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Commutators of contactomorphisms ☆

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Abstract

The group of volume preserving diffeomorphisms, the group of symplectomorphisms and the group of contactomorphisms constitute the classical groups of diffeomorphisms. The first homology groups of the compactly supported identity components of the first two groups have been computed by Thurston and Banyaga, respectively. In this paper we solve the long standing problem on the algebraic structure of the third classical diffeomorphism group, i.e. the contactomorphism group. Namely we show that the compactly supported identity component of the group of contactomorphisms is perfect and simple (if the underlying manifold is connected). The result could be applied in various ways.

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1. Introduction

Let (M,α) be a contact manifold, i.e. M is a C^∞ smooth paracompact manifold of dimension $m=2n+1$, $m\geqslant 3$, and α is a C^∞ 1-form on M such that $\alpha\wedge(d\alpha)^n$ is a volume form. A contactomorphism f of (M,α) is a C^∞ diffeomorphism of M such that $f^*\alpha=\lambda_f\alpha$, where λ_f is a smooth nowhere vanishing function on M depending on f. In other words, a contactomorphism f is a diffeomorphism whose tangent map Tf preserves the C^∞ contact hyperplane field

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$\mathcal{H} = \ker\alpha$. Notice that contactomorphisms of (M, α) are determined by the contact hyperplane field $\mathcal{H}$.

Let $\mathrm{Cont}(M, \alpha)$ denote the group of contactomorphisms of (M, α), and let $\mathrm{Cont}_c(M, \alpha)$ be its compactly supported subgroup. Observe that $\mathrm{Cont}(M, \alpha)$ carries the structure of an infinite dimensional Lie group (see, e.g., [9]). Then, in view of the local contractibility of $\mathrm{Cont}_c(M, \alpha)$, its identity component $\mathrm{Cont}_c(M, \alpha)_0$ coincides with all $f \in \mathrm{Cont}(M, \alpha)$ which can be joined with the identity by a smooth isotopy in $\mathrm{Cont}_c(M, \alpha)$. Our main result is the following

Theorem 1.1. *The group* $\mathrm{Cont}_c(M, \alpha)_0$ *is perfect, that is* $\mathrm{Cont}_c(M, \alpha)_0$ *is equal to its own commutator subgroup.*

Epstein in [4] proved that the commutator subgroup of a group of homeomorphisms satisfying some natural conditions is simple. It is easily checked that Epstein's conditions are satisfied by $\mathrm{Cont}_c(M, \alpha)_0$ (Section 10). Therefore we have

Corollary 1.2. *If M is connected then the group* $\mathrm{Cont}_c(M, \alpha)_0$ *is simple.*

The contactomorphism group is a classical group of diffeomorphisms. Since the well-known results of Herman [8], Thurston [17] and Mather [12] on the simplicity of $\mathrm{Diff}^r_c(M)_0$, $r = 1, \ldots, \infty$, $r \neq \dim(M) + 1$, the problem of the perfectness (or of computing the first homology group) of groups of diffeomorphisms have been studied in several papers. First of all such studies have been done on the classical groups of diffeomorphisms.

An essential feature of the geometry and topology of manifolds with a volume or a symplectic form is the existence of invariants, called the flux homomorphisms. According to the celebrated results by Thurston [16] and Banyaga [1] (see also [2]) the first homology groups $H_1(\mathrm{Diff}_c(M, \omega)_0)$, where ω is a volume or a symplectic form, can be expressed by means of the flux homomorphism and other invariants, and they depend also on the compactness of the underlying manifold. Notice that the results and the methods of their proofs in both cases are similar. In general, the compactly supported identity components of the volume preserving diffeomorphism group and the symplectomorphism group are not perfect. Note that Banyaga's theorem was generalized to the locally conformal symplectic structures in [7].

A basic reason that $\mathrm{Cont}_c(M, \alpha)_0$ is perfect is the fact that in the contact case there do not exist invariants analogous to the flux homomorphism and, consequently, a fragmentation property holds in its usual form. In view of this fact Theorem 1.1 was conjectured, e.g. in [2]. A main obstacle to find a proof similar to that of Thurston [16] is the lack of a canonical contact structure on the torus T^m, a clue ingredient of a hypothetical proof by this method. Canonical contact structures do exist, however, on the cylinders $\mathcal{W}^m_k = (\mathbb{S}^1)^k \times \mathbb{R}^{m-k}$, $k = 1, \ldots, n+1$, and this fact is essential in our proof.

The fragmentation property (Lemma 5.2) is, in fact, an indispensable ingredient of the proof. Nevertheless, it is probably not a sufficient tool to prove Theorem 1.1. My idea is to use in the proof also a fragmentation of contactomorphisms in a neighborhood of the identity of $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ (Section 5). I call it a fragmentation of the second kind. An essential advantage of such fragmentations is that the factors of the resulting decomposition are uniquely determined by the initial contactomorphism. Moreover, the norms of these factors are controlled by the norm of the initial contactomorphism in a convenient way.

The proof consists in an application of Schauder–Tichonoff's fixed point theorem to some operator in a functional space. The origins of this method were explained in Epstein [5], where it was used to give an alternative proof of the perfectness of $\mathrm{Diff}_c^\infty(M)_0$. We would like to stress, however, that several parts of the proof for diffeomorphisms cannot be carried over to the contact case and some new ideas and technical refinements in the proof of Theorem 1.1 are indispensable. Our construction of a fixed point operator consists of ten steps (cf. Section 9) and functional spaces on various domains must be considered in it.

A crucial step in the proof is the use of a rolling-up operator Ψ_A defined in Section 8 (Proposition 8.7). Such operators are used in [12] and [5], and analogous operators exist for the group $\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$, but only with respect to the first $n+1$ variables. However, they are useless since the property

$$\forall f\in \mathrm{dom}(\Psi_A),\quad \big[\Psi_A(f)\big]=[f]\quad \text{in } H_1\big(\mathrm{Cont}_c\big(\mathbb{R}^m,\alpha_{st}\big)_0\big),$$

a clue part of the proof in [12], does not hold in the contact category for very basic reasons. In this situation we construct a new rolling-up operator Ψ_A for $\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$ by means of auxiliary operators acting on contactomorphisms on the subsequent contact cylinders $\mathcal{W}_k^m$, $k=1,\dots,n$. An essential fact is that a "remainder" contactomorphism living on the last cylinder $\mathcal{W}_{n+1}^m$ possesses a representant in the commutator subgroup of $\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$ (Lemma 8.6). This ensures the above property for Ψ_A. Such an argument is no longer true for $\mathrm{Diff}_c^r(\mathbb{R}^m)_0$ and, consequently, the proof of Theorem 1.1 cannot be carried over to the case of diffeomorphisms.

The contact topology and geometry are intensively studied nowadays, cf. [6]. Theorem 1.1, which is a contact analog of the theorems of Thurston and Banyaga, could be possibly applicable in various ways. In the last section we indicate two directions of such applications. The most important seems to be the fact that due to Theorem 1.1 the commutator length is a conjugation-invariant norm on $\mathrm{Cont}_c(M,\alpha)_0$. For the significance of Banyaga's theorem in the symplectic topology, see, e.g., [14] and [3].

In Appendix A it is observed that the universal covering group of $\mathrm{Cont}_c(M,\alpha)_0$ is also perfect by an argument similar to that for $\mathrm{Cont}_c(M,\alpha)_0$.

2. The group of contactomorphisms

Let M be a smooth manifold with $\dim(M)=m=2n+1$ and let α be a contact form on M. A contact form α can be put into the following normal form. For any $p\in M$ there is a chart $(x_0,x_1,\dots,x_n,y_1,\dots,y_n): M\supset U\to u(U)\subset\mathbb{R}^m$, centered at p, such that $\alpha|_U=\mathrm{d}x_0-y_1\,\mathrm{d}x_1-\cdots-y_n\,\mathrm{d}x_n$.

The symbol $\mathfrak{X}(M,\alpha)$ will stand for the Lie algebra of all contact vector fields, i.e. $X\in\mathfrak{X}(M,\alpha)$ iff $L_X\alpha=\mu_X\alpha$ for some function $\mu_X\in C^\infty(M)$, where L is the Lie derivative. Let $\mathfrak{X}_c(M,\alpha)$ be the Lie subalgebra of compactly supported elements of $\mathfrak{X}(M,\alpha)$.

Let $h\in\mathrm{Cont}_c(M,\alpha)_0$ and $\{h_t\}_{t\in I}$ be a smooth isotopy such that $h_1=h$, $h_0=\mathrm{id}$ and each h_t stabilizes outside a fixed compact $K\subset M$. Of course, such a smooth contact isotopy determines a smooth family of contact vector fields $X_t\in\mathfrak{X}_c(M,\alpha)$, namely for $p\in M$ we have

$$\frac{\partial h_t}{\partial t}(p)=X_t\big(h_t(p)\big).\tag{2.1}$$

In fact, one has $L_{X_t}\alpha=\mu_{X_t}\alpha$ with $\mu_{X_t}=(\partial\ln\lambda_{h_t}/\partial t)h_t^{-1}$ where $h_t^*\alpha=\lambda_{h_t}\alpha$.

Let X_α denote the unique vector field satisfying $i_{X_\alpha}\alpha=1$ and $i_{X_\alpha}\mathrm{d}\alpha=0$. X_α is called the Reeb vector field. A vector field X is called *horizontal* if $i_X\alpha=0$. A dual concept is a *semibasic* form, i.e. any 1-form γ such that $\gamma(X_\alpha)=0$, and the duality is established by the isomorphism $d\alpha : X \mapsto i_X\,\mathrm{d}\alpha$. It follows the isomorphism of vector bundles

$$I_\alpha : TM \ni X \mapsto i_X\,\mathrm{d}\alpha + \alpha(X)\alpha \in T^*M. \tag{2.2}$$

As a consequence we have the existence of the isomorphism $\mathcal{I}_\alpha$ below (cf. Libermann [10]), an important tool in the contact geometry.

Proposition 2.1. *There is an isomorphism $\mathcal{I}_\alpha : \mathfrak{X}(M,\alpha) \to \mathrm{C}^\infty(M)$ by $\mathcal{I}_\alpha(X) = i_X\alpha$. For $H \in \mathrm{C}^\infty(M)$ we have*

$$\mathcal{I}_\alpha^{-1}(H) = HX_\alpha + (d\alpha)^{-1}\big((i_{X_\alpha}\mathrm{d}H)\alpha - \mathrm{d}H\big).$$

We will deal with the standard contact form $\alpha_{st} = \mathrm{d}x_0 - \sum_{i=1}^n y_i\,\mathrm{d}x_i$ on $\mathbb{R}^m$. Then we have $X_{\alpha_{st}} = \frac{\partial}{\partial x_0}$ and $\mathrm{d}\alpha_{st} = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$.

Notice that the isomorphism $I_{\alpha_{st}} : T\mathbb{R}^m \to T^*\mathbb{R}^m$ is independent of the variables x_i, $i = 0,\dots,n$. Likewise, the isomorphism $\mathcal{I}_{\alpha_{st}} : \mathfrak{X}(\mathbb{R}^m, \alpha_{st}) \to \mathrm{C}^\infty(\mathbb{R}^m)$ sends vector fields independent of x_i to functions independent of x_i and vice versa.

Observe that $\mathcal{H} = \ker(\alpha_{st})$ is generated by $Y_i = \frac{\partial}{\partial y_i}$ and $X_i = \frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial x_0}$, where $i = 1,\dots,n$.

Next it is easily seen that $d\alpha_{st}(Y_i) = -\mathrm{d}x_i$ and $d\alpha_{st}(X_i) = \mathrm{d}y_i$. Every contact vector field $X = u_0\frac{\partial}{\partial x_0} + \sum_{i=1}^n u_i\frac{\partial}{\partial x_i} + u_{n+i}\frac{\partial}{\partial y_i} \in \mathfrak{X}(\mathbb{R}^m, \alpha_{st})$ is identified by $\mathcal{I}_{\alpha_{st}}$ with the function $H = i_X\alpha_{st} = u_0 - \sum_{i=1}^n y_i u_i \in \mathrm{C}^\infty(\mathbb{R}^m)$. Conversely, in view of Proposition 2.1 and the above equalities, we have

$$X_H = \left(H - \sum_{i=1}^n y_i\frac{\partial H}{\partial y_i}\right)\frac{\partial}{\partial x_0} - \sum_{i=1}^n \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} + \sum_{i=1}^n\left(\frac{\partial H}{\partial x_i} + y_i\frac{\partial H}{\partial x_0}\right)\frac{\partial}{\partial y_i}, \tag{2.3}$$

where $X_H = \mathcal{I}_{\alpha_{st}}^{-1}(H)$ for all $H \in \mathrm{C}^\infty(\mathbb{R}^m)$. Now we wish to specify some elements in $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$. The following contact vector fields on $\mathbb{R}^m$ and their flows will be of use. Throughout we will often write x instead of $(x_1,\dots,x_n)$ and y instead of $(y_1,\dots,y_n)$.

(1) Let H_0 be the constant function 1. Then $X_{H_0} = \frac{\partial}{\partial x_0} = X_{\alpha_{st}}$ and its flow takes the translation form $\mathrm{Fl}_t^{H_0}(x_0,x,y) = (x_0+t, x, y)$.
(2) Put $H_i(x_0,x,y) = -y_i$ $(i = 1,\dots,n)$. Then $X_{H_i} = \frac{\partial}{\partial x_i}$ and its flow consists of the translations $\mathrm{Fl}_t^{H_i}(x_0,x,y) = (x_0, x+t\mathbf{1}_i, y)$.
(3) Let $H_{n+i}(x_0,x,y) = x_i$ $(i = 1,\dots,n)$. From (2.3) we obtain $X_{H_{n+i}} = \frac{\partial}{\partial y_i} + x_i\frac{\partial}{\partial x_0}$. Hence $\mathrm{Fl}_t^{H_{n+i}}(x_0,x,y) = (x_0+tx_i, x, y+t\mathbf{1}_i)$.
(4) For $H(x_0,x,y) = 2x_0 - \sum_{i=1}^n x_i y_i$, we have $X_H = 2x_0\frac{\partial}{\partial x_0} + \sum_{i=1}^n(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i})$, and its flow assumes the form $\mathrm{Fl}_t^H(x_0,x,y) = (e^{2t}x_0, e^t x, e^t y)$. By $\chi_a : \mathbb{R}^m \to \mathbb{R}^m$, where $\chi_a(x_0,x,y) = (a^2x_0, ax, ay)$, i.e. $\chi_a = \mathrm{Fl}_{\ln a}^H$, we will denote the resulting contact homothety.
(5) Let $\bar H(x,y,z) = x_0 - \sum_{i=1}^n x_i y_i$. Then $X_{\bar H} = \sum_{i=0}^n x_i\frac{\partial}{\partial x_i}$, and its flow satisfies $\mathrm{Fl}_t^{\bar H}(x_0,x,y) = (e^t x_0, e^t x, y)$. Let $\eta_a : \mathbb{R}^m \to \mathbb{R}^m$, where $\eta_a(x_0,x,y) = (ax_0, ax, y)$, i.e. $\eta_a = \mathrm{Fl}_{\ln a}^{\bar H}$, denote the resulting map.

Denote $\tau_{i,t} = \mathrm{Fl}_t^{H_i}$, $i = 0, 1, \dots, n$, and $\sigma_{i,t} = \mathrm{Fl}_t^{H_{n+i}}$, $i = 1, \dots, n$. The supports of $\tau_{i,t}$, $\sigma_{i,t}$, χ_a and η_a are not compact. But if we take the product of H_i, H, or $\bar{H}$ with a suitable bump function we will obtain elements of $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ which are equal to the previous contactomorphisms on a sufficiently large interval. Abusing the notation we will denote all these elements of $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ by the same letters as before. This ambiguity will not matter in the proof and we will not mention it in the sequel.

Observe that the translations along the y_i-axes are not contactomorphisms since they do not preserve the contact distribution.

Proposition 2.2.

(1) *A diffeomorphism f of $\mathbb{R}^m$ is a contactomorphism if and only if for $f = (f_0, f_1, \dots, f_{2n})$ we have*

$$\frac{\partial f_0}{\partial x_0} - \sum_{j=1}^{n} f_{n+j} \frac{\partial f_j}{\partial x_0} = \lambda_f,$$

$$\frac{\partial f_0}{\partial x_i} - \sum_{j=1}^{n} f_{n+j} \frac{\partial f_j}{\partial x_i} = -y_i \lambda_f, \quad i = 1, \dots, n,$$

$$\frac{\partial f_0}{\partial y_i} - \sum_{j=1}^{n} f_{n+j} \frac{\partial f_j}{\partial y_i} = 0, \quad i = 1, \dots, n.$$

(2) *If $f \in \mathrm{Cont}(\mathbb{R}^m, \alpha_{st})$ is independent of x_0 (i.e. $\frac{\partial(f-\mathrm{id})}{\partial x_0} = 0$) then $\lambda_f = 1$. If, in addition, f is independent of x_i, $i = 1, \dots, n$, then $f_{n+j}(x_0, x, y) = y_j$ for $j = 1, \dots, n$.*

Proof. (1) This is the equality $f^* \alpha_{st} = \lambda_f \alpha_{st}$ written in coordinates.

(2) It follows immediately from the first $n + 1$ equalities. □

A crucial idea in the proof of Theorem 1.1 is to consider groups of contactomorphisms on cylinders which admit standard contact structures. Let us denote

$$\mathcal{W}_k^m := (\mathbb{S}^1)^k \times \mathbb{R}^{m-k}, \tag{2.4}$$

and for a constant $A > 0$

$$E_A^{(k)} := (\mathbb{S}^1)^k \times [-A, A]^{m-k}, \quad k = 1, \dots, n+1. \tag{2.5}$$

It will be often convenient to write

$$\mathcal{W}_0^m := \mathbb{R}^m, \qquad E_A^{(0)} = E_A := [-A, A]^m. \tag{2.6}$$

The coordinates of $\mathcal{W}_k^m$ will be written $(\xi_0, \xi_1, \dots, \xi_n, y_1, \dots, y_n)$, where ξ_i is the natural coordinate on $\mathbb{S}^1$ for $i = 0, \dots, k-1$, and $\xi_i = x_i$ for $i = k, \dots, n$. For short, we will often write ξ instead of $(\xi_1, \dots, \xi_n)$, that is the natural coordinates on $\mathcal{W}_k^m$ will be denoted by (ξ_0, ξ, y). On the cylinder $\mathcal{W}_k^m$ we have the standard contact form given by $\alpha_{st} = \mathrm{d}\xi_0 - y_1\,\mathrm{d}\xi_1 - \cdots - y_n\,\mathrm{d}\xi_n$.

If $E \subset \mathcal{W}_k^m$ the symbol $\mathrm{Cont}_E(\mathcal{W}_k^m, \alpha_{st})$ stands for the totality of elements of $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$ with support included in E. The description of a chart in $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$ at the identity will be given in Section 4.

Observe that the translations $\tau_{i,t}$ still live on $\mathcal{W}_k^m$ whenever $k \leqslant i \leqslant n$.

