Semi-cardinal interpolation and difference equations: From cubic B-splines to a three-direction box-spline construction

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Abstract

This paper considers the problem of interpolation on a semi-plane grid from a space of box-splines on the three-direction mesh. Building on a new treatment of univariate semi-cardinal interpolation for natural cubic splines, the solution is obtained as a Lagrange series with suitable localization and polynomial reproduction properties. It is proved that the extension of the natural boundary conditions to box-spline semi-cardinal interpolation attains half of the approximation order of the cardinal case.

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1. Introduction

A treatment of cardinal interpolation (i.e. interpolation at the set $\mathbb{Z}$ of integers) with univariate polynomial splines was given by Schoenberg in [26], using the concept of B-spline functions. In [25,27], Schoenberg also considered the related problem of semi-cardinal interpolation (i.e. interpolation at the set $\mathbb{Z}_+$ of non-negative integers) from a space of univariate splines satisfying certain end-point conditions. The extension of cardinal interpolation to three-directional bivariate box-splines interpolating data on the grid $\mathbb{Z}^2$ was obtained by de Boor et al. [10].

The present paper introduces the problem of interpolation on the semi-plane grid $\mathbb{Z} \times \mathbb{Z}_+$ from a space of bivariate piecewise polynomial functions generated by the three-direction box-spline $M$ whose direction matrix has every multiplicity 2. This box-spline represents a genuine bivariate analog of the univariate cubic B-spline, and its utility for approximation and computer aided design has been established early in several studies by Frederickson [12–14], Sabin [20,21] Sablonnière [22–24], Chui and Wang [6].

The main idea of our bivariate extension of semi-cardinal interpolation to box-splines is to formulate certain automatic boundary conditions in terms of finite difference equations for box-spline coefficients. As demonstrated by Chui et al. [5], the presence of boundary conditions complicates the study of spline spaces even for simpler generating functions than $M$. Bringing in Fourier methods from the theory of bi-dimensional Wiener-Hopf difference equations, the model
of interpolation on a semiplane grid enables a full analysis of localization and polynomial reproduction properties of the proposed box-spline scheme.

Our approach is first illustrated for the univariate case in Section 2, in which the space of one-dimensional semi-cardinal cubic splines is regarded as a subspace of cardinal splines whose B-spline coefficients satisfy a system of ‘natural’ difference equations. The resulting construction is simpler than those obtained by Schoenberg in [25,27]. In Section 3, we propose a suitable extension of the natural boundary conditions to bivariate piecewise polynomials generated by $M$. The corresponding semi-cardinal interpolation problem with box-splines is then solved by constructing the set of fundamental functions and the associated Lagrange scheme. Our analysis is based on the explicit solution of bi-dimensional difference equations of Wiener–Hopf type. In Section 4 we prove that the ‘natural’ semi-cardinal box-spline scheme attains half of the approximation order of the corresponding cardinal scheme. The generalization of these results to other box-splines requires significantly different methods of proof in order to avoid explicit computations. This remains a problem for future research.

Note that a different multivariate extension of semi-cardinal interpolation was obtained in [2,3] for certain polyharmonic spline methods, using a Fourier transform treatment [1] of the univariate case. However, a complete analysis establishing the approximation order of the polyharmonic semi-cardinal schemes has yet to be achieved.

Notation: For a given integer $n$, the set of integers smaller than or equal to $n$ will be denoted by $\mathbb{Z} \leq n$. Also, $\mathbb{Z} \geq n := \mathbb{Z} \setminus \mathbb{Z} \leq n - 1$, $\mathbb{Z}^{+} := \mathbb{Z} \geq 0$, and $\mathbb{R}^{+} := [0, \infty)$.

2. Semi-cardinal interpolation with cubic B-splines

A cardinal cubic spline is a function $s : \mathbb{R} \to \mathbb{C}$, such that

(i) $s \in C^2(\mathbb{R})$, and
(ii) $s$ is a cubic polynomial on $[k, k + 1]$, for any $k \in \mathbb{Z}$.

The space of such functions will be denoted by $\mathcal{S}_3$. In [27], Schoenberg studies interpolation to data given on $\mathbb{Z}^{+}$—referred to as ‘semi-cardinal interpolation’—from the linear space $\mathcal{S}_3^{+}$ of functions $s : \mathbb{R} \to \mathbb{C}$ satisfying (i), as well as

(iii) $s$ is a cubic polynomial on $[k, k + 1]$, for any $k \in \mathbb{Z}^{+}$, and
(iv) $s''(x) = 0$, for $x \in (-\infty, 0]$ (i.e. $s$ is a linear polynomial on $(-\infty, 0]$).

Since (iv) is known as a ‘natural’ end condition, an arbitrary element of $\mathcal{S}_3^{+}$ will be called a natural semi-cardinal cubic spline.

Two methods are used in [27] in order to construct semi-cardinal interpolation from $\mathcal{S}_3^{+}$ and, more generally, from similar odd-degree spline spaces. The first one [27, Chapter I] is based on a set of ‘fundamental functions’ \{ $L_j : j \in \mathbb{Z}^{+}$ \} $\subset \mathcal{S}_3^{+}$ satisfying

\[ L_j(k) = \delta_{jk}, \quad j, k \in \mathbb{Z}^{+}, \]  

such that the corresponding Lagrange scheme

\[ s(x) = \sum_{j=0}^{\infty} y_j L_j(x), \quad x \in \mathbb{R}^{+}, \]  

is absolutely and uniformly convergent on compact subsets of $\mathbb{R}^{+}$ for any data sequence $\{ y_j \}_{j=0}^{\infty}$ of polynomial growth, and $s(j) = y_j, \ j \in \mathbb{Z}^{+}$. In turn, for each $j \in \mathbb{Z}^{+}$, the Lagrange function $L_j$ is defined as a linear combination of the shifted fundamental function for cardinal interpolation and a set of so-called eigenspline functions [27, (4.2)]. It can be noted that this linear combination belongs to the cardinal space $\mathcal{S}_3$, but this fact is not explicitly used or mentioned by Schoenberg.

The second method [27, Chapter II] is employed for a different class of data sequences and builds on the fact that, if $s \in \mathcal{S}_3^{+}$, then the second derivative $s''$ is a cardinal spline of degree one, determined by its linear B-spline series.
The present section proposes a simpler approach to semi-cardinal interpolation for cubic splines, based on two observations. Firstly, any natural semi-cardinal cubic spline is in particular a cardinal cubic spline, i.e. \( S^+ \subset S \). Secondly, for any function \( s \) satisfying (iv), we have
\[
s(x) - 2s(x + 1) + s(x + 2) = 0, \quad x \in (-\infty, -2].
\] (2.3)

As the following result shows, these two properties completely characterize \( S^+ \).

