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Deriving amplitude equations for weakly-nonlinear oscillators and their generalizations

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Dedicated to Professor Roderick S.C. Wong, Grand Master of Asymptotics, on the occasion of his 60th birthday

Abstract

Results by physicists on renormalization group techniques have recently sparked interest in the singular perturbations community of applied mathematicians. The survey paper, [Phys. Rev. E 54(1) (1996) 376–394], by Chen et al. demonstrated that many problems which applied mathematicians solve using disparate methods can be solved using a single approach. Analysis of that renormalization group method by Mudavanhu and O'Malley [Stud. Appl. Math. 107(1) (2001) 63–79; SIAM J. Appl. Math. 63(2) (2002) 373–397], among others, indicates that the technique can be streamlined. This paper carries that analysis several steps further to present an amplitude equation technique which is both well adapted for use with a computer algebra system and easy to relate to the classical methods of averaging and multiple scales.

Keywords: Singular perturbations; Amplitude equations; Multiple scales; Averaging

1. Introduction

Many different techniques exist for solving singular perturbation problems. Matched asymptotic expansions, the WKBJ method, the Poincaré–Lindstedt procedure, the methods of averaging and multiple

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scales, to name but a few, were all developed to solve special classes of singular perturbation problems. These asymptotic methods illustrate the varied history of applied mathematics, since they each evolved from the need to solve a specific type of problem of physical significance. The tendency to asymptotically solve many different types of problems using many different techniques survives to this day, even though more efficient and ubiquitous procedures may be available. This, unfortunately, gives the impression that our arsenal of perturbation techniques forms an ad hoc collection of clever tricks practiced primarily by the cognoscenti.

Chen et al. [4], present a renormalization group method which they use to solve many types of singular perturbation problems, superseding the traditional methods of applied mathematics named above. Their technique is based on generating a naive expansion from a regular perturbation procedure. Having obtained the naive expansion, whose validity is generally limited to an $\mathcal{O}(1)$ time interval, they proceed to eliminate "secular terms" in order to develop a more refined expansion which is uniformly valid on an appropriate extended interval. Essentially, this is done by replacing the constant initial value of the naive expansion with a slowly varying function that is asymptotically selected to cancel secularities and by making other appropriate higher-order modifications. As a basic framework for solving singular perturbation problems, this methodology is promising, though elucidating the underlying ideas requires improved exposition and implementation.

Mudavanhu and O'Malley [13,14], analyze the renormalization group technique of Chen et al. for weakly-nonlinear oscillators and show that it offers many benefits. The renormalization group technique is equivalent to the methods of averaging and multiple scales [15] and closely related to the more efficient use of amplitude equations or normal forms [5,21]. Mudavanhu and O'Malley also offer a simplification of notation and a clarification of methodology to make the renormalization group technique more transparent.

This paper continues the work of [13,14]. We bypass the need to calculate the naive expansion and proceed directly to a pair of equations to eliminate secular terms from the regular perturbation expansion. In the case of weakly nonlinear oscillators, this optimized technique is analogous to averaging and multiple scales. Indeed, the averaged equations and the amplitude equations coincide.

Secular terms often arise in the asymptotic solution of weakly nonlinear oscillators, causing the expansions to not be uniformly valid over (appropriately restricted) long time intervals. Uniform validity can also be destroyed when two or more terms in the asymptotic expansion achieve the same asymptotic order at large times, destroying the implicit assumption that successive terms in a perturbation series are of increasing order with respect to an asymptotic sequence. For autonomous, weakly nonlinear perturbations of harmonic oscillators, this most often occurs when terms proportional to $\sin(t)$ and $\cos(t)$ are present in the forcings of the differential equations that define higher-order terms in the expansion. More generally, resonance occurs when solutions of the leading-order linear operator are present in the forcings of differential equations satisfied by later coefficients.

We begin our analysis by noting that we can transform the weakly nonlinear oscillator equation

$$\ddot{y} + y + \varepsilon f(y, \dot{y}, \varepsilon) = 0$$

into the vector periodic standard form

$$\dot{\mathbf{x}} = \varepsilon \mathbf{F}(\mathbf{x}, t, \varepsilon).$$

There, the only solutions of the limiting leading-order linear system are constants. Thus to remove secular terms we must eliminate constants from the forcings for successive terms of the naive expansion for the solution. The method of averaging was designed to do precisely this by integrating the forcings over a period and removing any nontrivial constant averages that remain. Thus averaging provides a means to remove the troublesome secular terms. Our method refocuses attention on preventing secular terms and allows generalization to equations with a nonperiodic forcing.

Having identified how secular terms arise, we proceed by first introducing a near-identity transformation. This transformation allows the leading-order dynamics to vary on a slower timescale, rather than remaining constant. This new degree of freedom allows us to separate the secular dynamics in the resulting partial differential equation. These secular dynamics correspond to the constants in a Fourier expansion of $\mathbf{F}$. The nonlinear, autonomous evolution equation for the secular parts, which we shall call the *amplitude equation*, determines the primary evolution of the system on a slow timescale. Knowing how secular parts evolve, the differential system determines successive, periodic corrections to these dynamics, thus allowing the asymptotic expansion to remain uniformly valid on long time intervals.

2. Defining the amplitude equation method

Consider the first-order vector system in *standard form*,

$$\dot{\mathbf{x}} = \varepsilon \mathbf{F}(\mathbf{x}, t, \varepsilon) \tag{2.1}$$

on $t \geqslant 0$ with a prescribed initial vector $\mathbf{x}(0)$ and an asymptotically small positive parameter ε. Assume that $\mathbf{F}$ is analytic in $\mathbf{x}$, 2π-periodic in t and possesses an asymptotic series in ε, so that Taylor series and asymptotic power series expansions may both be used in our calculations.

