Journal of Pure and Applied Algebra 20 (1981) 195–225
North-Holland Publishing Company

TRANSFER ON ALGEBRAIC K-THEORY AND WHITEHEAD TORSION FOR PL FIBRATIONS

Hans J. MUNKHOLM*
Mathematics Department, University of Odense, DK-5230 Odense M, Denmark

Communicated by H. Bass
Received November 1979

0. Introduction

It is well known (Bass [3], Quillen [11]) that some ring homomorphisms $\varphi: A \to B$ give rise to a transfer map $\varphi^*: K_1(B) \to K_1(A)$ on algebraic K-theory. For group homomorphisms $\varphi: \pi \to \rho$ one has often used the fact that the ring homomorphism (still called φ) $\varphi: \mathbb{Z}\pi \to \mathbb{Z}\rho$ gives rise to a transfer map $\varphi^*: K_1(\mathbb{Z}\rho) \to K_1(\mathbb{Z}\pi)$ when φ is an inclusion of a subgroup of finite index. However, it seems to have gone unnoticed that φ^* is defined much more generally, namely when φ has kernel of type (FF) and image of finite index in ρ.

The classical homotopy-theoretic interpretation of $K_1(\mathbb{Z}\pi)$ goes via the quotient $\mathrm{Wh}(\pi) = K_1(\mathbb{Z}\pi)/\pm\pi$. We refer, of course, to the Whitehead torsion $\tau(B, A) \in \mathrm{Wh}(\pi_1(B))$, defined for any pair of finite simplicial complexes with $A \subset B$ a homotopy equivalence (see e.g. Milnor [8]).

In this paper we investigate the relationship between this homotopy-theoretic interpretation of $\mathrm{Wh}(\pi)$ and the algebraically defined transfer map $\varphi^*: K_1(\mathbb{Z}\rho) \to K_1(\mathbb{Z}\pi)$. The connecting tissue is made by PL fibrations (Hatcher [6]).

To describe our results we let $p: E \to B$ be a PL fibration with E and B *compact* polyhedra. Let F be the fiber, and

$$\pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B)$$

the induced sequence of fundamental groups. For any homotopy equivalence $A \subseteq B$ one has the Whitehead torsion

$$\tau(B, A) \in \mathrm{Wh}(\pi_1(B)).$$

Our general problem will be the computation of $\tau(E, p^{-1}(A))$ in terms of $\tau(B, A)$. The first result states that this problem is closely related to the transfer map.

* This work was done while the author visited Princeton University and was also partially supported by the Danish Natural Science Research Council.

Corollary A. *Assume that F is the classifying space, $B\nu$, for a group ν of type* (FF), *and that $i_*: \nu \to \pi_1(E)$ is monic. Then $\varphi = p_*: \pi_1(E) \to \pi_1(B)$ gives rise to a transfer map on K_1, which induces*

$$\varphi^*: \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E)).$$

Moreover,

$$\tau(E, p^{-1}(A)) = \varphi^*(\tau(B, A)).$$

Remark. When $\nu \to \pi \to \rho$, a short exact sequence of finitely presented groups with ν of type (FF) is given, then for any $\tau \in \mathrm{Wh}(\rho)$ one can find $A \subseteq B$ and $p: E \to B$ with fiber $B\nu$ such that $\pi_1(B\nu) \to \pi_1(E) \to \pi_1(B)$ is (isomorphic to) the given sequence and $\tau(B, A) = \tau$. Thus the homotopy-theoretic interpretation characterizes $\varphi^*: \mathrm{Wh}(\rho) \to \mathrm{Wh}(\pi)$ completely.

For the next result we recall that $H_i(F)$ is a $\mathbb{Z}\pi_1(B)$ module, so it represents an element of the integral representation ring $G(\pi_1(B))$. Also $\mathrm{Wh}(\pi_1(B))$ is a right $G(\pi_1(B))$ module.

Corollary B. *If p_* factors as shown here*

$$\begin{array}{ccc} \pi_1 E & \xrightarrow{\varphi} & \tilde{\pi} \\ & \searrow^{p_*} & \downarrow i \\ & & \pi_1 B \end{array}$$

with φ onto and i monic then in $\mathrm{Wh}(\tilde{\pi})$ *one has*

$$\varphi_*(\tau(E, p^{-1}A)) = i^*(\tau(B, A) \cdot \textstyle\sum (-1)^i [H_i(F_0)])$$

where F_0 is the base point component of F.

Remark. In the case where p is a fiber bundle, p_* is onto, and each $H_i(F)$ is free over $\mathbb{Z}$, this result is due to Anderson [1].

For the final corollary we let $\nu = \mathrm{Ker}(\varphi = p_*: \pi_1(E) \to \pi_1(B))$, and we let $\hat{F} \to F_0$ be the covering of the base point component of F corresponding to the subgroup

$$\mathrm{Ker}(\pi_1(F_0) \to \pi_1(E)) \subseteq \pi_1(F_0).$$

Then $\pi_1(E)$ acts on $H_i(\hat{F})$ in a natural way (described in Section 6) and one has the following.

Corollary C. *If $\nu = \mathrm{Ker}(p_*)$ is of type* (FF) *then there is a transfer map $\varphi^*: \mathrm{Wh}(\pi_1 B) \to \mathrm{Wh}(\pi_1(E))$ and one has*

$$\tau(E, p^{-1}(A)) = \varphi^*(\tau(B, A)) \cdot \textstyle\sum (-1)^i [H_i(\hat{F})].$$

All the above corollaries are derived from the main theorem which is stated and proved in Section 7. It computes $r_*\tau(E, p^{-1}A)$ whenever

$$\pi_1(E) \underset{r}{\to} \pi \underset{\varphi}{\to} \pi_1 B$$

is a factorization of p_* with r onto and Ker(φ) of type (FF).

In [2] Anderson proved that any PL fiber bundle $p:E \to B$ gives rise to a (geometrically defined) transfer map p^*: $\mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E))$.

According to Remark 2 of Ehrlich's paper [5] Anderson's p^* depends only on the restriction of p to the 2-skeleton of B. Our results may be viewed as an algebraic description (in terms of the algebraically defined transfer map and homology of various fibers) of some important instances of Anderson's geometrically defined transfer map. In this respect we should also mention that a recent preprint by Pedersen [10], shows that the geometrically defined transfer depends only on the "fundamental group and orientation" data of the PL fibre bundle $p:E \to B$. Presumably the proper setting for these results by Anderson, Ehrlich and Pedersen are PL fibrations rather than PL fiber bundles.

We now outline the contents of the various sections.

In Section 1 we generalize Milnor's treatment, in [8], of chain complexes of *based* A modules with *based* homology modules to the context of modules with a *given based, finite resolution*. The build-up follows Milnor's closely. In Section 2 we define the Whitehead torsion for suitably restricted spectral sequences, and we prove that for suitably filtered chain complexes C_* with spectral sequence E^*_{**} one has $\tau(C_*) = \tau(E^*_{**})$.

The results of Sections 1 and 2 overlap considerably with those of Maumary [7]. We have chosen to present our own approach because it seems better adapted to our applications.

In Section 3 we recall the definition of the transfer map $\varphi^*: K_1(B) \to K_1(A)$ for suitable ring homomorphisms $\varphi: A \to B$. And we prove the formula

$$\tau(\varphi^! C_*) = \varphi^*(\tau(C_*)) \in \bar{K}_1(B). \tag{D}$$

Here C_* is a based chain complex with based homology, over B. $\varphi^! C_*$ is C_*, viewed over A, via φ. The point of the formula is that $\varphi^! C_*$ (and its homology) becomes b.f. resolved in a natural way, so that $\tau(\varphi^! C_*)$ is defined.

In Section 4 we recall the integral representation ring, $G(\pi)$, admitting also representations N which are just finitely generated (and not free) over $\mathbb{Z}$. Also we recall how it acts on $K_1(\mathbb{Z}\pi)$ and $\mathrm{Wh}(\pi)$. This part owes much to Pedersen and Taylor [9]. We relate the $G(\pi)$ action to the torsion of chain complexes by showing that when C_* is b.f. resolved and free over $\mathbb{Z}$ while $H_*(C_*)$ is b.f. resolved, then $C_* \otimes N$ and $H_*(C_* \otimes N)$ inherit b.f.-resolutions with respect to which one has

$$\tau(C_* \otimes N) = \tau(C_*)[N] \tag{E}$$

in $\bar{K}_1(\mathbb{Z}\pi)$ or $\mathrm{Wh}(\pi)$.

In Section 5 we show that a group homomorphism $\varphi : \pi \to \rho$ gives rise to a transfer map

$$\varphi^* : K_1(\mathbb{Z}\rho) \to K_1(\mathbb{Z}\pi)$$

provided Ker(φ) is of type (FF) and Im φ is of finite index in ρ. We have not been able to prove, algebraically, that φ^* induces a map (still called)

$$\varphi^* : \mathrm{Wh}(\rho) \to \mathrm{Wh}(\pi).$$

This, however, follows from our main theorem, at least when all the groups are finitely presented.

In Section 6 we recall, in some detail, Hatcher's description of PL fibrations in terms of iterated mapping cylinder decomposition. We study the behaviour of such decompositions under passage to covering spaces. And we use the results to rederive the E^1-term of the Serre spectral sequence for homology. We need the explicit isomorphisms in order to compare, later on, different b.f.-resolutions.

Finally, in Section 7, we use all of the above to prove the main theorem. Essentially what happens is that formulas (D) and (E) allow one to compute the torsion of the E^1-term in the relevant spectral sequence, namely

$$E^1_{**} = \varphi^!(C_*(\tilde{B}, \tilde{A})) \otimes H_*(\hat{F}).$$

Also, from Section 2, one knows that this differs from $\tau(E, p^{-1}(A)) = \tau(C_*(\hat{E}, \hat{E}_A))$ by $\sum (-1)^s \tau(E^0_{s,*})$. To show that this latter term vanishes, one uses a generalization of Anderson's excision lemma [1], to express it as a product $\chi(B, A) i_* \tau(C_*(\hat{F}))$ where the Euler characteristic, of course, vanishes. This depends crucially on the fact (Hatcher [6]) that the maps entering into the iterated mapping cylinders which decompose a PL fibration are *simple* homotopy equivalences.

1. Based, finite resolutions

Let A be an associative ring with unit having the property that any free A module has a well-defined dimension. Unless something else is mentioned *module* will mean finitely generated left A module. Recall from [8] the group $\bar{K}_1(A)$ obtained from the infinite general linear group over A by abelianizing and factoring out the subgroup generated by ± 1. If two bases b and b' for a module M are given we let $[b/b'] \in \bar{K}_1(A)$ be the element represented by the matrix expressing b in terms of b'. We shall freely use the properties of [-/-] given in [8].

A *based finite resolution* ε for a module M is defined to be an exact sequence of modules

$$0 \to F_k(\varepsilon) \to F_{k-1}(\varepsilon) \to \cdots \to F_0(\varepsilon) \to M \to 0$$

together with a finite basis $f_i(\varepsilon)$ for each $F_i(\varepsilon)$. We shall use the abbreviation a "b.f.r." $F_*(\varepsilon) \to M$. If a b.f.r. for M has been specified then we shall say that M is *b-f-resolved*. It is the purpose of this section to redo Sections 2 and 3 of [8] for b-f-resolved modules instead of based (or stably based) modules.

If two b.f.r.'s for M, ε and ε', are given, then we define $\{\varepsilon/\varepsilon'\} \in \bar{K}_1(A)$ as follows. The identity on M lifts to a map φ_* of resolutions

$$\begin{array}{ccc} F_*(\varepsilon) & \longrightarrow & M \\ \downarrow{\scriptstyle \varphi_*} & & \| \\ F_*(\varepsilon') & \longrightarrow & M \end{array} \quad ;$$

the mapping cone $\mathrm{MC}(\varphi_*)$ of φ_* is in an obvious way a based, acyclic chain complex over A. We put

$$\{\varepsilon/\varepsilon'\} = \tau(\mathrm{MC}(\varphi_*)) \in \bar{K}_1(A),$$

the torsion of $\mathrm{MC}(\varphi_*)$ in the sense of [8]. To fix notation let us mention that $\mathrm{MC}(\varphi_*)$ has

$$\mathrm{MC}(\varphi_*)_n = F_{n-1}(\varepsilon) \oplus F_n(\varepsilon')$$

with differential given by the matrix

$$\begin{pmatrix} -d & 0 \\ \varphi_{n-1} & d \end{pmatrix}.$$

To see that $\{\varepsilon/\varepsilon'\}$ is well defined we note that a different choice φ'_* of lifting will be chain homotopic to φ_*. And from a chosen chain homotopy h, one readily constructs a chain isomorphism

$$\psi_* = \begin{pmatrix} 1 & 0 \\ h_* & 1 \end{pmatrix} : \mathrm{MC}(\varphi_*) \to \mathrm{MC}(\varphi'_*)$$

which is simple with respect to the given bases (i.e. ψ_n represents 0 in $\bar{K}_1(A)$). It follows that $\tau(\mathrm{MC}(\varphi_*)) = \tau(\mathrm{MC}(\varphi'_*))$.