**Proposition 2.1.** If \( s \in S \) satisfies condition (2.3), then \( s \in S^+ \).

**Proof.** We only have to verify condition (iv). For any negative integer \( k \leq -1 \), the cardinal cubic spline \( s \) can be expressed in Taylor form on the interval \([k, k + 1]\):
\[
s(x) = s(k) + s'(x - k) + \frac{1}{2} s''(x - k)^2 + \frac{1}{6} s'''(x - k)^3, \quad x \in [k, k + 1].
\] (2.4)

It is well-known from \([7,18]\) that the smoothness condition (i) implies
\[
s''(k) = 6[s(k + 1) - s(k)] - 4s'(k) - 2s'(k + 1), \quad k \leq -3,
\] (2.5)

\[
s''(k^+) = 12[s(k) - s(k + 1)] + 6s'(k) + 6s'(k + 1),
\] (2.6)

\[
s'(k - 1) + 4s'(k) + s'(k + 1) = 3[s(k + 1) - s(k - 1)].
\]

Using (2.4) to express \( s \) in (2.3) for \( x \in [k, k + 1] \) and \( k \leq -3 \), yields
\[
s(k) - 2s(k + 1) + s(k + 2) = 0,
\]
\[
s'(k) - 2s'(k + 1) + s'(k + 2) = 0.
\]

The last three relations imply
\[
s'(k) = s(k + 1) - s(k) = s'(k + 1), \quad k \leq -3,
\]

so \( s''(k) = s'''(k^+) = 0 \) for \( k \leq -3 \), by (2.5) and (2.6). Consequently, \( s \) is a linear polynomial on \((-\infty, -2]\) which, due to (2.3), shows that (iv) holds. \( \square \)

The new approach proposed here relies on the fact \([9]\) that any cardinal cubic spline admits a unique B-spline series expansion
\[
s(x) = \sum_{k \in \mathbb{Z}} a_k M^4_k(x - k), \quad x \in \mathbb{R},
\] (2.7)

where \( M^4_k := M^4(\cdot + 2) \) is the centered cubic B-spline, supported and non-negative in the interval \([-2, 2]\), and taking the values
\[
M^4_0(0) = \frac{2}{3}, \quad M^4_0(-1) = M^4_0(1) = \frac{1}{3}.
\]

Acting on (2.7) with the second order forward difference operator provides
\[
s(x) - 2s(x + 1) + s(x + 2)
\]
\[
= \sum_{k \in \mathbb{Z}} a_k [M^4_k(x - k) - 2M^4_k(x - k + 1) + M^4_k(x - k + 2)]
\]
\[
= \sum_{k \in \mathbb{Z}} (a_k - 2a_{k+1} + a_{k+2}) M^4_k(x - k),
\]

which, due to the local linear independence of the shifts \( \{M^4_k(\cdot - k) : k \in \mathbb{Z}\} \), shows that (2.3) is equivalent to the system of difference equations
\[
a_k - 2a_{k+1} + a_{k+2} = 0, \quad k \leq -1.
\] (2.8)
We will now use these remarks in order to construct, for each \( j \in \mathbb{Z}_+ \), a fundamental function \( L_j \in \mathcal{S}^3_+ \) such that \( L_j \) is bounded on the positive semi-axis and satisfies the Lagrange interpolation conditions (2.1). Clearly, this problem amounts to the determination of the sequence of coefficients \( \{\mu_{j,k} : k \in \mathbb{Z}\} \) such that the B-spline expansion

\[
L_j(x) := \sum_{k \in \mathbb{Z}} \mu_{j,k} M_4(x - k), \quad x \in \mathbb{R},
\]

has the required properties. By the above discussion, \( L_j \in \mathcal{S}^3_+ \) if and only if

\[
\mu_{j,k} - 2\mu_{j,k+1} + \mu_{j,k+2} = 0, \quad k \leq -1.
\]

On the other hand, since for any integer \( l \geq 0 \),

\[
L_j(l) = \sum_{k \in \mathbb{Z}} \mu_{j,k} M_4(l - k) = \frac{1}{6}(\mu_{j,l-1} + 4\mu_{j,l} + \mu_{j,l+1}),
\]

the interpolation conditions (2.1) are equivalent to

\[
\mu_{j,l-1} + 4\mu_{j,l} + \mu_{j,l+1} = 6\delta_{jl}, \quad l \in \mathbb{Z}_+.
\]

It is also straightforward to deduce from the stability property [9] of a cubic B-spline series that \( L_j \) is bounded on \( \mathbb{R}_+ \) if and only if \( \{\mu_{j,k}\}_{k=0}^\infty \) is a bounded sequence.

Letting \( l := 0 \) in (2.11) and \( k := -1 \) in (2.10), i.e. considering the extreme equations of these systems, we deduce

\[
\mu_{j,0} = \delta_{j0}.
\]

Hence replacing this value in (2.11) for \( l := 1 \), we obtain a system of linear recurrence relations in the unknowns \( \{\mu_{j,k} : k \geq 1\} \). The unique bounded solution of this system (see Lemma 3.5) is now explicitly given by

\[
\mu_{j,k} = \sqrt{3}\left(1 - \lambda^2\min[k,j] + \frac{\sqrt{3}}{3}\delta_{j0}\right)^{j-k}, \quad j \in \mathbb{Z}_+, \quad k \geq 1,
\]

where \( \lambda := \sqrt{3} - 2 \) is the root of modulus less than 1 of the characteristic equation

\[
\lambda^2 + 4\lambda + 1 = 0.
\]

With \( \mu_{j,1} \) and \( \mu_{j,0} \) determined, (2.10) provide by recurrence the remaining values of the cardinal B-spline coefficients of \( L_j \), which completes its construction and proves the first assertion of the following result.

**Proposition 2.2.** For each \( j \in \mathbb{Z}_+ \), there exists a unique function \( L_j \in \mathcal{S}^3_+ \), bounded on \( \mathbb{R}_+ \) and satisfying the semi-cardinal Lagrange interpolation conditions (2.1). In addition, there exists a constant \( C > 0 \) such that

\[
|L_j(x)| \leq C|\lambda|^{x-j}, \quad x \in \mathbb{R}_+, \quad j \in \mathbb{Z}_+.
\]

**Proof.** Formula (2.12) implies

\[
|\mu_{j,k}| \leq C_1|\lambda|^{k-j}, \quad j, k \in \mathbb{Z}_+.
\]

for some constant \( C_1 \). The decay estimate (2.13) now follows from (2.9) taking account of the compact support of \( M_4^c \). \( \Box \)

**Remark 2.3.** The B-spline coefficients (2.12) have also been obtained by Schoenberg [25] as limits of similar coefficients corresponding to interpolation on \( \mathbb{Z} \cap [0,n] \), for \( n \to \infty \). However, neither [25], nor [27] use the cardinal B-spline series (2.9) to represent the semi-cardinal Lagrange function \( L_j \).
3. Semi-cardinal interpolation with $M$

The three-direction mesh (or type-1 triangulation) in the plane is the union of all lines containing at least one point of integer coordinates and parallel to either one of the coordinate axes or to the first diagonal. Throughout this section, $M$ will denote the centered three-direction box-spline with direction matrix

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}.
$$

For the terminology and basic elements of box-spline theory used in this section, the reader is referred to the monographs by de Boor et al. [11], Chui [4], and Wang [28].

In analogy with the cubic B-spline, we have $M \in C^2(\mathbb{R}^2)$ and $M$ is a piecewise polynomial function (made of 24 pieces) on the three-direction mesh partition of the plane. Further, each piece of $M$ belongs to the 12-dimensional polynomial space spanned by the set

$$\{x_1^m x_2^n : m, n \in \mathbb{Z}_+, m + n \leq 3\} \cup \{x_1^4 - 2x_1^3 x_2, x_2^4 - 2x_2^3 x_1\},$$

where $x_1$ and $x_2$ are the two variable coordinates. Also, $M$ is non-negative, with compact support on the hexagon

$$\left\{ u \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + v \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + w \left( \begin{array}{c} 1 \\ 1 \end{array} \right) : u, v, w \in [-1, 1] \right\},$$

and the only non-zero values of $M$ on $\mathbb{Z}^2$ are

$$M(0, 0) = \frac{1}{12}, \quad M(-1, -1) = M(\pm 1, 0) = M(0, \pm 1) = M(1, 1) = \frac{1}{12}.$$ (3.3)

Let $\mathcal{S}_M$ denote the cardinal box-spline space

$$\mathcal{S}_M := \text{span}\{M(\cdot - k) : k \in \mathbb{Z}^2\}.$$

As a consequence of the (global) linear independence of the shifts of $M$ [11, (II.29)], any function $s \in \mathcal{S}_M$ admits a unique box spline series representation of the form

$$s(x) = \sum_{k \in \mathbb{Z}^2} a_k M(x - k), \quad x \in \mathbb{R}^2.$$ (3.4)

