JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 108, 403–408 (1985)

Measure of Non-compactness and Multivalued Mappings in Complete Metric Topological Vector Spaces

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1. INTRODUCTION

Let (X, d) be a complete metric space and $A \subset X$ a subset of X. The Kuratowski measure of non-compactness of A is by definition

$$\alpha(A) = \inf\left\{\varepsilon > 0: \exists X_i,\ i = 1,..., n X_i \subset X, \text{diam } X_i < \varepsilon,\ A = \bigcup_{i=1} X_i\right\}.$$

If A is a closed subset of X then $\alpha(A) = 0$ if and only if A is compact. If A is a subset of B then $\alpha(A) \leqslant \alpha(B)$.

In [12], Kuratowski proved the following result: if $(A_n)_{n \geqslant 1}$ is a decreasing sequence of non-empty closed subsets of X such that $\lim_{n \to \infty} \alpha(A_n) = 0$ then $\bigcap_{n \geqslant 1} A_n$ is not empty and is compact.

In Section 2 we give a generalization of Kuratowski's theorem. In Section 3 we give some applications to fixed points and non-void intersection theorems for multivalued mappings in complete metric topological vector-spaces. The latter results are generalizations of well-known theorems of Fan and El Mechaiekh, Deguire and Granas.

2. GENERALISATION OF KURATOWSKI'S THEOREM

Let X be a set and $(X_i)_{i \in I}$ a family of subsets of X, and let us recall that $(X_i)_{i \in I}$ has the finite intersection property if for any finite, non-empty subset J of I $\bigcap_{i \in J} X_i$ is not empty.

THEOREM 1. *Let $(X; d)$ be complete metric space and $(F_i)_{i \in I}$ be a family of closed subsets of X having the finite intersection property. If $\operatorname{Inf}_{i \in I} \alpha(F_i) = 0$ then $\bigcap_{i \in I} F_i$ is non-empty and compact.*

0022-247X/85 $3.00

Proof. For each $n > 1$ one can find $i(n) \in I$ such that $\alpha(F_{i(n)}) < 1/n$. Let $A_k = \bigcap_{n=1}^{k} F_{i(n)}$, then A_k is non-empty, closed and $\alpha(A_k) < 1/k$, furthermore $A_{k+1} \subset A_k$. From Kuratowski's theorem, it follows that $K = \bigcap_{k \geqslant 1} A_k$ is a non-empty compact subset of X.

Let J be any non-empty finite subset of I and define

$$F_{J,1} = \bigcap_{i \in J} F_i, \qquad F_{J,k} = \bigcap_{i \in J} (F_i \cap A_{k-1}) \qquad \text{if} \quad k \geqslant 2.$$

$F_{J,k}$ is a non-empty subset of X, $F_{J,k+1} \subset F_{J,k}$ and $\alpha(F_{J,k+1}) < 1/k$. Once more, it follows from Kuratowski's theorem that $\bigcap_{k \geqslant 1} F_{J,k}$ is non-empty and compact. But $\bigcap_{k \geqslant 1} F_{J,k} = \bigcap_{i \in J} F_i \cap \bigcap_{k \geqslant 1} A_k = \bigcap_{i \in J} F_i \cap K$, so taking $\hat{F}_i = F_i \cap K$ for each $i \in I$ we see that $(\hat{F}_i)_{i \in I}$ is a family of compact subsets having the finite intersection property, so $\bigcap_{i \in I} \hat{F}_i$ is not empty, and since $\bigcap_{i \in I} \hat{F}_i \subset \bigcap_{i \in I} F_i$ and $\alpha(\bigcap_{i \in I} F_i) = 0$ the proof is complete.

COROLLARY 1. *Let $(X; d)$ be a complete metric space and $f: X \to R$ a lower semi-continuous function such that* $\operatorname{Inf}_{x \in X} \alpha\{y \in X: f(y) \leqslant f(x)\} = 0$, *then f is bounded from below and there exists $x_0 \in X$ such that* $f(x_0) = \operatorname{Inf}_{x \in X} f(x)$.

Proof. Let $F(x) = \{y \in X: f(y) \leqslant f(x)\}$, then $(F(x))_{x \in X}$ is a family of closed subsets of X having the finite intersection property. Since $\operatorname{Inf}_{x \in X} \alpha(F(x)) = 0$ there exist $x_0 \in X$ such that $f(x_0) \leqslant f(x)$ for all x in X.

COROLLARY 2. *Let $(X; d)$ be a complete metric space and $g: X \to X$ a function such that*:

(i) $x \mapsto d(x, g(x))$ *is lower semi-continuous,*

(ii) $\operatorname{Inf}_{x \in X} d(x, g(x)) = 0$,

(iii) $\operatorname{Inf}_{x \in X} \alpha\{y \in X: d(y, g(y)) \leqslant d(x, g(x))\} = 0$.

Then there exist $x_0 \in X$ such that $g(x_0) = x_0$.