3. Basic estimates

Let r be a nonnegative integer. For $g \in \mathrm{C}^\infty(\mathbb{R}^m, \mathbb{R}^{m'})$ we define

$$\left\| D^r g \right\| = \sup_{p \in \mathbb{R}^m} \left| D^r g(p) \right| = \sup_{p \in \mathbb{R}^m} \sup_{|u_1| \leqslant 1, \ldots, |u_r| \leqslant 1} \left| D^r g(p)(u_1, \ldots, u_r) \right| \leqslant \infty,$$

where $D^0 g = g$. Next, for a diffeomorphism $f \in \mathrm{Diff}^\infty(\mathbb{R}^m)$ we put for any $r \geqslant 0$

$$\mu_r(f) = \left\| D^r(f - \mathrm{id}) \right\|,$$
$$M_r(f) = \max\left\{\mu_0(f), \mu_1(f), \ldots, \mu_r(f)\right\}.$$

If $\mathbf{f} = (f_1, \ldots, f_k)$ then we define

$$\mu_r(\mathbf{f}) = \sup_{i=1,\ldots,k} \mu_r(f_i), \qquad M_r(\mathbf{f}) = \sup_{i=1,\ldots,k} M_r(f_i).$$

We have $\mu_1(f) \leqslant \|Df\| + 1$, $\|Df\| \leqslant \mu_1(f) + 1$ and $\mu_r(f) = \|D^r f\|$ for $r \geqslant 2$. Let $E \subset \mathbb{R}^m$ be a closed set. We define

$$R_E = \sup_{p \in E} \mathrm{dist}\left(p, \overline{\mathbb{R}^m \setminus E}\right) \leqslant \infty. \tag{3.1}$$

Proposition 3.1. *Let $R_E < \infty$. If f is a diffeomorphism and* $\mathrm{supp}(f) \subset E$, *then*

$$\mu_r(f) \leqslant C \mu_{r+1}(f),$$

where $r \geqslant 0$ and the constant C depends on R_E.

In fact, the inequality is obtained by integrating partial derivatives of the map $f - \mathrm{id}$.

Let $f, g \in \mathrm{C}^\infty(\mathbb{R}^m, \mathbb{R}^m)$ and $r \geqslant 1$. Then we have

$$D(f \circ g) = (Df \circ g) Dg, \tag{3.2}$$

$$D^r(f \circ g) = \left(D^r f \circ g\right)(Dg \times \cdots \times Dg) + (Df \circ g) D^r g + \sum C_{i; j_1, \ldots, j_i} \left(D^i f \circ g\right)\left(D^{j_1} g \times \cdots \times D^{j_i} g\right). \tag{3.3}$$

It follows from (3.2) and (3.3) the equalities

$$D\left(f^{-1}\right) = (Df)^{-1} \circ f^{-1}, \tag{3.4}$$

$$D^r(f^{-1}) = D(f^{-1})(D^r f \circ f^{-1})(D(f^{-1}) \times \cdots \times D(f^{-1})) + D(f^{-1}) \sum C_{i;j_1,\ldots,j_i}(D^i f \circ f^{-1})(D^{j_1}(f^{-1}) \times \cdots \times D^{j_i}(f^{-1})). \quad (3.5)$$

In (3.3) and (3.5) the sum is taken over $1 < i < r$, $1 \leqslant j_s$, $j_1 + \cdots + j_i = r$ and $C_{i;j_1,\ldots,j_r}$ are positive integers. Note that in each term of the above sum there exists $j_s > 1$.

Definition 3.2. By polynomials we will understand polynomials with nonnegative coefficients.

An *admissible* polynomial is a polynomial without constant and linear terms. Admissible polynomials will be denoted by F with some indices. We will also consider polynomials without constant term. Such polynomials will be designated by P with some indices.

Convention 3.3. In order to avoid repeating that either polynomials, or constants depend on some values, we adopt the following convention. If, e.g., a polynomial P depends on ψ, r, and A, then we will write $P_{\psi,r,A}$, i.e. all the values determining a given object will appear as subscripts. The only exception is that we will not mention explicitly the dependence on $m = \dim(M)$.

In the sequel we will often omit the sign of composition $\circ$.

By using (3.2)–(3.5) and the induction argument we have the following lemma (cf. [12]).

Lemma 3.4.

(1) *For any* $f_1, \ldots, f_k \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$ *and* $\mathbf{f} = (f_1, \ldots, f_k)$

$$\mu_1(f_1 \circ \cdots \circ f_k) \leqslant k\mu_1(\mathbf{f})(1 + \mu_1(\mathbf{f}))^{k-1}.$$

(2) *For* $r, k \geqslant 2$ *there exists an admissible polynomial* $F_{r,k}$ *such that for any* $f_1, \ldots, f_k \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$, $\mathbf{f} = (f_1, \ldots, f_k)$, *one has*

$$\mu_r(f_1 \circ \cdots \circ f_k) \leqslant k\mu_r(\mathbf{f})(1 + \mu_1(\mathbf{f}))^{r(k-1)} + F_{r,k}(M_{r-1}(\mathbf{f})).$$

(3) *If* $f \in \mathrm{Diff}(\mathbb{R}^m)$ *with* $\mu_1(f) < 1$, *then*

$$\mu_1(f^{-1}) \leqslant \frac{\mu_1(f)}{1 - \mu_1(f)}.$$

(4) *For any* $r \geqslant 2$ *there exists an admissible polynomial* F_r *such that for any* $f \in \mathrm{Diff}(\mathbb{R}^m)$ *with* $\mu_1(f) < \frac{1}{2}$

$$\mu_r(f^{-1}) \leqslant \mu_r(f)(1 + 2\mu_1(f))^{r+1} + F_r(M_{r-1}(f)).$$

In the group $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ we will need more specified norms. For any $f \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})$ and $r \geqslant 0$ we put

$$\mu_r^*(f) = \max\{\|D^r(f - \mathrm{id})\|, \|D^r(\lambda_f - 1)\|\}$$

and

$$M_r^*(f) = \max\{\mu_0^*(f), \mu_1^*(f), \ldots, \mu_r^*(f)\}.$$

Here $\lambda_f \in C^\infty(\mathbb{R}^m)$ such that $f^*\alpha_{st} = \lambda_f \alpha_{st}$. We define $\mu_r^*(\mathbf{f})$ and $M_r^*(\mathbf{f})$ for $\mathbf{f} = (f_1, \ldots, f_k)$ analogously as above.

Proposition 3.5. *Let $R_E < \infty$. If $f \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})$, and $\mathrm{supp}(f) \subset E$, then*

$$\mu_r^*(f) \leqslant C\mu_{r+1}^*(f),$$

where $r \geqslant 0$ and C depends on R_E.

Indeed, the inequality follows from the definition of μ_r^* by integrating partial derivatives of the maps $f - \mathrm{id}$ and $\lambda_f - 1$.

Lemma 3.6.

(1) *For any $f_1, \ldots, f_k \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})$ and $\mathbf{f} = (f_1, \ldots, f_k)$ we have*

$$\mu_1^*(f_1 \circ \cdots \circ f_k) \leqslant k\mu_1^*(\mathbf{f})\big(\big(1 + \mu_0^*(\mathbf{f})\big)\big(1 + \mu_1^*(\mathbf{f})\big)\big)^{k-1}.$$

(2) *For $r, k \geqslant 2$ there exists an admissible polynomial $F_{r,k}$ such that for any $f_1, \ldots, f_k \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})$, $\mathbf{f} = (f_1, \ldots, f_k)$, one has*

$$\mu_r^*(f_1 \circ \cdots \circ f_k) \leqslant k\mu_r^*(\mathbf{f})\big(1 + \mu_0^*(\mathbf{f})\big)^{k-1}\big(1 + \mu_1^*(\mathbf{f})\big)^{r(k-1)} + F_{r,k}\big(M_{r-1}^*(\mathbf{f})\big).$$

(3) *If $f \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})$ with $\mu_0^*(f) < \frac{1}{2}$ and $\mu_1^*(f) < \frac{1}{2}$, then*

$$\mu_1^*\big(f^{-1}\big) \leqslant 8\mu_1^*(f).$$

(4) *For any $r \geqslant 2$ there exists an admissible polynomial F_r such that for any $f \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})$ with $\mu_0^*(f) < \frac{1}{2}$ and $\mu_1^*(f) < \frac{1}{2}$ one has*

$$\mu_r^*\big(f^{-1}\big) \leqslant 2^{r+2}\mu_r^*(f)\big(1 + 2\mu_1^*(f)\big)^{r+1} + F_r\big(M_{r-1}^*(f)\big).$$

Proof. First notice that

$$\lambda_{f_1 \circ \cdots \circ f_k} = (\lambda_{f_1} \circ f_2 \circ \cdots \circ f_k) \cdot (\lambda_{f_2} \circ f_3 \circ \cdots \circ f_k) \cdot \ldots \cdot \lambda_{f_k} \tag{3.6}$$

and

$$\lambda_{f^{-1}} = \frac{1}{\lambda_f \circ f^{-1}} \tag{3.7}$$

for any $f, f_1, \ldots, f_k \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})$. For $\mathbf{f} = (f_1, \ldots, f_k)$ denote $\lambda_{\mathbf{f}} = \sup_{i=1,\ldots,k} \|\lambda_{f_i}\|$. In order to show (1) observe that in view of (3.6) and (3.2) we have

$$
\begin{aligned}
&\|D\lambda_{f_1\circ\cdots\circ f_k}\| \\
&\quad\leqslant \lambda_{\mathbf{f}}^{k-1}\big(\|D(\lambda_{f_1}\circ f_2\circ\cdots\circ f_k)\| + \|D(\lambda_{f_2}\circ f_3\circ\cdots\circ f_k)\| + \cdots + \|D\lambda_{f_k}\|\big) \\
&\quad\leqslant \lambda_{\mathbf{f}}^{k-1}\big(\|D\lambda_{f_1}\|\|Df_2\|\cdots\|Df_k\| + \|D\lambda_{f_2}\|\|Df_3\|\cdots\|Df_k\| + \cdots + \|D\lambda_{f_k}\|\big) \\
&\quad\leqslant \lambda_{\mathbf{f}}^{k-1}\mu_1^*(\mathbf{f})\big(\big(1+\mu_1^*(\mathbf{f})\big)^{k-1} + \big(1+\mu_1^*(\mathbf{f})\big)^{k-2} + \cdots + 1\big) \\
&\quad\leqslant k\mu_1^*(\mathbf{f})\big(1+\mu_0(\mathbf{f})\big)^{k-1}\big(1+\mu_1^*(\mathbf{f})\big)^{k-1}.
\end{aligned}
$$

Here we used the inequalities $\|\lambda_{f_i}\| \leqslant 1+\mu_0^*(f_i)$, $\|D\lambda_{f_i}\| \leqslant \mu_1^*(f_i)$, and $\|Df_i\| \leqslant 1+\mu_1^*(f_i)$, for $i = 1,\dots,k$. Combining this with Lemma 3.4(1) we obtain (1). (2) follows analogously by (3.3), (3.6) and Lemma 3.4(2).

Next, (3) follows from the trivial inequality $\frac{\mu_1(f)}{1-\mu_1(f)} \leqslant \frac{\mu_1^*(f)}{1-\mu_1^*(f)}$ and

$$
\begin{aligned}
\|D\lambda_{f^{-1}}\| &\leqslant \|D(1/\lambda_f)\|\|Df^{-1}\| \leqslant 4\|D\lambda_f\|\big(1+\mu_1\big(f^{-1}\big)\big) \\
&\leqslant 4\mu_1^*(f)\big(1+2\mu_1(f)\big) \leqslant 8\mu_1^*(f)
\end{aligned}
$$

in view of (3.7), $\|\lambda_f\| > \frac{1}{2}$ and Lemma 3.4. Finally, in order to show (4) observe, in view of (3.7), (3.3) and Lemma 3.4, that

$$
\begin{aligned}
\|D^r\lambda_{f^{-1}}\| &\leqslant \|D^r(1/\lambda_f)\|\|Df^{-1}\|^r + \|D(1/\lambda_f)\|\|D^r f^{-1}\| + F_r^1\big(M_{r-1}^*(f)\big) \\
&\leqslant 4\|D^r\lambda_f\|\|Df^{-1}\|^r + 4\|D\lambda_f\|\|D^r f^{-1}\| + F_r^2\big(M_{r-1}^*(f)\big) \\
&\leqslant 2^{r+2}\mu_r^*(f)\big(1+2\mu_1^*(f)\big)^{r+1} + F_r\big(M_{r-1}^*(f)\big),
\end{aligned}
$$

as $\|\lambda_f\| > \frac{1}{2}$. Now (4) follows from the above inequality and Lemma 3.4(4). □

Remark 3.7. Note that Lemma 3.6 remains true for contactomorphisms on $\mathcal{W}_k^m$, cf. (2.4), from a sufficiently small C^1-neighborhood of the identity. The reason is that if we estimate the norms of these elements at a point then the r.h.s. of the inequalities in question may be written locally, that is in $\mathbb{R}^m$.

By a subinterval of $E_A^{(k)} \subset \mathcal{W}_k^m$, cf. (2.5), we understand a subset of $\mathcal{W}_k^m$ of the form $(\mathbb{S}^1)^k \times E'$, where E' is a subinterval of $[-A, A]^{m-k}$. If we put

$$
R_E = \sup_{p\in E} \operatorname{dist}\big(p, \overline{\mathcal{W}_k^m \setminus E}\big), \tag{3.8}
$$

then Proposition 3.5 still holds for $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$ instead of $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$.

4. Description of a chart

It is well-known that $\mathrm{Cont}(M,\alpha)$ admits an infinite dimensional Lie group structure (see Lychagin [11], or the elegant proof in Kriegl and Michor [9]). In particular, this group is locally contractible.

Observe that for an arbitrary diffeomorphism f of M endowed with a contact form α we may define $\lambda_f \in C^\infty(M)$ by

$$\lambda_f = i_{X_\alpha}\lambda_f\alpha = i_{X_\alpha}\big(f^*\alpha\big) = f^*(i_{f_*X_\alpha}\alpha), \tag{4.1}$$

where i designates the interior product. The construction of charts on the group $\mathrm{Cont}(M,\alpha)$ is based on the fact that a diffeomorphism f is a contactomorphism if and only if the graph of (f,λ_f),

$$\big\{\big(p, f(p), \lambda_f(p)\big)\colon\ p\in M\big\},$$

is a Legendrian submanifold of $(\tilde{M},\tilde{\alpha})$, where $\tilde{M} = M\times M\times\mathbb{R}\setminus 0$, $\tilde{\alpha} = t\,\mathrm{pr}_1^*\alpha - \mathrm{pr}_2^*\alpha$, $\mathrm{pr}_i : M\times M\times\mathbb{R}\setminus 0\to M$, $i=1,2$, is the projection onto the i-th factor, and t is the coordinate in $\mathbb{R}\setminus 0$.

Theorem 4.1. *(See [9,11].) If L is a Legendrian submanifold of a contact manifold (M,α) then there exist an open neighborhood U of L in M, an open neighborhood V of the zero section 0_L in $(T^*L\times\mathbb{R},\alpha_0)$, where $\alpha_0=\theta_L-\mathrm{d}t$ and θ_L is the canonical 1-form on T^*L, and a diffeomorphism $\varphi: U\to V$ such that $\varphi|_L=\mathrm{id}_L$ and $\varphi^*\alpha_0=\alpha$.*

Consequently, there is a smooth contactomorphism from a neighborhood of the graph of $(\mathrm{id}_M, 1_M)$ onto a neighborhood of zero in the space $J^1(M,\mathbb{R})$ of 1-jets of elements of $C^\infty(M)$. A Legendrian submanifold C^1-close to the graph of $(\mathrm{id}_M, 1_M)$ corresponds to the 1-jet of a smooth function on M C^2-close to zero.

Let $k=0,1,\dots,n+1$. We denote the coordinates in $\mathcal{W}_k^m\times\mathbb{R}^{m+1} = (\mathbb{S}^1)^k\times\mathbb{R}^{2n-k+1}\times\mathbb{R}^{2n+1}\times\mathbb{R}$ by $(\xi_0,\xi,y,\bar{x}_0,\bar{x},\bar{y},t)$, where we write $\bar{x}=(\bar{x}_1,\dots,\bar{x}_n)$, $\bar{y}=(\bar{y}_1,\dots,\bar{y}_n)$. We identify $\mathcal{W}_k^m\times\mathbb{R}^{m+1}$ with $T^*L\times\mathbb{R}$, where $L=\mathcal{W}_k^m\times 0\times 0\subset\mathcal{W}_k^m\times\mathbb{R}^{m+1}$. Then for the canonical 1-form α_0 on $T^*L\times\mathbb{R}$ we have

$$\alpha_0 = \bar{x}_0\,\mathrm{d}\xi_0 + \sum_{i=1}^n(\bar{x}_i\,\mathrm{d}\xi_i + \bar{y}_i\,\mathrm{d}y_i) - \mathrm{d}t.$$

Next, let U be a small open neighborhood of $L=\mathcal{W}_k^m\times 0\times 0\subset\mathcal{W}_k^m\times\mathbb{R}^m\times\mathbb{R}$. We have an embedding $\delta_{\mathcal{W}_k^m}: U\to\widetilde{\mathcal{W}}_k^m = \mathcal{W}_k^m\times\mathcal{W}_k^m\times\mathbb{R}\setminus 0$ given by

$$\delta_{\mathcal{W}_k^m}(\xi_0,\xi,y,\bar{x}_0,\bar{x},\bar{y},t) = (\xi_0,\xi,y,\xi_0+\bar{x}_0,\xi+\bar{x},y+\bar{y},t+1).$$

Since

$$\widetilde{\alpha_{st}} = t\left(\mathrm{d}\xi_0 - \sum_{i=1}^n y_i\,\mathrm{d}\xi_i\right) - \mathrm{d}\bar{x}_0 + \sum_{i=1}^n \bar{y}_i\,\mathrm{d}\bar{x}_i,$$

we obtain on U

$$\widehat{\alpha_{st}} := \delta_{\mathcal{W}_k^m}^*\widetilde{\alpha_{st}} = (t+1)\left(\mathrm{d}\xi_0 - \sum_{i=1}^n y_i\,\mathrm{d}\xi_i\right) - \mathrm{d}(\xi_0+\bar{x}_0) + \sum_{i=1}^n (y_i+\bar{y}_i)\,\mathrm{d}(\xi_i+\bar{x}_i).$$

Then L is a Legendrian submanifold w.r.t. both α_0 and $\widehat{\alpha_{st}}$.

Observe that a diffeomorphism f of $\mathcal{W}_k^m$, C^1- and C^0-close to the identity, is a contactomorphism iff its graph

$$\Gamma_f\big(\mathcal{W}_k^m\big) = \big\{\big(p, f(p) - p, \lambda_f(p) - 1\big):\ p \in \mathcal{W}_k^m\big\}$$

is a Legendrian submanifold of $(U, \widehat{\alpha_{st}})$.

From now on we will write for $A > 0$

$$\tilde{E}_A^{(0)} = \big[-A^2, A^2\big] \times [-A, A]^{2n} \times \mathbb{R}^{m+1}, \qquad \tilde{E}_A^{(k)} = \big(\mathbb{S}^1\big)^k \times [-A, A]^{m-k} \times \mathbb{R}^{m+1}, \tag{4.2}$$

where $k = 1, \ldots, n+1$. First we consider the case $k = 0$. Let $\varphi : U \to V$ be as in Theorem 4.1, where $U \subset \mathbb{R}^m \times \mathbb{R}^{m+1}$ is an open neighborhood of L as above and $\varphi^*\alpha_0 = \widehat{\alpha_{st}}$. Throughout we set

$$K_{\varphi,r} = \sup_{s=0,\ldots,r+1} \max\big\{\big\|D^s\varphi|_{U\cap\tilde{E}_1^{(0)}}\big\|, \big\|D^s(\varphi|_{U\cap\tilde{E}_1^{(0)}})^{-1}\big\|\big\}. \tag{4.3}$$

We have $\forall r \geqslant 1$, $K_{\varphi,r} < \infty$, as we may assume that $U \cap \tilde{E}_1^{(0)}$ is relatively compact.