We now proceed to formulate the problem of interpolation on the semi-plane grid $\mathbb{Z} \times \mathbb{Z}_+$ from a suitable subspace of $\mathcal{S}_M$. First, we need a bivariate finite-difference condition to replace the ‘natural’ condition (2.3). Note that the polynomial space (3.1) is the intersection of the null-spaces of the following three differential operators: $D_1^2 D_2^2$, $D_1^2 (D_1 + D_2)^2$, and $D_2^2 (D_1 + D_2)^2$, where $D_1$ and $D_2$ are the partial derivatives of order one. Since $D_1$ acts in the direction of the boundary of the semi-plane $\mathbb{R} \times \mathbb{R}_+$, in order to extend the univariate natural condition we discard the first two of the above differential operators and select ‘half’ of the third differential operator as the ‘natural’ operator, i.e. due to symmetry we let $D_2(D_1 + D_2)$ be the bivariate ‘natural’ differential operator. Thus, by discretization, we arrive at the ‘natural’ bivariate difference operator $\mathcal{N}$ defined by

$$\mathcal{N}s := s - s(\cdot + (0, 1)) - s(\cdot + (1, 1)) + s(\cdot + (1, 2)),$$

for any function $s$ of two variables.

**Definition 3.1.** A function $s \in \mathcal{S}_M$ is called a $\mathcal{N}$-type semi-cardinal box-spline if

$$s(x_1, x_2) - s(x_1, x_2 + 1) - s(x_1 + 1, x_2 + 1) + s(x_1 + 1, x_2 + 2) = 0,$$

$$\forall \ x = (x_1, x_2) \in \mathbb{R} \times (-\infty, -2].$$ (3.5)

The space of all $\mathcal{N}$-type semi-cardinal box-splines will be denoted by $\mathcal{S}^+_{M, \mathcal{N}}$.

The next result shows the effect of condition (3.5) on representation (3.4).
Proposition 3.2. A function \( s \in \mathcal{S}_M \) is a \( N \)-type semi-cardinal box-spline if and only if the coefficients of its box spline series (3.4) satisfy the conditions
\[
ak - a_{k+0,1} - a_{k+(1,1)} + a_{k+(1,2)} = 0, \quad k \in \mathbb{Z} \times \mathbb{Z} \leq -1. \tag{3.6}
\]
In this case, for any integer \( k_2 \leq 0 \), the restriction of \( s \) to \( \mathbb{R} \times \{k_2\} \) is a univariate cardinal cubic spline whose set of B-spline coefficients is \( \{a_{(k,k_2)} : k_1 \in \mathbb{Z}\} \), i.e.
\[
s(x_1, k_2) = \sum_{k_1 \in \mathbb{Z}} a_{(k_1,k_2)} M_4'(x_1 - k_1), \quad x_1 \in \mathbb{R}. \tag{3.7}
\]

Proof. Let \( s \in \mathcal{S}_M \) possess the cardinal box-spline series (3.4). Then
\[
\mathcal{N}s(x) = s(x) - s(x + (0, 1) - s(x + (1, 1)) + s(x + (1, 2))
= \sum_{k \in \mathbb{Z}^2} a_k (\mathcal{N}M)(x - k)
= \sum_{k \in \mathbb{Z}^2} [a_k - a_{k+0,1} - a_{k+(1,1)} + a_{k+(1,2)}] M(x - k), \quad x \in \mathbb{R}^2.
\]
By the local linear independence of the shifts \( \{M(\cdot - k) : k \in \mathbb{Z}^2\} \) (see [11, Chapter II]), and taking account of the support set (3.2) of \( M \), it follows that condition (3.5) is equivalent to the system (3.6) of difference equations.

For the second statement of the Proposition, it is sufficient to look at the case \( k_2 = 0 \). Without loss of generality, let \( x_1 \in [0, 1] \). Then the size of the support of \( M \) and the representation (3.4) imply
\[
s(x_1, 0) = a_{(-1,-1)} M(x_1 + 1, 1) + a_{(0,-1)} M(x_1, 1) + a_{(1,-1)} M(x_1 - 1, 1)
+ a_{(-1,0)} M(x_1 + 1, 0) + a_{(0,0)} M(x_1, 0) + a_{(1,0)} M(x_1 - 1, 0) + a_{(2,0)} M(x_1 - 2, 0)
+ a_{(0,1)} M(x_1, -1) + a_{(1,1)} M(x_1 - 1, -1) + a_{(2,1)} M(x_1 - 2, -1). \tag{3.8}
\]
Next, for \( x_1 \in [0, 1] \), the explicit formulae [6, p. 548] of the polynomial pieces of \( M \) provide
\[
M(x_1 + 1) = M(x_1, -1) = \frac{1}{12} (1 - x_1)^3 (1 + x_1) =: t(x_1),
M(x_1, 1) = M(x_1 - 1, -1) = \frac{1}{12} (1 + 2 x_1 - 4 x_1^3 + 2 x_1^4) =: u(x_1),
M(x_1 - 1, 1) = M(x_1 - 2, -1) = t(1 - x_1),
M(x_1 + 1, 0) = \frac{1}{12} (1 - x_1)^4 =: v(x_1),
M(x_1 - 2, 0) = v(1 - x_1),
M(x_1, 0) = \frac{1}{12} (6 - 12 x_1^2 + 8 x_1^3 - x_1^4) =: w(x_1),
M(x_1 - 1, 0) = w(1 - x_1).
\]
Thus, using (3.6) for \( k \in \{(-1,-1), (0,-1), (1,-1)\} \), it follows that the six coefficients whose second indices are \( \pm 1 \) can be eliminated from (3.8). It remains to note, for \( x_1 \in [0, 1] \), the identities
\[
t(x_1) + v(x_1) = M_5'(x_1 + 1),
t(x_1) + u(x_1) + w(x_1) = M_4'(x_1),
t(1 - x_1) + v(1 - x_1) = M_5'(x_1 - 2),
t(1 - x_1) + u(x_1) + w(1 - x_1) = M_4'(x_1 - 1),
\]
which imply
\[
s(x_1, 0) = a_{(-1,0)} M_5'(x_1 + 1) + a_{(0,0)} M_4'(x_1) + a_{(1,0)} M_5'(x_1 - 1) + a_{(2,0)} M_4'(x_1 - 2).
\]
This coincides with (3.7) for \( x_1 \in [0, 1] \) and \( k_2 = 0 \), finishing the proof. \( \square \)
We now construct the Lagrange functions for interpolation on the semiplane lattice $\mathbb{Z} \times \mathbb{Z}_+$ from the space $\mathcal{S}^+_{M, \mathcal{N}}$ of $\mathcal{N}$-type semi-cardinal box-splines.

**Theorem 3.3.** For each $j \in \mathbb{Z} \times \mathbb{Z}_+$, there exists a unique function $L_j \in \mathcal{S}^+_{M, \mathcal{N}}$ which is bounded on $\mathbb{R} \times [-1, \infty)$ and satisfies the semi-cardinal Lagrange interpolation conditions

$$L_j(k) = \delta_{jk}, \quad k \in \mathbb{Z} \times \mathbb{Z}_+. \tag{3.9}$$

In addition, there exist constants $C > 0$ and $r \in (0, 1)$ such that

$$|L_j(x)| \leq C r^{|x - j|^r}, \quad x \in \mathbb{R} \times \mathbb{R}_+, \quad j \in \mathbb{Z} \times \mathbb{Z}_+. \tag{3.10}$$

**Proof.** Let the cardinal box-spline series (3.4) of the required function $L_j$ be

$$L_j(x) = \sum_{k \in \mathbb{Z}^2} \mu_{jk} M(x - k), \quad x \in \mathbb{R}^2, \tag{3.11}$$

where each coefficient $\mu_{jk}$ depends on the bivariate indices $j = (j_1, j_2) \in \mathbb{Z} \times \mathbb{Z}_+$ and $k = (k_1, k_2) \in \mathbb{Z}^2$. In order to prove the first part of the theorem, we will show that, for each $j \in \mathbb{Z} \times \mathbb{Z}_+$, there exists a unique sequence $\{\mu_{jk}\}_{k \in \mathbb{Z}^2}$ such that the function $L_j$ defined by (3.11) belongs to $\mathcal{S}^+_{M, \mathcal{N}}$, satisfies (3.9), and is bounded on $\mathbb{R} \times [-1, \infty)$.