In case each φ_n is an isomorphism there is a based acyclic chain complex $C(n, \varphi_*)$ with $C(n, \varphi_*)_i = 0$ for $i \neq n, n+1$, $C(n, \varphi_*)_{n+1} = F_n(\varepsilon)$, $C(n, \varphi_*)_n = F_n(\varepsilon')$ and differential given by φ_n.

There is an obvious chain map

$$g(n): C(n, \varphi_*) \to \mathrm{MC}(\varphi_*)$$

sending $x \in C(n, \varphi_*)_{n+1}$ to $(x, 0)$ and $y \in C(n, \varphi_*)_n$ to $(-d\varphi_n^{-1}y, y)$. Furthermore

$$g = \sum g(n): \bigoplus C(n, \varphi_*) \to \mathrm{MC}(\varphi_*)$$

is a chain isomorphism, and simple in each degree. It follows that

$$\begin{aligned} \tau(\mathrm{MC}(\varphi_*)) &= \sum \tau(C(n, \varphi_*)) \\ &= \sum (-1)^n [f_n(\varepsilon)/\varphi_n^{-1}(f_n(\varepsilon'))]. \end{aligned} \tag{1.1}$$

Especially, we see that

$$\{\varepsilon/\varepsilon\}=0. \tag{1.2}$$

Next, let ε'' be a third b.f.r. for M. For brevity let $F_i=F_i(\varepsilon), F_i'=F_i(\varepsilon')$ and $F_i''=F_i(\varepsilon'')$. Let $\psi_*:F_*'\to F_*''$ be a lifting of the identity on M. Also let id be the identity map of F_*'. There is then a short exact sequence

$$0\to \mathrm{MC}(\varphi_*)\underset{\alpha}{\to}\mathrm{MC}(\psi_*\varphi_*\oplus \mathrm{id})\underset{\beta}{\to}\mathrm{MC}(\psi_*)\to 0$$

given in degree n by the matrices

$$\alpha_n=\begin{pmatrix}1 & 0\\ -\varphi_{n-1} & 0\\ 0 & \psi_n\\ 0 & -1\end{pmatrix},\qquad \beta_n=\begin{pmatrix}\varphi_{n-1} & 1 & 0 & 0\\ 0 & 0 & 1 & \psi_n\end{pmatrix}.$$

Moreover, up to simple isomorphisms, α_n and β_n are the natural inclusion and projection respectively. Therefore, Theorem 3.2 of [8] applies to give

$$\begin{aligned}\tau(\mathrm{MC}(\varphi_*))+\tau(\mathrm{MC}(\psi_*))&=\tau(\mathrm{MC}(\psi_*\varphi_*\oplus\mathrm{id}))\\ &=\tau(\mathrm{MC}(\psi_*\varphi_*))+\tau(\mathrm{MC}(\mathrm{id}))\\ &=\tau(\mathrm{MC}(\psi_*\varphi_*)).\end{aligned}$$

Thus we have

$$\{\varepsilon/\varepsilon'\}+\{\varepsilon'/\varepsilon''\}=\{\varepsilon/\varepsilon''\}. \tag{1.3}$$

The formulas (1.2) and (1.3) show that the relation $\varepsilon\sim\varepsilon'$ if $\{\varepsilon/\varepsilon'\}=0$ is an equivalence relation among b.f.r.'s for M.

Let $0\to M'\to M\to M''\to 0$ be a short exact sequence of modules with ε', ε'' b.f.r.'s for M', M''. We shall construct a b.f.r. $\varepsilon=\varepsilon'\varepsilon''$ for M. It has

$$F_i(\varepsilon)=F_i(\varepsilon')\oplus F_i(\varepsilon'')$$

and

$$f_i(\varepsilon)=(f_i(\varepsilon'),0)\cup(0,f_i(\varepsilon'')).$$

Furthermore, there exists a commutative diagram

$$\begin{array}{ccccccccc}0 & \longrightarrow & F_*(\varepsilon') & \longrightarrow & F_*(\varepsilon) & \longrightarrow & F_*(\varepsilon'') & \longrightarrow & 0\\ & & \downarrow & & \downarrow{\scriptstyle d_0} & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0\end{array}$$

It is well known that such $F_*(\varepsilon) \xrightarrow{d_0} M$ exists. Clearly d_0 and $d_i: F_i(\varepsilon) \to F_{i-1}(\varepsilon)$ are not unique. However, any new choice must have the form

$$d_0' = d_0\rho_0, \qquad d_i' = \rho_{i-1}^{-1} d_i \rho_i$$

where ρ_i is an automorphism of $F_i(\varepsilon') \oplus F_i(\varepsilon'')$ of the form

$$\rho_i = \begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix}.$$

Since ρ_i is simple it follows from (1.1) that the equivalence class of ε is independent of choices.

Suppose b.f.r.'s ε_1' and ε_1'' for M' and M'' are also given. Let

$$\varphi_*' : F_*(\varepsilon') \to F_*(\varepsilon_1'),$$

$$\varphi_*'' : F_*(\varepsilon'') \to F_*(\varepsilon_1''),$$

be liftings of the respective identities. One readily shows that the identity on M lifts to a chain map of the form

$$\varphi_* = \begin{pmatrix} \varphi_*' & \xi_* \\ 0 & \varphi_*'' \end{pmatrix} : F_*(\varepsilon') \oplus F_*(\varepsilon'') \to F_*(\varepsilon_1') \oplus F_*(\varepsilon_1'').$$

The mapping cones fit into an exact sequence

$$0 \to \mathrm{MC}(\varphi_*') \to \mathrm{MC}(\varphi_*) \to \mathrm{MC}(\varphi_*'') \to 0$$

with compatible bases. Therefore one has the formula

$$\{\varepsilon'\varepsilon''/\varepsilon_1'\varepsilon_1''\} = \{\varepsilon'/\varepsilon_1'\} + \{\varepsilon''/\varepsilon_1''\}. \tag{1.4}$$

A special case of this shows that changing ε' (and ε'') within its equivalence class leaves the class of $\varepsilon'\varepsilon''$ unchanged.

By abuse of notation we shall, henceforth, use ε, ε', etc. to denote equivalence classes of b.f.r.'s.

Following Milnor we consider the situation $M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3$, assuming that a b.f.r. ε_i for M_i/M_{i-1} is given ($i = 1, 2, 3$). One gets two b.f.r.'s for M_3/M_0, namely $(\varepsilon_1\varepsilon_2)\varepsilon_3$ corresponding to the exact sequences

$$0 \to M_1/M_0 \to M_2/M_0 \to M_2/M_1 \to 0,$$

$$0 \to M_2/M_0 \to M_3/M_0 \to M_3/M_2 \to 0,$$

and $\varepsilon_1(\varepsilon_2\varepsilon_3)$ coming from the exact sequences

$$0 \to M_1/M_0 \to M_3/M_0 \to M_3/M_1 \to 0,$$

$$0 \to M_2/M_1 \to M_3/M_1 \to M_3/M_2 \to 0.$$

It is not surprising, nor difficult to prove, that

$$\varepsilon_1(\varepsilon_2\varepsilon_3) \sim (\varepsilon_1\varepsilon_2)\varepsilon_3. \tag{1.5}$$

In fact by careful choice one can obtain

$$F_*(\varepsilon_1(\varepsilon_2\varepsilon_3)) = F_*((\varepsilon_1\varepsilon_2)\varepsilon_3).$$

Similarly, if $M_i \subseteq M$ and ε_i is a b.f.r. for $M_i/M_1 \cap M_2$, $i = 1, 2$ then (with $\{i, j\} = \{1, 2\}$) one has the b.f.r. $\varepsilon_i\varepsilon_j$ for $(M_1 + M_2)/(M_1 \cap M_2)$ corresponding to the exact sequence

$$0 \to M_i/(M_1 \cap M_2) \to (M_1 + M_2)/(M_1 \cap M_2) \to M_j/(M_1 \cap M_2) \to 0.$$

And there is the commutativity relation

$$\varepsilon_1\varepsilon_2 \sim \varepsilon_2\varepsilon_1. \tag{1.6}$$

In fact, using the obvious splitting $M_j/(M_1 \cap M_2) \to (M_1 + M_2)/(M_1 \cap M_2)$ one may take

$$F_*(\varepsilon_i\varepsilon_j) = F_*(\varepsilon_i) \oplus F_*(\varepsilon_j)$$

including differential and augmentation. The commutativity then amounts to the fact that the twisting isomorphism

$$F_*(\varepsilon_1) \oplus F_*(\varepsilon_2) \to F_*(\varepsilon_2) \oplus F_*(\varepsilon_1)$$

is simple with respect to the given bases. Here one needs to work with $\bar{K}_1(A)$ rather than $K_1(A)$.

If b is a basis for $M \oplus A^s$ ($s \geq 0$) then $0 \to A^s \to M \oplus A^s \to M \to 0$ with the standard basis for A^s and the basis b for $M \oplus A^s$ is a b.f.r., we call it β, for M. If we also have a basis b' for $M \oplus A^r$ then it is easily seen that $\{\beta/\beta'\} = [b/b']$. Thus our theory *is* an extension of Milnor's treatment of stably based modules.

2. Whitehead torsion for b.f.-resolved chain complexes and for spectral sequences

In this section we define the Whitehead torsion for certain spectral sequences, and (as a special case) for finite chain complexes C_* with each C_n and each H_nC b.f.-resolved. Our approach follows Milnor's [8] closely. There is considerable overlap with results of Maumary [7].

Lemma 2.1. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A modules, and if M and M'' admit b.f.r.'s then so does M'.*

Proof. Let ε and ε'' be the given b.f.r.'s. By 8.4 of Bass [3], M' admits a finite resolution by finitely generated projectives, say

$$0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M' \to 0.$$

Furthermore, one may assume that all P_i but P_k are free. In $K_0(A)$ one then has

$$\sum (-1)^i [P_i] + \sum (-1)^i [F_i(\varepsilon'')] = \sum (-1)^i [F_i(\varepsilon)].$$

This shows that $[P_k]$ is stably free. Thus

$$0 \to P_k \oplus A^s \to P_{k-1} \oplus A^s \to P_{k-2} \to \cdots \to P_0 \to M' \to 0$$

is the desired free resolution for M'.

Let C be a finite chain complex with a finite filtration by subcomplexes

$$0 = F_{-1}C \subseteq F_0 C \subseteq \cdots \subseteq F_l C = C.$$

Consider also the induced filtration on homology

$$F_p HC = \operatorname{Im}(HF_p C \to HC)$$

and the resulting spectral sequence $(E^r_{s,t}, d^r)$. This has

$$E^0_{s,t} = F_s C_{s+t} / F_{s-1} C_{s+t},$$

$$E^\infty_{s,t} = E^{l+1}_{s,t} \cong F_s H_{s+t} C / F_{s-1} H_{s+t} C.$$

We *assume that $E^0_{s,t}$ and $E^\infty_{s,t}$ have given b.f.r.'s $\varepsilon^0_{s,t}$ and $\varepsilon^\infty_{s,t} = \varepsilon^{l+1}_{s,t}$. Furthermore, we assume that the other modules $E^r_{s,t}$, $1 \leq r \leq l$, admit b.f.r's, say $\varepsilon^r_{s,t}$.*

Lemma 2.1 then applies to the exact sequences

$$0 \to B^r_{s,t} \to Z^r_{s,t} \to E^{r+1}_{s,t} \to 0,$$

$$0 \to Z^r_{s,t} \to E^r_{s,t} \to B^r_{s-r,t+r-1} \to 0,$$

to show (inductively on $s+t$) that $B^r_{s,t}$ admits a b.f.r., say $\beta^r_{s,t}$. Once $\beta^r_{s,t}$ is chosen one gets the b.f.r.

$$\beta^r_{s,t} \varepsilon^{r+1}_{s,t} \beta^r_{s-r,t+r-1} \quad \text{for } E^r_{s,t},$$

and one defines the torsion of the spectral sequence to be

$$\tau(E^*_{**}) = \sum (-1)^{s+t} \{\beta^r_{s,t} \varepsilon^{r+1}_{s,t} \beta^r_{s-r,t+r-1} / \varepsilon^r_{s,t}\}.$$

It is easily seen that this is independent of the choice of $\beta^r_{s,t}$ (all r, s, t) and of $\varepsilon^r_{s,t}$ ($1 \leq r \leq l$, all s, t).