Substitute the regular power series expansion

$$\mathbf{x}_\varepsilon(t) = \mathbf{x}_0(t) + \varepsilon \mathbf{x}_1(t) + \varepsilon^2 \mathbf{x}_2(t) + \cdots$$

into (2.1) and collect terms involving like powers of ε. Then the first few terms $\mathbf{x}_j(t)$ must satisfy the successive linear systems

$$\begin{aligned}
\dot{\mathbf{x}}_0 &= 0,\\
\dot{\mathbf{x}}_1 &= \mathbf{F}(\mathbf{x}_0, t, 0),\\
\dot{\mathbf{x}}_2 &= (\mathbf{x}_1 \cdot \bigtriangledown)\mathbf{F}(\mathbf{x}_0, t, 0) + \mathbf{F}_\varepsilon(\mathbf{x}_0, t, 0),\\
\dot{\mathbf{x}}_3 &= \tfrac{1}{2}(\mathbf{x}_1 \cdot \bigtriangledown)^2 \mathbf{F}(\mathbf{x}_0, t, 0) + (\mathbf{x}_2 \cdot \bigtriangledown)\mathbf{F}(\mathbf{x}_0, t, 0) + (\mathbf{x}_1 \cdot \bigtriangledown)\mathbf{F}_\varepsilon(\mathbf{x}_0, t, 0) + \tfrac{1}{2}\mathbf{F}_{\varepsilon\varepsilon}(\mathbf{x}_0, t, 0)
\end{aligned}$$

and so on, where the gradients are taken with respect to the components of $\mathbf{x}_0$. These equations can, in turn, be integrated directly to yield

$$\begin{aligned}
\mathbf{x}_0(t) &= \mathbf{x}(0),\\
\mathbf{x}_1(t) &= \int_0^t \mathbf{F}(\mathbf{x}_0(s), s, 0)\, \mathrm{d}s,\\
\mathbf{x}_2(t) &= \int_0^t ((\mathbf{x}_1(s) \cdot \bigtriangledown)\mathbf{F}(\mathbf{x}_0(s), s, 0) + \mathbf{F}_\varepsilon(\mathbf{x}_0(s), s, 0))\, \mathrm{d}s,
\end{aligned}$$

$$\mathbf{x}_3(t) = \int_0^t \left(\frac{1}{2}(\mathbf{x}_1(s) \cdot \bigtriangledown)^2 \mathbf{F}(\mathbf{x}_0(s), s, 0) + (\mathbf{x}_2(s) \cdot \bigtriangledown)\mathbf{F}(\mathbf{x}_0(s), s, 0) + (\mathbf{x}_1(s) \cdot \bigtriangledown)\mathbf{F}_\varepsilon(\mathbf{x}_0(s), s, 0) + \frac{1}{2}\mathbf{F}_{\varepsilon\varepsilon}(\mathbf{x}_0(s), s, 0) \right) \mathrm{d}s.$$

Note that if any of the integrands contains *secular* terms, i.e., constants with respect to t, the resulting integrals will become unbounded as $t \to \infty$. These constants are eigenfunctions of the linear operator $\mathrm{d}/\mathrm{d}t$, so their presence in the forcing causes a *resonance*, i.e., unboundedness of the solution. Since we can often prove that the exact solution is actually bounded, the unbounded terms obtained via the regular perturbation procedure are undesirable. Such unbounded asymptotic approximations give the incorrect impression that the true solution is also unbounded.

Experience with multiple scales and averaging suggests that a better assumption is that the leading-order term should not be a constant but instead a function of the *slow time*,

$$\sigma = \varepsilon t.$$

To account for this slow variation at leading order, we introduce the *near-identity* transformation

$$\mathbf{x}_\varepsilon(t, \sigma) = \mathbf{A}_\varepsilon(\sigma) + \varepsilon \mathbf{U}(\mathbf{A}_\varepsilon(\sigma), t, \varepsilon) \tag{2.2}$$

in place of the regular perturbation series. The function $\mathbf{A}_\varepsilon(\sigma)$ is a slowly varying function intended to replace the integration constant $\mathbf{x}(0)$ and $\mathbf{U}$ is a bounded correction. We shall require that $\mathbf{U}$ be analytic in $\mathbf{A}_\varepsilon$, 2π-periodic in t and possess an asymptotic series expansion with respect to ε. We also ask that $\mathbf{A}_\varepsilon(0) = \mathbf{x}(0)$ so that $\mathbf{U}(\mathbf{A}_\varepsilon(0), 0, \varepsilon) = 0$. The higher-order corrections will depend on the fast time, t, as well as the slow time, σ, through the amplitude $\mathbf{A}_\varepsilon(\sigma)$. We shall also require an *amplitude equation*

$$\frac{\mathrm{d}\mathbf{A}_\varepsilon}{\mathrm{d}\sigma} \equiv \mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon), \tag{2.3}$$

to be satisfied, for a yet to be determined function $\mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon)$.

As with multiple scales, we will treat the two timescales σ and t as independent variables. Substituting ansatz (2.2) into system (2.1) and applying the chain rule then implies the partial differential equation

$$\frac{\mathrm{d}\mathbf{A}_\varepsilon}{\mathrm{d}\sigma} + \varepsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\varepsilon} \frac{\mathrm{d}\mathbf{A}_\varepsilon}{\mathrm{d}\sigma} + \frac{\partial \mathbf{U}}{\partial t} = \mathbf{F}(\mathbf{A}_\varepsilon + \varepsilon \mathbf{U}, t, \varepsilon). \tag{2.4}$$

Substituting (2.3) into (2.4) and taking the average over $0 \leqslant t \leqslant 2\pi$ yields the integral equation

$$\mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \left(\mathbf{F}(\mathbf{A}_\varepsilon + \varepsilon \mathbf{U}, s, \varepsilon) - \varepsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\varepsilon} \mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon) \right) \mathrm{d}s, \tag{2.5}$$

since $\mathrm{d}\mathbf{A}_\varepsilon/\mathrm{d}\sigma$ is independent of t, while boundedness of $\mathbf{U}$ requires $\partial \mathbf{U}/\partial t$ to be periodic with a zero average. Alternatively,

$$\mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon) = \frac{1}{2\pi} \left(\mathbf{I} + \frac{\varepsilon}{2\pi} \int_0^{2\pi} \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\varepsilon}(\mathbf{A}_\varepsilon, s, \varepsilon)\, \mathrm{d}s \right)^{-1} \int_0^{2\pi} \mathbf{F}(\mathbf{A}_\varepsilon + \varepsilon \mathbf{U}, s, \varepsilon)\, \mathrm{d}s. \tag{2.6}$$

This isolates the constants which, when integrated, cause resonance. We then deal with them separately from the harmless, oscillatory remainder in (2.4).

