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3-colourability of Penrose kite-and-dart tilings

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Abstract

There are many tilings of the plain, some of them are periodic, others are aperiodic. A chromatic number of a tiling is defined as the minimum number of colours needed to colour the tiles in such a way that every two adjacent tiles have different colours. Determining the chromatic number of a periodic tiling is mostly easy but this problem has not been investigated for aperiodic tilings yet. In this paper the problem is solved for one of the most known aperiodic tiling called Penrose kite-and-dart tiling. This tiling is often used as a planar model for so called quasicrystals.

1. Properties of Penrose tilings

Let us have two types of tiles, the socalled kites and darts (see Fig. 1). The lengths of their edges are 1 and τ, where $\tau = (1+\sqrt{5})/2$ is the golden ratio; their vertices are marked by black and white dots. We can put the tiles to each other with the whole common edge only, and the colours of the vertices on the common edge must match.

These tiles with the above matching rules were discovered by Penrose [1]. He showed that these matching rules admit the tiling of the whole plane (without holes and overlaps) in infinitely many ways and every such a tiling is aperiodic, i.e. it admits no translation. Such tilings are called Penrose (kite-and-dart) tilings, for portions of these tilings see Figs. 6 and 3.

Bickford [2] posed the question to determine the chromatic number of Penrose tilings. The chromatic number is defined as the minimum number of colours we need to colour the tiles in such a way that every two adjacent (i.e. with a common edge) tiles have different colours. There are infinitely many Penrose tilings but this question has a good sense because of the next property. Every finite portion of any Penrose tiling is contained in every Penrose tiling, even infinitely many times. This implies due to compactness that the chromatic number is the same for all Penrose tilings.

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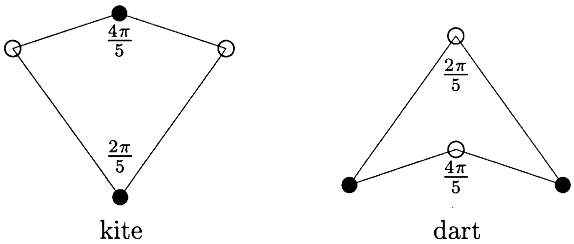


Fig. 1. Kite and dart.

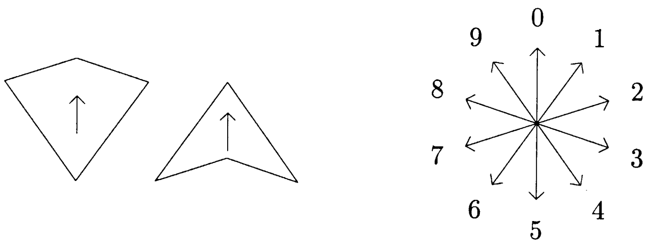
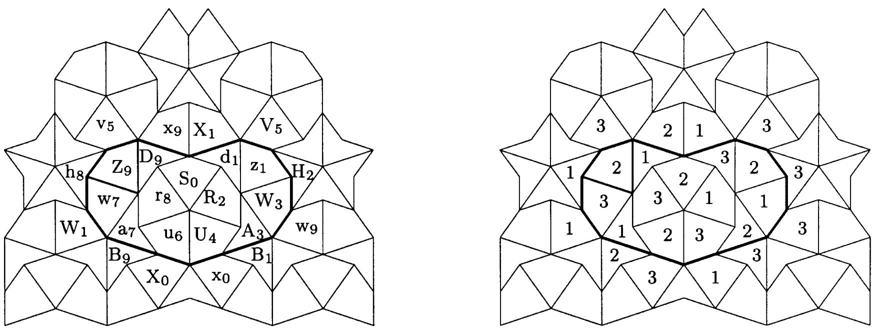


Fig. 2. Orientation of tiles.



(a) Names of tiles (b) Colours of tiles

Fig. 3. Neighbourhood of tile 'Z_9'.