Proposition 4.2. *Under the above notation we have:*

(1) *$\varphi = \varphi_0$ may be chosen so that it is independent of the variable x_i, $i = 0, 1, \ldots, n$, that is* $\frac{\partial(\varphi - \mathrm{id})}{\partial x_i} = 0$.
(2) *For any $A > 1$ there is a contactomorphism $\varphi_A : U' \to V'$, where U', V' are open neighborhoods of L, such that $\varphi_A|_L = \mathrm{id}_L$, and $\varphi_A^*\alpha_0 = \widehat{\alpha_{st}}$. Moreover, for $r = 0, 1, \ldots,$ one has $K_{\varphi,r,A} \leqslant A^2 K_{\varphi,r}$, where*

$$K_{\varphi,r,A} = \sup_{s=0,\ldots,r+1} \max\big\{\big\|D^s\varphi_A|_{U'\cap\tilde{E}_A^{(0)}}\big\|, \big\|D^s(\varphi_A|_{U'\cap\tilde{E}_A^{(0)}})^{-1}\big\|\big\}.$$

(3) *φ_A is independent of x_i, $i = 0, 1, \ldots, n$.*

Proof. (1) We appeal to Section 43.18 in [9]. Observe that the contact forms α_0 and $\widehat{\alpha_{st}}$ are independent of the variables x_i, $i = 0, \ldots, n$, and L is a Legendrian submanifold w.r.t. both of them. By an algebraic argument there is a vector bundle isomorphism $\gamma : T\mathbb{R}^{2m+1}|_L \to T\mathbb{R}^{2m+1}|_L$ such that $\gamma^*\alpha_0 = \widehat{\alpha_{st}}$ and γ is independent of x_i. Therefore there exists a diffeomorphism $\psi : U \to V$, where U, V are open neighborhoods of L in $\mathbb{R}^{2m+1}$ such that $\mathrm{d}\psi|_L = \gamma$, $\psi|_L = \mathrm{id}_L$, and ψ is independent of x_i. Denote $\alpha_1 = \psi^*\widehat{\alpha_{st}}$. Then α_1 and the contact form $\alpha_t = (1-t)\alpha_0 + t\alpha_1$ existing on a possibly smaller V are still independent of x_i.

Let $f_0 = \mathrm{id}$ and let f_t, $t \in \mathbb{R}$, be a smooth curve of diffeomorphisms in an open neighborhood of L such that $\mathrm{d}f_t|_{TL} = \mathrm{id}_{T\mathbb{R}^{2m+1}}|_{TL}$. Let X_t be the corresponding time dependent vector field, i.e. $\frac{\partial f_t}{\partial t} = X_t \circ f_t$. It follows that

$$\begin{aligned}\frac{\partial}{\partial t} f_t^*\alpha_t &= \frac{\partial}{\partial t} f_t^*\alpha_s\bigg|_{s=t} + f_s^*\frac{\partial}{\partial t}\alpha_t\bigg|_{s=t} = f_t^* L_{X_t}\alpha_t + f_t^*(\alpha_1 - \alpha_0)\\ &= f_t^*(i_{X_t}\,\mathrm{d}\alpha_t + \mathrm{d}i_{X_t}\alpha_t + \alpha_1 - \alpha_0).\end{aligned}$$

Therefore the proof consists in a construction of X_t such that

$$i_{X_t}\,\mathrm{d}\alpha_t + \mathrm{d}i_{X_t}\alpha_t + \alpha_1 - \alpha_0 = 0 \tag{4.4}$$

and such that X_t is independent of x_i. Indeed, then $\varphi = f_1^{-1} \circ \psi$ satisfies the claim.

We have $\alpha_0 = \alpha_1$ along L and $X_{\alpha_0} = \frac{\partial}{\partial t}$ is not tangent to L. Therefore $X_{\alpha_t} = X_{\alpha_0}$ along L and X_{α_t} is not tangent to L. Consequently, there exists a submanifold N of codim 1 in $\mathbb{R}^{2m+1}$ containing $L = \mathbb{R}^m \times 0$ such that N is transversal to the flow $\mathrm{Fl}^{X_{\alpha_t}}$ for all $t \in [0, 1]$. Define a time dependent $\mathbb{R}$-valued function u_t by

$$u_t\big(\mathrm{Fl}_s^{X_{\alpha_t}}(p)\big) = \int_0^s (\alpha_1 - \alpha_0)(X_{\alpha_t})\big(\mathrm{Fl}_\tau^{X_{\alpha_t}}(p)\big)\,d\tau$$

for $p \in N$. Hence u_t does not depend on x_i and it satisfies

$$\mathrm{d}u_t(X_{\alpha_t}) = i_{X_{\alpha_t}}(\alpha_1 - \alpha_0).$$

Now for the time dependent 1-form $\beta_t = \alpha_0 - \alpha_1 + \mathrm{d}u_t - u_t\alpha_t$, due to the existence of the isomorphism I_{α_t}, see (2.2), there is a unique time dependent vector field X_t, independent of x_i, such that $i_{X_t}\,\mathrm{d}\alpha_t + \alpha_t(X_t)\alpha_t = \beta_t$. Since $u_t = 0$ on L and $\mathrm{d}u_t|_{TL} = 0$, f_t is defined in a neighborhood of L in $\mathbb{R}^{2m+1}$ for all $t \in [0, 1]$. It follows that X_t satisfies (4.4).

(2) Let μ_A, ν_A be the diffeomorphisms of $\mathbb{R}^{2m+1}$ given by

$$\mu_A(x_0, x, y, \bar{x}_0, \bar{x}, \bar{y}, t) = \big(A^2x_0, Ax, Ay, A^2\bar{x}_0, A\bar{x}, A\bar{y}, t\big)$$

and

$$\nu_A(x_0, x, y, \bar{x}_0, \bar{x}, \bar{y}, t) = \big(A^2x_0, Ax, Ay, \bar{x}_0, A\bar{x}, A\bar{y}, A^2t\big).$$

We have $\mu_A^*\widehat{\alpha_{st}} = A^2\widehat{\alpha_{st}}$ and $\nu_A^*\alpha_0 = A^2\alpha_0$. That is, μ_A and ν_A are contactomorphisms w.r.t. $\widehat{\alpha_{st}}$ and α_0, resp., with $\lambda_{\mu_A} = \lambda_{\nu_A} = A^2$. We have $\mu_A(\tilde{E}_1^{(0)}) = \tilde{E}_A^{(0)}$ and $\nu_A(\tilde{E}_1^{(0)}) = \tilde{E}_A^{(0)}$. Put $U' = \mu_A(U)$, $V' = \nu_A(V)$, and $\varphi_A = \nu_A \circ \varphi \circ \mu_A^{-1}$. It follows that $\varphi_A^*\alpha_0 = \widehat{\alpha_{st}}$. Since $\|D^s\mu_A\| = \|D^s\nu_A\| = 0$ for $s > 1$, it is apparent from (3.3) that the inequality $K_{\varphi,r,A} \leqslant A^2K_{\varphi,r}$ holds.

(3) It is clear by definition that φ_A is independent of x_i if φ is so. □

Proposition 4.3. *Consider the contact form $\widehat{\alpha_{st}}$ in a neighborhood of $L \subset \mathcal{W}_k^m \times \mathbb{R}^{m+1}$, $k = 1, \ldots, n+1$. Then we have:*

(1) *For any $A > 1$ there is a contactomorphism $\varphi_A = \varphi_{A,k} : U' \to V'$, where U', V' are open neighborhoods of L, such that $\varphi_A|_L = \mathrm{id}_L$, $\varphi_A^*\alpha_0 = \widehat{\alpha_{st}}$, and for $r = 0, 1, \ldots$ one has $K_{\varphi,r,A} \leqslant A^2K_{\varphi,r}$, where*

$$K_{\varphi,r,A} = \sup_{k=1,\ldots,n+1}\ \sup_{s=0,\ldots,r+1} \max\big\{\big\|D^s\varphi_A|_{U'\cap\tilde{E}_A^{(k)}}\big\|, \big\|D^s(\varphi_A|_{U'\cap\tilde{E}_A^{(k)}})^{-1}\big\|\big\},$$

cf. (4.2), (4.3).

(2) *φ_A can be chosen so that it is independent of the variables ξ_i, $i = 0, 1, \ldots, n$.*

Proof. We will apply φ_A defined in Proposition 4.2. Since it is independent of x_i, it determines uniquely the map $\varphi_{A,k}$ which verifies all the requirements. □

Following the proof of Theorem 43.19 in [9] we can construct a chart at the identity in $\mathrm{Cont}_E(\mathcal{W}_k^m, \alpha_{st})$, where E is a subinterval of $E_A^{(k)}$, by means of φ_A from Propositions 4.2 and 4.3,

$$\Phi_A : \mathrm{Cont}_E\big(\mathcal{W}_k^m, \alpha_{st}\big) \supset \mathcal{U}_1 \ni f \mapsto u_f \in \mathcal{V}_2 \subset \mathrm{C}_E^\infty\big(\mathcal{W}_k^m\big),$$

where $\mathrm{C}_E^\infty(\mathcal{W}_k^m)$ is the totality of $\mathbb{R}$-valued functions on $\mathcal{W}_k^m$ compactly supported in E. Here $\mathcal{U}_1$ is a C^1-neighborhood of the identity in $\mathrm{Cont}_E(\mathcal{W}_k^m, \alpha_{st})$, $\mathcal{V}_2$ is a C^2-neighborhood of zero in $\mathrm{C}_E^\infty(\mathcal{W}_k^m)$, and $\Phi_A(\mathrm{id}) = 0_{\mathcal{W}_k^m}$.

Convention 4.4. In the subsequent steps of the proof of Theorem 1.1 the C^1 neighborhood $\mathcal{U}_1$ and the C^2 neighborhood $\mathcal{V}_2$ will be possibly shrunk several times and the resulting neighborhoods will depend on r, A, k, φ as above, and a smooth function ψ.

The chart Φ_A in the proof of Theorem 1.1 will be actually Φ_{A^5}, so in the sequel we will use in inequalities the coefficient A^β, β being a constant, rather than A^2, A^4, and so on.

The construction of Φ_A is the following. Let $\mathcal{U}_1$ be a small C^1-neighborhood of id in $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$. In particular, if $f \in \mathcal{U}_1$ then $\mu_0^*(f) < \frac{1}{4}$. For any $f \in \mathrm{Cont}_E(\mathcal{W}_k^m, \alpha_{st}) \cap \mathcal{U}_1$ let $\Gamma_f = (\mathrm{id}, f - \mathrm{id}, \lambda_f - 1) : \mathcal{W}_k^m \to \mathcal{W}_k^m \times \mathbb{R}^{m+1}$ be the corresponding graph map, that is $\Gamma_f(p) = (p, f(p) - p, \lambda_f(p) - 1)$ for all $p \in \mathcal{W}_k^m$. Then we set

$$\Phi_A(f) = u_f = \mathrm{pr}_3 \circ \varphi_A \circ \Gamma_f \circ (\mathrm{pr}_1 \circ \varphi_A \circ \Gamma_f)^{-1}, \tag{4.5}$$

where pr_i is the projection of $\mathcal{W}_k^m \times \mathbb{R}^m \times \mathbb{R}$ onto the i-th factor ($i = 1, 2, 3$), and we have

$$\mathrm{d}u_f = \mathrm{pr}_2 \circ \varphi_A \circ \Gamma_f \circ (\mathrm{pr}_1 \circ \varphi_A \circ \Gamma_f)^{-1}, \tag{4.6}$$

since $\varphi_A \circ \Gamma_f \circ (\mathrm{pr}_1 \circ \varphi_A \circ \Gamma_f)^{-1}$ is a section of pr_1 and a Legendre map w.r.t. α_0. Conversely, if $u = u_f \in \mathcal{V}_2$ then

$$\Phi_A^{-1}(u) - \mathrm{id} = f - \mathrm{id} = \mathrm{pr}_2 \circ \varphi_A^{-1} \circ \Gamma_u^{\mathrm{d}} \circ \big(\mathrm{pr}_1 \circ \varphi_A^{-1} \circ \Gamma_u^{\mathrm{d}}\big)^{-1} \tag{4.7}$$

and

$$\lambda_f - 1 = \mathrm{pr}_3 \circ \varphi_A^{-1} \circ \Gamma_u^{\mathrm{d}} \circ \big(\mathrm{pr}_1 \circ \varphi_A^{-1} \circ \Gamma_u^{\mathrm{d}}\big)^{-1}. \tag{4.8}$$

Here $\Gamma_u^{\mathrm{d}} : \mathcal{W}_k^m \to \mathcal{W}_k^m \times \mathbb{R}^{m+1}$ is given by $\Gamma_u^{\mathrm{d}}(p) = (p, \mathrm{d}u(p), u(p))$ for $u \in \mathrm{C}^\infty(\mathcal{W}_k^m)$ and $p \in \mathcal{W}_k^m$. It is easily seen that Φ_A^{-1} given by (4.7) is actually the inverse mapping of Φ_A given by (4.5).

From now on for a smooth function $h : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}$ and $r \geqslant 1$ we denote by $D_{(1)}^r h$ (resp. $D_{(2)}^r h$) the totality of partial derivatives of order r w.r.t. the first m variables (resp. the totality of partial derivatives of order r which contain at least one derivative w.r.t. the last $m + 1$ variables).

Consequently, we can write

$$D^r h = \big(D^r_{(1)} h, D^r_{(2)} h\big). \tag{4.9}$$

Lemma 4.5. *Suppose $r \geqslant 2$ and $k = 0, 1, \dots, n+1$. Under the notation of Propositions 4.2 and 4.3, there are constants β and $C_{\varphi,r}$, and $U' = U_{\varphi,r,A}$, an open neighborhood of L, such that for $i = 1, 2, 3$*

$$\big|D^r_{(1)}(\mathrm{pr}_i \circ \varphi_A)(p)\big| \leqslant A^\beta C_{\varphi,r} |p_2|$$

for any $p \in U' \cap \tilde{E}^{(k)}_A$. Here we denote $p = (p_1, p_2) \in \mathcal{W}^m_k \times \mathbb{R}^{m+1}$. The same is true if φ_A is replaced by φ_A^{-1}.

Proof. Observe that $D^r_{(1)}(\mathrm{pr}_i \circ \varphi_A)$ is a locally Lipschitz map and that in view of Propositions 4.2 and 4.3 the Lipschitz constant may be written in the form $A^2 C_{\varphi,r}$, since $\varphi_A|_L = \mathrm{id}_L$ and $D^r_{(1)}(\mathrm{pr}_i \circ \varphi_A) = 0$ on L by definition of φ_A. Consequently, $D^r_{(1)}(\mathrm{pr}_i \circ \varphi_A)|_{U' \cap \tilde{E}^{(k)}_A}$ is Lipschitz, where U' is an open neighborhood of L. The same is true for φ_A^{-1}. This implies the lemma. □

Proposition 4.6. *Let E be a subinterval of $E^{(k)}_A$. Under the above notation, for any $r \geqslant 2$ there is a C^1-neighborhood $\mathcal{U}_1$ of the identity in $\mathrm{Cont}_E(\mathcal{W}^m_k, \alpha_{st})$ such that for any $f \in \mathcal{U}_1$ one has*

(1) $\|D^{r+1} u_f\| \leqslant C_\varphi \mu^*_r(f) + A^\beta P_{\varphi,r}(M^*_{r-1}(f))$,
(2) $\mu^*_r(f) \leqslant C_\varphi \|D^{r+1} u_f\| + A^\beta P_{\varphi,r}(\sup_{i=0,\dots,r} \|D^i u_f\|)$,

where $P_{\varphi,r}$ has no constant term and β, C_φ are constants.

Proof. Set $\varphi_1 = \mathrm{pr}_1 \circ \varphi_A$, $\varphi_2 = \mathrm{pr}_2 \circ \varphi_A$.

(1) By (4.5), (4.6), (3.3) and Propositions 4.2 and 4.3 we have for $2 \leqslant s \leqslant r$

$$\big\|D^s(\varphi_2 \Gamma_f)\big\| \leqslant C_\varphi \mu^*_s(f) + A^\beta P_{\varphi,s}\big(M^*_{s-1}(f)\big). \tag{4.10}$$

In fact, the only nontrivial thing is to estimate $\|(D^s \varphi_2 \circ \Gamma_f) \cdot (D\Gamma_f \times \cdots \times D\Gamma_f)\|$ but, due to decomposition (4.9) and Lemma 4.5, we have

$$\begin{aligned}\big\|\big(D^s \varphi_2 \circ \Gamma_f\big) \cdot (D\Gamma_f \times \cdots \times D\Gamma_f)\big\| &\leqslant \big\|D^s_{(1)} \varphi_2 \circ \Gamma_f\big\| + \big\|D^s_{(2)} \varphi_2 \circ \Gamma_f\big\| \mu^*_1(f) \\ &\leqslant A^\beta C'_{\varphi,s} \mu^*_0(f) + C''_{\varphi,s} \mu^*_1(f) \\ &\leqslant A^\beta C_{\varphi,s} M^*_{s-1}(f).\end{aligned}$$

We have

$$\begin{aligned}\big\|D^{r+1} u_f\big\| = \big\|D^r(D u_f)\big\| &= \big\|D^r\big(\varphi_2 \circ \Gamma_f \circ (\varphi_1 \circ \Gamma_f)^{-1}\big)\big\| \\ &\leqslant C_\varphi \mu^*_r(f) + A^\beta P_{\varphi,r}\big(M^*_{r-1}(f)\big).\end{aligned}$$

Indeed, in view of (3.3), (3.5), (4.10) and Lemma 3.6(4), denoting $\varphi_{1f} = \varphi_1 \circ \Gamma_f$, the only nontrivial term to estimate is

$$\left\| D\varphi_{1f}^{-1} \cdot \left(\left(\left(D^r\varphi_1 \circ \Gamma_f\right) \cdot (D\Gamma_f \times \cdots \times D\Gamma_f)\right) \circ \varphi_{1f}^{-1}\right) \cdot \left(D\varphi_{1f}^{-1} \times \cdots \times D\varphi_{1f}^{-1}\right)\right\|,$$

and this can be obtained as above.

(2) We proceed analogously as in (1) and, in addition, we have to show that

$$\left\| D^r(\lambda_f - 1)\right\| \leqslant C_\varphi \left\| D^{r+1}u_f\right\| + A^\beta P_{\varphi,r}\Big(\sup_{i=0,\ldots,r} \left\| D^i u_f\right\|\Big).$$

This can be done as above in view of (4.8), (4.9) and Lemma 4.5. □

5. Two kinds of fragmentations

In most papers on the simplicity and perfectness of diffeomorphism groups a clue role is played by fragmentation properties. These properties enable usually to reduce the proof to the case $M = \mathbb{R}^m$. Contrary to the volume element case and the symplectic case (cf. [2]), in the contact case the fragmentation property takes its general form.

The following fragmentation property for infinitesimal contact automorphisms is a consequence of Proposition 2.1.

Lemma 5.1. *Let $X \in \mathfrak{X}_c(M,\alpha)$ with* $\mathrm{supp}(X) \subset \bigcup_{i=1}^k U_i$*, where U_i are open. Then there is a decomposition $X = X_1 + \cdots + X_k$ such that $X_i \in \mathfrak{X}_c(M,\alpha)$ and* $\mathrm{supp}(X_i) \subset U_i$*. The same is true for smooth curves in $\mathfrak{X}_c(M,\alpha)$ instead of elements of $\mathfrak{X}_c(M,\alpha)$.*

It follows the fragmentation property for $\mathrm{Cont}_c(M,\alpha)_0$.