By Proposition 3.2, $L_j \in \mathcal{S}^+_{M, \mathcal{N}}$ if and only if

$$\mu_{jk} - \mu_{jk+(0,1)} - \mu_{jk+(1,1)} + \mu_{jk+(1,2)} = 0, \quad k \in \mathbb{Z} \times \mathbb{Z} \leq -1. \tag{3.12}$$

On the other hand, since for any $l = (l_1, l_2) \in \mathbb{Z} \times \mathbb{Z}_+$ we have

$$L_j(l) = \sum_{k \in \mathbb{Z}^2} \mu_{jk} M(l - k) = \frac{1}{12} [\mu_{jl-(0,1)} + \mu_{jl-(1,1)} + \mu_{jl-(1,0)} + 6 \mu_{jl} + \mu_{jl+(0,1)} + \mu_{jl+(1,1)}],$$

conditions (3.9) translate into another system of difference equations

$$\mu_{jl-(0,1)} + \mu_{jl-(1,1)} + \mu_{jl-(1,0)} + 6 \mu_{jl} + \mu_{jl+(0,1)} + \mu_{jl+(1,1)} = 12 \delta_{jl}, \quad l \in \mathbb{Z} \times \mathbb{Z}_+. \tag{3.13}$$

The next lemma shows that the remaining condition, namely that $L_j$ should be bounded on $\mathbb{R} \times [-1, \infty)$, is also equivalent to a restriction on the coefficients of the box-spline representation (3.11).

**Lemma 3.4.** If $s \in \mathcal{S}^+_{M, \mathcal{N}}$, then $s$ is bounded on $\mathbb{R} \times [-1, \infty)$ if and only if the sequence of coefficients $\{a_k\}_{k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1}}$ from the expansion (3.4) is bounded.

**Proof.** For the first implication, assume that $s$ is bounded on $\mathbb{R} \times [-1, \infty)$. By Proposition 3.2 we know that, for $m \in [-1, 0]$, the restriction $s_{\mathbb{R} \times [m]}$ is a univariate cardinal cubic spline with B-spline coefficients $\{a_{(k_1, m)}\}_{k_1 \in \mathbb{Z}}$. Since by hypothesis $s_{\mathbb{R} \times [m]}$ is a bounded function, the stability property of the shifts of the cubic B-spline $M_k$ [9] implies that $\{a_{(k_1, m)}\}_{k_1 \in \mathbb{Z}}$ is a bounded sequence for $m \in [-1, 0]$. Moreover, using (3.6) for $k_2 = -1$ we deduce that $\{a_{(k_1, 1)}\}_{k_1 \in \mathbb{Z}}$ is also bounded.

We now invoke the stability of the shifts of the bivariate box-spline $M$ [8, Corollary 4.1]. In particular, this ensures the existence of a constant $c > 0$ such that, for any sequence of real numbers $\{b_k\}_{k \in \mathbb{Z}^2}$,

$$\sup_{k \in \mathbb{Z}^2} |b_k| \leq c \sup_{x \in \mathbb{R}^2} \left| \sum_{k \in \mathbb{Z}^2} b_k M(x - k) \right|. $$
Letting $b_k := a_k$ for $k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1}$, and $b_k := 0$ for $k \in \mathbb{Z} \times \mathbb{Z}_{\leq -2}$, we obtain
\begin{equation}
\sup_{k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1}} |a_k| \leq e \sup_{x \in \mathbb{R}^2} |T(x)|, \tag{3.14}
\end{equation}
where $T(x) := \sum_{k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1}} a_k M(x - k)$. Taking account of the support of $M$, we have $T(x) = s(x)$ for $x \in \mathbb{R} \times \mathbb{R}_+$, so the hypothesis implies that $T$ is bounded on $\mathbb{R} \times \mathbb{R}_+$. Also, note that $T$ is identically zero on $\mathbb{R} \times (-\infty, -3]$, and
\[
T(x) = \sum_{k \in \mathbb{Z} \times [-1, 0, 1]} a_k M(x - k), \quad x \in \mathbb{R} \times [-3, 0].
\]
Thus, since we have shown that $\{a_k\}_{k \in \mathbb{Z} \times [-1, 0, 1]}$ is a bounded sequence, we deduce that $T$ is also bounded on $\mathbb{R} \times [-3, 0]$. It follows that $T$ is bounded on $\mathbb{R}^2$, so (3.14) shows that the sequence $\{a_k\}_{k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1}}$ is bounded, as required. The converse implication is elementary. □

Consequently, the problem of constructing the Lagrange function $L_j$ with the properties required in the statement of the theorem is equivalent to finding a sequence $\{\mu_{jk}\}_{k \in \mathbb{Z}^2}$ which is bounded for $k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1}$ and satisfies the coupled systems of difference equations (3.12) and (3.13).

As in the one-dimensional situation, let us look at the extreme equations of the systems (3.12) and (3.13). Letting $I_2 := 0$ for the second component of the index $l$ in (3.13), and $k \in \{(l_1, -1), (l_1 - 1, -1)\}$ in (3.12), for each $l_1 \in \mathbb{Z}$ we deduce the equations
\[
\begin{align*}
\mu_{j(l_1, -1)} + \mu_{j(l_1 - 1, -1)} + 6 \mu_{j(l_1, 0)} + \mu_{j(l_1 + 1, 0)} + \mu_{j(l_1, 1)} + \mu_{j(l_1 + 1, 1)} &= 12 \delta_{j(l_1)} \delta_{j(0)}, \\
\mu_{j(l_1, -1)} - \mu_{j(l_1, 0)} - \mu_{j(l_1 + 1, 0)} + \mu_{j(l_1 + 1, 1)} &= 0, \\
\mu_{j(l_1, -1)} + \mu_{j(l_1 - 1, 0)} + \mu_{j(l_1, 0)} + \mu_{j(l_1 + 1, 0)} &= 0.
\end{align*}
\]
Hence, subtracting the above second and third equations from the first, we obtain
\[
\mu_{j(l_1, 0)} + 4 \mu_{j(l_1, 0)} + \mu_{j(l_1 + 1, 0)} + 6 \delta_{j(l_1)} \delta_{j(0)} = 0, \quad l_1 \in \mathbb{Z}.
\]
This one-dimensional system of difference equations admits a unique bounded solution given by
\[
\mu_{j(l_1, 0)} = \sqrt{3} \delta_{j(0)} \lambda^{l_1 - j_1}, \quad l_1 \in \mathbb{Z}, \tag{3.15}
\]
where $\lambda = \sqrt{3} - 2$.