In case $l = 0$ this specializes to the definition of $\tau(C)$ for any finite chain complex C with each C_n and each $H_n C$ b.f.-resolved.

In the general case ($l \geq 0$) C_n and $H_n C$ do get preferred b.f.r.'s, namely

$$\gamma_n = \varepsilon^0_{0,n} \varepsilon^0_{1,n-1} \cdots \varepsilon^0_{n,0},$$

$$\chi_n = \varepsilon^{l+1}_{0,n} \varepsilon^{l+1}_{1,n-1} \cdots \varepsilon^{l+1}_{n,0}.$$

Thus $\tau(C)$ is defined, and it is not surprising that one gets the following theorem.

Theorem 2.2. *Let C be a finite chain complex of A modules with a finite filtration*

$$0 = F_{-1}C \subseteq F_0 C \subseteq \cdots \subseteq F_l C = C$$

by subcomplexes of A modules. In the resulting spectral sequence assume that each $E^0_{s,t}$

and $E^{\infty}_{s,t}$ have a preferred b.f.r. and that each $E^{r}_{s,t}$ $(1\leq r\leq l)$ admits some b.f.r. Equip each C_n and H_nC with a b.f.r. as above. Then

$$\tau(E^{*}_{**})=\tau(C)\in\bar{K}_1(A).$$

The proof depends on the following generalization of Milnor's Theorem 3.2 [8]. Let $0\to C'\to C\to C''\to 0$ be a short exact sequence of b.f.-resolved chain complexes (over A) with b.f.-resolved homology modules. View the long exact homology sequence as an acyclic chain complex $\mathscr{H}$ of b.f.-resolved A modules. Then $\tau(\mathscr{H})$ is defined and one has the following.

Theorem 2.3. *In the above situation assume that the b.f.r.'s for C', C, and C'' have $\gamma_i\sim\gamma'_i\gamma''_i$. Then*

$$\tau(C)=\tau(C')+\tau(C'')+\tau(\mathscr{H}).$$

Proof. Replace the word "basis" by "b.f.r." and the symbol [-/-] by {-/-} everywhere in Milnor's Section 3 [8].

Proof of Theorem 2.2. We shall use induction on the length l of the filtration. If $l=0$ then one has $E^{r}_{s,t}=0$ except for $s=0$ and $E^{0}_{0,*}=C_*$, $E^{1}_{0,*}=H_*C=E^{2}_{0,*}=\cdots$, so the result is trivial.

Thus we assume that the theorem holds for the chain complex $\bar{C}=F_{l-1}C$ with the inherited filtration. We denote by a bar any quantity derived from $\bar{C}$. It is easily seen that one has

$$\bar{E}^{r}_{p,*}=\begin{cases}E^{r}_{p,*} & \text{if } p+r\leq l \text{ and } p<l,\\ 0 & \text{if } \qquad\qquad p\geq l.\end{cases}$$

Also there are straightforward exact sequences

$$0\to B^{l-p}_{p,*}\to\bar{E}^{r}_{p,*}\to E^{r}_{p,*}\to 0\quad \text{if } p+r>l \tag{2.4}$$

$$0\to E^{l-p+1}_{l,*}\to E^{l-p}_{l,*}\to B^{l-p}_{p,*}\to 0,\quad p=0,1,2,\ldots,l \tag{2.5}$$

$$0\to B^{l}_{0,*}\to E^{l}_{0,*}\to E^{l+1}_{0,*}\to 0. \tag{2.6}$$

We now choose b.f.r.'s for $\bar{E}^{0}_{**}$ and for $\bar{E}^{\infty}_{**}=\bar{E}^{l}_{**}$, namely

$$\begin{aligned}&\bar{\varepsilon}^{0}_{p,q}=\varepsilon^{0}_{p,q},\quad \text{if } p=0,1,\ldots,l-1, \text{ all } q,\\ &\bar{\varepsilon}^{l}_{p,q}=\beta^{l-p}_{p,q}\varepsilon^{l+1}_{p,q},\quad \text{if } l>p>0, \text{ all } q,\\ &\bar{\varepsilon}^{l}_{0,q}(=\varepsilon^{l}_{0,q})=\beta^{l}_{0,q}\varepsilon^{l+1}_{0,q},\quad \text{all } q.\end{aligned} \tag{2.7}$$

The resulting b.f.r.'s $\bar{\gamma}_n$ for $\bar{C}_n$, γ_n for C_n and the given one $\varepsilon^{0}_{l,n-l}$ for $E^{0}_{l,n-l}=C_n/\bar{C}_n$ are clearly related by

$$\gamma_n=\bar{\gamma}_n\varepsilon^{0}_{l,n-l}. \tag{2.8}$$

Therefore, Theorem 2.3 and the inductive hypothesis imply that

$$\tau(C)=\tau(\bar{E}^*_{**})+\tau(C/\bar{C})+\tau(\mathcal{H}), \tag{2.9}$$

where $\mathcal{H}$ is the long exact homology sequence of $\bar{C}\to C\to C/\bar{C}$. We shall finish the proof by identifying the various terms on the right-hand side of (2.9). For computing $\tau(E^*_{**})$ we take

$$\varepsilon^r_{p,q}=\beta^r_{p,q}\varepsilon^{r+1}_{p,q}\beta^r_{p-r,q+r-1}, \quad r=1,2,\dots,l. \tag{2.10}$$

Then, since $B^r_{p,q}=0$ for $r>l-p$,

$$\varepsilon^1_{p,q}=\beta^1_{p,q}\beta^2_{p,q}\cdots\beta^{l-p}_{p,q}\varepsilon^{l+1}_{p,q}\beta^l_{p-l,q+l-1}\beta^{l-1}_{p-l+1,q+l-2}\cdots\beta^1_{p-1,q} \tag{2.11}$$

and

$$\tau(E^*_{**})=\sum_{p=0}^{l}\sum_q(-1)^{p+q}\{\beta^0_{p,q}\varepsilon^1_{p,q}\beta^0_{p,q-1}/\varepsilon^0_{p,q}\}. \tag{2.12}$$

Similarly we may take

$$\bar{\varepsilon}^1_{p,q}=\bar{\beta}^1_{p,q}\bar{\beta}^2_{p,q}\cdots\bar{\beta}^{l-1-p}_{p,q}\bar{\varepsilon}^l_{p,q}\bar{\beta}^{l-1}_{p-l+1,q+l-2}\bar{\beta}^{l-2}_{p-l+2,q+l-3}\cdots\bar{\beta}^1_{p-1,q} \tag{2.13}$$

and then we have

$$\tau(\bar{E}^*_{**})=\sum_{p=0}^{l-1}\sum_q(-1)^{p+q}\{\bar{\beta}^0_{p,q}\bar{\varepsilon}^1_{p,q}\bar{\beta}^0_{p,q-1}/\bar{\varepsilon}^0_{p,q}\}. \tag{2.14}$$

Since

$$\bar{B}^r_{p,q}=B^r_{p,q} \quad \text{for } r+p\neq l \tag{2.15}$$

we can take

$$\bar{\beta}^r_{p,q}=\beta^r_{p,q} \quad \text{for } r+p\neq l. \tag{2.16}$$

It then follows from (2.11) and (2.13) together with (2.7) that

$$\bar{\varepsilon}^1_{p,q}=\varepsilon^1_{p,q} \quad \text{for } 0\le p\le l-1, \text{ all } q. \tag{2.17}$$

Using (2.17) and (2.16) for $r=0$, $p<l$, we see from (2.12) and (2.14) that

$$\begin{aligned}\tau(E^*_{**})&=\tau(\bar{E}^*_{**})+\sum_q(-1)^{l+q}\{\beta^0_{l,q}\varepsilon^1_{l,q}\beta^0_{l,q-1}/\varepsilon^0_{l,q}\}\\&=\tau(\bar{E}^*_{**})+\tau(C/\bar{C})\end{aligned} \tag{2.18}$$

where $H_n(C/\bar{C})=E^1_{l,n-l}$ has b.f.r. $\varepsilon^1_{l,n-l}$.

From (2.18) and (2.9) we see that we can finish the inductive step by showing that

$$\tau(\mathcal{H})=0. \tag{2.19}$$

Thus we consider the long exact sequence $\mathcal{H}$ together with its factorization into short exact sequences. By definition K_{i-1} is the cokernel of $H_i(C)\to H_i(C/\bar{C})$; the other ingredients in the factorization are as shown in Fig. 1. For the computation of $\tau(\mathcal{H})$

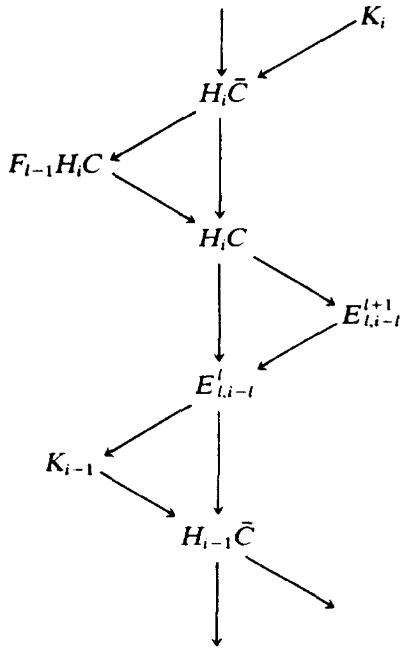


Fig. 1.

we are free to choose b.f.r.'s for $E^{l+1}_{l,i-l}$, K_{i-1} and $F_{l-1}H_{i-1}C$. For $E^{l+1}_{l,i-l}$ we choose $\varepsilon^{l+1}_{l,i-l}$, for $F_{l-1}H_{i-1}C$ we choose $\varepsilon^{l+1}_{0,i}\varepsilon^{l+1}_{1,i-1}\cdots\varepsilon^{l+1}_{l-1,i-l+1}$. Then, recalling how the b.f.r. for H_iC was chosen, we see that the contribution to $\tau(\mathscr{H})$ coming from the spot H_iC vanishes.

Next, consider the ladder of short exact sequences (Fig. 2), where $\bar{F}_s = F_sH_{i-1}\bar{C}$, $F_s = F_sH_{i-1}C$, F_sK_{i-1} is the kernel of the induced map $\bar{F}_s \to F_s$, and the index at each inclusion is the chosen b.f.r. for the cokernel of that inclusion. By

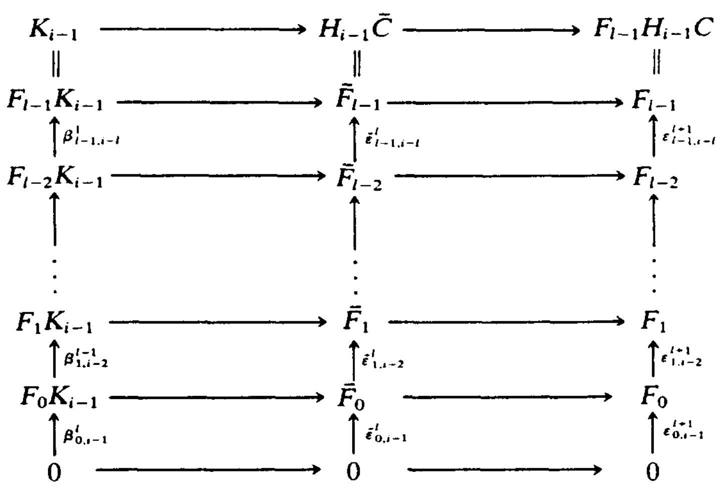


Fig. 2.