Next solve (2.4) for $\partial\mathbf{U}/\partial t$, insert (2.3) and integrate from 0 to t to obtain a second integral equation,

$$\mathbf{U}(\mathbf{A}_\varepsilon, t, \varepsilon) = \int_0^t \left(\mathbf{F}(\mathbf{A}_\varepsilon + \varepsilon\mathbf{U}, s, \varepsilon) - \left(\mathbf{I} + \varepsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\varepsilon} \right) \mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon) \right) \mathrm{d}s, \tag{2.7}$$

which asymptotically provides the bounded, oscillatory correction $\mathbf{U}$ in (2.2) since $\mathbf{H}$ is the average of the periodic function $\mathbf{F} - \varepsilon(\partial\mathbf{U}/\partial\mathbf{A}_\varepsilon)\mathbf{H}$. We shall iterate in this coupled pair of integral equations to define $\mathbf{H}$ and $\mathbf{U}$ asymptotically.

Assuming that $\mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon)$ and $\mathbf{U}(\mathbf{A}_\varepsilon, t, \varepsilon)$ have series expansions

$$\mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon) = \mathbf{H}_0(\mathbf{A}_\varepsilon) + \varepsilon\mathbf{H}_1(\mathbf{A}_\varepsilon) + \varepsilon^2\mathbf{H}_2(\mathbf{A}_\varepsilon) + \cdots \tag{2.8}$$

and

$$\mathbf{U}(\mathbf{A}_\varepsilon, t, \varepsilon) = \mathbf{U}_0(\mathbf{A}_\varepsilon, t) + \varepsilon\mathbf{U}_1(\mathbf{A}_\varepsilon, t) + \varepsilon^2\mathbf{U}_2(\mathbf{A}_\varepsilon, t) + \cdots, \tag{2.9}$$

$\mathbf{F}$ has the resulting expansion

$$\mathbf{F}(\mathbf{A}_\varepsilon + \varepsilon\mathbf{U}, t, \varepsilon) = \mathscr{F}_0(\mathbf{A}_\varepsilon, t) + \varepsilon\mathscr{F}_1(\mathbf{A}_\varepsilon, \mathbf{U}_0, t) + \varepsilon^2\mathscr{F}_2(\mathbf{A}_\varepsilon, \mathbf{U}_0, \mathbf{U}_1, t) + \cdots, \tag{2.10}$$

where the terms on the right hand side are given by the standard Taylor coefficients in ε. In particular, the first three terms of (2.10) are

$$\mathscr{F}_0(\mathbf{A}_\varepsilon, t) = \mathbf{F}(\mathbf{A}_\varepsilon, t, 0),$$

$$\mathscr{F}_1(\mathbf{A}_\varepsilon, \mathbf{U}_0, t) = (\mathbf{U}_0 \cdot \bigtriangledown)\mathbf{F}(\mathbf{A}_\varepsilon, t, 0) + \mathbf{F}_\varepsilon(\mathbf{A}_\varepsilon, t, 0)$$

and

$$\begin{aligned} \mathscr{F}_2(\mathbf{A}_\varepsilon, \mathbf{U}_0, \mathbf{U}_1, t) = {} & \tfrac{1}{2}(\mathbf{U}_0 \cdot \bigtriangledown)^2\mathbf{F}(\mathbf{A}_\varepsilon, t, 0) + (\mathbf{U}_1 \cdot \bigtriangledown)\mathbf{F}(\mathbf{A}_\varepsilon, t, 0) \\ & + (\mathbf{U}_0 \cdot \bigtriangledown)\mathbf{F}_\varepsilon(\mathbf{A}_\varepsilon, t, 0) + \tfrac{1}{2}\mathbf{F}_{\varepsilon\varepsilon}(\mathbf{A}_\varepsilon, t, 0), \end{aligned}$$

where the gradients refer to the components of $\mathbf{A}_\varepsilon$ and the partial derivative $\partial/\partial\varepsilon$ refers to the third argument of $\mathbf{F}$.

Lastly, we define the secular part of a periodic function to be its average over a period. To make the notation compact, we denote this value by

$$\langle g \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(s)\,\mathrm{d}s. \tag{2.11}$$

Then substituting the power series (2.8)–(2.10) into the integral equations (2.5) and (2.7), averaging and collecting terms with like powers of ε, one finds to leading order that

$$\mathbf{H}_0(\mathbf{A}_\varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{F}_0(\mathbf{A}_\varepsilon, s)\,\mathrm{d}s = \langle \mathscr{F}_0(\mathbf{A}_\varepsilon) \rangle \tag{2.12}$$

and

$$\mathbf{U}_0(\mathbf{A}_\varepsilon, t) = \int_0^t (\mathscr{F}_0(\mathbf{A}_\varepsilon, s) - \mathbf{H}_0(\mathbf{A}_\varepsilon))\,\mathrm{d}s. \tag{2.13}$$