3. Intersection and Fixed Point Theorems for Multivalued Mappings

In [3], Dugundji and Granas introduced the following definition: Let E be a real vector space, $X \subset E$ be an arbitrary subset. A multivalued mapping $G: X \to E$ is called a KKM map if for any finite subset A_0 of X $\operatorname{conv}(A_0) \subset \bigcup_{x \in A_0} G(x)$.[1]

They also proved the following fundamental result: If $G: X \to E$ is a

[1] $\operatorname{conv}(A_0)$ stands for the convex hull of A_0.

KKM map such that for any finite-dimensional subspace L of E and for any $x \in X$ $G(x) \cap L$ is closed in the euclidean topology of L then the family $\{G(x): x \in X\}$ has the finite intersection property. In what follows a complete metric topological vector space will be called for short a *Fréchet space*, and we do not assume local convexity.

DEFINITION. Let E be a Fréchet space, Y a closed convex subset of E, X a subset of Y and $R: X \to Y$ a multivalued mapping. An α-cover for R is family $\{A_i\}_{i \in I}$ of subsets of X such that:

(i) $\forall i \in I$ $\overline{\text{conv}}(A_i)$ is compact in Y,

(ii) $\forall i, j \in I \quad \exists k \in I \quad A_i \cup A_j \subset A_k$,

(iii) $\text{Inf}_{i \in I}\, \alpha(\bigcap_{x \in A_i} R(x)) = 0$.

THEOREM 2. *Let E be a Fréchet space, Y a closed convex subset of E, X a subset of Y and R: $X \to Y$ a KKM map with closed values.*

If there exists an α-cover for R then $\bigcap_{x \in X} R(x) \neq \varnothing$

Proof. Let $(A_i)_{i \in I}$ be an α-cover for R, and let Σ be the family of finite subsets of X.

For $j = (i, a) \in I \times \Sigma$ let $B_j = A_i \cup a$, then $(B_j)_{j \in J}$, where $J = I \times \Sigma$, is still an α-cover for R and furthermore $X = \bigcup_{j \in J} B_j$. Now let R_j: $B_j \to \overline{\text{conv}(B_j)}$ be the following map:

$$R_j(x) = R(x) \cap \overline{\text{conv}}(B_j).$$

This is a KKM map and since $R(x)$ is closed $R_j(x)$ is compact, the family $(R_j(x))_{x \in B_j}$ having the finite intersection property, and we can conclude that $\bigcap_{x \in B_j} R_j(x)$ is not empty, and consequently that $\bigcap_{x \in B_j} R(x)$ is closed and non-empty.

From the second property of an α-cover it folows that the family $(\bigcap_{x \in B_j} R(x))_{j \in J}$ has the finite intersection property since $\inf_{j \in J} \alpha(\bigcap_{x \in B_j} R(x)) = 0$. We conclude from Theorem 1 that

$$\bigcap_{j \in J} \bigcap_{x \in B_j} R(x) \neq \varnothing.$$

Since $\bigcup_{j \in J} B_j = X$ we just have to notice that $\bigcap_{j \in J} \bigcap_{x \in B_j} R(x) = \bigcap_{x \in X} R(x)$ to complete the proof.

The following corollary generalizes a result of Ky Fan.

COROLLARY 3. *Let E be a Fréchet space, Y a closed convex subset of E, X a subset of Y and R: $X \to Y$ a KKM with closed values. If there exists a subset $A \subset X$ such that $\overline{\text{conv}}(A)$ is compact in Y and $\bigcap_{x \in A} R(x)$ is compact then $\bigcap_{x \in X} R(x)$ is not empty.*

Proof. The family formed with the single element A is an α-cover for R.

Remark. The previous corollary is true even if E is not metrizable. A close look at the proof of Theorem 2 shows that under the assumption of Corollary 3 the measure of non-compactness is used only in a trivial way. Corollary 3 is due in a different form to Ky Fan [8].

Let us recall that a selection for a mapping $R: X \to Y$ is a single valued function $f: X \to Y$ such that $\forall x \in X\ f(x) \in R(x)$. For any multivalued mapping $R: X \to Y$ define $R^*: Y \to X$ by $R^*(y) = X \backslash R^{-1}(y)$ where $x \in R^{-1}(y)$ if $y \in R(x)$. It can be shown that if $X = Y$ is convex and if $\forall x \in X\ x \in R(x)$ and $R^*(x)$ is convex then $R: X \to X$ is a KKM map.

PROPOSITION 1. *Let C be a closed convex subset of a Fréchet space, $R: C \to C$ a multivalued mapping with closed values and $(A_i)_{i \in I}$ an α-cover for R such that:*

(i) *$\forall i \in I$ A_i is compact and convex and $C = \bigcup_{i \in I} A_i$,*

(ii) *$\forall i \in I$ there exists a continuous selection $s_i: A_i \to C$ of R,*

(iii) *$R^*: C \to C$ has convex values.*

Then $\bigcap_{x \in C} R(x) \neq \emptyset$.