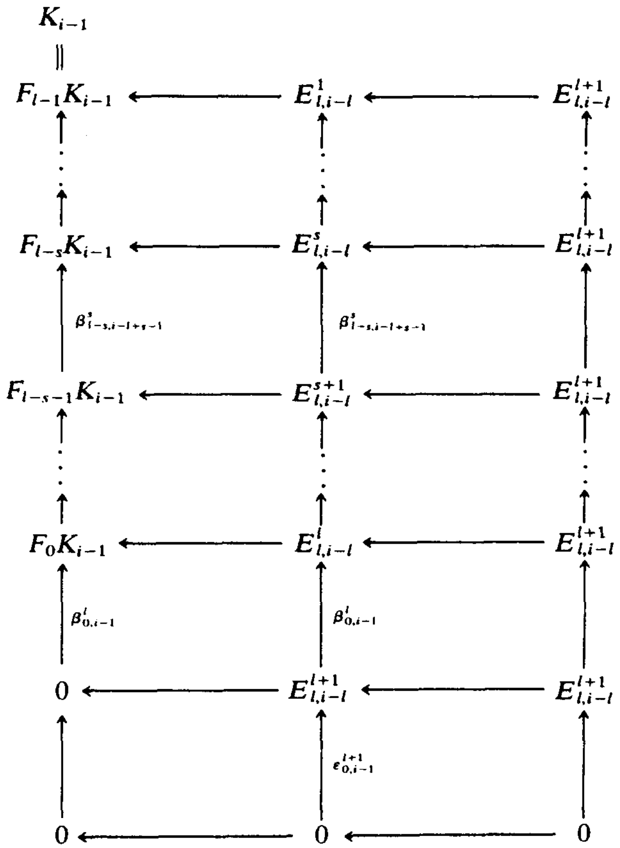


Fig. 3.

"concatenation" one gets the same b.f.r.'s for $H_{i-1}\bar{C}$ and for $F_{l-1}H_{i-1}C$ that were chosen before. And one gets the b.f.r. $\kappa_{i-1} = \beta^1_{l-1,i-l}\beta^2_{l-2,i-l+1} \cdots \beta^{l-1}_{1,i-2}\beta^l_{0,i-1}$ for K_{i-1}. Using (2.7) it is then easy to show that the contribution to $\tau(\mathcal{H})$ from the spot $H_{i-1}\bar{C}$ vanishes.

Finally we see that the filtration which defines κ_{i-1} and the one which defines $\varepsilon^1_{l,i-l}$ are compatible, i.e. there is a further commutative ladder of exact sequences (Fig. 3). It follows easily that $\kappa_{i-1}\varepsilon^{l+1}_{l,i-l} = \varepsilon^1_{l,i-l}$ and the contribution to $\tau(\mathcal{H})$ at the spot $E^1_{l,i-l}$ vanishes. This finishes the proof of Theorem 2.2.

3. The transfer map and b.f.-resolved chain complexes

We start by recalling from Quillen [11, p. 103] the *transfer map* $f^*: \tilde{K}_1(B) \to \tilde{K}_1(A)$ which is defined whenever $f: A \to B$ is a ring homomorphism with $f^!(B)$ (i.e. B viewed as A module via f) admitting a finite, finitely generated, projective resolution over A. In fact, let $P(B)$ be the category of finitely generated projective B modules, and $P_{<\infty}(B)$ the category of B modules admitting finite resolutions by objects of $P(B)$. The assumptions on $f: A \to B$ guarantee that $f^!$ is a functor from $P(B)$ to $P_{<\infty}(A)$ (see e.g. Bass [3]). One also has the inclusion $i: P(A) \to P_{<\infty}(A)$ which induces an isomorphism

$$K_1(A) = K_1(P(A)) \xrightarrow{i_*} K_1(P_{<\infty}(A)).$$

By definition, the transfer f^* is the composition

$$K_1(B) = K_1(P(B)) \xrightarrow{(f^!)_*} K_1(P_{<\infty}(A)) \xrightarrow{i_*^{-1}} K_1(P(A)) = K_1(A).$$

If the resolution of $f^*(B)$ can be chosen (finite, finitely generated, and) *free* then f^* sends the subgroup generated by -1 into "itself" so it induces a map (still called) $f^*: \bar{K}_1(B) \to \bar{K}_1(A)$.

We want to relate this transfer map to the notion of b.f.-resolved chain complexes. We need the following lemma.

Lemma 3.1. *Let $f: A \to B$ be a ring homomorphism such that $f^!(B)$ admits a finite, free, finitely generated resolution over A. Then $f^!(B)$ admits a b.f.r. $G_* \to f^!(B)$ with the following property: any automorphism φ of $f^!(B)^{(n)} = \bigoplus_{i=1}^n f^!(B)$ lifts to a map of resolutions $\varphi_*: G_*^{(n)} \to G_*^{(n)}$ which is an automorphism in each degree (any integral $n > 0$).*

Proof. Let

$$0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to f^!(B) \to 0$$

be a given b.f.r. for $f^!(B)$. Define $d_0: G_* \to f^!(B)$ as follows. Put

$$\tilde{G}_i = F_0 \oplus F_1 \oplus \cdots \oplus F_i, \quad \tilde{G}_{-1} = 0,$$

$$G_i = \tilde{G}_i \oplus \tilde{G}_{i-1} \oplus F_i, \quad 0 \le i < k,$$

$$G_k = \tilde{G}_{k-1} \oplus F_k.$$

Give each $\tilde{G}_i$ and each G_i the obvious basis; define the differential $d_i: G_i \to G_{i-1}$ and $d_0: G_0 \to f^!(B)$ in terms of the similar quantities for $F_* \to f^!(B)$ (also called d_i and d_0) as follows

$$d_0 = (0 \quad d_0), \qquad d_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_i \end{pmatrix}, \qquad d_k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & d_k \end{pmatrix}.$$

One checks easily that $G_* \to f^!(B)$ is a b.f.r. The lifting property can be proved as in Lemma 7.4 of Bass [4]. Actually G_* is just an explicit version of the resolution obtained there.

Now let m be a given basis for a B module M. View m as an isomorphism of A modules

$$m: f^!(B)^{(n)} \to f^!(M), \quad n = |m|.$$

Once a b.f.r. G_* as in the lemma has been chosen then one gets a b.f.r.

$$md_0^{(n)}: G_*^{(n)} \to f^!(M)$$

for the A module $f^!(M)$. Call that b.f.r. $\hat{m}$.

In this way, if C is a finite chain complex of based B modules with based homology modules the chain complex $f^!(C)$ becomes a b.f.-resolved chain complex of A modules with b.f.-resolved homology modules. Also, the terminology is not misleading, i.e. one has the following theorem.

Theorem 3.2. *Let $f: A \to B$ be a ring homomorphism with $f^!(B)$ admitting a b.f.r. over A. If C is a finite, based chain complex of B modules with based homology modules, then*

$$\tau(f^!(C)) = f^*(\tau(C)) \in \bar{K}_1(A).$$

Remark. The equivalence class of the b.f.r. $\hat{m}$ *does* depend on the choice of $G_* \to f^!(B)$. However, the theorem shows that $\tau(f^!(C))$ is independent of the choice.

Proof. We claim that it is suffices to prove the result when C has the form

$$0 \to 0 \to \cdots \to 0 \to M \xrightarrow{1} M.$$

In fact knowing the theorem for that case amounts to knowing that whenever M comes equipped with two B bases, m_1 and m_0, then

$$\{\hat{m}_1/\hat{m}_0\} = f^*([m_1/m_0]). \tag{3.3}$$

Thus equivalent bases for M give rise to equivalent b.f.r.'s for $f^!(M)$, and it easily follows that one has

$$\widehat{m'm''} = \hat{m}'\hat{m}'' \tag{3.4}$$

whenever m' and m'' are those given bases for B modules M' and M'' appearing in a short exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

And from (3.3) and (3.4) the general case of the theorem follows by a purely formal manipulation, at least if the modules of boundaries of C are free over B. If they are not, then at least they are stably free, and then—by adding to C a number of chain complexes of the form

$$0 \to \cdots \to 0 \to B \xrightarrow{\text{id}} \mathrm{B} \to 0 \to \cdots \to 0,$$

an operation that changes neither $\tau(C)$ nor $\tau(f^!(C))$—one may make them free over B.

Thus we concentrate on proving (3.3). To compute $\{\hat{m}_1/\hat{m}_0\}$ we need a lifting φ_* in the diagram

$$\begin{array}{ccccc}
G_*^{(n)} & \xrightarrow{d_0^{(n)}} & f^!(B)^{(n)} & \xrightarrow{m_1} & f^!(M) \\
\Big\downarrow{\scriptstyle \varphi_*} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow \\
G_*^{(n)} & \longrightarrow & f^!(B)^{(n)} & \xrightarrow[m_0]{} & f^!(M)
\end{array}$$

where $\varphi = m_0^{-1} m_1$. By Lemma 3.1, a φ_* with each φ_i and A automorphism exists. But, by the description of

$$(i_*)^{-1} : K_1(P_{<\infty}(A)) \to K_1(P(A))$$

given by Bass [3], this means that

$$(i_*)^{-1}[f^!(B)^{(n)}, \varphi] = \sum_i (-1)^i [G_i^{(n)}, \varphi_i]. \tag{3.5}$$

Here we use $[P, \psi]$ to represent the element of $K_1(\mathfrak{A})$, $\mathfrak{A}$ a category, represented by the automorphism ψ of the object P of $\mathfrak{A}$.

If $g_i^{(n)}$ is the given basis for $G_i^{(n)}$, then

$$[G_i^{(n)}, \varphi_i] = [g_i^{(n)} / \varphi_i^{-1}(g_i^{(n)})]$$

so (1.1) shows that the right-hand side of (3.5) is $\tau(\mathrm{MC}(\varphi_*)) = \{\hat{m}_1/\hat{m}_0\}$. Since also $[B^{(n)}, \varphi] = [m_1/m_0]$ the left-hand side is $f^*([m_1/m_0])$.

This finishes the proof of (3.3).

4. The action of the representation ring on the torsion of chain complexes

Let R be a *principal ideal domain for which every finitely generated R torsion module is finite*; of course $R = \mathbb{Z}$ and $R = \mathbb{Z}/k\mathbb{Z}$ are important examples. Let π be a group and let $A = R\pi$ be the group ring for π over R.

Recall that $G_R(\pi)$ is the abelian group with one generator $[N]$ for each $R\pi$ module which is finitely-generated-and-free over R and one relation $[N] = [N'] + [N'']$ for each short exact sequence $N' \to N \to N''$ of such $R\pi$ modules. Also $G(\pi)$ is the similar construction where one gives up the requirement that N be free over R. Pedersen and Taylor [9] show that the obvious map $G_R(\pi) \to G(\pi)$ is an isomorphism of rings, the product in both cases being induced by $[N][N_1] = [N \otimes_R N']$ where π acts diagonally on the tensor product.

If $P \in P(R\pi)$ and $f \in \mathrm{Aut}_{R\pi}(P)$ then $[P, f]$ is the typical generator for $K_1(R\pi)$, and the definition

$$[P, f][N] = [P \otimes_R N, f \otimes 1]$$

makes $K_1(R\pi)$ into a *right* $G_R(\pi)$ module. The quotients $\bar{K}_1(R\pi)$ and $\mathrm{Wh}(R\pi) = \bar{K}_1(R\pi)/\pi$ inherit $G_R(\pi)$ module structures. We use the above isomorphism $G_R(\pi) \to G(\pi)$ to consider $K_1(R\pi)$, $\bar{K}_1(R\pi)$ and $\mathrm{Wh}(R\pi)$ right $G(\pi)$ modules.

Now, let M be a b.f.-resolved $R\pi$ module which is R-torsion free. Let $\varepsilon : F_*(\varepsilon) \to M$ be the given b.f.r. Also let N be an $R\pi$ module which is finitely generated over R. *We shall construct a b.f.r. $\hat{\varepsilon}$ for $M \otimes_R N$ (diagonal π action).* Since there are short exact sequences of $R\pi$ modules

$$0 \to T \to N \to F \to 0 \tag{4.1}$$

$$0 \to M \otimes_R T \to M \otimes_R N \to M \otimes_R F \to 0 \tag{4.2}$$

(the latter induced from the former) where T is the R-torsion part of N and F is R-torsion free, we can treat the cases $N = T$ and $N = F$ separately.

Since T is finite, so is $\rho = \mathrm{Aut}_R(T)$. Also the action of π on T is a homomorphism $\alpha : \pi \to \rho$. Now choose a free finitely generated $R\rho$ module $\bar{F}_0$ and an epimorphism $\bar{F}_0 \to T$. Let $\bar{F}_1 \to \bar{F}_0$ be its kernel. Let $F_i = \alpha^! \bar{F}_i$. Then one has the exact sequence

$$0 \to F_1 \to F_0 \to T \to 0$$

of $R\pi$ modules, and F_0, F_1 are R-torsion free, i.e. they represent elements of $G_R(\pi)$. Under the abovementioned isomorphism $G(\pi) \to G_R(\pi)$, $[T]$ maps to $[F_0]-[F_1]$.