We now seek a bounded solution to the system determined by all those equations (3.13) for which the second component of the index $l$ satisfies $l_2 \geq 1$. Using the known values $\{\mu_{j(l_1, 0)}\}_{l_1 \in \mathbb{Z}}$, we can rewrite this system as
\[
\sum_{k \in \mathbb{Z} \times \mathbb{Z}_{\geq 1}} M(l - k) \mu_{jk} = v_{jl}, \quad l \in \mathbb{Z} \times \mathbb{Z}_{\geq 1}, \tag{3.16}
\]
where
\[
\mu_{jl} := \begin{cases} 
\delta_{jl}, & l_2 \geq 2, \\
\delta_{jl} - \frac{1}{12} \mu_{j(l_1, 0)}, & l_2 = 1.
\end{cases}
\]
In this form, (3.16) is a so-called Wiener–Hopf system of difference equations on the semiplane lattice $\mathbb{Z} \times \mathbb{Z}_{\geq 1}$. The ‘symbol’ or ‘characteristic function’ of this Wiener–Hopf system is, by definition, the periodic function
\[
\tilde{M}(x_1, x_2) := \sum_{k \in \mathbb{Z}^2} M(-k)e^{ik_1x_1}e^{ik_2x_2}, \quad x_1, x_2 \in \mathbb{R}, \tag{3.17}
\]
which is also associated to the problem of cardinal interpolation with the box spline $M$ (see [11, Chapter IV]). From the classical theory developed by Goldenstein and Gohberg [15,16] for such systems, it is known that, if: (1) $\tilde{M}$ does not vanish, and (2) for each fixed $x_1$, the winding number (or index) of the periodic univariate function $\tilde{M}(x_1, \cdot)$ is zero, then, for each $p \in [1, \infty]$, the Wiener–Hopf system admits a unique solution in $\ell^p$ whenever its right-hand side is in $\ell^p$. 


Further, multiplying (3.13) by $e^{\frac{70}{2}}$, we obtain

\[
\tilde{\mu}_{j\ell}(\theta) := \sum_{l_1 \in \mathbb{Z}} \mu_{j_1,j_2} e^{-i\ell_1 \theta}, \quad \theta \in [-\pi, \pi],
\]

the series being absolutely convergent. For $l_2 = 0$, (3.15) implies

\[
\tilde{\mu}_{j0}(\theta) = \sqrt{3} \delta_{j0} \sum_{l_1 \in \mathbb{Z}} \hat{\beta}^{l_1} e^{-i\ell_1 \theta} = \beta(\theta) e^{-i\ell_1 \theta} \delta_{j0}, \quad \theta \in [-\pi, \pi],
\]

where

\[
\beta(\theta) := \frac{3}{2 + \cos \theta}.
\] (3.18)

Further, multiplying (3.13) by $e^{-i\ell_1 \theta}$ and summing over $l_1 \in \mathbb{Z}$, we obtain

\[
(1 + e^{-i\ell_1 \theta}) \tilde{\mu}_{j_1,j_{1}-1}(\theta) + (e^{-i\ell_1 \theta} + 6 + e^{i\ell_1 \theta}) \tilde{\mu}_{j_1,j_{2}}(\theta) + (1 + e^{i\ell_1 \theta}) \tilde{\mu}_{j_1,j_{1}+1}(\theta)
\]

\[
= 12 \delta_{j_2} e^{-i\ell_1 \theta}, \quad \theta \in [-\pi, \pi], \quad l_2 \geq 1,
\]

which is a linear difference system depending on the parameter $\theta$. Note also that the absolute summability of the sequence $\{\mu_{jk}\}_{jk \in \mathbb{Z} \times \mathbb{Z} \geq 1}$ implies in particular the boundedness of the set $\{\max_{\theta \in [-\pi, \pi]} |\tilde{\mu}_{j_2}(\theta)| : l_2 \geq 1\}$.

In order to solve explicitly the difference system (3.19), we fix $m \in \mathbb{Z}_+$ and consider the more general recurrence relations

\[
a v_{n+1} + b v_n + c v_{n-1} = \sigma \delta_{mn}, \quad n \geq 1,
\] (3.20)

where $a, b, c, \sigma$ are complex numbers such that $ac \neq 0$. The proof of the following lemma is straightforward.

**Lemma 3.5.** If $v_0 = x \delta_{m0}$ for some complex number $x$ and if the characteristic equation associated to (3.20) has one root $\rho$ with $|\rho| < 1$, the other root being of modulus strictly greater than 1, then there exists a unique bounded sequence $\{v_n\}_{n \in \mathbb{Z}_+}$ that satisfies (3.20). For $n \geq 1$, this bounded solution is given by the formula

\[
v_n = \begin{cases} 
\frac{x \rho^n}{b^2 - 4ac} \left(1 - \left(\frac{\sigma}{c \rho^2}\right)^{\min[m,n]} \right) \left(\frac{\sigma}{c}\right)^{\max[0,m-n]} \rho^{m-n} & \text{if } m = 0, \\
\frac{(b + 2a \rho) \sigma}{b^2 - 4ac} \left[1 - \left(\frac{\sigma}{c \rho^2}\right)^{\min[m,n]} \right] \left(\frac{\sigma}{c}\right)^{\max[0,m-n]} \rho^{m-n} & \text{if } m \neq 0.
\end{cases}
\] (3.21)

For $-\pi < \theta < \pi$, we now apply Lemma 3.5 with

\[
a = a(\theta) := 1 + e^{i\theta}, \quad b = b(\theta) := e^{-i\theta} + 6 + e^{i\theta},
\]

\[
c = c(\theta) := 1 + e^{-i\theta}, \quad \sigma = \sigma(\theta) := 12 e^{-i\ell_1 \theta},
\]

\[
\alpha = \alpha(\theta) := \beta(\theta) e^{-i\ell_1 \theta}.
\]

Since $b^2(\theta) - 4a(\theta)c(\theta) = 4(\cos^2 \theta + 4 \cos \theta + 7) > 0$, we have

\[
\rho = \rho(\theta) := \frac{-2 \cos \theta e^{-i\theta}}{3 + \cos \theta + \sqrt{\cos^2 \theta + 4 \cos \theta + 7}}.
\] (3.22)
We thus obtain, for \( \theta \in (-\pi, \pi) \), the bounded solution \( \{ \hat{\mu}_{jk_2}(\theta) : k_2 \geq 1 \} \) of the recurrence (3.19) given by

\[
\hat{\mu}_{jk_2}(\theta) = \begin{cases} 
\beta(\theta)\rho^{k_2}(\theta)e^{-ij_1\theta} & \text{if } j_2 = 0, \\
\gamma(\theta)G_{j_2,k_2}(\theta)\rho^{k_2-j_2}(\theta)e^{-ij_1\theta} & \text{if } j_2 > 0,
\end{cases}
\]

(3.24)

where

\[
G_{j_2,k_2}(\theta) := \left[1 - (e^{i\theta}\rho(\theta))^{\min(j_2,k_2)}\right]e^{i\theta\max(0,j_2-k_2)}.
\]

On the other hand, if \( \theta = \pm \pi \), then (3.19) provides

\[
\hat{\mu}_{jk_2}(\pm \pi) = 3\delta_{j_2j_1}(-1)^{j_1}, \quad l_2 \geq 1,
\]

the same values being obtained from (3.24) by continuity extension as \( \theta \to \pm \pi \).

Consequently, (3.24) holds for \( \theta \in [-\pi, \pi] \), and the solution \( \{ \mu_{jk} \}_{k \in \mathbb{Z} \times \mathbb{Z} \geq 1} \) of the Wiener–Hopf system (3.16) is obtained by computing the Fourier coefficients of the set of periodic and continuous functions \( \{ \hat{\mu}_{jk_2} : k_2 \in \mathbb{Z} \geq 1 \} \), i.e.

\[
\mu_{j,k_1,k_2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\mu}_{jk_2}(\theta)e^{ik_1\theta}d\theta, \quad k_1 \in \mathbb{Z}.
\]

(3.25)

The proof of (3.10) will follow from the next property.