These are, of course, well-known results of the method of averaging [19,22]. Equating terms at $\mathcal{O}(\varepsilon^n)$, for each $n \geqslant 1$, we determine the convenient corresponding pair of recursion relations

$$\mathbf{H}_n(\mathbf{A}_\varepsilon) = \langle \mathscr{F}_n(\mathbf{A}_\varepsilon, \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}) \rangle - \sum_{k=0}^{n-1} \left\langle \frac{\partial \mathbf{U}_k}{\partial \mathbf{A}_\varepsilon} \right\rangle \mathbf{H}_{n-k-1}(\mathbf{A}_\varepsilon) \tag{2.14}$$

and

$$\mathbf{U}_n(\mathbf{A}_\varepsilon, t) = \int_0^t \left(\mathscr{F}_n(\mathbf{A}_\varepsilon, \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}, s) - \mathbf{H}_n(\mathbf{A}_\varepsilon) - \sum_{k=0}^{n-1} \frac{\partial \mathbf{U}_k}{\partial \mathbf{A}_\varepsilon} \mathbf{H}_{n-k-1}(\mathbf{A}_\varepsilon) \right) \mathrm{d}s. \tag{2.15}$$

These relations asymptotically determine both bounded, higher-order approximations to the solution (2.2) and higher-order terms in the amplitude equation (2.3). To completely succeed, all that remains is to *solve* the nonlinear initial value problem

$$\frac{\mathrm{d}\mathbf{A}_\varepsilon}{\mathrm{d}\sigma} = \mathbf{H}(\mathbf{A}_\varepsilon, \varepsilon), \quad \mathbf{A}_\varepsilon(0) = \mathbf{x}(0). \tag{2.16}$$

To asymptotically solve (2.16), we insert a regular perturbation series for $\mathbf{A}_\varepsilon(\sigma)$, i.e.

$$\mathbf{A}_\varepsilon(\sigma) = \mathbf{A}_0(\sigma) + \varepsilon \mathbf{A}_1(\sigma) + \varepsilon^2 \mathbf{A}_2(\sigma) + \cdots. \tag{2.17}$$

The limiting problem,

$$\frac{\mathrm{d}\mathbf{A}_0}{\mathrm{d}\sigma} = \mathbf{H}_0(\mathbf{A}_0), \quad \mathbf{A}_0(0) = \mathbf{x}(0), \tag{2.18}$$

is generally fully nonlinear. Wherever it can be solved, later terms $\mathbf{A}_k$, for $k \geqslant 1$, follow successively as solutions of a sequence of linearized problems

$$\frac{\mathrm{d}\mathbf{A}_k}{\mathrm{d}\sigma} = \frac{\partial \mathbf{H}_0}{\partial \mathbf{A}_\varepsilon}(\mathbf{A}_0)\mathbf{A}_k + \boldsymbol{\alpha}_k(\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1}), \quad \mathbf{A}_k(0) = 0, \tag{2.19}$$

where $\boldsymbol{\alpha}_k(\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1})$ is known from inserting the regular perturbation series (2.17) into (2.16) and Taylor expanding for small ε. Using the fundamental matrix $\partial \mathbf{A}_0 / \partial \mathbf{x}(0)$, we have

$$\mathbf{A}_k(\sigma) = \frac{\partial \mathbf{A}_0(\sigma)}{\partial \mathbf{x}(0)} \int_0^\sigma \left[\left(\frac{\partial \mathbf{A}_0(r)}{\partial \mathbf{x}(0)} \right)^{-1} \boldsymbol{\alpha}_k(\mathbf{A}_0(r), \mathbf{A}_1(r), \ldots, \mathbf{A}_{k-1}(r)) \right] \mathrm{d}r, \tag{2.20}$$

defined whenever $\mathbf{A}_0(\sigma)$ is. Its behavior as $\sigma \to \infty$ follows according to various stability hypotheses on $\mathbf{A}_0$. By comparison with the related work of [18,20,21], we should expect the series for $\mathbf{H}$ and $\mathbf{U}$ to diverge as Gevrey series.

If this regular perturbation method should be unsatisfactory, e.g., if some $\mathbf{A}_j(\sigma)$ blows up, one may instead attempt to rescale the independent and dependent variables. Such a rescaled problem may then be solved using the regular perturbation method. As before, the rescaled amplitude equation will be nonlinear, while remaining terms in the series will follow directly in turn from linear equations.

If neither the regular perturbation method nor the rescaling method gives satisfactory results, one could instead integrate the approximate amplitude equation (2.16) numerically. Though typically a last resort for many practitioners in perturbation theory, the numerical integration of (2.16) should be more effective than integrating the original equation directly because the amplitude evolves on a slower time scale. The amplitude equation (2.16) and its truncations are, of course, the results of our analysis and were not initially obvious.

We are only guaranteed to have a local solution to the limiting amplitude equation. This reflects the well-known limitation that results found from multiple scales and averaging are generally restricted to time intervals where σ is finite. The possibility of performing further transformations on the amplitude equation suggests that the solution, $\mathbf{x}_\varepsilon$, might be obtainable on an even longer time interval where $\mathbf{A}_\varepsilon$ thereby becomes defined.

Before proceeding, we want to point out one way to obtain the standard form (2.1) from a given perturbed system of equations. Suppose one is given

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, t) + \varepsilon \mathbf{g}(\mathbf{z}, t, \varepsilon), \tag{2.21}$$

where both $\mathbf{f}$ and $\mathbf{g}$ are smooth. The basic existence theorem implies that when a solution for $\varepsilon = 0$ exists, it can be represented as

$$\mathbf{z} = \mathbf{z}(t, \mathbf{z}(0)), \tag{2.22}$$

where the parameter $\mathbf{z}(0)$ is the initial vector.