If $N = F$ is R-torsion free then one takes $F_1 = 0$, $F_0 = F$; thus in both cases we have the resolution

$$0 \to F_1 \to F_0 \to N \to 0 \tag{4.3}$$

with F_i R-torsion free. Then F_i is R free; *choose a basis f_i for F_i.*

When one equips the module $F_{i-1}(\varepsilon) \otimes F_1 \oplus F_i(\varepsilon) \otimes F_0$ with the $R\pi$ basis $f_{i-1}(\varepsilon) \otimes f_1 \cup f_i(\varepsilon) \otimes f_0$ the resolution

$$F_*(\varepsilon) \otimes_R F_* \to M \otimes_R N$$

represents a b.f.r. for $M \otimes_R N$ which we call $\hat{\varepsilon}$.

Theorem 4.4. *Let C be a finite chain complex of $R\pi$ modules. Assume that C_n and H_nC are b.f.-resolved and R-torsion free. Also let N be an $R\pi$ module finitely generated over R. Then under the action of $G(\pi) \cong G_R(\pi)$ on $\bar{K}_1(R\pi)$ one has*

$$\tau(C \otimes_R N) = \tau(C)[N]$$

when $C \otimes_R N$ and its homology are given b.f.r.'s as above.

Proof. We claim that it suffices to prove this for the case where C has the form

$$0 \to 0 \to \cdots \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0.$$

In fact, knowing the formula in that case amounts to knowing that

$$\{\hat{\mu}/\hat{\mu}_1\} = \{\mu/\mu_1\}[N] \tag{4.5}$$

holds whenever μ and μ_1 are b.f.r.'s for M (M R-torsion free $R\pi$ module) and $[N]$ is as above.

But if (4.5) holds then we see that the equivalence class of $\hat{\mu}$ depends only on that of μ (and on $[N] \in G(\pi)$). It then easily follows that

$$\widehat{\mu'\mu''} \sim \hat{\mu}'\hat{\mu}'' \tag{4.6}$$

holds whenever μ' and μ'' are b.f.r.'s for R-torsion free $R\pi$ modules that fit into a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of $R\pi$ modules.

And, using (4.5) and (4.6), the proof of Theorem 4.4 is purely formal. Thus we concentrate on proving (4.5). If

$$\varphi_*: F_*(\mu) \to F_*(\mu_1)$$

lifts the identity on M, then

$$\varphi_* \otimes_R F_*: F_*(\mu) \otimes_R F_* \to F_*(\mu_1) \otimes_R F_*$$

lifts the identity on $M \otimes_R N$. Also

$$\mathrm{MC}(\varphi_* \otimes_R F_*) \cong \mathrm{MC}(\varphi_*) \otimes_R F_*$$

by an isomorphism which is simple in each degree (the map is not just a permutation map; it also involves sign changes). Finally, there is an exact sequence of based, acyclic chain complexes

$$0 \to \mathrm{MC}(\varphi_*) \otimes_R F_0 \to \mathrm{MC}(\varphi_*) \otimes_R F_* \to \mathrm{MC}(\varphi_*) \otimes_R F_1 \to 0$$

with the projection having degree -1. It follows that

$$\{\hat{\mu}/\hat{\mu}_1\} = \tau(\mathrm{MC}(\varphi_*) \otimes_R F_0) - \tau(\mathrm{MC}(\varphi_*) \otimes_R F_1).$$

But for a based, acyclic chain complex like $\mathrm{MC}(\varphi_*)$ and an $R\pi$ module F_i, which is free over R, one easily sees that

$$\tau(\mathrm{MC}(\varphi_*) \otimes_R F_i) = \tau(\mathrm{MC}(\varphi_*))[F_i].$$

Since $[N] = [F_0] - [F_1]$ this finishes the proof of (4.5).

5. The transfer map for group rings

Recall that a group ν is said to be of type (FF) if $\mathbb{Z}$, viewed as $\mathbb{Z}\nu$ module with trivial ν action, admits some b.f.r.

Consider a group homomorphism $\varphi: \pi \to \rho$ with $\mathrm{Im}(\varphi) = \bar{\pi} \subseteq \rho$ and $\mathrm{Ker}(\varphi) = \nu \subseteq \pi$. Also, let $\varphi: \mathbb{Z}\pi \to \mathbb{Z}\rho$ be the corresponding ring homomorphism.

Proposition 5.1. *Let*

$$1 \to \nu \to \pi \xrightarrow{\varphi} \rho$$

be an exact sequence of groups. Assume that

(i) *ν is of type* (FF),

(ii) *the index $[\rho : \mathrm{Im}(\varphi)]$ is finite.*

Then $\varphi: \mathbb{Z}\pi \to \mathbb{Z}\rho$ gives rise to a transfer map

$$\varphi^*: \bar{K}_1(\mathbb{Z}\rho) \to \bar{K}_1(\mathbb{Z}\pi).$$

Moreover, if all the groups are finitely presented then φ^* induces a transfer map on

Whitehead groups

$$\varphi^*:\mathrm{Wh}(\rho)\to\mathrm{Wh}(\pi).$$

Proof. If $P_*\to\mathbb{Z}$ is a b.f.r. for $\mathbb{Z}$ over $\mathbb{Z}\nu$, then

$$\mathbb{Z}\pi\otimes_{\mathbb{Z}\nu}P_*\to\mathbb{Z}\pi\otimes_{\mathbb{Z}\nu}\mathbb{Z}=\mathbb{Z}\bar{\pi}$$

is a b.f.r. for $\mathbb{Z}\bar{\pi}$ over $\mathbb{Z}\pi$. Since $\mathbb{Z}\rho$ is a direct sum of $[\rho:\bar{\pi}]$ copies of $\mathbb{Z}\bar{\pi}$, as a $\mathbb{Z}\pi$ module, we have the desired transfer map on $\bar{K}_1$.

We have not found any direct algebraic proof that φ^* induces a map on Whitehead groups. This, however, follows from the main result of Section 7 (Remark 1 after Theorem 7.1).

6. Iterated mapping cylinder structures and the Serre spectral sequence

Let us start by recalling some results from Hatcher's paper [6]. If

$$F_0\xrightarrow{f_1}F_1\xrightarrow{f_2}\cdots\xrightarrow{f_s}F_s$$

is a string of PL maps between polyhedra then the iterated mapping cylinder $M(f_1,\ldots,f_s)$ is defined as follows: $M(f_1)$ is the ordinary mapping cylinder of f_1. Assume that $M(f_1,\ldots,f_{s-2})$ is defined together with the induced PL map $f'_{s-1}:M(f_1,\ldots,f_{s-2})\to F_{s-1}$; one then lets

$$M(f_1,\ldots,f_{s-1})=M(f'_{s-1})=M(f_1,\ldots,f_{s-1})\times I\cup F_{s-1}$$

and one defines f'_s by

$$f'_s(x,t)=f_sf'_{s-1}(x),\qquad x\in M(f_1,\ldots,f_{s-1}),\quad t\in I,$$

$$f'_s(y)=f_s(y),\qquad y\in F_{s-1}.$$

Clearly this is functorial in the sense that a commutative ladder of PL maps

$$\begin{array}{ccccccc}
F_0 & \xrightarrow{f_1} & F_1 & \longrightarrow & \cdots & \xrightarrow{f_s} & F_s\\
\downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & & & \downarrow{\scriptstyle\varphi_s}\\
G_0 & \xrightarrow{g_1} & G_1 & \longrightarrow & \cdots & \xrightarrow{g_s} & G_s
\end{array}$$

induces a PL map $M(\varphi_0,\ldots,\varphi_s):M(f_1,\ldots,f_s)\to M(g_1,\ldots,g_s)$.

Letting each G_i be a point one gets the projection

$$M(f_1,\ldots,f_s)\xrightarrow{\pi}\Delta[s]$$

onto the standard s-simplex. It is easily seen that the restriction of π to the ith face $\Delta^{(i)}[s]$ is (identifiable with)

$$M^{(i)}(f_1, \ldots, f_s) = \begin{cases} M(f_2, \ldots, f_s), & i = 0, \\ M(f_1, \ldots, f_i f_{i+1}, \ldots, f_s), & 1 \leq i < s, \\ M(f_1, \ldots, f_{s-1}), & i = s \end{cases}$$

(and its projection onto $\Delta^{(i)}[s]$).

Now let $p: E \to B$ be a PL map of compact polyhedra. Hatcher proves that p admits an *iterated mapping cylinder decomposition*; this means that we have the following:

(i) triangulations for E and B with respect to which p is simplicial;

(ii) an ordering of the vertices of each simplex σ (of the triangulation) of B; call the vertices $\sigma(0), \sigma(1), \ldots, \sigma(s)$ in this ordering ($s = \dim(\sigma)$);

(iii) simplicial maps $f_{i,\sigma}: p^{-1}(\sigma(i-1)) \to p^{-1}(\sigma(i))$ ($i = 1, 2, \ldots, s = \dim(\sigma)$; σ simplex of B);

(iv) PL homeomorphisms

$$\psi_\sigma : M(f_{1,\sigma}, f_{2,\sigma}, \ldots, f_{s,\sigma}) \to p^{-1}(\sigma) \subseteq E.$$

These data are subject to the following compatibility relations.

(i) When we view σ as an affine map from $\Delta[s]$ onto σ then

$$\begin{array}{ccc} M(f_{1,\sigma}, \ldots, f_{s,\sigma}) & \xrightarrow{\psi_\sigma} & p^{-1}(\sigma) \\ \downarrow & & \downarrow \\ \Delta[s] & \xrightarrow{\sigma} & \sigma \end{array}$$

commutes.

(ii) The orderings of the vertices of the simplices are compatible under face operations, i.e. if $\sigma^{(i)}$ is the ith face of σ, then

$$\begin{aligned} \sigma^{(i)}(j) &= \sigma(j) && \text{if } j < i \\ &= \sigma(j+1) && \text{if } j \geq i. \end{aligned}$$

(iii) The homeomorphisms ψ_σ are compatible with face operators, i.e.

$$f_{j,\sigma^{(i)}} = \begin{cases} f_{j,\sigma}, & j < i, \\ f_{j+1,\sigma} f_{j,\sigma}, & j = i, \\ f_{j+1,\sigma}, & j > i, \end{cases}$$

and

$$\psi_{\sigma^{(i)}} = \psi_\sigma \,|\, M^{(i)}(f_{1,\sigma^{(i)}}, \ldots, f_{s-1,\sigma^{(i)}}).$$

(iv) Finally, for any vertex v, $\psi_v : p^{-1}(v) \to p^{-1}(v)$ is the identity.

Moreover, Hatcher also proves that p is a PL fibration precisely if $f_{i,\sigma}^{-1}(x)$ is contractible for each $x \in p^{-1}(\sigma(i))$ and all i, σ. Hence for PL fibrations all the $f_{i,\sigma}$ are onto; this implies that the iterated mapping cylinder $M(f_{1,\sigma}, \ldots, f_{s,\sigma})$ is a quotient of $\Delta[s] \times p^{-1}(\sigma(0))$ by the relation $\sim$ generated by

$$(t_0, \ldots, t_s, x) \sim (t_0, \ldots, t_s, y),$$

if

$$t_0 = t_1 = \cdots = t_i = 0$$

and

$$f_{i+1,\sigma} f_{i,\sigma} \cdots f_{1,\sigma}(x) = f_{i+1,\sigma} f_{i,\sigma} \cdots f_{1,\sigma}(y).$$

We shall let

$$\mathrm{pr}: \Delta[s] \times p^{-1}(\sigma(0)) \to M(f_{1,\sigma}, \ldots, f_{s,\sigma})$$

be the projection.

