\psi_0, b, E}} = \left\| g(t) \sup_{0 < s < t} \psi_0(t) f^*(s) \right\|_E
\] (4.5)

where \( \pi_E > \rho_G \) if \( 0 \leq \rho_G < 1 \) or \( \frac{1}{(u+1)^{\rho_G}} \in E' \) if \( \rho_G < 0 \). Then

(i) \( D_{\psi_0, b, E} \) is a r.i. space with the norm equivalent to the quasinorm (4.5); this norm can be obtained via replacement of \( f^* \) in (4.5) by \( f^{**} \).
(ii) This space has the Boyd indices $\pi_D = \pi_{\psi_0}$, $\rho_D = \rho_{\psi_0}$ and the fundamental function

$$\varphi_D(\lambda) \sim \psi_0(\lambda) g(\lambda) \varphi_E \left( \frac{e \lambda}{\ln e \lambda} \right).$$

(iii) The embeddings $L_{\psi_0, g \ln e^t} \subset D_{\psi_0, g} \subset L_{\psi_0, g}$ hold for any $E$ and are strict whenever $\lim_{t \to \infty} \varphi_E(t) = \infty$.

Now let us finish the paper with the theorem on optimal interpolation near the left endpoint of weak interpolation.

**Theorem 4.7.** Let $\phi_0, \psi_0, \phi_1, \psi_1$ be quasi-power functions with positive extension indices $\pi_{\phi_0/\phi_1}, \pi_{\psi_0/\psi_1}$. Let $E$ be any r.i. space with the Fatou property and such that $\pi_E > \rho_G$ if $0 \leq \rho_G < 1$ or $\frac{1}{(u+1)g(e^{-u})} \in E'$ if $\rho_G < 0$. Then, for any linear operator $T \in W(\phi_0, \psi_0; \phi_1, \psi_1)$, one has that

(i) $T : G_{\phi_0, g} \to D_{\psi_0, g}$,

(ii) $T : G_{\phi_0, g} \to L_{\psi_0, g}$,

(iii) $T : L_{\phi_0, g \ln e^t} \to D_{\psi_0, g}$,

(iv) $T : L_{\phi_0, g \ln e^t} \to L_{\psi_0, g}$.

Moreover, $G_{\phi_0, g}$ is the upper optimal (co-orbit) space for $L_{\psi_0, g}$ and $D_{\psi_0, g}$ is the lower optimal (orbit) space for $L_{\phi_0, g \ln e^t}$.

By analogy with Corollary 3.5 we obtain

**Corollary 4.8.** Under conditions of Theorem 4.7 the space $D_{\psi_0, g}$ is the orbit space of $G_{\phi_0, g}$ and the space $G_{\phi_0, g}$ is the co-orbit space of $D_{\psi_0, g}$, i.e., the interpolation triples $(\Lambda_{\phi_0}, \Lambda_{\psi_1}; G_{\phi_0, g})$ and $(M_{\psi_0}, M_{\psi_1}; D_{\psi_0, g})$ are optimal for each other.

**References**