To transform (2.21), we exploit the nonlinear version of variation of parameters [16]. The essence of this procedure is to assume that the integration constant, $\mathbf{x}$, in

$$\mathbf{z} = \mathbf{z}(t, \mathbf{x}) \tag{2.23}$$

varies with time. This suggests a change of variables from $\mathbf{z}$ to $\mathbf{x}$. Inserting (2.23) into (2.21), the chain rule implies that

$$\frac{\partial \mathbf{z}}{\partial t} + \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{z}(t, \mathbf{x}), t) + \varepsilon \mathbf{g}(\mathbf{z}(t, \mathbf{x}), t, \varepsilon),$$

where the matrix, $\partial \mathbf{z}/\partial \mathbf{x}$, of first partial derivatives is (at least) locally invertible, due to uniqueness. Since $\mathrm{d}\mathbf{z}/\mathrm{d}t = \mathbf{f}(\mathbf{z}, t)$, we obtain

$$\dot{\mathbf{x}} = \varepsilon \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right)^{-1} \mathbf{g}(\mathbf{z}(t, \mathbf{x}), t, \varepsilon),$$

which is in the standard form (2.1). Related *osculating element* techniques occur frequently in the classical celestial mechanics literature [1].

3. Weakly nonlinear oscillators

The general autonomous, weakly nonlinear oscillator equation,

$$\ddot{y} + y + \varepsilon f(y, \dot{y}, \varepsilon) = 0, \tag{3.1}$$

is a differential equation for which the methods of multiple scales and averaging are well adapted. The examples below show that our amplitude equation technique is at least as accurate as they. We shall, without loss of generality, use the initial values $y(0) = 1$ and $\dot{y}(0) = 0$. Josic and Peles [9], consider synchronization of an array of such oscillators.

Before applying the amplitude equation method, we transform (3.1) to the standard form (2.1). This can be accomplished either by utilizing the variation of parameters procedure already described or by following van der Pol's lead and changing to polar coordinates [8]. Choosing the latter, let

$$y = \rho\cos(\eta + \phi), \quad \dot{y} = -\rho\sin(\eta + \phi), \tag{3.2}$$

where ρ and ϕ vary (slowly) with t and $\eta \equiv (1 + \varepsilon^2\omega(\varepsilon))t$ is a strained coordinate for the fast scale. We will choose the coefficients of the power series $\omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \cdots$ to ensure that higher-order approximate solutions behave nicely at infinity. By enforcing the consistency condition $(\mathrm{d}/\mathrm{d}t)y = \dot{y}$ and using (3.1), one gets two first-order ODEs which are solved as a linear algebraic system. This yields an equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \varepsilon\mathbf{F}(\mathbf{x}, \eta, \varepsilon), \tag{3.3}$$

which is in standard form for $\mathbf{x} = \binom{\rho}{\phi}$, where the forcing is given by

$$\mathbf{F}(\mathbf{x}, \eta, \varepsilon) = \begin{pmatrix} \sin(\eta + \phi) \\ \dfrac{1}{\rho}\cos(\eta + \phi) \end{pmatrix} f(\rho\cos(\eta + \phi), -\rho\sin(\eta + \phi), \varepsilon) - \varepsilon\begin{pmatrix} 0 \\ \omega(\varepsilon) \end{pmatrix}. \tag{3.4}$$

We note that the addition of the strained coordinate η necessitates only a small change in the solution procedure. In particular, as a result of the chain rule, one must replace the recursion formula (2.15) with the modified formula

$$\begin{aligned} \mathbf{U}_n(\mathbf{A}_\varepsilon, \eta) = \int_0^\eta \Bigg(& \mathscr{F}_n(\mathbf{A}_\varepsilon, \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}, s) - \mathbf{H}_n(\mathbf{A}_\varepsilon) \\ & - \sum_{k=0}^{n-1} \frac{\partial \mathbf{U}_k}{\partial \mathbf{A}_\varepsilon} \mathbf{H}_{n-k-1}(\mathbf{A}_\varepsilon) \Bigg) \mathrm{d}s - \sum_{k=0}^{n-2} \mathbf{U}_k(\mathbf{A}_\varepsilon, \eta)\, \omega_{n-k-2}, \end{aligned} \tag{3.5}$$

when equating coefficients at $\mathcal{O}(\varepsilon^n)$ for $n \geqslant 2$.

Consider as a first example the Duffing equation

$$\ddot{y} + y + \varepsilon y^3 = 0. \tag{3.6}$$

After transforming (3.6) into (3.4) for the polar coordinates ρ and ϕ and applying the recursion formulas (2.14) and (2.15), one finds that the corresponding amplitude, $\mathbf{A}_\varepsilon = \binom{R_\varepsilon}{\Phi_\varepsilon}$, evolves according to the system

$$\frac{\mathrm{d}R_\varepsilon}{\mathrm{d}\sigma} = \mathcal{O}(\varepsilon^4), \tag{3.7}$$

$$\frac{\mathrm{d}\Phi_\varepsilon}{\mathrm{d}\sigma} = \frac{3}{8}R_\varepsilon^2 - \varepsilon\left(\omega_0 + \frac{21}{256}R_\varepsilon^4\right) - \varepsilon^2\left(\omega_1 - \frac{81}{2048}R_\varepsilon^6\right) - \varepsilon^3\left(\omega_2 + \frac{6549}{262144}R_\varepsilon^8\right) + \mathcal{O}(\varepsilon^4). \tag{3.8}$$

Solving the amplitude equations (3.7), (3.8) is particularly simple. Using $R_\varepsilon(0)=1$, one finds that

$$R_\varepsilon(\sigma)=1+\mathcal{O}(\varepsilon^4).$$

Then substituting this into the phase equation (3.8), we find we must choose $\omega_0=-\frac{21}{256}$, $\omega_1=\frac{81}{2048}$ and $\omega_2=-\frac{6549}{262144}$ in order to simplify the phase equation and ensure no more than linear growth in the phase as $\sigma\to\infty$. This leaves the equation

$$\frac{d\Phi_\varepsilon}{d\sigma}=\frac{3}{8}+\mathcal{O}(\varepsilon^4),$$

whose solution with $\Phi_\varepsilon(0)=0$ is

$$\Phi_\varepsilon(\sigma)=\tfrac{3}{8}\sigma+\mathcal{O}(\varepsilon^4\sigma).$$