Proof. Define, for each $i \in I$, a multivalued mapping $S_i: A_i \to A_i$ by $S_i(x) = s_i^{-1}(R(x))$.

Obviously $S_i(x)$ is compact, $x \in S_i(x)$ and $S_i^*(x) = R^*(s_i(x)) \cap A_i$ is convex, so S_i is KKM; if $x_0 \in \bigcap_{x \in A_i} S_i(x)$ then $s_i(x_0) \in \bigcap_{x \in A_i} R(x)$. Since $\bigcap_{x \in A_i} R(x)$ is closed and since $\{A_i\}_{i \in I}$ is an α-cover of R such that $C = \bigcup_{i \in I} A_i$ one can see that $\bigcap_{x \in C} R(x) = \bigcap_{i \in I} \bigcap_{x \in A_i} R(x) \neq \emptyset$.

Our next result is a generaliztion of Ky Fan's fixed point theorem as presented by El Mechaiekh, Deguire and Granas in [1]. Let C be convex subset of a topological vector space and $S: C \to C$ a multivalued mapping. In [1] the following class is introduced: $S \in F(C, C)$ if

(i) $\forall x \in C$ $S(x)$ is open in C,

(ii) $\forall x \in C$ $S^{-1}(x)$ is convex and non-empty.

COROLLARY 4. *Let C be a closed convex subset of a Frechet space, $S \in F(C, C)$ and $f: C \to C$ a single valued mapping. If there exists a family $(A_i)_{i \in I}$ of compact convex subset of C such that*

(i) $C = \bigcup_{i \in I} A_i$ *and* $\forall i, j \in I\ \exists k \in K\ A_i \cup A_j \subset A_k$,

(ii) $\text{Inf}_{i \in I}\, \alpha(C \backslash \bigcup_{x \in A_i} S(x)) = 0$,

(iii) *$\forall i \in I$ the restriction of f to A_i is continuous,* *then there exist* $x_0 \in C$ *such that* $f(x_0) \in S(x_0)$. *In particular S has a fixed point.*

Proof. If for every $x \in C$ $f(x)$ is not in $S(x)$ then f is a selection of $R(x) = C \backslash S(x)$. Since $R^*(x) = S^{-1}(x)$ is convex and $R(x)$ is closed the hypotheses of Proposition 1 are verified. Take $x_0 \in \bigcap_{x \in C} R(x)$, then $S^{-1}(x_0)$ is empty and this contradicts $S \in F(C, C)$. The proof is complete.

The following corollary is a generalization of Ky Fan's minimax theorem [1, 6, 7].

COROLLARY 5. *Let C be a closed convex subset of a Fréchet space and* $f: C \times C \to R$ a function such that $\forall x \in C$ $y \to f(x, y)$ *is lower semi-continuous. If there exists a family* $\{A_i\}_{i \in I}$ *of compact convex subsets of C such that*

(i) $C = \bigcup_{i \in I} A_i$,

(ii) $\forall i, j \in I \; \exists k \in I \; A_i \cup A_j \subset A_k$,

(iii) $\mathrm{Inf}_{i \in I} \alpha\{y \in C: \sup_{x \in A_i} f(x, y) \leqslant \sup_{x \in C} f(x, x)\} = 0$,

(iv) $\forall y \in C \quad \{x \in C: f(x, y) > \sup_{x \in C} f(x, x)\}$ *is convex, then* $\mathrm{Inf}_{y \in C} \sup_{x \in C} f(x, y) \leqslant \sup_{x \in C} f(x, x)$.

Proof. We can assume that $\mu = \sup_{x \in C} f(x, x)$ is finite otherwise there is nothing to prove.

Let $R(x) = \{y \in C: f(x, y) \leqslant \mu\}$. Proposition 1 gives a point y_0 belonging to $\bigcap_{x \in C} R(x)$. We have $\forall x \in C$ $f(x, y_0) \leqslant \sup_{x \in C} f(x, x)$. This completes the proof.

Our last result is a generalization of Ky Fan's coincidence theorem as given by Granas in [9].

COROLLARY 6. *Let C be a closed convex subset of a Fréchet space E,* $S: C \to C$ *a multivalued mapping belonging to* $F(C, C)$ *and* $(A_i)_{i \in I}$ *a family of compact convex subsets of C with properties* (i) *and* (ii) *of corollary* 4, *and let* $R: C \to C$ *be a multivalued mapping.*

If $R^{-1} \in F(C, C)$ *or if E is locally convex and R is lower semi-continuous then there exist* $x_0 \in C$ *such that* $S(x_0) \cap R(x_0) \neq \varnothing$.

Proof. It can be shown that if C is paracompact, which is the case here, and if $R^{-1} \in F(C, C)$, then there is a continuous selection $f: C \to C$ for R. If E is locally convex and R lower semi-continuous the existence of a continuous selection follows from the Michael theorem [2]. From Corollary 4 we know that $f(x_0) \in S(x_0)$ for some point $x_0 \in C$, since if f is a selection of R, $f(x_0)$ is also in $R(x_0)$, and the proof is complete.

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