Now let $p: E \to B$ be a PL fibration with basepoints $e_0 \in E$, $b_0 = p(e_0) \in B$ and fiber $F = p^{-1}(b_0)$. Assume that F, E, and B are connected and that

$$\pi_1(E, e_0) \xrightarrow{s} \pi \xrightarrow{\varphi} \rho = \pi_1(B, b_0)$$

is a factorization, with s onto, of

$$p_*: \pi_1(E, e_0) \to \pi_1(B, b_0).$$

Let $\nu \subset \pi$ be the kernel of φ. We then have a commutative diagram

$$\begin{array}{ccccc}
F & = & F & \xleftarrow{q_F} & \hat{F} \\
\downarrow & & \downarrow & & \downarrow \\
E & \xleftarrow{\bar{q}} & \bar{E} & \xleftarrow{\hat{q}} & \hat{E} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle \hat{p}} \\
B & \xleftarrow[q]{} & \tilde{B} & = & \tilde{B}
\end{array}$$

where q is the universal covering map for B, $\bar{q}$ the induced covering of E, $\hat{q}$ the covering of $\bar{E}$ corresponding to the subgroup $\mathrm{Ker}(s)$ of $\pi_1(\bar{E}, \bar{e}_0) = \mathrm{Ker}(\varphi s) \subseteq \pi_1(E, e_0)$; $\bar{p}$ and $\hat{p}$ are PL fibrations and the fiber $\hat{F} = \hat{p}^{-1}(\tilde{b}_0)$ is the covering of F induced from $\bar{q}$. Here, of course, base points are chosen compatibly, and we use them to identify the various fundamental groups with the relevant covering transformation groups. This means that the covering transformation groups are as follows:

$$\mathrm{Cov}(q_F) = \mathrm{Cov}(\hat{q}) = \nu \subseteq \pi = \mathrm{Cov}(\bar{q}\hat{q}),$$

$$\mathrm{Cov}(q) = \rho = \mathrm{Cov}(\bar{q}).$$

Also, let $A \subseteq B$ be a subcomplex; put $\tilde{A} = q^{-1}(A)$, $E_A = p^{-1}(A)$, $\hat{E}_A = \hat{p}^{-1}(\tilde{A})$. We want to describe the Serre spectral sequence for $(\hat{E}, \hat{E}_A) \to (\tilde{B}, \tilde{A})$ including its $\mathbb{Z}\pi$ module structure. This we do by means of iterated mapping cylinder decompositions.

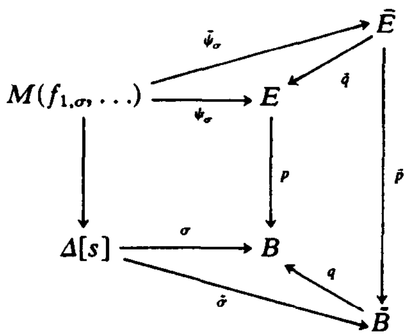
Thus let $\{\psi_\sigma, f_{i,\sigma}\}$ be such a decomposition for $p: E \to B$. For each vertex v of B choose a lifted vertex $\tilde{v}$ of $\tilde{B}$. Then each simplex σ of B is covered by a unique simplex $\tilde{\sigma}$ of $\tilde{B}$ having $\tilde{\sigma}(0) = \widetilde{\sigma(0)}$. And there is a unique $h(\sigma) \in \rho$ with

$$\widetilde{\sigma^{(0)}} = h(\sigma)\tilde{\sigma}^{(0)}$$

while

$$\widetilde{\sigma^{(i)}} = \tilde{\sigma}^{(i)}, \quad i > 0.$$

There are unique maps $\bar{\psi}_\sigma$ making



commute. It easily follows that

$$\bar{\psi}_{\sigma^{(i)}} = \bar{\psi}_\sigma | M^{(i)}(f_{1,\sigma}, \ldots).$$

Therefore, if one uses $\bar{q}: \bar{E} \to E$ to identify $\bar{p}^{-1}(\tilde{v})$ with $p^{-1}(p\tilde{v})$ for all vertices $\tilde{v}$ of $\tilde{B}$, then it is seen that $\bar{p}: \bar{E} \to \tilde{B}$ has an iterated mapping cylinder structure consisting of the obvious triangulations of $\bar{E}$ and $\tilde{B}$ and the maps

$$f_{i,g\tilde{\sigma}} = f_{i,\sigma}: \bar{p}^{-1}(g\tilde{\sigma}(i-1)) \to \bar{p}^{-1}(g\tilde{\sigma}(i)),$$

$$\psi_{g\tilde{\sigma}} = g\bar{\psi}_\sigma: M(f_{1,\sigma}, \ldots) \to \bar{E}.$$

Here σ varies over the simplices of B and g varies over $\rho = \mathrm{Cov}(\bar{q})$.

In order to lift this decomposition up to one for $\hat{p}: \hat{E} \to \tilde{B}$ we need the following lemma.

Lemma 6.1. *Let*

$$F_0 \xrightarrow{f_1} F_1 \to \cdots \xrightarrow{f_s} F_s$$

and a map $c: M(f_1, \ldots, f_s) \to B\nu$ be given (ν any discrete group). Let $\hat{F}_i \to F_i$ be the covering classified by $c\,|F_i$. Then there are unique maps $\hat{f}_i: \hat{F}_{i-1} \to \hat{F}_i$, covering f_i, such that the covering of $M(f_1, \ldots, f_s)$ classified by c is $M(\hat{f}_1, \ldots, \hat{f}_s)$.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} \hat{F}_0 & \longrightarrow & \hat{M}(f_1,\dots) & \longrightarrow & E\nu \\ {\scriptstyle i_0}\downarrow & \overset{\text{Pr}}{\nearrow} & \downarrow & & \downarrow \\ \Delta[s]\times\hat{F}_0 \longrightarrow \Delta[s]\times F_0 & \xrightarrow[\text{pr}]{} & M(f_1,\dots) & \xrightarrow[0]{} & B\nu \end{array}$$

where $\hat{M}(f_1,\dots)$ is the covering considered, and where $i_0(x)=(1,0,\dots,0,x)$, $x\in\hat{F}_0$. The dotted arrow, Pr, can be filled in because of standard properties of cofibrations versus fibrations. Actually the fill-in is unique; therefore it is equivariant with respect to the actions of ν. Also, the uniqueness permits one to prove the following: the restriction of Pr to (the ith vertex of $\Delta[s]$)$\times\hat{F}_0$ is a map $\text{Pr}_i:\hat{F}_0\to\hat{F}_i\subseteq\hat{M}(f_1,\dots)$; moreover $\text{Pr}_i=\hat{f}_i\,\text{Pr}_{i-1}$ for a unique map $\hat{f}_i:\hat{F}_{i-1}\to\hat{F}_i$, and $\hat{f}_i$ covers f_i. Finally, Pr represents $\hat{M}(f_1,\dots)$ as the quotient $M(\hat{f}_1,\dots)$ of $\Delta[s]\times\hat{F}_0$.

Returning to the PL fibration $\hat{p}:\hat{E}\to\tilde{B}$ we apply the lemma to the composite maps

$$c\psi_{g\tilde{\sigma}}:M(f_{1,\sigma},\dots)\to\bar{E}\to B\nu$$

where c classifies $\hat{E}$. There result maps

$$\hat{f}_{i,g\tilde{\sigma}}:\hat{p}^{-1}(g\tilde{\sigma}(i-1))\to\hat{p}^{-1}(g\tilde{\sigma}(i))$$

covering

$$f_{i,g\tilde{\sigma}}:\bar{p}^{-1}(g\tilde{\sigma}(i-1))\to\bar{p}^{-1}(g\tilde{\sigma}(i))$$

and maps

$$\hat{\psi}_{g\tilde{\sigma}}:M(\hat{f}_{1,g\tilde{\sigma}},\dots)\to\hat{E}.$$

It is straightforward to prove that the set $\{\hat{f}_{i,g\tilde{\sigma}},\hat{\psi}_{g\tilde{\sigma}}\}$ gives an iterated mapping cylinder decomposition of $\hat{p}:\hat{E}\to\tilde{B}$ when g varies over ρ and σ varies over the simplices of B.

With this decomposition available we return to the Serre spectral sequence for the homology of $(\hat{E},\hat{E}_A)$. We let C_* denote the ordered simplicial chain functor. Then $C_*(\hat{E},\hat{E}_A)$ is filtered by the subcomplexes

$$F_sC_*(\hat{E},\hat{E}_A)=C_*(\hat{E}_s,\hat{E}_A)$$

where

$$\hat{E}_s=\hat{p}^{-1}((\tilde{B},\tilde{A})^{[s]}),\quad(\tilde{B},\tilde{A})^{[s]}\text{ the }s\text{-skeleton of }(\tilde{B},\tilde{A}).$$

In the resulting spectral sequence one has

$$E^0_{s,*}=C_{s+*}(\hat{E}_s,\hat{E}_{s-1}),\ E^1_{s,*}=H_{s+*}(\hat{E}_s,\hat{E}_{s-1}).$$

If we let σ vary over the s-simplices of B, not in A, and we let g vary over ρ, then we have a relative homomorphism

$$\hat{\psi}:\coprod_{g,\sigma}(M(\hat{f}_{1,g\tilde{\sigma}},\dots),\dot{M}(\hat{f}_{1,g\tilde{\sigma}},\dots)\to(\hat{E}_s,\hat{E}_{s-1})\tag{6.2}$$

which is $\hat{\psi}_{g\tilde{\sigma}}$ on the summand corresponding to (g, σ). Also there is the sum of the projections

$$\mathrm{pr}: \coprod_{g,\sigma} (\Delta[s], \dot{\Delta}[s]) \times \hat{p}^{-1}(g\tilde{\sigma}(0) \to \coprod_{g,\sigma} (M(\hat{f}_{1,g\tilde{\sigma}}, \ldots), \dot{M}(\hat{f}_{1,g\tilde{\sigma}}, \ldots)). \tag{6.3}$$

The composition $\hat{\psi}(\mathrm{pr})$ is a homology isomorphism, so if we let ι_s be the generator of $H_s(\Delta[s], \dot{\Delta}[s])$ we get an isomorphism

$$\theta: \coprod_{g,\sigma} \{g\tilde{\sigma}\} \times H_*(\hat{p}^{-1}(g\tilde{\sigma}(0))) \to E^1_{s,*} \tag{6.4}$$

given by

$$\theta(g\tilde{\sigma}, x) = \hat{\psi}_*(\mathrm{pr})_*(\iota_s \times x), \quad x \in H_*(\hat{p}^{-1}(g\tilde{\sigma}(0))). \tag{6.5}$$

It is easily seen that under the isomorphism θ the differential d^1 corresponds to the map ∂ given by

$$\partial(g\tilde{\sigma}, x) = (g\tilde{\sigma}^{(0)}, (\hat{f}_{1,g\tilde{\sigma}})_*(x)) + \sum_{i=1}^{s} (-1)^i (g\tilde{\sigma}^{(i)}, x). \tag{6.6}$$

What we have recovered is, of course, just the well-known isomorphism

$$\theta: C_*(\tilde{B}, \tilde{A}; \mathcal{H}_*(\hat{F})) \to E^1_{**} \tag{6.7}$$

between E^1_{**} and the (ordered, simplicial) chains of $(\tilde{B}, \tilde{A})$ with local coefficients $\mathcal{H}_*(\hat{F})$.

If $h \in \pi = \mathrm{Cov}(\tilde{q}\hat{q})$ has image $\bar{h} \in \rho$, then under the isomorphism θ the action of h on E^1_{**} becomes

$$h(g\tilde{\sigma}, x) = (\bar{h}g\tilde{\sigma}, h_* x) \tag{6.8}$$

where

$$h_*: H_*(\hat{p}^{-1}(g\tilde{\sigma}(0)) \to H_*(\hat{p}^{-1}(\bar{h}g\tilde{\sigma}(0))).$$

Any edge path ω, from a vertex $\tilde{u}$ to a vertex $\tilde{v}$, in $\tilde{B}$ gives rise to an isomorphism

$$\omega_*: H_*(\hat{p}^{-1}(\tilde{u})) \to H_*(\hat{p}^{-1}(\tilde{v}))$$

which is composed of maps of the form $(\hat{f}_{1,\tau})_*$ and their inverses (τ an edge in $\tilde{B}$). Since $\tilde{B}$ is simply connected this isomorphism depends only on the endpoints. There results an isomorphism

$$\chi: C_*(\tilde{B}, \tilde{A}) \otimes H_*(\hat{F}) \to C_*(\tilde{B}, \tilde{A}, \mathcal{H}_*(\hat{F})) \tag{6.9}$$

sending $g\tilde{\sigma} \otimes x$ to $(g\tilde{\sigma}, \omega_*(x))$ where ω is any path in $\tilde{B}$ from $\tilde{b}_0$ to $g\tilde{\sigma}(0)$.