2. Three-colourability of Penrose tilings

In the previous section we have discussed that the chromatic number χ do not depend on the concrete choice of a Penrose tiling. The upper bound $\chi \leqslant 4$ follows from the well known four-colour theorem for planar graphs or from the Brook's Theorem. On the contrary, the lower bound $\chi \geqslant 3$ is easy to see. Then, the problem is to determine whether $\chi = 3$ or 4.

Theorem. *The chromatic number of every Penrose kite-and-dart tiling is* 3.

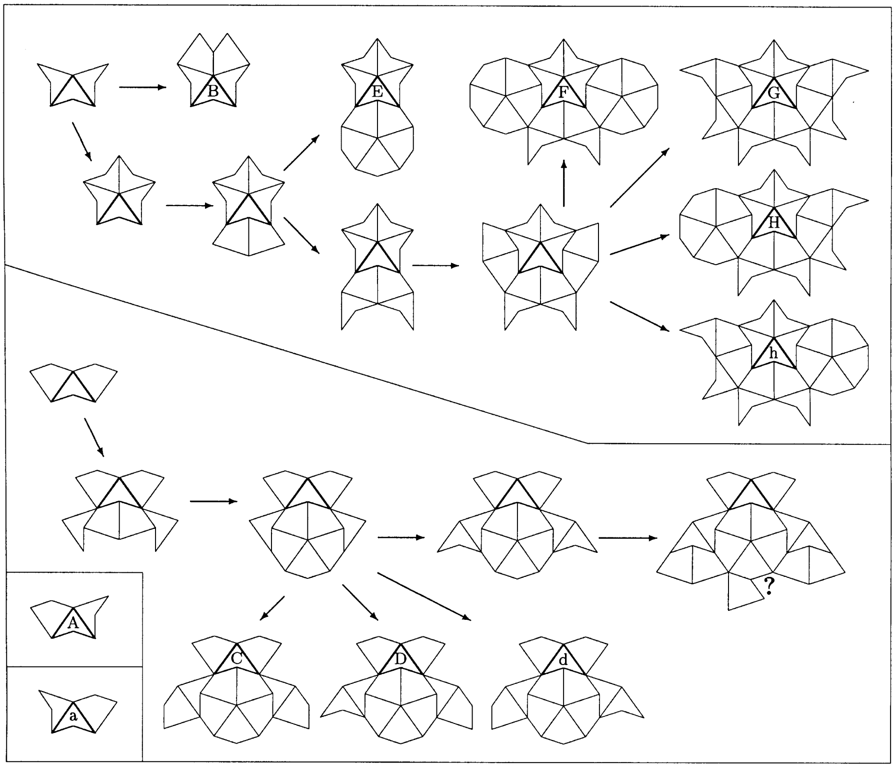


Fig. 4. Generation of dart classes.

Proof. The main idea of the proof is a distribution of the tiles into 280 classes depending on their neighbourhood and orientation and then trying to colour every tile in one class with the same colour.

Every tile has one of the 28 possible neighbourhoods shown in Fig. 6. These cases are denoted by capitals and small letters. Two neighbourhood classes, which are mutually symmetrical, are indicated by the capital and small form of the same letter.

A generation of the neighbourhood classes is shown in Fig. 4 for darts and in Fig. 5 for kites. Every dart lies in one of the four closest neighbourhoods depending on one tile on the left side and one tile on the right side. Two of these closest neighbourhoods are chosen as classes ('A' and 'a'), for the remaining two we explore wider neighbourhoods as shown in the figure. Not every finite part of a tiling is extendable to the tiling of the whole plane, for example if there exists a place where no tile can be placed correctly. Such places are marked by question marks. If only one extension is shown for some neighbourhood, then this extension is uniquely determined by the previous part. Simpler non-extendable cases are not shown. The same principle is applied for a generation of the kite classes. The small-letter classes u, v, w, x, y, z are symmetrical to the capital-letter classes U, V, W, X, Y, Z and are obtained in the symmetrical way.