**Lemma 3.6.** There exist constants \( C > 0 \) and \( r \in (0, 1) \) such that the above coefficients of the box-spline representation (3.11) satisfy

\[
|\mu_{jk}| \leq Cr^{\|k-j\|}, \quad j \in \mathbb{Z} \times \mathbb{Z}_+, \ k \in \mathbb{Z} \times \mathbb{Z} \geq -1.
\]

(3.26)

**Proof.** Formula (3.22) shows that \( |\rho(\theta)| \) is bounded above by a constant smaller than 1 for \( \theta \in [-\pi, \pi] \), and that \( \rho \) admits an analytic and \( 2\pi \)-periodic extension to an open strip containing the real axis. Therefore there exist \( r_0 \in (0, 1) \) and \( \varepsilon > 0 \) such that

\[
|\rho(\theta + i\eta)| < r_0, \quad \theta \in [-\pi, \pi], \ \eta \in (-\varepsilon, \varepsilon), \ (3.27)
\]

\( \rho \) is an analytic function on \( [-\pi, \pi] \times (-\varepsilon, \varepsilon) \), and \( r_0^2e^{\varepsilon} < 1 \). If necessary, \( \varepsilon \) can be decreased in order that the periodic functions \( \beta \) and \( \gamma \), from (3.18) and (3.23) respectively, also admit analytic and periodic extensions to the strip \( [-\pi, \pi] \times (-\varepsilon, \varepsilon) \). Thus, by Cauchy’s integral theorem, the integral in (3.25) may be replaced by the integral

\[
\int_{-\pi}^{\pi} \hat{\mu}_{jk_2}(\theta + i\eta_0)e^{ik_1(\theta+i\eta_0)}d\theta,
\]

where \( \eta_0 = (\varepsilon/2) \text{ sgn}(k_1 - j_1) \).

For \( j_2 = 0 \), we estimate the last integral using the boundedness of \( \beta \) on the strip \( [-\pi, \pi] \times [-\varepsilon/2, \varepsilon/2] \), together with (3.27) and

\[
|e^{(k_1-j_1)(\theta+i\eta_0)}| = e^{-(\varepsilon/2)|k_1-j_1|},
\]

in order to obtain

\[
|\mu_{(j_1,0),k}| \leq C_1r_0^{k_2}e^{-(\varepsilon/2)|k_1-j_1|}, \quad j_1 \in \mathbb{Z}, \ k \in \mathbb{Z} \times \mathbb{Z}_+.
\]

(3.28)

If \( 0 < j_2 \leq k_2 \), then a similar estimate using the boundedness of \( \gamma \) on the strip \( [-\pi, \pi] \times [-\varepsilon/2, \varepsilon/2] \) and

\[
|e^{(\theta+i\eta_0)}\rho^2(\theta+i\eta_0)| \leq e^{-\eta_0r_0^2} < e^{\varepsilon^2/4} < 1,
\]
provides
\[ |\mu_{jk}| \leq C_2 r_0^{k_2-j_2} e^{-(\kappa/2)|k_1-j_1|}. \] (3.29)
If \( j_2 > k_2 \geq 0 \), we use
\[ |\rho(\theta + i\eta_0)e^{i(\theta+i\eta_0)}| \leq r_0 e^{-\eta_0} \leq r_0 e^{\kappa/2}, \]
and analogously obtain
\[ |\mu_{jk}| \leq C_2 (r_0 e^{\kappa/2})^{j_2-k_2} e^{-(\kappa/2)|k_1-j_1|}. \] (3.30)
Therefore, for all \( j, k \in \mathbb{Z} \times \mathbb{Z}_+ \), (3.26) follows from (3.28), (3.29) and (3.30) by setting \( r := \max\{r_0 e^{\kappa/2}, e^{-\kappa/2}\} \) and \( C := \max\{C_1, C_2\}. \)

Using (3.12) for \( k_2 = -1 \), we deduce that (3.26) also remains valid (with an updated constant \( C \)) for \( k \in \mathbb{Z} \times \{-1\}. \)

The last lemma and the fact that \( M \) has compact support readily imply (3.10), completing the proof of Theorem 3.3. □

The method of proof of Theorem 3.3 can also be applied when the coupled systems (3.12) and (3.13) are replaced respectively by (3.6) and
\[ \sum_{k \in \mathbb{Z}^2} a_k M(j-k) = y_j, \quad j \in \mathbb{Z} \times \mathbb{Z}_+. \] (3.31)
for an arbitrary bounded data sequence \( \{y_j : j \in \mathbb{Z} \times \mathbb{Z}_+\} \). In this case, we obtain the following result.

**Corollary 3.7.** For any bounded sequence \( \{y_j\}_{j \in \mathbb{Z} \times \mathbb{Z}_+} \), there exists a unique solution \( \{a_k\}_{k \in \mathbb{Z}^2} \) of the coupled systems of difference equations (3.6) and (3.31) such that \( \{a_k\}_{k \in \mathbb{Z} \times \mathbb{Z}_+} \) is bounded. The corresponding semi-cardinal interpolant \( s := s_y \) defined by (3.4) admits the Lagrange representation
\[ s_y(x) = \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} y_j L_j(x), \quad x \in \mathbb{R} \times \mathbb{R}_+, \] (3.32)
and satisfies
\[ \|s_y\|_{L^p(\mathbb{R} \times \mathbb{R}_+)} \leq C \|y\|_p, \quad y \in L^p(\mathbb{Z} \times \mathbb{Z}_+), \quad p \in [1, \infty], \] (3.33)
for some constant \( C \) that depends only on \( p \).

**Proof.** Using (3.3), the system (3.31) is equivalent to
\[ a_{k-(0,1)} + a_{k-(1,1)} + a_{k-(1,0)} + 6a_k + a_{k+(1,0)} + a_{k+(0,1)} + a_{k+(1,1)} = 12y_k, \quad k \in \mathbb{Z} \times \mathbb{Z}_+. \]
Therefore the first statement of the corollary follows by the same arguments—based on the strict positivity of the symbol (3.17)—that proved the existence of a unique solution, bounded on \( \mathbb{Z} \times \mathbb{Z}_+ \), of the coupled systems of difference equations (3.12) and (3.13).

Next, for a bounded sequence \( \{y_j\}_{j \in \mathbb{Z} \times \mathbb{Z}_+} \), by Theorem 3.3 the series of the right-hand side of (3.32), denoted by \( S_y \), is absolutely and uniformly convergent on compact sets of \( \mathbb{R} \times \mathbb{R}_+ \), and satisfies \( S_y(j) = y_j, j \in \mathbb{Z} \times \mathbb{Z}_+ \). In addition, using the representation (3.11), it follows that \( S_y \) admits a box-spline series representation on \( \mathbb{R} \times \mathbb{R}_+ \), in which the coefficient of index \( k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1} \) is given by the absolutely convergent series
\[ \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} y_j \mu_{jk}. \]
By Lemma 3.6 and the boundedness of \( \{y_j\}_{j \in \mathbb{Z} \times \mathbb{Z}_+} \), the sequence of box-spline series coefficients of \( S_y \) supported on \( \mathbb{Z} \times \mathbb{Z}_+ \) is bounded. On the other hand, the box-spline representation of \( S_y \) can uniquely be augmented with translates...
\(M(\cdot - k), k \in \mathbb{Z} \times \mathbb{Z}_{\leq -2}\), whose coefficients satisfy (3.6). By the unicity property from the first part of the corollary, we deduce that \(s_j = S_k\) on \(\mathbb{R} \times \mathbb{R}_+\), i.e. (3.32) holds.

It is now straightforward to establish (3.33) by arguments similar to those used in [11, (IV.12)] for the corresponding cardinal interpolation problem. □

4. Approximation order

In this section, the approximation order under scaling of the semi-cardinal Lagrange scheme (3.32) will be obtained from two ingredients: the localization relations (3.10) and the polynomial reproduction of linear polynomials by the scheme (3.32). To investigate the latter property, note that even when the data sequence from two ingredients: the localization relations (3.10) and the polynomial reproduction of linear polynomials by the scheme (3.32). For any \(x \in \mathbb{R} \times \mathbb{R}_+\) and any pair \((m_1, m_2) \in \mathbb{Z}_+^2\), we have

\[
\sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} j_1^{m_1} j_2^{m_2} L_j(x) = x_1^{m_1} x_2^{m_2}, \quad x \in \mathbb{R} \times \mathbb{R}_+.
\]

Moreover, the formula does not hold for \((m_1, m_2) = (0, 2)\).