Any $h \in \pi$ maps $M(\hat{f}_{1,g\tilde{\sigma}}, \ldots)$ onto $M(\hat{f}_{1,\bar{h}g\tilde{\sigma}}, \ldots)$, hence

$$\begin{array}{ccc}
\hat{p}^{-1}(g\tilde{\sigma}(0)) & \xrightarrow{\hat{f}_{1,g\tilde{\sigma}}} & \hat{p}^{-1}(g\tilde{\sigma}(1)) \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
\hat{p}^{-1}(\bar{h}g\tilde{\sigma}(0)) & \xrightarrow[\hat{f}_{1,\bar{h}g\tilde{\sigma}}]{} & \hat{p}^{-1}(\bar{h}g\tilde{\sigma}(1))
\end{array}$$

commutes. It follows that there is a π-action on $H_*(\hat{F})$ which lets $h \in \pi$ act as the composition

$$H_*(\hat{F}) \xrightarrow{h_*} H_*(\hat{p}^{-1}(\bar{h}(\tilde{b}_0))) \xrightarrow{\omega(b)_*} H_*(\hat{F})$$

where $\omega(b)$ is any edge path in $\tilde{B}$ from $\bar{h}(\tilde{b}_0)$ to $\tilde{b}_0$.

Also, when one transfers the π action on $C_*(\tilde{B}, \tilde{A}, \mathcal{H}_*(\hat{F}))$ back to $C_*(\tilde{B}, \tilde{A}) \otimes H_*(\hat{F})$ via the isomorphism χ, then one gets

$$h(g\tilde{\sigma} \otimes y) = \bar{h}g\tilde{\sigma} \otimes h(y), \quad (h \in \pi,\ g\tilde{\sigma} \otimes y \in C_*(\tilde{B}, \tilde{A}) \otimes H_*(\hat{F})). \tag{6.10}$$

This means that we have shown the following proposition.

Proposition 6.11. *With the above π-action on $H_*(\hat{F})$ and diagonal action on the tensor product one has a π-equivariant isomorphism of chain complexes*

$$\chi\theta : \varphi^! C_*(\tilde{B}, \tilde{A}) \otimes H_*(\hat{F}) \to E^1_{**}.$$

7. The main theorem

Let $p : E \to B$ be a PL fibration with

$$\pi_1(E, e_0) \underset{r}{\to} \pi \underset{\varphi}{\to} \rho = \pi_1(B, b_0)$$

a given factorization of the induced map p_*, with r onto. Let F_0 be the base point component of the fiber F. It is easily seen that there is a PL fibration $p_0 : E \to B_0$ where $B_0 = \tilde{B}/\bar{\pi} \to B$ is the (possibly irregular) covering of B corresponding to the subgroup $\bar{\pi} = \mathrm{Im}(\varphi)$ of ρ; moreover the fiber of p_0 is F_0. Applying the considerations of Section 6 one gets the diagram

$$\begin{array}{ccccc}
F & \supseteq & F_0 & \longleftarrow & \hat{F}_0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
E & = & E & \longleftarrow & \hat{E} \\
\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p_0} & & \big\downarrow{\scriptstyle \hat{p}_0} \\
B & \underset{q_0}{\longleftarrow} & B_0 & \longleftarrow & \tilde{B}
\end{array}$$

Then π acts on $H_*(\hat{F}_0)$, so we get an element $\sum(-1)^i[H_i(\hat{F}_0)] \in G(\pi)$. Our main result is the following theorem.

Theorem 7.1. *In the above situation assume that v is of type* (FF) *and that we have a deformation retract A of B. Then the transfer map $\varphi^* : \bar{K}_1(\mathbb{Z}\rho) \to \bar{K}_1(\mathbb{Z}\pi)$ is defined, and if $\beta \in \bar{K}_1(\mathbb{Z}\rho)$ represents $\tau(B, A) \in \mathrm{Wh}(\rho)$ then $r_*\tau(E, p^{-1}A) \in \mathrm{Wh}(\pi)$ is* *represented by the product*

$$\varphi^*(\beta)\cdot\sum(-1)^i[H_i(\hat{F}_0)]$$

(using the $G(\pi)$ *module structure on* $\bar{K}_1(\mathbb{Z}\pi)$*).*

Remarks. (1) If ν is of type (FF) and $\nu\to\pi\xrightarrow{\bar{\varphi}}\bar{\pi}$ is a short exact sequence of finitely presented groups then we take $p_A\colon E_A\to A$ to be the restriction of $B\pi\to B\bar{\pi}$ to the two-skeleton of $B\bar{\pi}$. For given $\tau\in\mathrm{Wh}(\bar{\pi})$ there is then a finite complex $B\supseteq A$ such that A is a deformation retract of B and $\tau(B,A)=\tau$. Letting $d\colon B\to A$ be the deformation retraction we take $p\colon E\to B$ to be the pull-back of $p_A\colon E_A\to A$ via d. When one applies the theorem to an arbitrary representative $\beta\in\bar{K}_1(\mathbb{Z}\bar{\pi})$ for $\tau\in\mathrm{Wh}(\bar{\pi})$ one notes that $\bar{\varphi}^*(\beta)\in\bar{K}_1(\mathbb{Z}\pi)$ represents $\tau(E,E_A)\in\mathrm{Wh}(\pi)$. Since the latter is independent of the choice of β it follows that $\bar{\varphi}^*$ does induce a map from $\mathrm{Wh}(\bar{\pi})$ to $\mathrm{Wh}(\pi)$. This proves the "moreover part" of Proposition 5.1 for the case where $\varphi\colon\pi\to\rho$ is onto. The general case follows since it is very easy, algebraically, to prove that the transfer map i^* corresponding to the inclusion $i\colon\bar{\pi}\to\rho$ induces a map on Whitehead groups.

(2) If $r=$ identity and $F=B\nu$, then the theorem specializes to Corollary A of the introduction.

(3) If $r=p_*$, so $\pi=\mathrm{Im}(p_*)\subseteq\rho$, then we get Corollary B.

(4) Finally Corollary C results when one takes $r=$ identity.

Proof of Theorem 7.1. First note that one may assume $p_*\colon\pi_1(E,e_0)\to\pi_1(B,b_0)$ onto. Indeed, referring to the diagram above, one has

$$\tau(E,p^{-1}A)=\tau(E,p_0^{-1}(q_0^{-1}A)),$$

$$\tau(B_0,q_0^{-1}(A))=i^*\tau(B,A),$$

where $i\colon\bar{\pi}\to\pi_1(B,b_0)$ is the inclusion. Since also $\varphi^*=\bar{\varphi}^*i^*$, when $\varphi=i\bar{\varphi}$, the theorem for p_0 implies the one for p.

Thus, from now on, p_* is onto, and (consequently) $F_0=F$. Then $\beta=\tau(C_*(\tilde{B},\tilde{A}))$ where the preferred $\mathbb{Z}\rho$ basis is obtained by taking one lifted simplex $\tilde{\sigma}$ for every simplex σ of (B,A). Also $r_*\tau(E,E_A)$ is represented by $C_*(\hat{E},\hat{E}_A)$ similarly based over $\mathbb{Z}\pi$.

In the Serre spectral sequence for $C_*(\hat{E},\hat{E}_A)$ the E^0_{**}-term

$$E^0_{s,t}=C_{s+t}(\hat{E}_s,\hat{E}_{s-1})$$

is based in a similar way as (and compatibly with) $C_*(\hat{E},\hat{E}_A)$. Also we may use Proposition 6.11 to transport onto $E^1_{s,t}$ the b.f.r. on $\varphi^! C_s(\tilde{B},\tilde{A})\otimes H_t(\hat{F})$ which results from the given $\mathbb{Z}\rho$ basis in $C_s(\tilde{B},\tilde{A})$ by using Theorems 3.2 and 4.4. Also one then has, using first Theorem 2.2, then the definition of $\tau(E^*_{**})$ and finally Theorems

3.2 and 4.4,

$$\begin{aligned}\tau(C_*(\hat{E}, \hat{E}_A)) &= \tau(E^*_{**}) \\ &= \sum (-1)^s \tau(E^0_{s,*}) + \sum (-1)^t \tau(E^1_{*,t}) \\ &= \sum (-1)^s \tau(E^0_{s,*}) + \varphi^*(\beta) \cdot \sum (-1)^t [H_t(\hat{F})].\end{aligned}$$

Thus the proof will be completed by showing that $\sum (-1)^s \tau(E^0_{s,*}) = 0$. This occupies the rest of the present section. The main tool is the following generalization of Anderson's excision lemma [1].

Lemma 7.2. *Let $\hat{E} \to E$ be a regular covering with covering transformation group π and E' a subcomplex of E with induced covering $\hat{E}' \subseteq \hat{E}$. Assume given a* PL *relative homeomorphism $\varphi: (M, M') \to (E, E')$ and a pull-back diagram*

$$\begin{array}{ccc}(\hat{M}, \hat{M}') & \xrightarrow{\hat{\varphi}} & (\hat{E}, \hat{E}') \\ \downarrow & & \downarrow \\ (M, M') & \xrightarrow{\varphi} & (E, E')\end{array}$$

If $H_*(\hat{M}, \hat{M}')$ and $H_*(\hat{E}, \hat{E}')$ are given b.f.r.'s over $\mathbb{Z}\pi$ with respect to which $\hat{\varphi}_*$ is simple, and if $C_*(\hat{M}, \hat{M}')$, $C_*(\hat{E}, \hat{E}')$ are given the obvious bases then

$$\tau(C_*(\hat{M}, \hat{M}')) = \tau(C_*(\hat{E}, \hat{E}'))$$

in $\mathrm{Wh}(\pi)$.

Proof. Anderson's proof [1] carries over directly to the present situation.

The lemma applies to the pull-back diagram

$$\begin{array}{ccc}\pi \times_\nu \coprod_\sigma (M(\hat{f}_{1,\tilde{\sigma}}, \ldots), \dot{M}(\hat{f}_{1,\tilde{\sigma}}, \ldots)) & \xrightarrow{\hat{\varphi}} & (\hat{E}_s, \hat{E}_{s-1}) \\ \downarrow & & \downarrow \\ \coprod_\sigma (M(f_{1,\sigma}, \ldots), \dot{M}(f_{1,\sigma}, \ldots)) & \xrightarrow{\varphi} & (E_s, E_{s-1})\end{array}$$

Here $\varphi | M(f_{1,\sigma}, \ldots) = \psi_\sigma$ in some iterated mapping cylinder structure for $p: E \to B$. Thus φ is a PL relative homomorphism when σ varies over the s-simplices of (B, A). Also $\hat{\varphi}$ restricted to $1 \times_\nu M(f_{1,\tilde{\sigma}}, \ldots)$ is $\hat{\psi}_{\tilde{\sigma}}$ (see Section 6) and $\hat{\varphi}$ is π-equivariant.

For brevity let $M_{g\tilde{\sigma}} = M(\hat{f}_{1,g\tilde{\sigma}}, \ldots)$ and $\hat{F}_{g\tilde{\sigma}} = \hat{p}^{-1}(g\tilde{\sigma}(0))$. One then has an isomorphism

$$\alpha: \coprod_\sigma \mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} H_*(\hat{F}) \to H_*\Big(\pi \times_\nu \coprod_\sigma (M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})\Big)$$

which sends $g \otimes x \in \mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} H_*(\hat{F})$ into

$$\mathrm{pr}_*(\iota_s \times \omega_*(x)) \in H_*(g \times_\nu (M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})) \subseteq H_*(\pi \times_\nu \coprod (M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}}))$$

where ι_s generates $H_s(\Delta[s], \dot{\Delta}[s])$, pr is the projection $\Delta[s]\times \hat{F}_{\tilde{\sigma}} \to M_{\tilde{\sigma}}$, and ω is a path in $\tilde{B}$ from $\tilde{b}_0$ to $\tilde{\sigma}(0)$.

We now fix a b.f.r. for $H_*(\hat{F})$ over $\mathbb{Z}\nu$, say $\varepsilon_{\hat{F}}$. Then $\coprod_\sigma \mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} \varepsilon_{\hat{F}}$ is a b.f.r. for $\coprod_\sigma \mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} H_*(\hat{F})$; we use α to transport it over to become a b.f.r. for $H_*(\pi \times_\nu \coprod_\sigma (M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}}))$. Since the basis for $C_*(\pi \times_\nu \coprod_\sigma (M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}}))$ is obtained from $\mathbb{Z}\nu$ bases for $C_*(M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})$ by applying $\coprod_\sigma(\mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} \text{-})$ we see that

$$\tau\Big(C_*\Big(\pi \times_\nu \coprod_\sigma (M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})\Big)\Big) = \sum_\sigma i_* \tau(C_*(M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})) \tag{7.3}$$

where $i: \nu \to \pi$ is the inclusion. To finish the proof we need the following two lemmas.