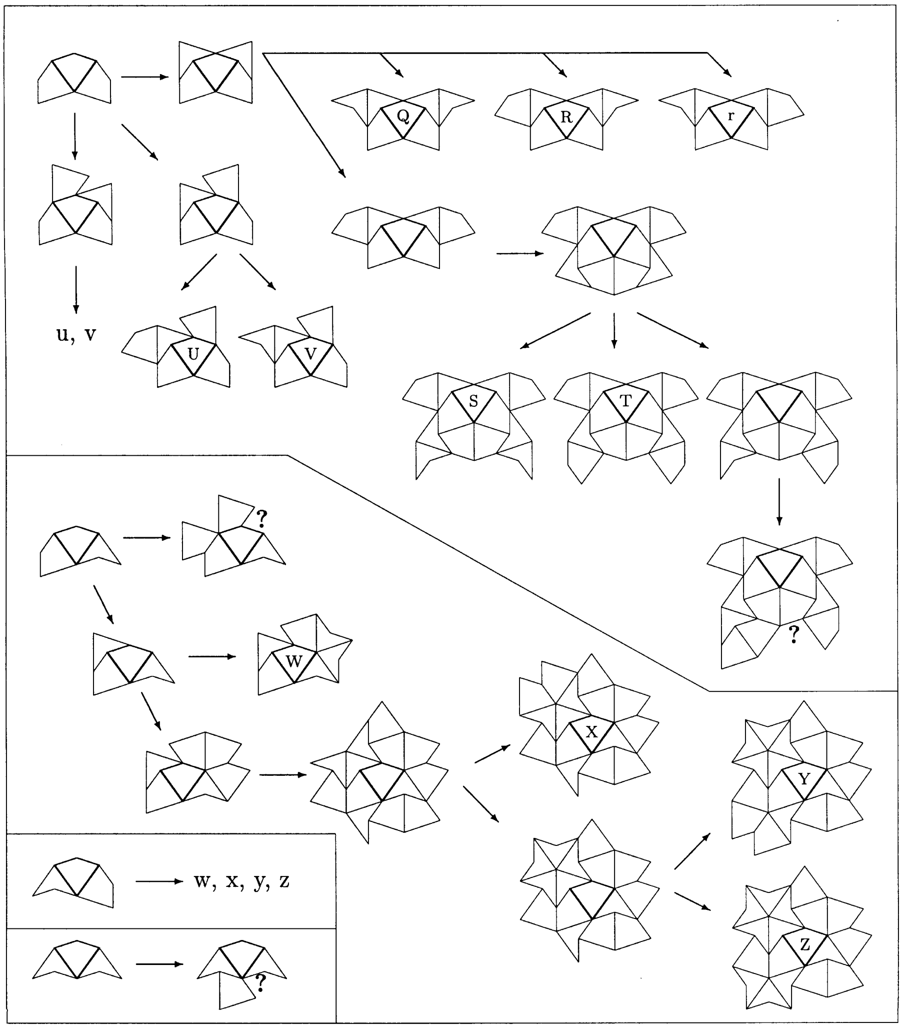


Fig. 5. Generation of kite classes.

Every tile is oriented in one of the 10 directions, the orientation is determined by the arrows drawn in the tiles in Fig. 2. The orientations are denoted by digits 0–9. By the combination of 28 neighbourhoods and 10 orientations we have made 280 classes. Then, every class is denoted by the combination of a letter and a digit (as a subscript). Fig. 7 shows the possible neighbours for every type of tile. The orientation is meant mod 10. Again, all other possibilities cannot be extended to a correct tiling of the whole plane.

We can now try to assign every type of tiles with one of the three colours. This assignment is shown in Fig. 8. It is not hard (but it is tedious) to check (with Fig. 7) that this colouring is in good order with only one exception, that is the colour of the class 'Z_9'.