Proof. As already observed, the above sum is absolutely convergent due to (3.10). For any \(x \in \mathbb{R} \times \mathbb{R}_+\) and any pair \((m_1, m_2) \in \mathbb{Z}_+^2\), we have

\[
\sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} j_1^{m_1} j_2^{m_2} L_j(x) = \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} j_1^{m_1} j_2^{m_2} \sum_{k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1}} \mu_{jk} M(x - k) = \sum_{k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1}} \sigma_k^{(m_1, m_2)} M(x - k),
\]

where

\[
\sigma_k^{(m_1, m_2)} := \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} j_1^{m_1} j_2^{m_2} \mu_{jk}, \quad k \in \mathbb{Z} \times \mathbb{Z}_{\geq -1},
\]

is absolutely convergent by Lemma 3.6. Note that the interchange of summation signs in the second equality of (4.2) is permitted due to the compact support of \(M\), which causes the sums over \(k\) to have at most twelve terms.

Consider the case \(m_1 = 0, m_2 \in \{0, 1\}\). For each \(k \in \mathbb{Z} \times \mathbb{Z}_+\), the absolute convergence of (4.3) validates the re-arrangement

\[
\sigma_k^{(0, m_2)} = \sum_{j_2=0}^{\infty} j_2^{m_2} \sum_{j_1 \in \mathbb{Z}} \mu_{j(k_1, k_2)},
\]

In order to compute the sum over \(j_1\), we use (3.24) and (3.25). Note that, since \(\hat{\mu}_{jk_2}(\theta) e^{ij_1\theta}\) is an expression independent of \(j_1\), it follows that, for each fixed \(j_2 \geq 0\), the set \(\{\mu_{j(k_1, k_2)}\}_{j_1 \in \mathbb{Z}}\) is the absolutely summable sequence of Fourier coefficients of this expression. Thus we deduce

\[
\sum_{j_1 \in \mathbb{Z}} \mu_{j(k_1, k_2)} = [e^{ij_1\theta} \hat{\mu}_{jk_2}(\theta)]_{\theta=0} = \gamma(0) \left[ 1 - \rho^2 \min[k_2, j_2](0) + \frac{\beta(0) \delta_{j_20}}{\gamma(0)} \right] \rho^{j_2 - k_21}(0).
\]

\[
\sum_{j_1 \in \mathbb{Z}} \mu_{j(k_1, k_2)} = \gamma(0) \left[ 1 - \rho^2 \min[k_2, j_2](0) + \frac{\beta(0) \delta_{j_20}}{\gamma(0)} \right] \rho^{j_2 - k_21}(0).
\]
Since $\gamma(0) = \sqrt{3}$, $\rho(0) = \sqrt{3} - 2 = \lambda$ and $\beta(0) = 1$, comparing (4.5) with formula (2.12), we have

$$
\sum_{j_1 \in \mathbb{Z}} \mu_j(k_1,k_2) = \mu_{j_2,k_2},
$$

(4.6)

where in the notation of Section 2, $\mu_{j_2,k_2}$ is the B-spline coefficient of a univariate semi-cardinal Lagrange function. Thus, for any $k \in \mathbb{Z} \times \mathbb{Z}_+$, (4.4) and (2.12) imply

$$
\sigma_{k_{0,m_2}}(j_2) = \sum_{j_2=0}^{\infty} j_2 m_2 \mu_{j_2,k_2} = k_2 m_2, \quad m_2 \in \{0, 1\}.
$$

(4.7)

By summing Eqs. (3.12) over $j \in \mathbb{Z} \times \mathbb{Z}_+$, we deduce that the last equality also holds for $k \in \mathbb{Z} \times \{-1\}$.

Next, let $(m_1, m_2) = (1, 0)$ and, for each $k \in \mathbb{Z} \times \mathbb{Z}_+$, choose the re-arrangement

$$
\sigma_{k_{0,0}}(j_2) = \sum_{j_1 \in \mathbb{Z}} j_1 \sum_{j_2=0}^{\infty} \mu_{j_1,k_2}.
$$

(4.8)

If $k_2 = 0$, then using (3.24), (3.25) and $\tilde{\mu}_{j_0}(\theta) \equiv 0$ for $j_2 \geq 1$, we obtain

$$
\sum_{j_2=0}^{\infty} \mu_{j_1,0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(\theta) e^{i(k_1-j_1)\theta} d\theta.
$$

Thus, since $\beta(0) = 1$, $\beta'(0) = 0$, and both $\beta$ and its derivative $\beta'$ admit absolutely convergent Fourier expansions, we deduce

$$
\sum_{j_1 \in \mathbb{Z}} j_1 \sum_{j_2=0}^{\infty} \mu_{j_1,0} = -i \frac{d}{d\theta} [\beta(\theta) e^{i k_1 \theta}]_{\theta=0} = -i \beta'(0) + k_1 \beta(0) = k_1.
$$

(4.9)

On the other hand, if $k_2 \geq 1$, then (3.24) and (3.25) imply

$$
\sum_{j_2=0}^{\infty} \mu_{j_1,k_2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(\theta) \rho^{k_2}(\theta) e^{i(k_1-j_1)\theta} d\theta + \frac{1}{2\pi} \sum_{j_2=1}^{\infty} \int_{-\pi}^{\pi} \tilde{\mu}_{j_2}(\theta) e^{i k_1 \theta} d\theta.
$$

From (3.24) it follows that $\sum_{j_2=1}^{\infty} \tilde{\mu}_{j_2}(\theta)$ is a uniformly and absolutely convergent series of functions for $\theta \in [-\pi, \pi]$, since $|\rho(\theta)|$ is bounded above by a constant smaller than 1. Thus, after permuting the summation sign with the integral in the last term of the above display, we use (3.24) to compute

$$
\sum_{j_2=1}^{\infty} G_{j_2,k_2}(\theta) \rho^{k_2-j_2}(\theta) = \sum_{j_2=1}^{k_2} + \sum_{j_2=k_2+1}^{\infty}
$$

$$
= - \left[ 1 - \rho^{k_2}(\theta) \right] \frac{2(1 + e^{-i\theta}) + (e^{-i\theta} + 6 + e^{i\theta}) \rho(\theta)}{2(e^{-i\theta} + 6 + e^{i\theta}) \rho(\theta)}.
$$

Hence, setting

$$
g(\theta) := \beta(\theta) \rho^{k_2}(\theta) - \gamma(\theta)[1 - \rho^{k_2}(\theta)] \frac{2(1 + e^{-i\theta}) + (e^{-i\theta} + 6 + e^{i\theta}) \rho(\theta)}{2(e^{-i\theta} + 6 + e^{i\theta}) \rho(\theta)},
$$

we obtain

$$
\sum_{j_2=0}^{\infty} \mu_{j_1,k_2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{i(k_1-j_1)\theta} d\theta.
$$
Note that \( g \) and its derivative \( g' \) admit absolutely convergent Fourier expansions, which implies

\[
\sum_{j_1 \in \mathbb{Z}} \sum_{j_2=0}^{\infty} \mu_{j(k_1,k_2)} = -i \frac{d}{d\theta}[g(\theta)e^{ik_1\theta}]_{\theta=0} = -ig'(0) + k_1g(0) = k_1,
\]