Lemma 5.2. *Let $f \in \mathrm{Cont}_c(M,\alpha)_0$ and let $\{U_i\}_{i=1}^k$ be an open cover of M. Then there exist $f_j \in \mathrm{Cont}_c(M,\alpha)_0$, $j = 1,\ldots,l$, with $f = f_1 \circ \cdots \circ f_l$ such that* $\mathrm{supp}(f_j) \subset U_{i(j)}$ *for all j. The same is true for isotopies of contactomorphisms instead of contactomorphisms.*

The proof exploits the correspondence between isotopies in $\mathrm{Cont}_c(M,\alpha)_0$ and smooth curves in $\mathfrak{X}_c(M,\alpha)$ given by (2.1) combined with Lemma 5.1.

The fragmentation in Lemma 5.2 is said to be of the *first kind*. This lemma enables to replace $\mathrm{Cont}_c(M,\alpha)_0$ by $\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$ in the proof of Theorem 1.1. However, we need in this proof also the *second kind* of fragmentations. Such fragmentations exist in a C^1-neighborhood of the identity in the groups $\mathrm{Cont}_c(\mathcal{W}_k^m,\alpha_{st})_0$, $k = 0,1,\ldots,n+1$. Moreover, we claim that the norms of the factors of a given fragmentation are estimated by the norm of the initial contactomorphism in a convenient way and that the fragmentation itself is uniquely determined.

Definition 5.3. Suppose E is a subinterval of $E_A^{(k)}$. Let $\psi : \mathcal{W}_k^m \to [0,1]$ be a smooth function. It follows from Proposition 4.6 that there exists a C^1-neighborhood of the identity $\mathcal{U}_{\varphi,\psi,A} \subset \mathcal{U}_1$ such that for any $f \in \mathcal{U}_{\varphi,\psi,A}$ with $\mathrm{supp}(f) \subset E$ the contactomorphism

$$f^\psi := \Phi_A^{-1}\big(\psi\Phi_A(f)\big) = \Phi_A^{-1}(\psi u_f)$$

is well-defined and $\mathrm{supp}(f^\psi) \subset E$.

In fact, for any $r \geqslant 1$ there is a polynomial without constant term $P_{\psi,r}$ such that for all $u \in C_c^\infty(\mathcal{W}_k^m)$

$$\begin{aligned}\|D^{r+1}(\psi u)\| &\leqslant \|D^{r+1}u\| + \sum_{j=1}^{r+1} C_{r,j}\|D^j\psi\|\|D^{r+1-j}u\| \\ &\leqslant \|D^{r+1}u\| + P_{\psi,r}\Big(\sup_{s=0,\dots,r}\|D^s u\|\Big). \qquad (5.1)\end{aligned}$$

In particular we may ensure that $\psi u_f \in \mathcal{V}_2$. The following is obvious.

Proposition 5.4. *One has* $\operatorname{supp}(f^\psi) \subset \operatorname{supp}(\psi)$ *and* $f^\psi = f$ *on any open* $U \subset \mathcal{W}_k^m$ *such that* $\psi = 1$ *on* U.

Lemma 5.5. *Under the above notation, for any* $r \geqslant 2$ *there are polynomials* $P_{\varphi,\psi,r}$ *without constant term and constants* β, $C_{\varphi,\psi}$ *such that*

$$\mu_r^*(f^\psi) \leqslant C_{\varphi,\psi}\mu_r^*(f) + A^\beta P_{\varphi,\psi,r}(M_{r-1}^*(f)),$$

whenever $f \in \mathcal{U}_{\varphi,\psi,A}$ *and* $\operatorname{supp}(f) \subset E$. *In particular, if* $R_E \leqslant 2$ *(cf.* (3.1) *and* (3.8)*) there exists a constant* $C_{\varphi,\psi,r}$ *such that* $\mu_r^*(f^\psi) \leqslant A^\beta C_{\varphi,\psi,r}\mu_r^*(f)$ *for all* $f \in \mathcal{U}_{\varphi,\psi,A}$ *with* $\operatorname{supp}(f) \subset E$.

Proof. The first assertion follows from Proposition 4.6 and (5.1). The second is a consequence of Proposition 3.5. □

In particular, we obtain fragmentations of the second kind on large intervals in $\mathbb{R}^m$.

Proposition 5.6. *Let* $2A > 1$ *be an even integer, and let* $\psi : [0,1] \to [0,1]$ *be a smooth function such that* $\psi = 1$ *in a neighborhood of* $[0, \frac{1}{4}]$ *and* $\psi = 0$ *on* $[\frac{3}{4}, 1]$. *Then there exists a* C^1*-neighborhood* $\mathcal{U}_{\varphi,\psi,A}$ *of the identity in* $\operatorname{Cont}_{E_{2A}}(\mathbb{R}^m, \alpha_{st})_0$, *cf.* (2.6)*, such that for any* $f \in \mathcal{U}_{\varphi,\psi,A}$ *there exists a decomposition* $f = f_1 \dots f_{4A+1}$, *uniquely determined by* φ, ψ *and* A, *where each* $\operatorname{supp}(f_\kappa)$ *is contained in an interval of the form* $([k - \frac{3}{4}, k + \frac{3}{4}] \times \mathbb{R}^{2n}) \cap E_{2A}$, *with* $k \in \mathbb{Z}$, $|k| \leqslant 2A$, *and the inequalities*

(1) $\mu_r^*(f_\kappa) \leqslant C_{\varphi,\psi}\mu_r^*(f) + A^\beta P_{\varphi,\psi,r}(M_{r-1}^*(f))$,
(2) $\mu_r^*(f_\kappa) \leqslant A^\beta C_{\varphi,\psi,r}\mu_r^*(f)$, *whenever* $\operatorname{supp}(f) \subset E \subset E_{2A}$ *with* $R_E \leqslant 2$,

hold for all $\kappa = 1, \dots, 4A + 1$ *and* $r \geqslant 2$. *Analogous decompositions can be obtained w.r.t. the variables* x_i *and* y_i, $i = 1, \dots, n$.

Proof. By abusing the notation we extend ψ to the function $\psi : [-1,1] \to [0,1]$ given by $\psi(x) = \psi(-x)$ on $[-1,0]$, and, finally, to the periodic function $\psi : \mathbb{R} \to [0,1]$ of period 2.

Let $\psi_1 = \psi \circ \operatorname{pr}_1 : \mathbb{R}^m \to [0,1]$, where $\operatorname{pr}_1(x_0, x, y) = x_0$. Let $f \in \operatorname{Cont}_{E_{2A}}(\mathbb{R}^m, \alpha_{st})_0$ be sufficiently C^1-close to the identity and let f^{ψ_1} be defined as in Definition 5.3. Then we have

(1) $f^{\psi_1} = \prod_{k=-A}^{A} f_{2k}$, with $\operatorname{supp}(f_{2k}) \subset [2k - \frac{3}{4}, 2k + \frac{3}{4}] \times \mathbb{R}^{2n}$, and
(2) $f(f^{\psi_1})^{-1} = \prod_{k=-A}^{A-1} f_{2k+1}$ with $\operatorname{supp}(f_{2k+1}) \subset [2k + \frac{1}{4}, 2k + \frac{7}{4}] \times \mathbb{R}^{2n}$.

The inequalities follow from Lemmas 3.6 and 5.5. For convenience we renumerate f_κ. □

By applying Proposition 5.6 consecutively to all variables we get

Proposition 5.7. *Under the above assumptions, there exists a C^1-neighborhood of the identity $\mathcal{U}_{\varphi,\psi,A} \subset \operatorname{Cont}_{E_{2A}}(\mathbb{R}^m, \alpha_{st})_0$ such that for any $f \in \mathcal{U}_{\varphi,\psi,A}$ there exists a decomposition $f = f_1 \dots f_{a_m}$, uniquely determined by φ, ψ and A, where $a_m = (4A+1)^m$ and where each* $\operatorname{supp}(f_\kappa)$ *is contained in an interval of the form $([k_1 - \frac{3}{4}, k_1 + \frac{3}{4}] \times \cdots \times [k_m - \frac{3}{4}, k_m + \frac{3}{4}]) \cap E_{2A}$, with $k_i \in \mathbb{Z}$, $|k_i| \leqslant 2A$, for $i = 1, \dots, m$. Moreover, for all $\kappa = 1, \dots, a_m$ and $r \geqslant 2$*

(1) $\mu_r^*(f_\kappa) \leqslant C_{\varphi,\psi}\mu_r^*(f) + A^{\beta(m)} P_{\varphi,\psi,r}(M_{r-1}^*(f))$,
(2) $\mu_r^*(f_\kappa) \leqslant A^{\beta(m)} C_{\varphi,\psi,r}\mu_r^*(f)$, *whenever* $\operatorname{supp}(f) \subset E \subset E_{2A}$ *with* $R_E \leqslant 2$.

6. Shifting supports of contactomorphisms

From now on we set for $A > 1$

$$I_A = [-2, 2]^{n+1} \times [-2A, 2A]^n. \tag{6.1}$$

In this section we will describe the procedure of shifting supports of contactomorphisms on $\mathbb{R}^m$ in the y_i-directions. Fortunately, this can be done by using the contactomorphisms $\sigma_{i,t}$, $i = 1, \dots, n$, introduced in Section 2. Fix $1 \leqslant i \leqslant n$ and put $\sigma_t = \sigma_{i,t}$. Recall that $\sigma_t(x_0, x, y) = (x_0 + tx_i, x, y + t\mathbf{1}_i)$. Notice that for any $t \in \mathbb{R}$ we have $\|D\sigma_t\| = 1 + |t|$, and $\|D^r\sigma_t\| = 0$ for all $r > 1$. Next we define $\rho_{A,t} = \eta_A \circ \chi_A \circ \sigma_t$, see Section 2.

Under the assumption $A > 5n$, observe that

$$\operatorname{supp}\bigl(\rho_{A,t} \circ f \circ \rho_{A,t}^{-1}\bigr) \subset J_A, \tag{6.2}$$

where

$$J_A = \bigl[-A^5, A^5\bigr]^{n+1} \times [-2A, 2A]^n, \tag{6.3}$$

for all $f \in \operatorname{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ with support in $[-2, 2]^{n+i} \times [k-1, k+1] \times [-2, 2]^{n-i}$ with $|k| \leqslant 2A$ and suitable t. Likewise, the inclusion (6.2) holds for any $f \in \operatorname{Cont}_{I_A}(\mathbb{R}^m, \alpha_{st})_0$ with $\operatorname{supp}(f) \subset \mathbb{R}^{n+1} \times [k_1 - 1, k_1 + 1] \times \cdots \times [k_n - 1, k_n + 1]$ and $|k_i| \leqslant 2A$, where $i = 1, \dots, n$, with $\rho_{A,t}$ replaced by $\tilde{\rho}_{A,\mathbf{t}}$ given by

$$\tilde{\rho}_{A,\mathbf{t}} = \eta_A \circ \chi_A \circ \tilde{\sigma}_{\mathbf{t}},$$

where $\mathbf{t} = (t_1, \dots, t_n)$, and $\tilde{\sigma}_{\mathbf{t}} = \sigma_{1,t_1} \circ \cdots \circ \sigma_{n,t_n}$, with suitably chosen t_i so that $|t_i| \leqslant 2A$ for $i = 1, \dots, n$.

Proposition 6.1. *If* $|t_i| \leqslant 2A$ *for* $i = 1, \dots, n$ *and* $f \in \mathrm{Cont}_{I_A}(\mathbb{R}^m, \alpha_{st})_0$ *then, for any* $r \geqslant 2$

$$\mu_r^*\big(\tilde\rho_{A,\mathbf{t}} \circ f \circ \tilde\rho_{A,\mathbf{t}}^{-1}\big) \leqslant A^{4-r}(3n)^{r+1}\mu_r^*(f).$$

Proof. We have

$$\big(\rho_{A,t}^{-1}\big)(x_0, x, y) = \sigma_{-t}\big(A^{-3}x_0, A^{-2}x, A^{-1}y\big) = \big(A^{-3}x_0 - tA^{-2}x_i, A^{-2}x, A^{-1}y - t\mathbf{1}_i\big).$$

It follows that $\|D\tilde\rho_{A,\mathbf{t}}^{-1}\| \leqslant 3nA^{-1}$, as $|t| \leqslant 2A$. Likewise $\|D\tilde\rho_{A,\mathbf{t}}\| \leqslant 3nA^4$. Therefore for $r \geqslant 2$

$$\begin{aligned}\mu_r^*\big(\tilde\rho_{A,\mathbf{t}} \circ f \circ \tilde\rho_{A,\mathbf{t}}^{-1}\big) &\leqslant \|D\tilde\rho_{A,\mathbf{t}}\| \big\|D^r f\big\| \big\|D\tilde\rho_{A,\mathbf{t}}^{-1}\big\|^r \\ &\leqslant 3nA^4\big\|D^r f\big\|\big(3nA^{-1}\big)^r \leqslant A^{4-r}(3n)^{r+1}\mu_r^*(f),\end{aligned}$$

in view of (3.3) and the fact that $D^s\tilde\rho_{A,\mathbf{t}} = 0$ whenever $s > 1$.

Next, notice that $\lambda_{\chi_A} = A^2$, $\lambda_{\eta_A} = A$ and $\lambda_{\sigma_t} = 1$. Consequently, $\lambda_{\tilde\rho_{A,\mathbf{t}}} = A^3$ and by (3.6) $\lambda_{\tilde\rho_{A,\mathbf{t}} \circ f \circ \tilde\rho_{A,\mathbf{t}}} = \lambda_f \circ \tilde\rho_{A,\mathbf{t}}$. It follows from (3.3) that

$$\begin{aligned}\big\|D^r\lambda_{\tilde\rho_{A,\mathbf{t}} \circ f \circ \tilde\rho_{A,\mathbf{t}}^{-1}}\big\| = \big\|D^r\big(\lambda_f \circ \tilde\rho_{A,\mathbf{t}}^{-1}\big)\big\| &\leqslant \big\|D^r\lambda_f\big\|\big\|D\tilde\rho_{A,\mathbf{t}}^{-1}\big\|^r \\ &\leqslant \big\|D^r\lambda_f\big\|\big(3nA^{-1}\big)^r \leqslant A^{-r}(3n)^r\mu_r^*(f).\end{aligned}$$

Combining the above inequalities we obtain the claim. □

7. Construction of a correcting contactomorphism on $\mathcal{W}_k^m$

In this section for any sufficiently C^1-small contactomorphism on $\mathcal{W}_{k+1}^m$, $k = 0, \dots, n$, we construct a correcting contactomorphism which is indispensable in the construction of auxiliary rolling-up operators $\Psi_A^{(k)}$ (Proposition 8.5). The reason is that, given $f \in \mathrm{Cont}_{E_A^{(k)}}(\mathcal{W}_k^m, \alpha_{st})_0$, we wish to ensure that the norm $\mu_r^*(\Psi_A^{(k)}(f))$ of the rolled-up contactomorphism $\Psi_A^{(k)}(f)$ would be controlled by $\mu_r^*(f)$. The procedure of rolling-up contactomorphisms will be described in the next section.

For a smooth function $h : \mathcal{W}_{k+1}^m \to \mathbb{R}^l$ by $D_{[k]}^r h$ we denote the system of all partial derivatives of order r of h with at least one derivative w.r.t. ξ_k.

Lemma 7.1. *Let* $h \in \mathrm{C}^\infty(\mathcal{W}_{k+1}^m, \mathbb{R}^l)$. *Then we have* $\|D_{[k]}^s h\| \leqslant \|D_{[k]}^r h\|$ *for all* $1 \leqslant s \leqslant r$.

Proof. Let $h = (h_1, \dots, h_l)$ and $\gamma = (\gamma_0, \dots, \gamma_{2n}) \in \mathbb{N}_0^m$ with $|\gamma| = s$ and $\gamma_k > 0$. Set $\bar\gamma = \gamma + \mathbf{1}_k$. Then $|\bar\gamma| = s + 1$ and we may integrate $D^{\bar\gamma}h_i$, $i = 1, \dots, l$, w.r.t. ξ_k and use the fact that $D^\gamma h_i$ vanishes at a point (ξ_0, ξ, y) for any fixed $(\xi_0, \dots, \xi_{k-1}, \xi_{k+1}, \dots, y_n)$ to obtain $\|D^\gamma h_i\| \leqslant \|D^{\bar\gamma}h_i\|$. It follows $\|D_{[k]}^s h\| \leqslant \|D_{[k]}^{s+1}h\|$. The claim follows by induction. □

However, Lemma 7.1 does not hold for $s = 0$.

Observe that we may lift uniquely any $g \in \mathrm{Cont}_{E_A^{(k+1)}}(\mathcal{W}_{k+1}^m, \alpha_{st})_0$ sufficiently C^1-close to the identity to a contactomorphism $\tilde g \in \mathrm{Cont}_{E_A^{(k)}}(\mathcal{W}_k^m, \alpha_{st})_0$ which is periodic with period 1 (that is, $\tilde{g} - \mathrm{id}$ is periodic as a function with period 1) w.r.t. the variable ξ_k. Notice that $\tilde{g}$ depends continuously on g and $\mu_r^*(\tilde{g}) = \mu_r^*(g)$.

Let us denote for $l = 1, \dots, n+1$

$$\mathrm{Cont}_c\big(\mathcal{W}_k^m, \alpha_{st}\big)_0^{(l)} = \big\{f \in \mathrm{Cont}_c\big(\mathcal{W}_k^m, \alpha_{st}\big)_0\colon\ D_{\xi_i}(f - \mathrm{id}) = 0,\ i = 0, 1, \dots, l-1\big\}, \quad (7.1)$$

where $D_{\xi_i} = \frac{\partial}{\partial \xi_i}$. That is, $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0^{(l)}$ is the subgroup of $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$ consisting of all its elements which are independent of ξ_i, $i = 0, 1, \dots, l-1$. Further, denote $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0^{(0)} = \mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$.

Let $f \in \mathrm{Cont}_{E_A^{(k+1)}}(\mathcal{W}_{k+1}^m, \alpha_{st})_0^{(k)}$ will be sufficiently C^1-close to the identity. By using the chart Φ_A, we put $u_f = \Phi_A(f)$. Then $u_f \in \mathrm{C}_c^\infty(\mathcal{W}_{k+1}^m)$ is independent of $\xi_0, \dots, \xi_{k-1}$ in view of Proposition 4.3. Define $v_f \in \mathrm{C}_c^\infty(\mathcal{W}_{k+1}^m)$, independent of $\xi_0, \dots, \xi_k$, by fixing ξ_k to be equal to 0, that is

$$v_f(\xi_{k+1}, \dots, \xi_n, y) = u_f(0, \xi_{k+1}, \dots, \xi_n, y).$$

Then $u_f, v_f \in \mathcal{V}_2$ and we define

$$\hat{f} = \Phi_A^{-1}(v_f). \quad (7.2)$$

Notice that $\hat{f}$ is independent of $\xi_0, \dots, \xi_k$ as v_f is so. Next we put

$$w_f = \Phi_A\big(f\hat{f}^{-1}\big). \quad (7.3)$$

Observe that the equality

$$\hat{f}(\xi_{k+1}, \dots, \xi_n, y) = f(0, \xi_{k+1}, \dots, \xi_n, y)$$

is not true, since $\hat{f}$ defined by it does not fulfil the equalities in Proposition 2.2, provided f does.