Then using the corrections $\mathbf{U}_i(\mathbf{A}_\varepsilon,\eta)$, for $i=0,1,2$, we may define the attracting limit cycle to $\mathcal{O}(\varepsilon^3)$ as

$$\begin{aligned} y(t;\varepsilon)=\ &\cos(\eta+\Phi_\varepsilon)-\varepsilon(\tfrac{1}{32}\cos(\eta+\Phi_\varepsilon)-\tfrac{1}{32}\cos(3(\eta+\Phi_\varepsilon)))\\ &+\varepsilon^2\left(\frac{23}{1024}\cos(\eta+\Phi_\varepsilon)-\frac{3}{128}\cos(3(\eta+\Phi_\varepsilon))+\frac{1}{1024}\cos(5(\eta+\Phi_\varepsilon))\right)\\ &-\varepsilon^3\left(\frac{547}{32768}\cos(\eta+\Phi_\varepsilon)-\frac{297}{16384}\cos(3(\eta+\Phi_\varepsilon))\right.\\ &\left.+\frac{3}{2048}\cos(5(\eta+\Phi_\varepsilon))-\frac{1}{32768}\cos(7(\eta+\Phi_\varepsilon))\right)+\mathcal{O}(\varepsilon^4), \end{aligned}\tag{3.9}$$

where

$$\eta+\Phi_\varepsilon=\left(1+\frac{3}{8}\varepsilon-\frac{21}{256}\varepsilon^2+\frac{81}{2048}\varepsilon^3-\frac{6549}{262144}\varepsilon^4\right)t+\mathcal{O}(\varepsilon^5t).$$

This approximation is valid for bounded $\sigma=\varepsilon t$, in precise agreement with the approximations found using multiple scales or the Poincaré–Lindstedt method. The exact solution of the Duffing equation can be expressed in terms of elliptic functions and is periodic with an ε-dependent frequency. Because we only approximate the frequency with a power series, we attain only orbital stability, rather than asymptotic stability, as $t\to\infty$.

As a second example, consider the van der Pol equation

$$\ddot{y}+y+\varepsilon(y^2-1)\dot{y}=0.\tag{3.10}$$

After introducing polar coordinates and applying the recursion formulas, one finds that the amplitude, $\mathbf{A}_\varepsilon=\binom{R_\varepsilon}{\Phi_\varepsilon}$, evolves as the triangular system

$$\frac{dR_\varepsilon}{d\sigma}=\frac{1}{2}\left(R_\varepsilon-\frac{1}{4}R_\varepsilon^3\right)+\varepsilon^2\left(-\frac{5}{128}R_\varepsilon^3+\frac{17}{768}R_\varepsilon^5-\frac{37}{12288}R_\varepsilon^7\right)+\mathcal{O}(\varepsilon^4),\tag{3.11}$$

$$\frac{d\Phi_\varepsilon}{d\sigma}=-\varepsilon\left(\omega_0+\frac{1}{8}-\frac{11}{32}R_\varepsilon^2+\frac{21}{256}R_\varepsilon^4\right)-\varepsilon^2\omega_1+\mathcal{O}(\varepsilon^3).\tag{3.12}$$

The first correction term, $\mathbf{U}_0(\mathbf{A}_\varepsilon, \eta)$, defines the $\mathcal{O}(\varepsilon)$ solution

$$\rho = R_\varepsilon + \varepsilon(-\tfrac{1}{4}R_\varepsilon \sin(2(\eta + \Phi_\varepsilon)) + \tfrac{1}{32}R_\varepsilon^3 \sin(4(\eta + \Phi_\varepsilon))) + \mathcal{O}(\varepsilon^2), \tag{3.13}$$

$$\phi = \Phi_\varepsilon + \varepsilon(\tfrac{1}{4} - \tfrac{5}{32}R_\varepsilon^2 - \tfrac{1}{4}(1 - \tfrac{1}{2}R_\varepsilon^2)\cos(2(\eta + \Phi_\varepsilon)) + \tfrac{1}{32}R_\varepsilon^2 \cos(4(\eta + \Phi_\varepsilon))) + \mathcal{O}(\varepsilon^2), \tag{3.14}$$

where the amplitude equations may again be asymptotically solved by introducing a regular perturbation series

$$R_\varepsilon(\sigma) = R_0(\sigma) + \mathcal{O}(\varepsilon^2). \tag{3.15}$$

The amplitude equations (3.11), (3.12) are actually sufficient to define an $\mathcal{O}(\varepsilon^2)$ approximation (when we also include $\mathbf{U}_1$, which is also needed to get the amplitude equations above), but we omit the extra algebra as it adds little insight. Once we obtain R_ε, the phase Φ_ε will follow immediately by integration because of the decoupling. Using $R_0(0) = 1$, one finds that the limiting amplitude is

$$R_0(\sigma) = \frac{2}{\sqrt{1 + 3\mathrm{e}^{-\sigma}}}, \tag{3.16}$$

the solution of the limiting Bernoulli equation. Note that taking the limit of the right-hand side of (3.12) as $\sigma \to \infty$ (equivalently, letting $R_0 \to 2$) implies that the phase will become unbounded unless we choose $\omega_0 = -\frac{1}{16}$. Similarly, we determine $\omega_1 = 0$ and the phase is then given asymptotically by $\Phi_\varepsilon = \varepsilon\Phi_1 + \mathcal{O}(\varepsilon^3\sigma)$, where

$$\Phi_1 = \varepsilon(\tfrac{21}{64}(R_0^2 - 1) - \tfrac{1}{8}\ln(R_0)). \tag{3.17}$$

Substituting (3.15) and (3.17) into (3.13) and (3.14) provides

$$y(t;\varepsilon) = R_0 \cos\eta + [(-\tfrac{11}{64}R_0 - \tfrac{7}{64}R_0^3 + \tfrac{1}{8}R_0\ln(R_0))\sin\,\eta - \tfrac{1}{32}R_0^3 \sin 3\eta] + \mathcal{O}(\varepsilon^2),$$

where

$$\eta = (1 - \tfrac{1}{16}\varepsilon^2)t + \mathcal{O}(\varepsilon^4 t).$$

This approximation agrees exactly with the multiple scales solution and is valid for bounded σ.