Lemma 7.4. *With the above notation*

$$\tau(C_*(M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})) = (-1)^s \tau(C_*(\hat{F}_{\tilde{\sigma}})) = (-1)^s \tau(C_*(\hat{F})).$$

Lemma 7.5. *With the chosen b.f.r.'s*

$$\hat{\varphi}_*: H_*\Big(\pi \times_\nu \coprod_\sigma (M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})\Big) \to H_*(\hat{E}_s, \hat{E}_{s-1}) = E^1_{s,*-s}$$

is simple.

Indeed, Lemma 7.4 shows that the excision lemma is applicable. And the excision lemma in its turn shows that (7.3) is $\tau(C_*(\hat{E}_s, \hat{E}_{s-1}))$. Since obviously $\tau(E^0_{s,*}) = (-1)^s \tau(C_*(\hat{E}_s, \hat{E}_{s-1}))$ one gets

$$\begin{aligned}\sum_s (-1)^s \tau(E^0_{s,*}) &= \sum_s \tau(C_*(\hat{E}_s, \hat{E}_{s-1}))\\ &= \sum_{s,\sigma} i_* \tau(C_*(M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}}))\\ &= \sum_{s,\sigma} (-1)^s i_* \tau(C_*(\hat{F}))\\ &= \chi(B, A) i_* \tau(C_*(\hat{F}))\\ &= 0\end{aligned}$$

because the Euler characteristic $\chi(B, A)$ vanishes.

Proof of Lemma 7.4. For the first equality it suffices to show that

$$\tau(C_*(M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})) = -\tau(C_*(M_{\tilde{\sigma}^{(s)}}, \dot{M}_{\tilde{\sigma}^{(s)}})). \tag{7.6}$$

Let us write $M^{[i]}_{\tilde{\sigma}}$ for the part of $M_{\tilde{\sigma}}$ which sits above $\tilde{\sigma}^{(0)} \cup \tilde{\sigma}^{(1)} \cup \cdots \cup \tilde{\sigma}^{(i)}$. Hence, especially, $M^{[s]}_{\tilde{\sigma}} = \dot{M}_{\tilde{\sigma}}$. We start by showing

$$\tau(C_*(M_{\tilde{\sigma}}, M^{[i]}_{\tilde{\sigma}}) = 0 \quad \text{for } 0 \leq i \leq s-1. \tag{7.7}$$

This is done by induction on s and for fixed s by induction on i. First note that $H_*(M_{\tilde\sigma}, M_{\tilde\sigma}^{[i]}) = 0$ so that, at least, the desired torsion element is defined. For $s = 0$ the statement is empty. For general s, and $i = 0$, one notes that $M_{\tilde\sigma}$ is the mapping cone of the obvious map $\hat F_{\tilde\sigma(0)} \to M_{\tilde\sigma}^{[0]}$ so Lemma 7.5 of Milnor [8] implies the vanishing. For $0 < i \leq s-1$ we have the short exact sequence of acyclic, based $\mathbb{Z}\nu$ module chain complexes

$$C_*(M_{\tilde\sigma}^{[i]}, M_{\tilde\sigma}^{[i-1]}) \to C_*(M_{\tilde\sigma}, M_{\tilde\sigma}^{[i-1]}) \to C_*(M_{\tilde\sigma}, M_{\tilde\sigma}^{[i]}).$$

When one applies Theorem 2.3 and the inductive hypothesis one gets

$$\tau(C_*(M_{\tilde\sigma}, M_{\tilde\sigma}^{[i]}) = -\tau(C_*(M_{\tilde\sigma}^{[i]}, M_{\tilde\sigma}^{[i-1]}))$$

but the latter vanishes by the inductive hypothesis and the excision lemma $((M_{\tilde\sigma}^{[i]}, M_{\tilde\sigma}^{[i-1]})$ is replaced by $(M_{\tilde\sigma^{(i)}}, M_{\tilde\sigma^{(i)}}^{[i-1]}))$.

With (7.7) established we proceed to prove (7.6). The short exact sequence

$$C_*(\dot M_{\tilde\sigma}, M_{\tilde\sigma}^{[s-1]}) \to C_*(M_{\tilde\sigma}, M_{\tilde\sigma}^{[s-1]}) \to C_*(M_{\tilde\sigma}, \dot M_{\tilde\sigma})$$

has $H_*(M_{\tilde\sigma}, M_{\tilde\sigma}^{[s-1]}) = 0$. Also, when we give $H_*(M_{\tilde\sigma}, M_{\tilde\sigma}^{[s-1]})$ a b.f.r. coming from the one on $H_*(\hat F)$ by the isomorphism

$$H_*(\hat F) \xrightarrow{\omega_*} H_*(\hat F_{\tilde\sigma}) \cong H_*(\dot\Delta, \Delta^{[s-1]}) \otimes H_*(\hat F_{\tilde\sigma}) \xrightarrow[\mathrm{pr}_*]{} H_*(\dot M_{\tilde\sigma}, M_{\tilde\sigma}^{[s-1]})$$

then the boundary isomorphism in the homology sequence becomes simple (both in the technical, and the non-technical sense), so Theorem 2.3 implies that

$$\tau(C_*(M_{\tilde\sigma}, \dot M_{\tilde\sigma})) = -\tau(C_*(\dot M_{\tilde\sigma}, M_{\tilde\sigma}^{[s-1]})).$$

And now an obvious excision argument finishes the proof of (7.6).

For the second equality in Lemma 7.4 it suffices to prove that

$$\tau(C_*(\hat F_{\tilde\sigma(0)})) = \tau(C_*(\hat F_{\tilde\sigma(1)}). \tag{7.8}$$

We let $\tilde\tau$ be the edge of $\tilde\sigma$ connecting $\tilde u = \tilde\sigma(0)$ to $\tilde v = \tilde\sigma(1)$. Then we have the short exact sequence

$$C_*(\hat F_{\tilde u} \cup \hat F_{\tilde v}, \hat F_{\tilde u}) \to C_*(M_{\tilde\tau}, \hat F_{\tilde u}) \to C_*(M_{\tilde\tau}, \dot M_{\tilde\tau}).$$

Theorem 2.3 is applicable and gives

$$\tau(C_*(M_{\tilde\tau}, \hat F_{\tilde u})) = \tau(C_*(M_{\tilde\tau}, \dot M_{\tilde\tau})) + \tau(C_*(\hat F_{\tilde u} \cup \hat F_{\tilde v}, \hat F_{\tilde u})).$$

By definition, the left-hand side is the image in $\mathrm{Wh}(\nu)$ (under the obvious projection $\pi_1(F, e_0) \to \nu$) of the Whitehead torsion of $f_{1,\sigma}: p^{-1}(\sigma(o)) \to p^{-1}(\sigma(1))$. Since $f_{1,\sigma}$ is simple, by Hatcher [6], we see that the left-hand side vanishes.

The first term on the right-hand side is $-\tau(C_*(\hat F_{\tilde\sigma(0)}))$ by the first part of the theorem. And the last term on the right-hand side is $\tau(C_*(\hat F_{\tilde\sigma(1)}))$. Thus the proof is complete.

Proof of Lemma 7.5. We have the following commutative diagram

$$\begin{array}{ccc} \coprod_{\sigma} \mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} H_*(\hat{F}) & \xrightarrow{\alpha} & H_*(\pi \times_\nu \coprod_{\sigma} (M_{\tilde{\sigma}}, \dot{M}_{\tilde{\sigma}})) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \hat{\varphi}_*} \\ \varphi^! C_*(\tilde{B}, \tilde{A}) \otimes H_*(\hat{F}) & \xrightarrow{\chi\theta} & H_*(\hat{E}_s, \hat{E}_{s-1}) \end{array}$$

where μ restricted to the summand corresponding to σ is given by

$$\mu_\sigma(g \otimes x) = \bar{g}\tilde{\sigma} \otimes g(x).$$

Here $\bar{g} \in \rho$ is the image of $g \in \pi$ under φ and $g(x)$ refers to the action of π on $H_*(\hat{F})$ defined earlier. Since the b.f.r.'s involved are transports by means of α and $\chi\theta$ of those on the left-hand modules, and since μ is the direct sum of the μ_σ it suffices to prove that

$$\mu_\sigma: \mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} H_*(\hat{F}) \to \varphi^!(\mathbb{Z}\rho) \otimes H_*(\hat{F}) \tag{7.9}$$

is simple.

Here π acts from the left only on the source, and diagonally on the target. Also one has $\mu_\sigma(g \otimes x) = \bar{g} \otimes g(x)$.

Let $P_* \to \mathbb{Z}$ be the chosen b.f.r. for $\mathbb{Z}$ over $\mathbb{Z}\nu$, $Q_* \to \mathbb{Z}$ the derived b.f.r. as in the proof of Lemma 3.1. Then $\mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} P_* \to \varphi^!(\mathbb{Z}\rho)$ is a b.f.r. for $\varphi^!(\mathbb{Z}\rho)$ and the proof of Lemma 3.1 gives $\mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} Q_* \to \varphi^!(\mathbb{Z}\rho)$ as our preferred b.f.r. for $\varphi^!(\mathbb{Z}\rho)$. Also let

$$0 \to R_1 \to R_0 \to H_i(\hat{F}) \to 0$$

be a resolution of $H_i(\hat{F})$ by $\mathbb{Z}\pi$ modules, free and finitely generated over $\mathbb{Z}$ (as in (4.3)). Then

$$(\mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} Q_*) \otimes R_* \to \varphi^!(\mathbb{Z}\rho) \otimes H_i(\hat{F}) \tag{7.10}$$

is the chosen b.f.r. for the right-hand side of (7.9). The corresponding $\mathbb{Z}\pi$ basis is $1 \otimes q_* \otimes r_*$ where q_* is a $\mathbb{Z}\nu$ basis for R_* and r_* is a $\mathbb{Z}$ basis for R_*. Moreover, in (7.10) $h \in \pi$ acts as follows

$$h((g \otimes q) \otimes r) = (hg \otimes q) \otimes hr. \tag{7.11}$$

On the other hand, as the preferred b.f.r. for $H_i(\hat{F})$, over $\mathbb{Z}\nu$, we have

$$Q_* \otimes R_* \to \mathbb{Z} \otimes H_i(\hat{F}) = H_i(\hat{F})$$

with bases $q_* \otimes r_*$. So the chosen b.f.r. for $\mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} H_i(\hat{F})$ is

$$\mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} (Q_* \otimes R_*) \to \mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} H_i(\hat{F}) \tag{7.12}$$

with bases $1 \otimes q_* \otimes r_*$. Also the module structure is given by

$$h(g \otimes (q \otimes r)) = hg \otimes q \otimes r. \tag{7.13}$$

The map μ_σ lifts to the following map of resolutions

$$\mu_{\sigma^*}: \mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} (Q_* \otimes R_*) \to (\mathbb{Z}\pi \otimes_{\mathbb{Z}\nu} Q_*) \otimes R_*$$

with

$$\mu_{\sigma^*}(g \otimes q \otimes r) = g \otimes q \otimes g(r).$$

It is easily seen that μ_{0^*} is well defined and a lifting of μ_σ. Since μ_{σ^*} preserves the preferred bases the proof is finished.

References

[1] D.R. Anderson, The Whitehead torsion of the total space of a fiber bundle, Topology 11 (1972) 179–194.

[2] D.R. Anderson, The Whitehead torsion of a fiber homotopy equivalence, Michigan Math. J. 21 (1974) 171–180.

[3] H. Bass, Algebraic *K*-theory (W.A. Benjamin, New York, 1968).

[4] H. Bass, Introduction to some methods of algebraic *K*-theory, Regional Conference Series in Mathematics 20 (American Mathematical Society, Providence, RI, 1974).

[5] K. Ehrlich, Fibrations and transfer map in algebraic K-theory, J. Pure Appl. Algebra 14 (1979) 131–136.

[6] A.E. Hatcher, Higher simple homotopy theory, Ann. of Math. 102 (1975) 101–137.

[7] S. Maumary, Contributions á la théorie du type simple d'homotopie, Comment. Math. Helv. 44 (1969) 410–437.

[8] J.W. Milnor, Whitehead torsion, Bull. Amer. Math. Soc. 72 (1966) 358–426.

[9] E.K. Pedersen and L.R. Taylor, The Wall finiteness obstruction for a fibration, Amer. J. Math. 100 (1978) 887–896.

[10] E.K. Pedersen, Universal geometric examples for transfer maps in algebraic K- and L-theory, J. Pure Appl. Alg., to appear.

[11] D.G. Quillen, Higher algebraic K-theory I, Lecture Notes in Mathematics 341 (Springer-Verlag, Berlin, 1973).