Finally, denote for $r \geqslant 1$

$$\nu_r^*(f) = C_\varphi K^r \mu_r^*(f) + F_{\varphi,r}\big(M_{r-1}^*(f)\big),$$

where K, C_φ are constants, $F_{\varphi,r}$ is an admissible polynomial and $F_{\varphi,1} = 0$.

Proposition 7.2. *Let E be a subinterval of $E_A^{(k+1)} \subset \mathcal{W}_{k+1}^m$, $k = 0, 1, \dots, n$. There exist constants and polynomials as above and a constant β such that if f belongs to a sufficiently small C^1-neighborhood $\mathcal{U}_1 = \mathcal{U}_{\varphi,A}$ of the identity in $\mathrm{Cont}_E(\mathcal{W}_{k+1}^m, \alpha_{st})_0^{(k)}$ then we have for all $r \geqslant 1$:*

(1) $\hat{f} \in \mathrm{Cont}_E(\mathcal{W}_{k+1}^m, \alpha_{st})_0^{(k+1)}$ *and* $\lambda_{\hat{f}} = 1$.
(2) $\forall 1 \leqslant s \leqslant r+1$, $\|D_{[k]}^s u_f\| \leqslant C_\varphi \mu_r^*(f)$.
(3) $\forall 0 \leqslant s \leqslant r$, $\mu_s^*(f\hat{f}^{-1}) \leqslant \nu_r^*(f)$.
(4) $\mu_r^*(\hat{f}) \leqslant \nu_r^*(f)$.
(5) $\forall 0 \leqslant s \leqslant r+1$, $\|D^s w_f\| \leqslant A^\beta \nu_r^*(f)$.

Proof. For short we will write $\varphi_i = \mathrm{pr}_i \circ \varphi_A$ and $\bar{\varphi}_i = \mathrm{pr}_i \circ \varphi_A^{-1}$ for $i = 1, 2, 3$, that is $\varphi_A = (\varphi_1, \varphi_2, \varphi_3)$ and $\varphi_A^{-1} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$, cf. (4.5)–(4.8). Further we will denote $\varphi_{if} = \varphi_i \circ \Gamma_f$, $\bar{\varphi}_{iu} = \bar{\varphi}_i \circ \Gamma_u^{\mathrm{d}}$, $i = 1, 2, 3$.

Let $\mathcal{M}_m$ be the set of all nonsingular matrices of deg m. By the Lipschitz property of the inverse mapping in $\mathcal{M}_m$ there are a neighborhood U of id in $\mathcal{M}_m$ and a constant L such that for all $m_1, m_2 \in U$

$$\left|m_1^{-1} - m_2^{-1}\right| \leqslant L|m_1 - m_2|. \tag{7.4}$$

(1) As we stated above $\hat{f}$ is independent of $\xi_0, \dots, \xi_k$. Since $X_{\alpha_{st}} = \frac{\partial}{\partial \xi_0}$, we have $\hat{f}_* X_{\alpha_{st}} = X_{\alpha_{st}}$. Consequently, by (4.1), $\lambda_{\hat{f}} = \hat{f}^*(i_{X_{\alpha_{st}}} \alpha_{st}) = 1$.

(2) First note that $D_{[k]}\varphi_{1f} = \mathbf{1}_k + l_f$, where for any $p \in \mathcal{W}_k^m$, $l_f(p) \in \mathbb{R}^m$ is such that $\|l_f\| \leqslant C_\varphi \|D_{[k]}(f - \mathrm{id})\|$. Then by Lemma 7.1 $\|l_f\| \leqslant C_\varphi \mu_r^*(f) \leqslant \nu_r^*(f)$. Thanks to (7.4) and the formula for inverse matrix, the same property possesses $D_{[k]}\varphi_{1f}^{-1}$.

Observe that

$$(D_{\xi_k}\varphi_A)_i = \delta_{ik}, \quad i = 1, \dots, 2m + 1, \tag{7.5}$$

due to Proposition 4.3. It follows that

$$\begin{aligned} \|D_{[k]}\varphi_{3f}\| &= \left\|(D\varphi_3 \circ \Gamma_f) \cdot D_{[k]}\Gamma_f\right\| \leqslant \|D_{\xi_k}\varphi_3\| + \|D\varphi_3\| \left\|D_{\xi_k}(f - \mathrm{id})\right\| \\ &= C_\varphi \left\|D_{\xi_k}(f - \mathrm{id})\right\| \leqslant C_\varphi \mu_r^*(f) \leqslant \nu_r^*(f), \end{aligned}$$

by using (7.5) and Lemma 7.1. Now, due to (4.5), (7.4) and the above arguments,

$$\|D_{[k]}u_f\| = \left\|\left(D\varphi_{3f} \circ \varphi_{1f}^{-1}\right) \cdot D_{[k]}\varphi_{1f}^{-1}\right\| \leqslant \|D_{[k]}\varphi_{3f}\| + \|l_f\| \leqslant C_\varphi \mu_r^*(f) \leqslant \nu_r^*(f),$$

where l_f corresponds to $D_{[k]}\varphi_{1f}^{-1}$.

For $s > 1$ we have $D_{[k]}^s \varphi_A = 0$ by (7.5). We use (4.6), (3.3), (3.5), (7.4), and the proof is similar.

(3) From the definition of v_f we have

$$\|u_f - v_f\| \leqslant \|D_{\xi_k}u_f\|, \qquad \|Du_f - Dv_f\| = \|D_{\xi_k}u_f\|. \tag{7.6}$$

First we show (3) for $s = 0$. Observe that $\|f\hat{f}^{-1} - \mathrm{id}\| = \|f - \hat{f}\|$. By (4.7) we obtain

$$\begin{aligned} \|f - \hat{f}\| &= \left\|\bar{\varphi}_{2u_f}\bar{\varphi}_{1u_f}^{-1} - \bar{\varphi}_{2v_f}\bar{\varphi}_{1v_f}^{-1}\right\| \\ &= \left\|\bar{\varphi}_{2u_f}\bar{\varphi}_{1u_f}^{-1} - \bar{\varphi}_{2u_f}\bar{\varphi}_{1v_f}^{-1}\right\| + \left\|\bar{\varphi}_{2u_f}\bar{\varphi}_{1v_f}^{-1} - \bar{\varphi}_{2v_f}\bar{\varphi}_{1v_f}^{-1}\right\| \\ &\leqslant L_\varphi \left\|\Gamma_{u_f}^{\mathrm{d}} - \Gamma_{v_f}^{\mathrm{d}}\right\| \leqslant C_\varphi \mu_r^*(f), \end{aligned}$$

due to (2), (7.6) and the Lipschitz property.

Next, in view of (1) and (3.6) we have $\|\lambda_{f\hat{f}^{-1}} - 1\| = \|\lambda_f - \lambda_{\hat{f}}\|$. Hence by (4.8) and a similar argument, $\|\lambda_{f\hat{f}^{-1}} - 1\| \leqslant C_\varphi \mu_r^*(f)$.

By (4.7) and (3.2) we get

$$D(f-\hat{f}) = \big(D\bar{\varphi}_{2u_f}\circ\bar{\varphi}_{1u_f}^{-1}\big)\cdot D\bar{\varphi}_{1u_f}^{-1} - \big(D\bar{\varphi}_{2v_f}\circ\bar{\varphi}_{1v_f}^{-1}\big)\cdot D\bar{\varphi}_{1v_f}^{-1}$$
$$= \big(\big(D\bar{\varphi}_{2u_f}\circ\bar{\varphi}_{1u_f}^{-1}\big)\cdot D\bar{\varphi}_{1u_f}^{-1} - \big(D\bar{\varphi}_{2u_f}\circ\bar{\varphi}_{1v_f}^{-1}\big)\cdot D\bar{\varphi}_{1v_f}^{-1}\big)$$
$$+ \big(\big(D\bar{\varphi}_{2u_f}\circ\bar{\varphi}_{1v_f}^{-1}\big)\cdot D\bar{\varphi}_{1v_f}^{-1} - \big(D\bar{\varphi}_{2v_f}\circ\bar{\varphi}_{1v_f}^{-1}\big)\cdot D\bar{\varphi}_{1v_f}^{-1}\big). \tag{7.7}$$

It follows that $\|D(f-\hat{f})\| \leqslant C_\varphi \mu_r^*(f)$, due to (2), (7.6) and the Lipschitz property.

Let $1 < s \leqslant r$. In view of (3.3), (7.6), (7.7), the Leibniz rule, the Lipschitz property and, again, (2) we have

$$\big\|D^s(f-\hat{f})\big\| \leqslant \sup_{|\gamma|=s-1}\big\|D^\gamma\big(D(f-\hat{f})\big)\big\| \leqslant \nu_r^*(f). \tag{7.8}$$

Likewise

$$\big\|D^s(\lambda_f - \lambda_{\hat{f}})\big\| \leqslant \nu_r^*(f). \tag{7.9}$$

Now, since we have

$$D^s\big(f\hat{f}^{-1}-\mathrm{id}\big) = D^{s-1}\big(D\big(f\hat{f}^{-1}-\mathrm{id}\big)\big)$$
$$= D^{s-1}\big(\big(Df\circ\hat{f}^{-1}\big)\cdot D\hat{f}^{-1} - \big(D\hat{f}\circ\hat{f}^{-1}\big)\cdot D\hat{f}^{-1}\big),$$

(3) for $s \geqslant 1$ follows from (3.3), the Leibniz rule, (7.8) and (7.9).

(4) It is an immediate consequence of (7.8) and (7.9).

(5) To simplify notation let $g = f\hat{f}^{-1}$. In view of (4.5) and (7.3) we have $w_f = \varphi_3\Gamma_g(\varphi_1\Gamma_g)^{-1} = \varphi_{3g}\varphi_{1g}^{-1}$. For $1 \leqslant s \leqslant r$ we have

$$\big\|D^{s+1}\varphi_{3g}\big\| \leqslant C_\varphi\big(\mu_s^*(g) + A^2\mu_0^*(g)\big) + C_{\varphi,s}\sup\big\|\big(D^i\varphi_3\circ\Gamma_g\big)\cdot\big(D^{j_1}\Gamma_g\times\cdots\times D^{j_i}\Gamma_g\big)\big\|$$
$$\leqslant A^\beta\nu_r^*(f),$$

where sup is taken over $i = 2,\dots,s-1$, with $j_1+\cdots+j_i = s$, $j_l \geqslant 1$ for $l = 1,\dots,i$, and $j_l > 1$ for some l. In fact, it follows from (3) and (4) above, (3.3) and Lemma 4.5. In order to obtain (5) for $s \geqslant 2$, in view of (3.3) and (3.5), it suffices to show that $\|D^s\varphi_{1g}\| \leqslant A^\beta\nu_r^*(f)$, and this can be done analogously as above.

Finally, to obtain (5) for $s = 0$ and $s = 1$ we integrate $D^2 w_f$ w.r.t. y_1 twice or once, bearing in mind that $\mathrm{supp}(w_f) \subset E_A^{(k+1)}$. □

Corollary 7.3. *If f belongs to a sufficiently small C^{r-1}-neighborhood $\mathcal{U}_1 = \mathcal{U}_{\varphi,r,A}$ of the identity in $\mathrm{Cont}_E(\mathcal{W}_{k+1}^m,\alpha_{st})_0^{(k)}$ then we have for all $r \geqslant 1$:*

(1) $\mu_r^*(\hat{f}) \leqslant C_{\varphi,r}\mu_r^*(f)$.
(2) $\forall 0 \leqslant s \leqslant r+1$, $\|D^s w_f\| \leqslant A^\beta C_{\varphi,r}\mu_r^*(f)$.

In fact, we can rewrite (4) as

$$\mu_r^*(\hat{f}) \leqslant C_\varphi K^r \mu_r^*(f) + F_{\varphi,r,A}\big(M_{r-1}^*(f)\big),$$

and use Definition 3.2. Similarly, we can proceed with (5).

8. Rolling-up contactomorphisms

A possible application of Mather's rolling-up operators $\Psi_{i,A}$ (cf. [12]) to the contact case fails completely in the y_i-directions. But even in the "good" directions x_i, $i = 0, \dots, n$, the operators $\Psi_{i,A}$ do not apply verbatim. The next and greater difficulty is that for a contactomorphism f the class $[\Psi_{i,A}(f)]$ need not be equal to $[f]$ in the abelianization $H_1(\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0)$. Roughly speaking, the reason is that given $f \in \mathrm{Cont}(\mathbb{R}^m, \alpha_{st})_0$ with $\mathrm{supp}(f) \subset \mathbb{R} \times [-A, A]^{2n}$, any $g \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ such that $g = f$ on $[-A, A]^m$ must depend on y_i, cf. (2.3). This fact seems to spoil any possible proof that $[\Psi_{0,A}(f)] = [f]$, and the same is for $i = 1, \dots, n$.

In the present section we define a new rolling-up operator which works in the contact category (Proposition 8.7). To this end we will use the contact cylinders $(\mathcal{W}_k^m, \alpha_{st})$, $k = 1, \dots, n+1$. The correcting contactomorphisms defined in the previous section enable us to define auxiliary rolling-up operators $\Psi_A^{(k)}$ acting on $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$. A clue observation is that a "remainder" contactomorphism living on $\mathcal{W}_{n+1}^m$ admits a representant in the commutator subgroup of $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$.

Observe that the application of the rolling-up operator is indispensable in the proof. In fact, we cannot apply the procedure described in Section 5 (the fragmentation of the second kind) to the group $\mathrm{Cont}_{J_A}(\mathbb{R}^m, \alpha_{st})_0$, considered in the proof of Theorem 1.1 (Section 9), since in this case a coefficient of the form A^{Cr} would appear in Proposition 5.7(2) and the proof would be no longer valid.

In this section A is a large positive integer. Throughout we denote

$$\begin{aligned}
J_A^{(0)} &= J_A = \big[-A^5, A^5\big]^{n+1} \times [-2A, 2A]^n, \\
J_A^{(k)} &= \big(\mathbb{S}^1\big)^k \times \big[-A^5, A^5\big]^{n-k+1} \times [-2A, 2A]^n, \quad k = 1, \dots, n, \\
K_A^{(0)} &= K_A = [-2, 2] \times \big[-A^5, A^5\big]^n \times [-2A, 2A]^n, \\
K_A^{(k)} &= \big(\mathbb{S}^1\big)^k \times [-2, 2] \times \big[-A^5, A^5\big]^{n-k} \times [-2A, 2A]^n, \quad k = 1, \dots, n.
\end{aligned} \tag{8.1}$$

Observe that $R_{K_A^{(k)}} = 2$ and $R_{J_A^{(k)}} = 2A$ (cf. (3.1) and (3.8)).

Denote by $\pi_k : \mathcal{W}_k^m \to \mathcal{W}_{k+1}^m$, $k = 0, 1, \dots, n$, the canonical projection. In other words, π_k is induced by the canonical projection $\pi : \mathbb{R} \to \mathbb{S}^1$ on the $(k+1)$-st factor of $\mathcal{W}_k^m$.

Let $f \in \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^m, \alpha_{st})_0 \cap \mathcal{U}_1$, where $\mathcal{U}_1$ is a sufficiently C^1-small neighborhood of the identity in $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$, with $\mu_0(f) \leqslant \frac{1}{2}$. For $q \in \mathcal{W}_{k+1}^m$ we choose $p = (\xi_0, \xi, y) \in \mathcal{W}_k^m$ with $\pi_k(p) = q$ and $\xi_k < -A^5$. Let $\tau_k = \tau_{k,1}$ be the unit translation along the x_k-axis (cf. Section 2). Then we choose $l \in \mathbb{N}$ such that $((\tau_k f)^l(p))_k > A^5$. We define $\Theta_A^{(k)}(f) : \mathcal{W}_{k+1}^m \to \mathcal{W}_{k+1}^m$ by

$$\Theta_A^{(k)}(f)(q) = \pi_k\big((\tau_k f)^l(p)\big).$$

The definition is independent of the choice of l and p.

Proposition 8.1. *Let $k = 0, 1, \dots, n$. Possibly shrinking $\mathcal{U}_1$, the mapping*

$$\Theta_A^{(k)} : \mathrm{Cont}_{J_A^{(k)}}\big(\mathcal{W}_k^m, \alpha_{st}\big)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{J_A^{(k+1)}}\big(\mathcal{W}_{k+1}^m, \alpha_{st}\big)_0$$

satisfies the following conditions:

(1) $\Theta_A^{(k)}$ *is continuous and it preserves the identity.*
(2) $\Theta_A^{(k)}(\mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^m, \alpha_{st})_0^{(k)}) \subset \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^m, \alpha_{st})_0^{(k)}$, *cf.* (7.1).
(3) *There exist constants β, K, and admissible polynomials $F_{r,A}$ for all $r \geqslant 1$ such that*

$$\mu_r^*\big(\Theta_A^{(k)}(g)\big) \leqslant A^\beta K^r \mu_r^*(g) + F_{r,A}\big(M_{r-1}^*(g)\big),$$

for any $g \in \mathrm{dom}(\Theta_A^{(k)})$. Moreover, we may have $F_{1,A} = 0$.

Proof. (1) and (2) are obvious. A standard proof for (3) follows by virtue of Lemma 3.6 and Remark 3.7. □

In order to define the rolling-up operator Ψ_A first we introduce

$$\Xi_A^{(k)} : \mathrm{Cont}_{J_A^{(k+1)}}\big(\mathcal{W}_{k+1}^m, \alpha_{st}\big)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{K_A^{(k)}}\big(\mathcal{W}_k^m, \alpha_{st}\big)_0,$$

where $k = 0, 1, \dots, n$ and $\mathcal{U}_1$ is a C^1-neighborhood of id in $\mathrm{Cont}_c(\mathcal{W}_{k+1}^m, \alpha_{st})_0$.

Let $\psi : \mathbb{S}^1 \to [0, 1]$ be a smooth function such that $\psi = 1$ in a neighborhood of $[-\frac{1}{8}, \frac{1}{8}]$ and $\psi = 0$ on $[\frac{3}{8}, \frac{5}{8}]$. Abusing the notation, let $\psi : \mathcal{W}_{k+1}^m \to [0, 1]$ such that $\psi(\xi_0, \xi, y) = \psi(\xi_k)$. For $g \in \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^m, \alpha_{st})_0 \cap \mathcal{U}_1$ we define

$$g^\psi = \Phi_A^{-1}\big(\psi \Phi_A(g)\big) = \Phi_A^{-1}(\psi u_g),$$

as in Definition 5.3. For short, set $\mathcal{E}_{A,n,k} = [-A^5, A^5]^{n-k} \times [-2A, 2A]^n$. Then $g^\psi = g$ on $(\mathbb{S}^1)^k \times [-\frac{1}{8}, \frac{1}{8}] \times \mathcal{E}_{A,n,k}$ and $\mathrm{supp}(g^\psi) \subset (\mathbb{S}^1)^k \times [-\frac{3}{8}, \frac{3}{8}] \times \mathcal{E}_{A,n,k}$, in view of Proposition 5.4.

Let g_1^ψ (resp. g_2^ψ) be the unique lift of $(g^\psi)^{-1}g$ (resp. g^ψ) to $\mathcal{W}_k^m$. Then g_1^ψ and g_2^ψ are periodic contactomorphisms supported in $(\mathbb{S}^1)^k \times \mathbb{R} \times \mathcal{E}_{A,n,k}$. For small enough $\mathcal{U}_1$ there is $\varepsilon > 0$ such that $g_1^\psi = g$ on $(\mathbb{S}^1)^k \times [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \times \mathcal{E}_{A,n,k}$ and $g_2^\psi = g$ on $(\mathbb{S}^1)^k \times [1 - \varepsilon, 1 + \varepsilon] \times \mathcal{E}_{A,n,k}$.