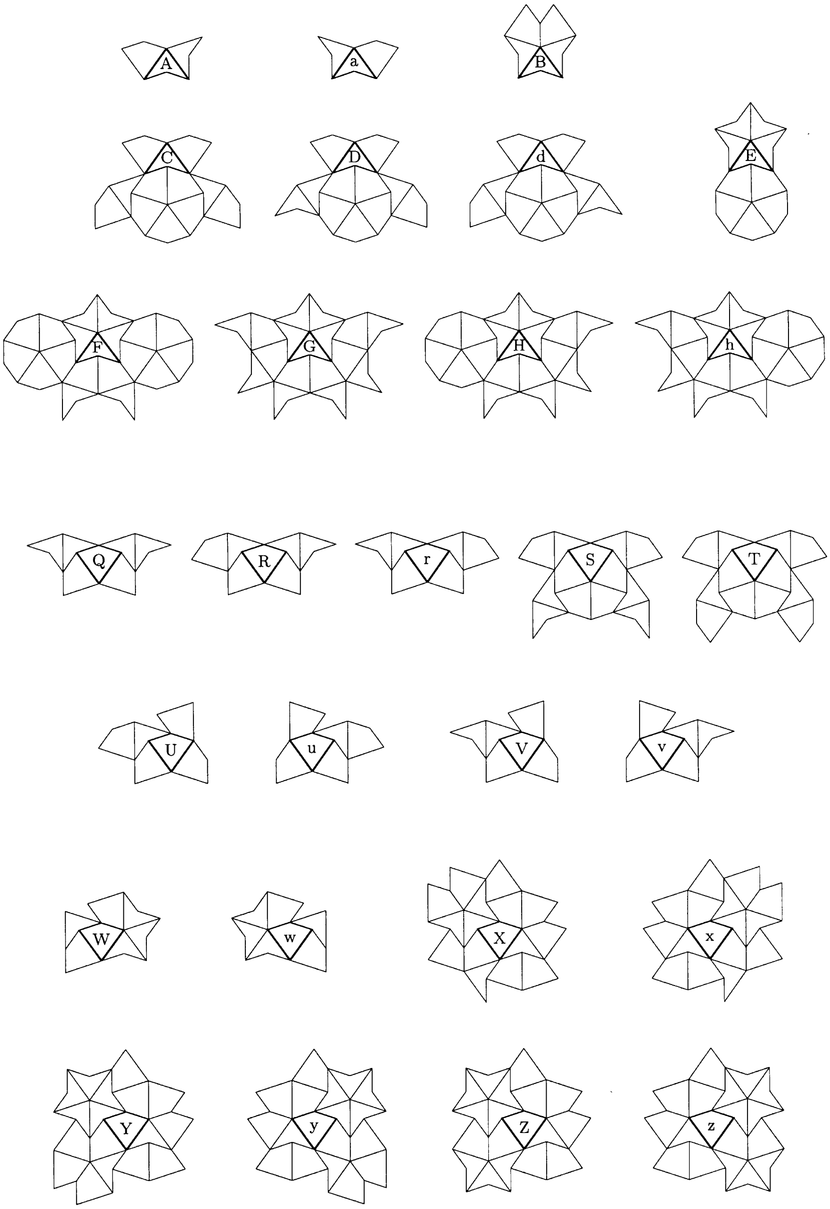


Fig. 6. Possible neighbourhoods of tiles.

How to colour the tiles of this class? In Fig. 3a the larger neighbourhood of the tile of the class 'Z_9' is shown. It is uniquely determined by its neighbourhood in the definition (in Fig. 6). The colouring of such tiles is possible with some local changes as shown in Fig. 3b.