(4.10)
since \( g(0) = 1 \) and \( g'(0) = 0 \). The last value is obtained from

\[
g'(0) = k_2 \lambda^{k_2-1} \rho'(0) - \gamma'(0)[1 - \lambda^{k_2}] \frac{1 + 2\lambda}{3\lambda} + \sqrt{3} k_2 \lambda^{k_2-1} \rho'(0) \frac{1 + 2\lambda}{3\lambda} - \frac{\sqrt{3}}{2}[1 - \lambda^{k_2}] \frac{d}{d\theta}\left[ \frac{2(1 + e^{-i\theta}) + (e^{-i\theta} + 6 + e^{i\theta})\rho(\theta)}{(e^{-i\theta} + 4 + e^{i\theta})\rho(\theta)} \right]_{\theta=0},
\]

using \( \rho'(0) = -i\lambda/2, \gamma'(0) = 0 \), and

\[
\frac{d}{d\theta}\left[ \frac{2(1 + e^{-i\theta}) + (e^{-i\theta} + 6 + e^{i\theta})\rho(\theta)}{(e^{-i\theta} + 4 + e^{i\theta})\rho(\theta)} \right]_{\theta=0} = 0.
\]

Therefore (4.8), (4.9) and (4.10) imply

\[
s^1(0) = k_1,
\]

(4.11)
for each \( k = (k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}_+ \), and, as for (4.7), we can use the recurrences (3.12) to deduce that the last equality also holds for \( k \in \mathbb{Z} \times \{ -1 \} \).

We now invoke the polynomial reproduction formula

\[
\sum_{k \in \mathbb{Z}^2} k_1^{m_1} k_2^{m_2} M(x - k) = x_1^{m_1} x_2^{m_2}, \quad x \in \mathbb{R}^2, \quad (m_1, m_2) \in \{(0, 0), (0, 1), (1, 0)\},
\]

(4.12)
which is a Marsden identity for the box-spline \( M \) (see [11, Chapter III]). Therefore the reproduction property (4.1) is the consequence of (4.2), (4.7), (4.11) and (4.12).

Turning to the second assertion of the proposition, let \( m_1 = 0 \) and \( m_2 = 2 \). Then, for any \( k \in \mathbb{Z} \times \mathbb{Z}_+ \), (4.6) and (2.12) imply

\[
s^{(0,2)} = \sum_{j_2=0}^{\infty} \sum_{j_1 \in \mathbb{Z}} \mu_{j(k_1,k_2)} = \sum_{j_2=0}^{\infty} j_2^2 \mu_{j,k} = k_2^2 + \frac{1}{3}(\lambda^{k_2} - 1).
\]

Hence from (4.2) and the Marsden identity [11, Chapter III]

\[
x_2^2 = \sum_{k \in \mathbb{Z}^2} \left( \lambda^{k_2} - \frac{1}{3} \right) M(x - k), \quad x \in \mathbb{R}^2,
\]

we obtain

\[
\sum_{j_2 \times \mathbb{Z}_+} j_2^2 L_j(x) = x_2^2 + \frac{1}{3} \sum_{j \in \mathbb{Z}_+} \lambda^{k_2} M(x - k), \quad x \in \mathbb{R} \times [1, \infty) \text{.}
\]

The conclusion is now a consequence of the linear independence of the translates \( M(\cdot - k), k \in \mathbb{Z}^2 \) (see [11, (II.29)]), which implies that the last infinite sum cannot be identically zero for \( x \in \mathbb{R} \times [1, \infty) \). \( \square \)

Let \( f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) be a continuous data function such that \( |f(x)| \) is of polynomial growth as \( x \in \mathbb{R} \times \mathbb{R}_+ \) and \( \|x\| \to \infty \). For each \( h > 0 \), consider the associated scaled scheme

\[
s_{f,h}(x) = \sum_{j_2 \times \mathbb{Z}_+} f(hj)L_j(h^{-1}x), \quad x \in \mathbb{R} \times \mathbb{R}_+.
\]

(4.13)
absolutely and uniformly convergent on compact subsets of \( \mathbb{R} \times \mathbb{R}_+ \) by (3.10). The following result shows that the approximation order of the semi-cardinal interpolation (4.13) is 2, i.e. half of the known approximation order for cardinal interpolation with the box-spline \( M \) [11, (IV.60)].

**Theorem 4.2.** If \( f \in C^2(\mathbb{R} \times \mathbb{R}_+) \) has all its partial derivatives of total order 2 bounded on \( \mathbb{R} \times \mathbb{R}_+ \), then there exists a constant \( c_f > 0 \) such that

\[
\sup_{x \in \mathbb{R} \times \mathbb{R}_+} |f(x) - s_{f,h}(x)| \leq c_f h^2,
\]

for sufficiently small \( h \). Moreover, there exist functions \( f \in C^\infty(\mathbb{R} \times \mathbb{R}_+) \), for which

\[
\sup_{x \in \mathbb{R} \times \mathbb{R}_+} |f(x) - s_{f,h}(x)| \neq o(h^2) \quad \text{as} \quad h \to 0.
\]

**Proof.** We adapt a convergence technique developed by Jackson [17, Chapter 5] and Powell [19, Section 8] in the context of radial basis functions. Let \( x \in \mathbb{R} \times \mathbb{R}_+ \) be fixed and denote by \( p \) the linear polynomial that matches \( f \) and its first-order partial derivatives at \( x \). Since the partial derivatives of total order 2 of \( f \) are bounded on \( \mathbb{R} \times \mathbb{R}_+ \), we deduce

\[
|p(z) - f(z)| \leq C_1\|z - x\|^2, \quad z \in \mathbb{R} \times \mathbb{R}_+,
\]

for some constant \( C_1 \) that depends on \( f \), but not on \( x \). In particular, it follows that \(|f(x)|\) is of polynomial growth as \( x \in \mathbb{R} \times \mathbb{R}_+ \) and \( \|x\| \to \infty \), so \( s_{f,h} \) is well-defined.

On the other hand, (3.10) implies the existence of a constant \( C_2 \) such that

\[
|L_j(z)| \leq C_2(1 + \|z - j\|)^{-5}, \quad z \in \mathbb{R} \times \mathbb{R}_+, \ j \in \mathbb{Z} \times \mathbb{Z}_+.
\]

Thus, using the polynomial reproduction property (4.1), we have

\[
|f(x) - s_{f,h}(x)| = |p(x) - s_{f,h}(x)| = \left| \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} [p(hj) - f(hj)]L_j(h^{-1}x) \right| \leq C_1 C_2 \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} \|hj - x\|^2(1 + \|h^{-1}x - j\|)^{-5} \leq C_1 C_2 h^2 \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} (1 + \|h^{-1}x - j\|)^{-3}.
\]

Since \( h^{-1}x \in \mathbb{R} \times \mathbb{R}_+ \) for \( h > 0 \), and

\[
C_3 := \sup_{z \in \mathbb{R} \times \mathbb{R}_+} \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} (1 + \|z - j\|)^{-3} < \infty,
\]

we obtain the estimate (4.14) with \( c_f := C_1 C_2 C_3 \).

For the second part of the proof, let \( q(x) := x_2^2 \) and assume that (4.15) does not hold for \( f := q \). Then fixing \( x \neq 0 \) in \( \mathbb{R} \times \mathbb{R}_+ \), it follows in particular that

\[
|q(hx) - s_{q,h}(hx)| = o(h^2) \quad \text{as} \quad h \to 0.
\]

Therefore (4.13) and the homogeneity property \( q(hx) = h^2 q(x) \) imply

\[
\left| q(x) - \sum_{j \in \mathbb{Z} \times \mathbb{Z}_+} q(j)L_j(x) \right| = o(1) \quad \text{as} \quad h \to 0.
\]

The left-hand side of this relation is independent of \( h \), so it must be zero. Since \( x \) was arbitrary in \( \mathbb{R} \times \mathbb{R}_+ \), this contradicts the second assertion of Proposition 4.1 and establishes (4.15) for \( f := q \). \( \square \)
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References