Next we put $E_k^- = \{(\xi_0, \xi, y) \in \mathcal{W}_k^m\colon -1 \leqslant \xi_k \leqslant 0\}$ and $E_k^+ = \{(\xi_0, \xi, y) \in \mathcal{W}_k^m\colon \frac{1}{2} \leqslant \xi_k \leqslant \frac{3}{2}\}$, and we define $\Xi_A^{(k)}(g)$ by the conditions

$$\Xi_A^{(k)}(g)\big|_{E_k^-} = g_1^\psi\big|_{E_k^-}, \qquad \Xi_A^{(k)}(g)\big|_{E_k^+} = g_2^\psi\big|_{E_k^+}, \tag{8.2}$$

and $\Xi_A^{(k)}(g) = \mathrm{id}$ on $\mathcal{W}_k^m \setminus (E_k^- \cup E_k^+)$.

Proposition 8.2. *Taking $\mathcal{U}_1$ small enough, the mapping $\Xi_A^{(k)}$ satisfies the following conditions:*

(1) $\Xi_A^{(k)}$ *is continuous and it preserves the identity.*

(2) $\Xi_A^{(k)}(\mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^m,\alpha_{st})_0^{(k)}) \subset \mathrm{Cont}_{K_A^{(k)}}(\mathcal{W}_k^m,\alpha_{st})_0^{(k)}$.

(3) *There are constants $C_{\varphi,\psi}$, β and K, and for any $r \geqslant 2$ there is a polynomial with no constant term $P_{\varphi,\psi,r}$ such that for any $g \in \mathrm{dom}(\Xi_A^{(k)})$ one has*

$$\mu_r^*\big(\Xi_A^{(k)}(g)\big) \leqslant K^r C_{\varphi,\psi}\mu_r^*(g) + A^\beta P_{\varphi,\psi,r}\big(M_{r-1}^*(g)\big).$$

In particular, $\mu_r^*(\Xi_A^{(k)}(g)) \leqslant A^\beta C_{\varphi,\psi,r}\mu_r^*(g)$ whenever $\mathrm{supp}(g) \subset E$ with $R_E \leqslant 2$.

(4) *For any $g \in \mathrm{dom}(\Xi_A^{(k)})$ one has $\Theta_A^{(k)}\Xi_A^{(k)}(g) = g$.*

Proof. The properties (1) and (4) can be deduced from the definition. To check (2) we use Proposition 4.3. Finally, as $\mu_r^*(\Xi_A^{(k)}(g)) \leqslant \max\{\mu_r^*(g_1^\psi), \mu_r^*(g_2^\psi)\}$, (3) follows from Lemmas 5.5 and 3.6. □

It will be useful to introduce operators

$$\Theta^{(k)} : \mathrm{Cont}_c\big(\mathcal{W}_k^m,\alpha_{st}\big)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_c\big(\mathcal{W}_{k+1}^m,\alpha_{st}\big)_0, \quad k = 0,\dots,n,$$

obtained by gluing-up $\Theta_A^{(k)}$, $\Theta^{(k)} = \bigcup \Theta_A^{(k)}$. Now, let us return to the "hat" operation defined by (7.2). For $f \in \mathrm{Cont}_{E_A^{(k)}}(\mathcal{W}_k^m,\alpha_{st})_0^{(k)} \cap \mathcal{U}_1$ denote $\hat{\Theta}_A^{(k)}(f) = \widehat{\Theta_A^{(k)}(f)}$. We set $\hat{\Theta}^{(k)} = \bigcup \hat{\Theta}_A^{(k)}$ and we have operators

$$\hat{\Theta}^{(k)} : \mathrm{Cont}_c\big(\mathcal{W}_k^m,\alpha_{st}\big)_0^{(k)} \cap \mathcal{U}_1 \to \mathrm{Cont}_c\big(\mathcal{W}_{k+1}^m,\alpha_{st}\big)_0^{(k+1)}, \quad k = 0,\dots,n.$$

Likewise $\Xi^{(k)} = \bigcup \Xi_A^{(k)}$, that is we have

$$\Xi^{(k)} : \mathrm{Cont}_c\big(\mathcal{W}_{k+1}^m,\alpha_{st}\big)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_c\big(\mathcal{W}_k^m,\alpha_{st}\big)_0, \quad k = 0,\dots,n.$$

Lemma 8.3. *If $f, g \in \mathrm{dom}(\Theta^{(k)})$ and $\Theta^{(k)}(f) = \Theta^{(k)}(g)$ then $[f] = [g]$ in $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m,\alpha_{st})_0)$.*

Proof. Let us define a contactomorphism $\Lambda_k = \Lambda_k(f,g)$ by

$$\Lambda_k(p) = (\tau_k g)^l(\tau_k f)^{-l}(p), \tag{8.3}$$

where $p \in \mathcal{W}_k^m$, $\tau_k = \tau_{k,1}$ is the translation, and l is a positive integer so large that $[(\tau_k f)^{-l}(p)]_k < -A^5$. Clearly, Λ_k does not depend on l, and $\Lambda_k \in \mathrm{Cont}_c(\mathcal{W}_k^m,\alpha_{st})_0$ in view of the definition of $\Theta^{(k)}$ and the assumption. From (8.3) we have $\Lambda_k\tau_k f\Lambda_k^{-1} = \tau_k g$ and, consequently, $[f] = [g]$. □

Lemma 8.4. *Let $k = 0, 1, \dots, n$.*

(1) *If $\Theta^{(k)}(f_i) = g_i$, $i = 1,\dots,l$, then there are $\bar{f}_i \in \mathrm{Cont}_c(\mathcal{W}_k^m,\alpha_{st})_0$ such that $\Theta^{(k)}(f) = g_1\dots g_l$, where $f = \bar{f}_1\dots\bar{f}_l$, and $[\bar{f}_i] = [f_i]$ in $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m,\alpha_{st})_0)$ for all i. Moreover, we can have $\bar{f}_1 = f_1$.*

(2) *If* $g_1, g_2, g_1g_2 \in \mathrm{dom}(\Xi^{(k)})$ *then* $[\Xi^{(k)}(g_1g_2)] = [\Xi^{(k)}(g_1)\Xi^{(k)}(g_2)]$ *in the group* $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0)$.

(3) *If* $g \in \mathrm{Cont}_c(\mathcal{W}_{k+1}^m, \alpha_{st})_0$ *with* $[g] = e$ *in* $H_1(\mathrm{Cont}_c(\mathcal{W}_{k+1}^m, \alpha_{st})_0)$ *then there is* $f \in \mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$ *such that* $\Theta^{(k)}(f) = g$ *and* $[f] = e$ *in* $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0)$.

Proof. (1) We may shift supports of f_i by the translations $\tau_{k,t}$ to obtain $\bar{f}_i$ such that the family $\{\bar{f}_i\}$ has pairwise disjoint supports. Clearly $[\bar{f}_i] = [f_i]$. Moreover, by definition of $\Theta^{(k)}$ we can arrange $\bar{f}_i$ in the way that $\Theta_A^{(k)}(f) = g_1 \ldots g_l$ for $f = \bar{f}_1 \ldots \bar{f}_l$ and $\bar{f}_1 = f_1$.

(2) Put $f_i = \Xi^{(k)}(g_i)$, $i = 1, 2$. In view of (1) there is $f \in \mathrm{dom}(\Theta^{(k)})$ such that $[f] = [f_1f_2]$ and $\Theta^{(k)}(f) = g_1g_2$. By Proposition 8.2(4), $\Theta^{(k)}\Xi^{(k)}(g_1g_2) = g_1g_2 = \Theta^{(k)}(f)$. Therefore, from Lemma 8.3

$$\big[\Xi^{(k)}(g_1g_2)\big] = [f] = [f_1f_2] = \big[\Xi^{(k)}(g_1)\Xi^{(k)}(g_2)\big].$$

(3) First we define an operator

$$\bar{\Xi}^{(k)} : \mathrm{Cont}_c\big(\mathcal{W}_{k+1}^m, \alpha_{st}\big)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_c\big(\mathcal{W}_k^m, \alpha_{st}\big)_0, \quad k = 0, \ldots, n,$$

with $\mathrm{dom}(\bar{\Xi}^{(k)}) = \mathrm{dom}(\Xi^{(k)})$ such that for any $g \in \mathrm{dom}(\bar{\Xi}^{(k)})$ we have $[\bar{\Xi}^{(k)}(g)] = [\Xi^{(k)}(g)^{-1}]$ and $\Theta^{(k)}\bar{\Xi}^{(k)}(g) = g^{-1}$.

Namely, let us return to the definition of $\Xi^{(k)}(g) = \Xi_A^{(k)}(g)$. We have the decomposition $g = g_2^{\psi} g_1^{\psi}$, where ψ is a suitable smooth function. Now, we define $\bar{\Xi}^{(k)}(g)$ by changing (8.2) as follows

$$\bar{\Xi}_A^{(k)}(g)\big|_{E_k^-} = \tau_{k,\frac{3}{2}}^{-1} \circ \big(g_2^{\psi}\big)^{-1}\big|_{E_k^+} \circ \tau_{k,\frac{3}{2}}, \qquad \bar{\Xi}_A^{(k)}(g)\big|_{E_k^+} = \tau_{k,\frac{3}{2}} \circ \big(g_1^{\psi}\big)^{-1}\big|_{E_k^-} \circ \tau_{k,\frac{3}{2}}^{-1},$$

and $\bar{\Xi}_A^{(k)}(g) = \mathrm{id}$ on $\mathcal{W}_k^m \setminus (E_k^- \cup E_k^+)$.

By assumption there are $h_j \in \mathrm{Cont}_c(\mathcal{W}_{k+1}^m, \alpha_{st})_0$, $j = 1, \ldots, 2l$, such that

$$g = [h_1, h_2] \ldots [h_{2l-1}, h_{2l}].$$

For all j we may write a decomposition $h_j = h_{j,1} \cdots h_{j,l(j)}$, where the factors are C^1-small.

Put $f_{j,s} = \Xi^{(k)}(h_{j,s})$ and $f_{j,s}^* = \bar{\Xi}^{(k)}(h_{j,s})$, $j = 1, \ldots, 2l$, $s = 1, \ldots, l(j)$. Let us define $f_j = \bar{f}_{j,1} \cdots \bar{f}_{j,l(j)}$ and $f_j^* = \bar{f}_{j,l(j)}^* \cdots \bar{f}_{j,1}^*$ as in the proof of (1). In particular, $\Theta^{(k)}(f_j) = h_j$, $\Theta^{(k)}(f_j^*) = h_j^{-1}$, and $[f_j^*] = [f_j^{-1}]$ for $j = 1, \ldots, 2l$. Therefore, in view of (1), the claim follows. □

Next we introduce the auxiliary rolling-up operators.

Proposition 8.5. *Let* $r \geqslant 2$ *and let* $k = 0, 1, \ldots, n$. *There exist a* C^r*-neighborhood* $\mathcal{U}_1 = \mathcal{U}_{\varphi,\psi,r,A,k}$ *of the identity in* $\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0$ *and a mapping* $\Psi_A^{(k)} = \Psi_{\varphi,\psi,r,A,k}$ *such that*

$$\Psi_A^{(k)} : \mathrm{Cont}_{J_A^{(k)}}\big(\mathcal{W}_k^m, \alpha_{st}\big)_0^{(k)} \cap \mathcal{U}_1 \to \mathrm{Cont}_{K_A^{(k)}}\big(\mathcal{W}_k^m, \alpha_{st}\big)_0^{(k)},$$

cf. (7.1)*, which satisfies the following conditions:*

(1) $\Psi_A^{(k)}$ *is continuous and* $\Psi_A^{(k)}(\mathrm{id}) = \mathrm{id}$.
(2) *There are constants* C_φ, β *and* K, *and for any* $\rho \geqslant 2$ *polynomials with no constant term* $P_{\varphi,\psi,\rho}$ *such that for any* $g \in \mathrm{dom}(\Psi_A^{(k)})$ *one has*

$$\mu_\rho^*\big(\Psi_A^{(k)}(g)\big) \leqslant A^\beta K^\rho C_\varphi \mu_\rho^*(g) + A^\beta P_{\varphi,\psi,\rho}\big(M_{\rho-1}^*(g)\big).$$

(3) *There is a constant* $C_{\varphi,\psi,r}$ *such that for all* $g \in \mathrm{dom}(\Psi_A^{(k)})$ *one has*

$$\mu_r^*\big(\Psi_A^{(k)}(g)\big) \leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*(g).$$

(4) *For any* $g \in \mathrm{dom}(\Psi_A^{(k)})$ *we have* $[\Psi_A^{(k)}(g) \cdot \Xi^{(k)}\hat{\Theta}^{(k)}(g)] = [g]$ *in the group* $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_{st})_0)$.

Proof. Let $g \in \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^m, \alpha_{st})_0^{(k)} \cap \mathcal{U}_1$. Define

$$\Psi_A^{(k)}(g) := \Xi_A^{(k)}\big(\Theta_A^{(k)}(g) \cdot \hat{\Theta}_A^{(k)}(g)^{-1}\big) = \Xi^{(k)}\big(\Theta^{(k)}(g) \cdot \hat{\Theta}^{(k)}(g)^{-1}\big),$$

cf. (7.2). By virtue of Propositions 8.1 and 8.2 the definition is correct and (1) holds true. To show (2) and (3) denote $h = \Theta^{(k)}(g) \cdot \hat{\Theta}^{(k)}(g)^{-1}$. Then $u_h = w_{\Theta^{(k)}(g)}$, cf. (7.3). According to (5.1) and Propositions 4.6 and 7.2(5) we have

$$\mu_\rho^*\big(h^\psi\big) \leqslant A^\beta K^\rho C_\varphi \mu_\rho^*\big(\Theta_A^{(k)}(g)\big) + A^\beta P_{\varphi,\psi,\rho}\big(M_{\rho-1}^*\big(\Theta_A^{(k)}(g)\big)\big). \tag{8.4}$$

Next, by (5.1) and Corollary 7.3

$$\forall 0 \leqslant s \leqslant r+1, \quad \big\|D^s(\psi u_h)\big\| \leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*\big(\Theta_A^{(k)}(g)\big). \tag{8.5}$$

Now, (2) follows from (8.4), Lemma 3.6 and Propositions 8.1 and 8.2. On the other hand, by (8.5) with Propositions 4.6 and 8.1 we obtain

$$\mu_r^*\big(h^\psi\big) \leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*(g) + F_{r,A}\big(M_{r-1}^*(g)\big).$$

In view of the definition of $\Xi_A^{(k)}$, Lemma 3.6 and Definition 3.2, the claim (3) follows by shrinking possibly $\mathcal{U}_1$.

(4) We have by Lemma 8.4(2)

$$\begin{aligned}\big[\Psi_A^{(k)}(g) \cdot \Xi^{(k)}\hat{\Theta}^{(k)}(g)\big] &= \big[\Xi^{(k)}\big(\Theta^{(k)}(g) \cdot \hat{\Theta}^{(k)}(g)^{-1}\big) \cdot \Xi^{(k)}\hat{\Theta}^{(k)}(g)\big] \\ &= \big[\Xi^{(k)}\Theta^{(k)}(g)\big]. \end{aligned} \tag{8.6}$$

Notice that in view of Proposition 8.2(4) we get $\Theta^{(k)}\Xi^{(k)}\Theta^{(k)}(g) = \Theta^{(k)}(g)$. It follows from Lemma 8.3 that $[\Xi^{(k)}\Theta^{(k)}(g)] = [g]$. Combining this with (8.6), the claim follows. □

From now on we set for $k=0,1,\dots,n$

$$\tilde{\Theta}^{(k)}=\Theta^{(k)}\circ\dots\circ\Theta^{(0)},\qquad \Theta_*^{(k)}=\hat{\Theta}^{(k)}\circ\dots\circ\hat{\Theta}^{(0)},\qquad \tilde{\Xi}^{(k)}=\Xi^{(0)}\circ\dots\circ\Xi^{(k)}.$$

Notice that the image of $\Theta_*^{(k)}$ is in $\mathrm{Cont}_c(\mathcal{W}_{k+1}^m,\alpha_{st})_0^{(k+1)}$.

In the proof of Theorem 1.1 the following fact is crucial.

Lemma 8.6. *Suppose $\mathcal{U}_1$ is a sufficiently small C^1-neighborhood of the identity in* $\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$. *Then for all $f\in\mathcal{U}_1$:*

(1) *If $\tilde{\Theta}^{(n)}(f)=\tilde{f}$ then $[f]=[\tilde{\Xi}^{(n)}(\tilde{f})]$ in $H_1(\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0)$.*
(2) *$[\tilde{\Xi}^{(n)}\Theta_*^{(n)}(f)]=e$ in $H_1(\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0)$.*

Proof. (1) In view of Proposition 8.2(4) and Lemma 8.3 one has $[\tilde{\Theta}^{(n-1)}(f)]=[\Xi^{(n)}(\tilde{f})]$. Hence there is h_n in the commutator subgroup of $\mathrm{Cont}_c(\mathcal{W}_n^m,\alpha_{st})_0$ such that $\tilde{\Theta}^{(n-1)}(f)h_n=\Xi^{(n)}(\tilde{f})$. By the above argument and Lemmas 8.3, 8.4(3) and 8.4(1), there is h_{n-1} in the commutator subgroup of $\mathrm{Cont}_c(\mathcal{W}_{n-1}^m,\alpha_{st})_0$ such that $\tilde{\Theta}^{(n-2)}(f)h_{n-1}=\Xi^{(n-1)}\Xi^{(n)}(\tilde{f})$. Continuing this procedure we obtain the claim.

(2) For $f\in\mathcal{U}_1$ put

$$f^*=\Theta_*^{(n)}(f)\quad\text{and}\quad g=\tilde{\Xi}^{(n)}\Theta_*^{(n)}(f)=\tilde{\Xi}^{(n)}\big(f^*\big).$$

Notice that in view of Proposition 2.2, $f^*(\xi_0,\xi,y)=(\xi_0+f_0^*(y),\xi+f_1^*(y),y)$. It follows from the definition of $\tilde{\Xi}^{(n)}$ that $g(\xi_0,\xi,y)=(\xi_0+f_0^*(y),\xi+f_1^*(y),y)$ if $(\xi_0,\xi)\in([-\frac{1}{2}-\varepsilon,-\frac{1}{2}+\varepsilon]\cup[1-\varepsilon,1+\varepsilon])^{n+1}$ for some $\varepsilon>0$. Furthermore, $\mathrm{supp}(g)\subset([-1,0]\cup[\frac{1}{2},\frac{3}{2}])^{n+1}\times[-2A,2A]^n$ and, due to Proposition 8.2(4), $\tilde{\Theta}^{(n)}(g)=f^*$. We have to show that $[g]=e$.

Let us define $g_2\in\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$ such that

$$[g]=[g_2]=\big[g^{2^{n+2}}\big] \tag{8.7}$$

in $H_1(\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0)$. In the definition we will use the contactomorphisms $\eta_2,\tau_{i,t}\in\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$, $i=0,\dots,n$, defined in Section 2.