Although no exact solution of the van der Pol equation is known, the solution asymptotically exponentially approaches a stable limit cycle with radius $R_\varepsilon(\infty) = 2 + \mathcal{O}(\varepsilon^2)$. This limiting periodic solution can be approximated using the Poincaré–Lindstedt procedure. Because we only know the frequency asymptotically, we will again have orbital stability as $t \to \infty$.

A third, more intricate example is

$$\ddot{y} + y + \varepsilon(\dot{y}^3 + 3\varepsilon\dot{y}) = 0. \tag{3.18}$$

Historically, (3.18) was introduced by Morrison [12] as a counterexample to the method of multiple scales, which fails to capture the solution's extremely slow decay to its trivial rest point. In [10, Section 4.2.4], Kevorkian and Cole acknowledge the method's inability to effectively resolve such behavior when the forcing, f, in (3.1) depends on ε.

Using polar coordinates and the recursions, one finds that the amplitude, $\mathbf{A}_\varepsilon = \binom{R_\varepsilon}{\Phi_\varepsilon}$, evolves according to the decoupled system

$$\frac{\mathrm{d}R_\varepsilon}{\mathrm{d}\sigma} = -\frac{3}{8}R_\varepsilon^3 - \varepsilon\frac{3}{2}R_\varepsilon - \varepsilon^2\frac{45}{4096}R_\varepsilon^7 - \varepsilon^3\frac{9}{256}R_\varepsilon^5 + \mathcal{O}(\varepsilon^4), \tag{3.19}$$

$$\frac{\mathrm{d}\Phi_\varepsilon}{\mathrm{d}\sigma} = -\varepsilon\left(\omega_0 + \frac{45}{256}R_\varepsilon^4\right) - \varepsilon^2\left(\omega_1 + \frac{27}{32}R_\varepsilon^2\right) - \varepsilon^3\left(\omega_2 + \frac{9}{8} - \frac{6021}{262144}R_\varepsilon^8\right) + \mathcal{O}(\varepsilon^4). \tag{3.20}$$

We use the first correction, $\mathbf{U}_0(\mathbf{A}_\varepsilon, \eta)$, to express the solution as

$$\rho = R_\varepsilon + \varepsilon R_\varepsilon^3(\tfrac{1}{4}\sin(2(\eta + \Phi_\varepsilon)) - \tfrac{1}{32}\sin(4(\eta + \Phi_\varepsilon))) + \mathcal{O}(\varepsilon^2) \tag{3.21}$$

$$\phi = \Phi_\varepsilon + \varepsilon R_\varepsilon^2(-\tfrac{3}{32} + \tfrac{1}{8}\cos(2(\eta + \Phi_\varepsilon)) - \tfrac{1}{32}\cos(4(\eta + \Phi_\varepsilon))) + \mathcal{O}(\varepsilon^2). \tag{3.22}$$

Using a regular perturbation series $R(\sigma) = R_0(\sigma) + \varepsilon R_1(\sigma) + \mathcal{O}(\varepsilon^2)$, the leading-order solution of the amplitude equation (3.19),

$$R_0(\sigma) = \frac{2}{\sqrt{3\sigma + 4}},$$

decays algebraically to 0 as $\sigma \to \infty$. The first-order solution, $R_1(R_0) = R_0^3 - 1/R_0$, however, blows up like $\sqrt{\sigma}$ because of the term $1/R_0$. This blowup, which is identical with the result given by multiple scales, is unsatisfactory. If we now rescale the amplitude equations by introducing new variables

$$Z = \varepsilon^\alpha R, \quad \kappa = \varepsilon^\beta \sigma$$

and use dominant balance arguments, we find the only consistent balance leading to a new timescale is the choice $\alpha = -\frac{1}{2}$ and $\beta = 1$. This implies the rescaled amplitude equations

$$\frac{\mathrm{d}Z}{\mathrm{d}\kappa} = -\frac{3}{8}Z^3 - \frac{3}{2}Z + \varepsilon^4\left(\frac{81}{128}Z^3 - \frac{9}{256}Z^5 - \frac{45}{4096}Z^7\right) + \mathcal{O}(\varepsilon^5), \tag{3.23}$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\kappa} = -\omega_0 - \varepsilon\omega_1 + \varepsilon^2\left(\omega_2 + \frac{9}{8} + \frac{27}{32}Z^2 + \frac{45}{256}Z^4\right) - \varepsilon^3\omega_3 - \varepsilon^4\omega_4 + \mathcal{O}(\varepsilon^5), \tag{3.24}$$

where we have redefined the phase as $\Psi(\kappa) \equiv \Phi(\frac{\kappa}{\varepsilon})$. Now we solve the rescaled amplitude equations using the regular perturbation series

$$Z(\kappa) = Z_0(\kappa) + \varepsilon^4 Z_1(\kappa) + \mathcal{O}(\varepsilon^8).$$

The leading-order initial value problem is

$$\frac{\mathrm{d}Z_0}{\mathrm{d}\kappa} = -\frac{3}{8}Z_0^3 - \frac{3}{2}Z_0, \quad Z_0(0) = \frac{1}{\sqrt{\varepsilon}}.$$