class	—	neighbour classes					
A_n	—	B_{n-2}	Q_{n-1}	R_{n-1}	U_{n+1}	W_n	
a_n	—	B_{n+2}	Q_{n+1}	r_{n+1}	u_{n-1}	w_n	
B_n	—	A_{n+2}	a_{n-2}	X_{n+1}	x_{n-1}		
C_n	—	T_{n-1}	T_{n+1}	X_n	x_n	Y_n	y_n
D_n	—	r_{n-1}	S_{n+1}	x_n	y_n	Z_n	
d_n	—	R_{n+1}	S_{n-1}	X_n	Y_n	z_n	
E_n	—	F_{n-2}	F_{n+2}	H_{n-2}	h_{n+2}	V_{n+1}	v_{n-1}
F_n	—	E_{n+2}	E_{n-2}	Y_{n+1}	y_{n-1}		
G_n	—	G_{n+2}	G_{n-2}	H_{n+2}	h_{n-2}	W_{n+1}	w_{n-1}
H_n	—	E_{n+2}	G_{n-2}	h_{n-2}	W_{n+1}	z_{n-1}	
h_n	—	E_{n-2}	G_{n+2}	H_{n+2}	w_{n-1}	Z_{n+1}	
Q_n	—	A_{n+1}	a_{n-1}	U_{n+2}	u_{n-2}		
R_n	—	A_{n+1}	d_{n-1}	S_{n-2}	U_{n+2}		
r_n	—	a_{n-1}	D_{n+1}	S_{n+2}	u_{n-2}		
S_n	—	D_{n-1}	d_{n+1}	R_{n+2}	r_{n-2}		
T_n	—	C_{n+1}	C_{n-1}	T_{n+2}	T_{n-2}		
U_n	—	A_{n-1}	Q_{n-2}	R_{n-2}	u_{n+2}	v_{n+2}	x_{n-4}
u_n	—	a_{n+1}	Q_{n+2}	r_{n+2}	U_{n-2}	V_{n-2}	X_{n+4}
V_n	—	E_{n-1}	u_{n+2}	v_{n-2}	y_{n-4}	z_{n-4}	
v_n	—	E_{n+1}	U_{n-2}	V_{n+2}	Y_{n+4}	Z_{n+4}	
W_n	—	A_n	G_{n-1}	H_{n-1}	w_{n-4}	w_{n-2}	z_{n-2}
w_n	—	a_n	G_{n+1}	h_{n+1}	W_{n+4}	W_{n+2}	Z_{n+2}
X_n	—	B_{n-1}	C_n	d_n	u_{n-4}	x_{n-2}	
x_n	—	B_{n+1}	C_n	D_n	U_{n+4}	X_{n+2}	
Y_n	—	C_n	d_n	F_{n-1}	v_{n-4}	y_{n-2}	
y_n	—	C_n	D_n	F_{n+1}	V_{n+4}	Y_{n+2}	
Z_n	—	D_n	h_{n-1}	v_{n-4}	w_{n-2}		
z_n	—	d_n	H_{n+1}	V_{n+4}	W_{n+2}		

Fig. 7. Neighbourhoods of classes of tiles.

	AaB	CDdE	FGHh	QRrST	UuVv	WwXx	YyZz
0	113	2212	3232	11111	3111	2231	3112
1	333	3333	3333	33313	2113	1111	1111
2	232	1232	3333	22222	2222	1223	2312
3	111	1111	1111	22212	1311	2232	2222
4	111	3132	3113	33333	3333	3322	2232
5	223	2222	3232	22231	3333	3133	1111
6	322	3322	3223	11111	1112	1112	1213
7	233	3331	3313	11112	1212	3211	2222
8	313	3321	3111	22222	3333	2311	1123
9	122	3332	2112	33321	2233	3322	12-2

Fig. 8. Colouring of classes of tiles.

The colours outside the marked region are in harmony with Fig. 8, the colours inside are changed to a correct colouring. It is easy to check that the tiles inside the marked region and their neighbours are all of the different classes, so the problematic regions cannot overlap.

Thus the proof is complete.

3. Remarks

The proof was made with the help of a computer. In every trial, I made a list of classes (as in Fig. 6) and a list of neighbourhoods of classes (as in Fig. 7) — this makes a graph. The computer program tried to colour properly such a graph with three colours. I started with 20 classes and with the almost-proper colouring (with one error for 'Z_9') I obtained for 320 classes. Then, I reduced this number to the presented 280 classes.

Probably this method of the proof cannot be much shorter, at least a shorter proof would demand a new idea.

There are many other aperiodic tilings for which the question of their chromatic number has not been solved yet.

References

[1] R. Penrose, Pentaplexity, Bull Inst. Math. Appl. 10 (1974) 266–271.
[2] M. Bickford, internet web page `http://www.ics.uci.edu/~eppstein/junkyard/color.html`