First let $h=\eta_2^{-1}g\eta_2$. Then $\mathrm{supp}(h)\subset[-\frac{1}{2},0]\cup[\frac{1}{4},\frac{3}{4}]\times\mathcal{I}_{n,A}$, where $\mathcal{I}_{n,A}=([-\frac{1}{2},0]\cup[\frac{1}{4},\frac{3}{4}])^n\times[-2A,2A]^n$. Let us denote

$$f_{1/2}^*(\xi_0,\xi,y)=\left(\xi_0+\frac{1}{2}f_0^*(y),\xi+\frac{1}{2}f_1^*(y),y\right).$$

To simplify notation, put $\mathcal{J}_{l,\varepsilon}=([-\frac{1}{4}-\varepsilon,-\frac{1}{4}+\varepsilon]\cup[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon])^l\times\mathbb{R}^n$. There is $\varepsilon>0$ such that for $(\xi_0,\xi,y)\in\mathcal{J}_{n+1,\varepsilon}$ one has $h(\xi_0,\xi,y)=f_{1/2}^*(\xi_0,\xi,y)$.

We can write $h=\bar{h}_0\hat{h}_0$, where $\bar{h}_0=h$ on $[-\frac{1}{2},0]\times\mathbb{R}^{2n}$, $\bar{h}_0=\mathrm{id}$ off $[-\frac{1}{2},0]\times\mathbb{R}^{2n}$, $\hat{h}_0=h$ on $[\frac{1}{4},\frac{3}{4}]\times\mathbb{R}^{2n}$, $\hat{h}_0=\mathrm{id}$ off $[\frac{1}{4},\frac{3}{4}]\times\mathbb{R}^{2n}$. Put $h_0=\hat{h}_0\tau_{0,\frac{1}{2}}\bar{h}_0\tau_{0,\frac{1}{2}}^{-1}$. Clearly $[h_0]=[h]$. Observe that $h_0=f_{1/2}^*$ on $[\frac{1}{4}-\varepsilon,\frac{1}{2}+\varepsilon]\times\mathcal{J}_{n,\varepsilon}$ and $\mathrm{supp}(h_0)\subset[0,\frac{3}{4}]\times\mathcal{I}_{n,A}$.

In view of the equalities $\eta_2^{-1} f^* \eta_2 = f^*_{1/2}$ (here f^* is regarded as an element of $\mathrm{Cont}(\mathbb{R}^m, \alpha_{st})$) and $\eta_2^{-1}\tau_{0,1}\eta_2 = \tau_{0,\frac{1}{2}}$, we have $h_0\tau_{0,\frac{1}{2}}h_0 = f^*_{1/2}$ on $[0, \frac{1}{4}] \times \mathcal{J}_{n,\varepsilon}$ and, moreover, by the definition of $\Xi^{(n)}$ we get $h_0\tau_{0,\frac{1}{2}}h_0 = \eta_2^{-1}\tilde{\Xi}^{(n-1)}(f^*)\eta_2$ on $[0, \frac{1}{4}] \times \mathbb{R}^{2n}$. Here $\tilde{\Xi}^{(n-1)}(f^*) \in \mathrm{Cont}(\mathcal{W}^m_1, \alpha_{st})$ is viewed as an element of $\mathrm{Cont}(\mathbb{R}^m, \alpha_{st})$ with period 1 w.r.t. x_0, so that $\eta_2^{-1}\tilde{\Xi}^{(n-1)}(f^*)\eta_2$ is well-defined and can be also regarded as an element of $\mathrm{Cont}(\mathcal{W}^m_1, \alpha_{st})$ with period $\frac{1}{2}$ w.r.t. ξ_0.

Next we define $k_0 = h_0\tau_{0,\frac{1}{2}}h_0\tau^{-1}_{0,\frac{1}{2}}$. We have $\mathrm{supp}(k_0) \subset [0, \frac{5}{4}] \times \mathcal{I}_{n,A}$ and $k_0 = f^*_{1/2}$ on $[\frac{1}{4} - \varepsilon, 1 + \varepsilon] \times \mathcal{J}_{n,\varepsilon}$. Analogously as above, $k_0\tau_{0,1}k_0 = f^*_{1/2}$ on $[0, \frac{1}{4}] \times \mathcal{J}_{n,\varepsilon}$ and $k_0\tau_{0,1}k_0 = \eta_2^{-1}\tilde{\Xi}^{(n-1)}(f^*)\eta_2$ on $[0, \frac{1}{4}] \times \mathbb{R}^{2n}$.

It follows from the definition of $\Theta^{(0)}$ that $\Theta^{(0)}(k_0) = f^*_{1/2}$ on $\mathbb{S}^1 \times \mathcal{J}_{n,\varepsilon}$ and $\Theta^{(0)}(k_0) = \eta_2^{-1}\tilde{\Xi}^{(n-1)}(f^*)\eta_2$ on $\mathcal{W}^m_1$. One has also that $[k_0] = [h_0^2] = [h^2] = [g^2]$.

Next, starting with k_0, we define $\bar{h}_1, \hat{h}_1, h_1$ and k_1 analogously as before, but now with respect to the variable ξ_1. It follows that $\tilde{\Theta}^{(1)}(k_1) = f^*_{1/2}$ on $(\mathbb{S}^1)^2 \times \mathcal{J}_{n-1,\varepsilon}$, and $\tilde{\Theta}^{(1)}(k_1) = \eta_2^{-1}\tilde{\Xi}^{(n-2)}(f^*)\eta_2$ on $\mathcal{W}^m_2$. Moreover, $[k_1] = [k_0^2] = [h^4] = [g^4]$.

Continuing this procedure we obtain $h_2, k_2, \ldots, h_n, k_n \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ such that $[k_n] = [k^2_{n-1}] = [k^4_{n-2}] = \cdots = [k_0^{2^n}] = [g^{2^{n+1}}]$. Moreover, we have that $\tilde{\Theta}^{(n)}(k_n) = f^*_{1/2}$ on $\mathcal{W}^m_{n+1}$.

Thus, in order to define g_2 satisfying $\tilde{\Theta}^{(n)}(g_2) = f^*$ we have to double k_n and we set $g_2 = \tau k_n \tau^{-1} k_n$, where τ is a suitable translation, as in Lemma 8.4(1). It follows that $[g_2] = [k_n^2]$ and, in view of (1) of the present lemma, the equalities (8.7) hold.

Observe that the above procedure may be repeated for any integer $a > 2$ by making use of η_a and suitable translations $\tau_{i,t}$. As a result there exists $g_a \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ such that $\tilde{\Theta}^{(n)}(g_a) = f^*$ and $[g^{a^{n+2}}] = [g_a]$. Moreover, by (1) we have $[g_a] = [g]$.

Let $l_0 > 0$ be the least positive integer such that $[g^{l_0}] = e$. Then for any integers $a, b > 0$ the number $a^{n+2} - b^{n+2}$ is divided by l_0. If $l_0 > 1$ then l_0 divides $l_0^{n+2} - 1$, a contradiction. Thus $l_0 = 1$, as required. □

Proposition 8.7. *Let $r \geqslant 2$. If $\mathcal{U}_1 = \mathcal{U}_{\varphi,\psi,r,A}$ is a small C^r-neighborhood of the identity in $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$, there is a mapping $\Psi_A = \Psi_{\varphi,\psi,r,A}$, called the rolling-up operator,*

$$\Psi_A : \mathrm{Cont}_{J_A}\big(\mathbb{R}^m, \alpha_{st}\big)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{K_A}\big(\mathbb{R}^m, \alpha_{st}\big)_0,$$

which satisfies the following conditions:

(1) *Ψ_A is continuous and $\Psi_A(\mathrm{id}) = \mathrm{id}$.*
(2) *There are constants C_φ, β and K, and for any $\rho \geqslant 2$ there are polynomials with no constant term $P_{\varphi,\psi,\rho}$ such that for any $g \in \mathrm{dom}(\Psi_A)$*

$$\mu^*_\rho\big(\Psi_A(g)\big) \leqslant A^\beta K^\rho C_\varphi \mu^*_\rho(g) + A^\beta P_{\varphi,\psi,\rho}\big(M^*_{\rho-1}(g)\big).$$

(3) *There are constants β and $C_{\varphi,\psi,r}$, and an admissible polynomial $F_{r,A}$ such that for any $g \in \mathrm{dom}(\Psi_A)$*

$$\mu^*_r\big(\Psi_A(g)\big) \leqslant A^\beta C_{\varphi,\psi,r} \mu^*_r(g) + F_{r,A}\big(M^*_{r-1}(g)\big).$$

(4) *For any $g \in \mathrm{dom}(\Psi_A)$ one has $[\Psi_A(g)] = [g]$ in $H_1(\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0)$.*

Proof. Let $g \in \mathrm{Cont}_{J_A}(\mathbb{R}^m, \alpha_{st})_0 \cap \mathcal{U}_1$. Define $\Psi_A(g) = g_0 g_1 \dots g_n$, where $g_0 = \Psi_A^{(0)}(g)$ and, for $k = 1, \dots, n$,

$$g_k = \tilde{\Xi}^{(k-1)} \Psi_A^{(k)} \Theta_*^{(k-1)}(g).$$

In order to show (2) and (3), our first observation is that it suffices to have for $k = 0, 1, \dots, n$

$$\mu_\rho^*(g_k) \leqslant A^\beta K^\rho C_\varphi \mu_\rho^*(g) + A^\beta P_{\varphi,\psi,\rho}\big(M_{\rho-1}^*(g)\big), \tag{8.8}$$

for all $\rho \geqslant 2$, and

$$\mu_r^*(g_k) \leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*(g), \tag{8.9}$$

and to apply Lemma 3.6(2). For $k = 0$ it is just Proposition 8.5.

For $k = 1, \dots, n$, in view of Propositions 8.2 and 8.5 we get

$$\mu_\rho^*(g_k) \leqslant A^\beta K^\rho C_\varphi \mu_\rho^*\big(\Theta_*^{(k-1)}(g)\big) + A^\beta P_{\varphi,\psi,\rho}\big(M_{\rho-1}^*\big(\Theta_*^{(k-1)}(g)\big)\big). \tag{8.10}$$

On the other hand, by Propositions 7.2(4) and 8.1(3) we have

$$\begin{aligned}
\mu_\rho^*\big(\Theta_*^{(k-1)}(g)\big) &\leqslant K^\rho C_\varphi \mu_\rho^*\big(\Theta^{(k-1)} \Theta_*^{(k-2)}(g)\big) + P_{\varphi,\rho}\big(\Theta^{(k-1)} \Theta_*^{(k-2)}(g)\big) \\
&\leqslant A^\beta K_1^\rho C'_\varphi \mu_\rho^*\big(\Theta_*^{(k-2)}(g)\big) + P'_{\varphi,\rho}\big(M_{\rho-1}^*\big(\Theta_*^{(k-2)}(g)\big)\big) \\
&\quad \dots \\
&\leqslant A^{\beta'} K_2^\rho C''_\varphi \mu_r^*(g) + P''_{\varphi,\rho}\big(M_{\rho-1}^*(g)\big).
\end{aligned}$$

Combining this with (8.10) we obtain (8.8). In order to show (8.9) for $k = 1, \dots, n$ we proceed analogously, using Propositions 8.2, 3.5, 8.5 and 8.1, and Corollary 7.3, and possibly changing constants and shrinking $\mathcal{U}_1$

$$\begin{aligned}
\mu_r^*(g_k) &\leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*\big(\Psi^{(k)} \Theta_*^{(k-1)}(g)\big) \\
&\leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*\big(\Theta_*^{(k-1)}(g)\big) \\
&\leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*\big(\Theta^{(k-1)} \Theta_*^{(k-2)}(g)\big) \\
&\leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*\big(\Theta_*^{(k-2)}(g)\big) \\
&\quad \dots \\
&\leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*\big(\Theta^{(0)}(g)\big) \\
&\leqslant A^\beta C_{\varphi,\psi,r} \mu_r^*(g).
\end{aligned}$$

(4) By Lemmas 8.6(2) and 8.4(2), and Proposition 8.5(4), we have

$$\begin{aligned}
\big[\Psi_A(g)\big] &= [g_0 g_1 \cdots g_n] \\
&= \big[g_0 g_1 \cdots g_n \cdot \tilde{\Xi}^{(n)} \Theta_*^{(n)}(g)\big] \\
&= \big[g_0 g_1 \cdots g_{n-1} \cdot \tilde{\Xi}^{(n-1)} \Psi_A^{(n)} \Theta_*^{(n-1)}(g) \cdot \tilde{\Xi}^{(n-1)} \Xi^{(n)} \hat{\Theta}^{(n)} \Theta_*^{(n-1)}(g)\big] \\
&= \big[g_0 g_1 \cdots g_{n-1} \cdot \tilde{\Xi}^{(n-1)} \big(\Psi_A^{(n)} \Theta_*^{(n-1)}(g) \cdot \Xi^{(n)} \hat{\Theta}^{(n)} \Theta_*^{(n-1)}(g)\big)\big] \\
&= \big[g_0 g_1 \cdots g_{n-1} \cdot \tilde{\Xi}^{(n-1)} \Theta_*^{(n-1)}(g)\big] \\
&\quad \cdots \\
&= \big[g_0 \cdot \Xi^{(0)} \Theta_*^{(0)}(g)\big] \\
&= \big[\Psi_A^{(0)}(g) \cdot \Xi^{(0)} \hat{\Theta}^{(0)}(g)\big] = [g]. \qquad \square
\end{aligned}$$

Remark 8.8. It is easy to check that the proof of Lemma 8.6(2) and, consequently, of Proposition 8.7(4) fails in the case $\mathrm{Diff}^r_c(\mathbb{R}^m)_0$, since Proposition 2.2 is not true for diffeomorphisms. Thus the proof of Theorem 1.1 is not valid for $\mathrm{Diff}^r_c(\mathbb{R}^m)_0$.

9. Proof of Theorem 1.1

Let A be a large positive integer which will be fixed later on, and let I_A, J_A and K_A be the intervals in $\mathbb{R}^m$ given by (6.1), (6.3) and (8.1), resp. Let us define

$$\mathcal{L} = \big\{u \in \mathrm{C}^\infty_{I_A}\big(\mathbb{R}^m\big):\ \big\|D^{r+1}u\big\| \leqslant \epsilon_r,\ \forall r \geqslant r_0\big\},$$

where r_0 (large), ϵ_{r_0} (small), and ϵ_r for $r > r_0$ (large) will be fixed in due course.

Observe that $\mathcal{L}$ is a convex and compact subset of a locally convex space. Consequently, in view of Schauder–Tichonoff's theorem every continuous map $\vartheta : \mathcal{L} \to \mathcal{L}$ has a fixed point.

Let $f_0 \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$. We have to show that f_0 belongs to the commutator subgroup of $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$. According to Lemma 5.2 we may assume that $\mathrm{supp}(f_0) \subset I_A$. Furthermore, since $\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ is a topological group, we may have $\mu^*_{r_0}(f_0)$ arbitrarily small.

Now we will define a continuous operator $\vartheta : \mathcal{L} \to \mathcal{L}$ in the following ten steps:

(1) For any $u \in \mathcal{L}$ take $f \in \mathrm{Cont}_{I_A}(\mathbb{R}^m, \alpha_{st})_0$ such that $\Phi_A(f) = u_f = u$.
(2) Compose f with f_0.
(3) Use a fragmentation of the second kind for $g = f f_0$ (Proposition 5.7). We have a decomposition $g = g_1 \cdots g_{a_n}$, where $a_n = (4A+1)^n$, and each g_κ is supported in some interval

$$\big([-2,2]^{n+1} \times [k_1 - 1, k_1 + 1] \times \cdots \times [k_n - 1, k_n + 1]\big) \cap I_A,$$

with integers k_i such that $|k_i| \leqslant 2A$, $i = 1, \ldots, n$.
(4) Use the operation of shifting supports of contactomorphisms described in Section 6. For any $\kappa = 1, \ldots, a_n$ define

$$\tilde{g}_\kappa = \sigma_{n,t_n} \sigma_{n-1,t_{n-1}} \cdots \sigma_{1,t_1} g_\kappa \sigma^{-1}_{1,t_1} \cdots \sigma^{-1}_{n-1,t_{n-1}} \sigma^{-1}_{n,t_n},$$

for suitable $(t_1, \ldots, t_n) \in \mathbb{R}^n$ depending on κ in such a way that $\mathrm{supp}(\tilde{g}_\kappa) \subset [-A^2, A^2] \times [-2,2]^{2n}$ for all κ. Here we assume that $|t_i| \leqslant 2A$, $i = 1, \ldots, n$, and $A > 5n$.

(5) For any $\kappa = 1, \dots, a_n$ define $h_\kappa = \eta_A \chi_A \tilde{g}_\kappa \chi_A^{-1} \eta_A^{-1}$. It follows that $\operatorname{supp}(h_\kappa) \subset J_A$.
(6) Use the rolling-up operator Ψ_A described in Proposition 8.7, and define $\bar{h}_\kappa = \Psi_A(h_\kappa)$. Observe that $\operatorname{supp}(\bar{h}_\kappa) \subset K_A$.
(7) Make a fragmentation of the second kind in K_A in the x_i-directions, $i = 1, \dots, n$, cf. Proposition 5.7. We write for $\bar{a}_n = a_n^5$

$$\bar{h}_\kappa = \prod_{\iota=1}^{\bar{a}_n} \bar{h}_{\kappa\iota}.$$

(8) Use the operation of shifting supports of contactomorphisms in the x_i-directions by means of the translations τ_i, $i = 1, \dots, n$ (cf. Section 2). For any κ and ι define $\tilde{h}_{\kappa\iota}$ instead of $\bar{h}_{\kappa\iota}$ with $\operatorname{supp}(\tilde{h}_{\kappa\iota}) \subset I_A$. All the norms of $\tilde{h}_{\kappa\iota}$ are the same as the norms of $\bar{h}_{\kappa\iota}$ as we used translations.
(9) Take the product $h = \prod_{\kappa=1}^{a_n} \prod_{\iota=1}^{\bar{a}_n} \tilde{h}_{\kappa\iota}$.
(10) Take $u_h = \Phi_A(h)$.

Then we put $\vartheta(u) = u_h$. In view of the description of particular steps of the construction, ϑ is continuous. It remains to show that for a suitable choice of r_0, A, and ϵ_r for $r \geqslant r_0$, the operator ϑ takes $\mathcal{L}$ into itself.

In fact, suppose that $u = u_f \in \mathcal{L}$ is a fixed point of ϑ, i.e., $u_h = u_f$. Then $h = f$ and we have in $H_1(\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0)$

$$\begin{aligned}
[f f_0] &= [g] = [g_1 \cdots g_{a_n}] = [g_1] \cdots [g_{a_n}] = [\tilde{g}_1] \cdots [\tilde{g}_{a_n}] \\
&= [h_1] \cdots [h_{a_n}] = [\bar{h}_1] \cdots [\bar{h}_{a_n}] = [\bar{h}_{11}] \cdots [\bar{h}_{a_n \bar{a}_n}] \\
&= [\tilde{h}_{11}] \cdots [\tilde{h}_{a_n \bar{a}_n}] = [\tilde{h}_{11} \cdots \tilde{h}_{a_n \bar{a}_n}] = [h] = [f],
\end{aligned}$$

and therefore $[f_0] = e$. This means that f_0 is a product of commutators.

Now we wish to define r_0, A and ϵ_r for $r \geqslant r_0$. This will be done in view of the properties of the consecutive operations in the construction of ϑ.

Suppose $r_0 \geqslant 2$. In view of Propositions 4.6, 3.5, 5.7, 6.1 and 8.7, and Lemma 3.6 it follows the existence of a C^2-neighborhood $\mathcal{V}_2 = \mathcal{V}_{\varphi,\psi,r_0,A}$ of zero in $C^\infty_{I_A}(\mathbb{R}^m)$, of constants C_{φ,ψ,r_0} and $\beta = \beta(m) > 0$, and of admissible polynomials $F^i_{r_0,A}$, $i = 1, 3$, and $F^2_{\varphi,\psi,r_0,A}$, such that for a sufficiently small ϵ_{r_0} we have

$$\begin{aligned}
\|D^{r_0+1} u_h\| &\leqslant A^{\beta - r_0} C_{\varphi,\psi,r_0} \|D^{r_0+1} u\| + F^1_{r_0,A}\big(\mu^*_{r_0}(f)\big) \\
&\quad + F^2_{\varphi,\psi,r_0,A}\Big(\sup_\kappa \mu^*_{r_0}(h_\kappa)\Big) + F^3_{r_0,A}\Big(\sup_{\kappa,\iota} \mu^*_{r_0}(\tilde{h}_{\kappa\iota})\Big),
\end{aligned} \tag{9.1}$$

for all $u \in \mathcal{V}_2$ with $\|D^{r_0+1} u\| \leqslant \epsilon_{r_0}$. Here we assume that $\mu^*_{r_0}(f_0)$ is small enough. We assume as well that $\sup_{\kappa,\iota} \mu^*_i(\tilde{h}_{\kappa\iota}) < (A^{20n} r_0)^{-1}$, where $i = 0, 1$, by choosing ϵ_{r_0} sufficiently small. Then we have

$$\Big(\Big(1 + \sup_{\kappa,\iota} \mu^*_0(\tilde{h}_{\kappa\iota})\Big)\Big(1 + \sup_{\kappa,\iota} \mu^*_1(\tilde{h}_{\kappa\iota})\Big)\Big)^{A^{20n} r_0} < 6, \tag{9.2}$$

and we may apply Lemma 3.6(2) in order to obtain (9.1).

Fix $r_0 > \beta$ and choose A so large that $A^{\beta-r_0}C_{\varphi,\psi,r_0} < \frac{1}{4}$. It follows from Definition 3.2 that, possibly taking ϵ_{r_0} smaller, we have $F^1_{r_0,A}(\mu^*_{r_0}(f)) < \frac{\epsilon_{r_0}}{4}$, $F^2_{\varphi,\psi,r_0,A}(\sup_\kappa \mu^*_{r_0}(h_\kappa)) < \frac{\epsilon_{r_0}}{4}$, and $F^3_{r_0,A}(\sup_{\kappa,\sigma} \mu^*_{r_0}(\tilde{h}_{\kappa\iota})) < \frac{\epsilon_{r_0}}{4}$, whenever $\|D^{r_0+1}u\| \leqslant \epsilon_{r_0}$. We may also assume that $\|D^{r_0+1}u\| < \epsilon_{r_0}$ yields $u \in \mathcal{V}_2$. Then by (9.1) $\|D^{r_0+1}u_h\| \leqslant \epsilon_{r_0}$, if $\|D^{r_0+1}u\| \leqslant \epsilon_{r_0}$.

Next, we define ϵ_r for all $r > r_0$ inductively. Suppose we have defined $\epsilon_{r_0}, \ldots, \epsilon_{r-1}$.

In view of Propositions 4.6, 3.5, 5.7, 6.1 and 8.7, Lemma 3.6, and the inequality (9.2) rewritten for r with 6^r on the r.h.s., there exist constants $\beta > 0$ and $K = K_{\varphi,\psi}$, and polynomials $P_{\varphi,\psi,r,A}$ without constant term such that for all $u \in \mathcal{V}_2$ we have

$$\left\|D^{r+1}u_h\right\| \leqslant A^{\beta-r}K^r\left\|D^{r+1}u\right\| + P_{\varphi,\psi,r,A}\Big(\sup_{s=0,1,\ldots,r}\left\|D^s u\right\|\Big). \tag{9.3}$$

Enlarging A if necessary, suppose $A > K^{r_0}$. Hence we have $A^{\beta-r}K^r < \frac{1}{4}$. Put $b_r = P_{\varphi,\psi,r,A} \times (\sup_{s=0,1,\ldots,r}\|D^s u\|)$, where $\|D^{s+1}u\| \leqslant \epsilon_s$ for $s = r_0, \ldots, r-1$. Then (9.3) can be rewritten as

$$\left\|D^{r+1}u_h\right\| \leqslant \frac{1}{4}\left\|D^{r+1}u\right\| + b_r.$$

Define $\epsilon_r = 2b_r$. It follows that $\|D^{r+1}u_h\| \leqslant \epsilon_r$ whenever $\|D^{r+1}u\| \leqslant \epsilon_r$, as required.

10. Proof of Corollary 1.2

We have to check Epstein's axioms [4] for some basis of open sets $\mathcal{U}$ of M and $G = \mathrm{Cont}_c(M,\alpha)_0$:

(1) If $U \in \mathcal{U}$ and $g \in G$ then $g(U) \in \mathcal{U}$.
(2) G acts transitively on $\mathcal{U}$.
(3) Let $g \in G$, $U \in \mathcal{U}$ and let $\mathcal{V} \subset \mathcal{U}$ be a covering of M. Then there are $s \geqslant 1$, $g_1, \ldots, g_s \in G$ and $V_1, \ldots, V_s \in \mathcal{V}$ such that $g = g_1 \ldots g_s$, $\mathrm{supp}(g_i) \subset V_i$ and $\mathrm{supp}(g_i) \cup g_{i-1} \ldots g_1(\overline{U}) \neq M$ for $i = 1, \ldots, s$.

In fact, let U be any open ball in M and $\mathcal{U} = \{g(U)\colon\ g \in G\}$. By using χ_A, $\tau_{i,t}$, $i = 0, \ldots, n$, and $\sigma_{i,t}$, $i = 1, \ldots, n$, see Section 2, it is easily seen that $\mathcal{U}$ is a basis and (2) is fulfilled. In view of Lemma 5.2 a standard reasoning shows (3). Thus, due to [4] and Theorem 1.1, $\mathrm{Cont}_c(M,\alpha)_0$ is simple.

11. Final remarks

Let G be a group and let $g \in [G,G]$. The commutator length $cl_G(g)$ of g is 0 if $g = e$, and is the least positive integer N such that $g = [g_1,h_1]\cdots[g_N,h_N]$ for some $g_i, h_i \in G$, $i = 1, \ldots, N$, otherwise. Then cl_G is a conjugation-invariant norm on $[G,G]$, cf. [3]. In the paper [3] by Burago, Ivanov and Polterovich and in certain references therein a description of a role played by conjugation-invariant norms on groups of geometric origin is given.

As a trivial consequence of Theorem 1.1 we have

Corollary 11.1. *The commutator length is a conjugation-invariant norm on* $\mathrm{Cont}_c(M,\alpha)_0$.

It is known from several recent papers that the theorem of Banyaga [1] plays a clue role in the symplectic topology and geometry in the sense that some invariants are expressed in terms of the commutator length of related groups. It seems that, thanks to Corollary 11.1, a similar role could be played by $cl_{\mathrm{Cont}_c(M,\alpha)_0}$ in the contact topology and geometry.

Recall that a group is said to be *bounded* if it is bounded w.r.t. any bi-invariant metric on it or, equivalently, any conjugation-invariant norm on it is bounded. Recently, the problem of boundedness was solved in many cases of $\mathrm{Diff}_c(M)_0$, and the solutions depend on the topology of M (cf. [3,20]). In view of Corollary 11.1 it is interesting to know whether $\mathrm{Cont}_c(M,\alpha)_0$ is bounded and how it depends on M.

Another possible applications are related to Haefliger's classifying spaces of contact foliations. Let $B\,\overline{\mathrm{Cont}}_c(M,\alpha)$ be the classifying space for the foliated C^∞ products with compact support with transverse contact form. It is well-known that $B\,\overline{\mathrm{Cont}}_c(M,\alpha)$ is the homotopy fiber of the mapping

$$B\,\mathrm{Cont}_c(M,\alpha)_0^\delta \to B\,\mathrm{Cont}_c(M,\alpha)_0,$$

where the superscript δ denotes the discrete topology. By an argument similar to the proof of Theorem 1.1 we have the following

Theorem 11.2. $H_1(B\,\overline{\mathrm{Cont}}_c(M,\alpha);\mathbb{Z}) = H_1(\widetilde{\mathrm{Cont}_c(M,\alpha)}) = 0$*, where tilde indicates the universal covering group.*

For the proof, see Appendix A.

Up to my knowledge no version of the Thurston–Mather isomorphism (cf. [15,13,2,18,19]) is known for $\mathrm{Cont}_c(M,\alpha)_0$. It seems likely that such a version could be established, but a possible proof seems to be hard. This would give information on the connectedness of Haefliger's classifying space for contact foliations.

In [18] and [19] Tsuboi discussed the problem of the connectedness of the Haefliger classifying spaces. It is likely that Theorem 1.1 is still true for the group $\mathrm{Cont}^r_c(M,\alpha)$ of contactomorphisms of class C^r with r large.

Observe that Theorem 11.2 reveals further fundamental difference between the symplectic and the contact geometries. As it was mentioned in the introduction the flux homomorphism plays a crucial role in the geometry of symplectic forms [1,2,14] (as well as in case of regular Poisson manifolds, cf. [15], and of locally conformal symplectic manifolds, cf. [7]). The domain of the flux is the universal covering group of the group in question. In view of Theorem 11.2 a possible analog of such a homomorphism is necessarily trivial in the contact case.

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Appendix A. The proof of Theorem 11.2

Since the first equality is well-known it suffices to show the second.

Let G be a topological group. Denote by $\mathcal{P}G$ the totality of paths $\gamma : I \to G$ with $\gamma(0) = e$, where $I = [0, 1]$. The path group $\mathcal{P}G$ is a topological group with the compact-open topology. Likewise, for a locally convex vector space V let $\mathcal{P}V$ be the totality of paths $\gamma : I \to V$ with $\gamma(0) = 0$. Then $\mathcal{P}V$ is a locally convex vector space. If $X \subset G$ (resp. $Y \subset V$) are subsets containing e (resp. 0) then the subsets $\mathcal{P}X \subset \mathcal{P}G$ (resp. $\mathcal{P}Y \subset \mathcal{P}V$) are defined in the obvious way.

Next, the symbol $\mathcal{P}_0G$ (resp. $\mathcal{P}_0V$) will stand for the totality of $\{f_t\}_{t\in I} \in \mathcal{P}G$ (resp. $\{f_t\}_{t\in I} \in \mathcal{P}V$) such that $f_t = e$ (resp. $f_t = 0$) for $0 \leqslant t \leqslant \frac{1}{2}$. The elements of $\mathcal{P}_0G$ and $\mathcal{P}_0V$ will be called *special* paths. Note that the subsets $\mathcal{P}_0X \subset \mathcal{P}_0G$ (resp. $\mathcal{P}_0Y \subset \mathcal{P}_0V$) are well-defined for subsets $X \subset G$ (resp. $Y \subset V$) with $e \in X$ (resp. $0 \in Y$).

We have to show that $\widetilde{\mathrm{Cont}_c(M,\alpha)} = \mathcal{P}\mathrm{Cont}_c(M,\alpha)_0/{\sim}$ is a perfect group. Here $\sim$ denotes the relation of the homotopy rel. endpoints. It is clear that for every $[\{g_t\}]_\sim, [\{h_t\}]_\sim \in \widetilde{\mathrm{Cont}_c(M,\alpha)}$, the classes of them in $H_1(\widetilde{\mathrm{Cont}_c(M,\alpha)})$ are equal whenever $[\{g_t\}] = [\{h_t\}]$ in $H_1(\mathcal{P}\mathrm{Cont}_c(M,\alpha))$. Take arbitrarily $[\{h_t\}]_\sim \in \widetilde{\mathrm{Cont}_c(M,\alpha)}$, where $\{h_t\} \in \mathcal{P}\mathrm{Cont}_c(M,\alpha)_0$. In view of Lemma 5.2 we may and do assume that $\{h_t\} \in \mathcal{P}\mathrm{Cont}_{I_A}(\mathbb{R}^m,\alpha_{st})_0$. Observe that Lemma 5.2 is still valid for the group $\mathcal{P}_0\mathrm{Cont}_c(M,\alpha)_0$ instead of $\mathcal{P}\mathrm{Cont}_c(M,\alpha)_0$ and this fact is also used in the proof of Lemma 8.4(3) for special paths.

In order to show that $[\{h_t\}]_\sim$ belongs to the commutator subgroup of $\widetilde{\mathrm{Cont}_c(\mathbb{R}^m},\alpha_{st})_0$ we introduce suitable changes in the subsequent sections.

In Section 2 we single out special elements of $\mathcal{P}\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$ as follows (cf. (1)–(5) in Section 2). Abusing the notation they will be designated as before. Namely, $\tau_{i,t} = \{(\tau_{i,t})_s\}_{s\in I}$, $\sigma_{i,t} = \{(\sigma_{i,t})_s\}_{s\in I}$, $\chi_a = \{(\chi_a)_s\}_{s\in I}$, $\eta_a = \{(\eta_a)_s\}_{s\in I}$ are fixed elements of $\mathcal{P}\mathrm{Cont}_c(\mathbb{R}^m,\alpha_{st})_0$ such that $(\tau_{i,t})_s = \tau_{i,t}$, $(\sigma_{i,t})_s = \sigma_{i,t}$, $(\chi_a)_s = \chi_a$ and $(\eta_a)_s = \eta_a$ for all $\frac{1}{2} \leqslant s \leqslant 1$.

In Section 4 the chart $\Phi_A : \mathrm{Cont}_E(\mathcal{W}_k^m,\alpha_{st}) \supset \mathcal{U}_1 \ni f \mapsto u_f \in \mathcal{V}_2 \subset \mathrm{C}_E^\infty(\mathcal{W}_k^m)$ induces the homeomorphism

$$\mathcal{P}\Phi_A : \mathcal{P}\mathrm{Cont}_E\big(\mathcal{W}_k^m,\alpha_{st}\big) \supset \mathcal{P}\mathcal{U}_1 \ni \{f_t\} \mapsto \{u_{f_t}\} \in \mathcal{P}\mathcal{V}_2 \subset \mathcal{P}\mathrm{C}_E^\infty\big(\mathcal{W}_k^m\big).$$

Notice that $\mathcal{P}\Phi_A$ preserves the subspaces of special paths. We may and do assume that $\mathcal{U}_1^{-1}\cdot\mathcal{U}_1$ is contained in a contractible neighborhood of the identity.

In Section 5 by making use of $\mathcal{P}\Phi_A$ we define $\{f_t\}^\psi$ for $\{f_t\} \in \mathcal{P}\mathcal{U}_1$ by putting $\{f_t\}^\psi = \{f_t^\psi\}$. Observe that $\{f_t\}^\psi \in \mathcal{P}_0\mathcal{U}_1$ whenever $\{f_t\} \in \mathcal{P}_0\mathcal{U}_1$, and Proposition 5.4 holds. Proposition 5.7 holds for isotopies in the sense that there is a decomposition for isotopies and the estimates (1), (2) are satisfied for the corresponding members of isotopies with the same constants and polynomials. Next, Proposition 6.1 and the inclusion (6.2) are still valid for $\mathcal{P}_0\mathrm{Cont}_{I_A}(\mathbb{R}^m,\alpha_{st})_0$ in view of our new definition of $\tau_{i,t}$, $\sigma_{i,t}$, χ_a and η_a (with an analogous remark as for 5.7). Also for any $\{f_t\} \in \mathcal{P}_0\mathrm{Cont}_{E_A^{(k+1)}}(\mathcal{W}_{k+1}^m,\alpha_{st})_0^{(k)}$ there is $\{\hat{f}_t\} \in \mathcal{P}_0\mathrm{Cont}_{E_A^{(k+1)}}(\mathcal{W}_{k+1}^m,\alpha_{st})_0^{(k+1)}$ as in Section 7.

In Section 8 we have the operators $\mathcal{P}\Theta^{(k)}$ and $\mathcal{P}\Xi^{(k)}$ on the relevant spaces of paths induced by $\Theta^{(k)}$ and $\Xi^{(k)}$, resp. It is important that these operators descend to the operators $\mathcal{P}_0\Theta^{(k)}$ and $\mathcal{P}_0\Xi^{(k)}$ on the corresponding spaces of special paths. Lemmas 8.3, 8.4 and 8.6 remain valid on the spaces of special paths and their proofs are completely analogous. All these prerequisites lead to the rolling-up operator

$$\mathcal{P}_0\Psi_A : \mathcal{P}_0\,\mathrm{Cont}_{J_A}\big(\mathbb{R}^m, \alpha_{st}\big)_0 \cap \mathcal{P}_0\mathcal{U}_1 \to \mathcal{P}_0\,\mathrm{Cont}_{K_A}\big(\mathbb{R}^m, \alpha_{st}\big)_0,$$

which satisfies an analogue of Proposition 8.7 (with a similar remark as the above for 5.7). In particular, for any $\{g_t\} \in \mathrm{dom}(\mathcal{P}_0\Psi_A)$ one has $[\mathcal{P}_0\Psi_A(\{g_t\})] = [\{g_t\}]$ in $H_1(\mathcal{P}\,\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0)$.

In the proof of Theorem 11.2 we will use spaces of special paths and the proof is completely analogous. Fix A, r_0 and ϵ_r for $r \geqslant r_0$ as in Section 9. Suppose that $\mathcal{L}$ is as in Section 9. Then $\mathcal{P}_0\mathcal{L}$ is a convex subset of the locally convex vector space $\mathcal{P}_0\mathrm{C}^\infty_{I_A}(\mathbb{R}^m)$. We may and do assume that $\sup_{t\in I} \mu^*_{r_0}(\{h_t\})$ is sufficiently small since $\mathcal{P}\,\mathrm{Cont}_{I_A}(\mathbb{R}^m, \alpha_{st})_0$ is a topological group. Moreover, there is $\{\hat{h}_t\} \in \mathcal{P}_0\,\mathrm{Cont}_{I_A}(\mathbb{R}^m, \alpha_{st})_0$ such that $\sup_{t\in I} \mu^*_{r_0}(\{\hat{h}_t\})$ is also sufficiently small and $[\{\hat{h}_t\}]_\sim = [\{h_t\}]_\sim$.

We define $\mathcal{P}_0\vartheta : \mathcal{P}_0\mathcal{L} \to \mathcal{P}_0\mathcal{L}$ by the formula $\mathcal{P}_0\vartheta(\{u_t\}) = \{\vartheta_t(u_t)\}$, where $\vartheta_t : \mathcal{L} \to \mathcal{L}$ is determined by $\hat{h}_t$. Then there exists $\{f_t\} \in (\mathcal{P}\Phi_A)^{-1}\mathcal{P}_0\mathcal{L}$ such that $u_{f_1} = \Phi_A(f_1)$ is a fixed point of ϑ_1, and there is $\{g_t\}$ in the commutator subgroup of $\mathcal{P}\,\mathrm{Cont}_c(\mathbb{R}^m, \alpha_{st})_0$ such that

$$\{k_t\} := (\mathcal{P}\Phi_A)^{-1}\mathcal{P}_0\vartheta\mathcal{P}\Phi_A\big(\{f_t\}\big) = \{f_t\}\cdot\{\hat{h}_t\}\cdot\{g_t\}$$

is an isotopy in $\mathcal{U}_1$. Since $\Phi_A^{-1}\vartheta_1\Phi_A(f_1) = f_1$, it follows that $\{f_t\}^{-1}\cdot\{k_t\}$ is a contractible loop. Therefore, $[\{\hat{h}_t\}]_\sim = [\{g_t\}^{-1}]_\sim$ so that the class of $[\{\hat{h}_t\}]_\sim$ is equal to e in $H_1(\widetilde{\mathrm{Cont}_c(\mathbb{R}^m}, \alpha_{st})_0)$, as claimed.

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