Its solution,

$$Z_0(\kappa) = \frac{2}{\sqrt{(1+4\varepsilon)\mathrm{e}^{3\kappa} - 1}},$$

decays exponentially to the trivial rest point as $\kappa \to \infty$. To eliminate unbounded growth in the phase, we set $\omega_0 = \omega_1 = \omega_3 = \omega_4 = 0$ and $\omega_2 = -\frac{9}{8}$. Integrating (3.24) then implies that

$$\Psi(\kappa) = \varepsilon^2\left(\tfrac{15}{64}(Z_0^2 - 1) + \tfrac{3}{16}\ln\left(\tfrac{1}{5}(Z_0^2 + 4)\right)\right) + \mathcal{O}(\varepsilon^6\kappa).$$

We may then write the solution as

$$y(t;\varepsilon) = \sqrt{\varepsilon}Z_0\left(\cos\eta + \varepsilon^2\left(\left(\tfrac{33}{64}Z_0^2 + \tfrac{81}{64} + \tfrac{3}{16}\ln\left(\tfrac{1}{5}(Z_0^2+4)\right)\right)\sin\eta + \tfrac{1}{32}Z_0^3\sin 3\eta\right) + \mathcal{O}(\varepsilon^4)\right),$$

where

$$\eta = \left(1 - \tfrac{9}{8}\varepsilon^4\right)t + \mathcal{O}(\varepsilon^8 t).$$

The expansion is not singular in the $\varepsilon \to 0$ limit since $\sqrt{\varepsilon}Z_0$ is bounded. Since the solutions to the rescaled amplitude equations decay exponentially to their trivial rest point, the approximation is valid for all $t \geqslant 0$. A more extensive discussion of rescaling in the asymptotic solution of systems of ordinary differential equations is given in [17].

4. Oscillators with slowly varying coefficients

We now generalize the weakly nonlinear oscillator (3.1) by considering

$$\ddot{y} + \omega^2(\sigma)y + \varepsilon f(y, \dot{y}, \sigma, \varepsilon) = 0, \tag{4.1}$$

where the frequency ω is no longer constant, but rather a function of the slow time $\sigma = \varepsilon t$. We also require $\omega(\sigma) > 0$ to avoid turning points.

To apply the amplitude equation method, we first seek a new fast timescale $\tau = h(t;\varepsilon)$ to replace the original fast time, t, so that the limiting system's frequency becomes one on the τ-scale. Using $\tau = h(t;\varepsilon)$ in (4.1), one obtains the traditional fast time

$$\tau = h(t;\varepsilon) \equiv \frac{1}{\varepsilon}\int_0^\sigma \omega(s)\,\mathrm{d}s. \tag{4.2}$$

On this timescale, (4.1) becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + y + \frac{\varepsilon}{\omega^2(\sigma)}\left(\omega'(\sigma)\frac{\mathrm{d}y}{\mathrm{d}\tau} + f\left(y, \omega(\sigma)\frac{\mathrm{d}y}{\mathrm{d}\tau}, \sigma, \varepsilon\right)\right) = 0, \tag{4.3}$$

where the prime denotes differentiation with respect to σ. Observe that τ is nearly constant for bounded t and that the change of timescale adds a small, linear damping in the equation. Eq. (4.3) can now be put in the standard form and the amplitude equation method can be applied directly, though one must be careful handling the two timescales τ and σ because the slow scale, σ, is no longer simply ε times the fast scale, τ.

More generally, consider the system

$$\frac{d\mathbf{x}}{d\tau} = \varepsilon \mathbf{F}(\mathbf{x}, \tau, \sigma, \varepsilon), \tag{4.4}$$

where $\mathbf{F}$ is smooth in all its arguments and 2π periodic in the fast scale τ determined by a specified $\omega(\sigma)$. The common situation where the frequency $\omega(\sigma)$ must be obtained asymptotically could be treated analogously. To account for the "extra" σ-dependence, we modify the near-identity transformation (2.2) to read

$$\mathbf{x}_\varepsilon(\tau, \sigma) = \mathbf{A}_\varepsilon(\sigma) + \varepsilon \mathbf{U}(\mathbf{A}_\varepsilon(\sigma), \tau, \sigma, \varepsilon), \tag{4.5}$$

where $\mathbf{U}$ is smooth and 2π-periodic with respect to τ. Substituting (4.5) into (4.4), we find the analog of (2.4) for (4.5) to be

$$\frac{d\mathbf{A}_\varepsilon}{d\sigma} + \varepsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\varepsilon} \frac{d\mathbf{A}_\varepsilon}{d\sigma} + \omega(\sigma) \frac{\partial \mathbf{U}}{\partial \tau} + \varepsilon \frac{\partial \mathbf{U}}{\partial \sigma} = \omega(\sigma) \mathbf{F}(\mathbf{A}_\varepsilon + \varepsilon \mathbf{U}, \tau, \sigma, \varepsilon), \tag{4.6}$$

since $d\sigma/d\tau = \varepsilon/\omega(\sigma)$, where $\mathbf{U}$ now has a separate, explicit dependence on σ.

In this more general case, we seek a nonautonomous amplitude equation

$$\frac{d\mathbf{A}_\varepsilon}{d\sigma} \equiv \omega(\sigma) \mathbf{H}(\mathbf{A}_\varepsilon, \sigma, \varepsilon) = \omega(\sigma)(\mathbf{H}_0(\mathbf{A}_\varepsilon, \sigma) + \varepsilon \mathbf{H}_1(\mathbf{A}_\varepsilon, \sigma) + \cdots), \tag{4.7}$$

which reduces to (2.3) when $\omega(\sigma) = 1$. Again, we take the average of (4.6) to define an integral equation for the secular part, $\mathbf{H}$. Next we solve (4.6) for $\partial \mathbf{U}/\partial \tau$ and integrate to define an integral equation for the correction $\mathbf{U}$. The resulting linked pair of integral equation is

$$\mathbf{H}(\mathbf{A}_\varepsilon, \sigma, \varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \left(\mathbf{F}(\mathbf{A}_\varepsilon + \varepsilon \mathbf{U}, s, \sigma, \varepsilon) - \frac{\varepsilon}{\omega(\sigma)} \frac{\partial \mathbf{U}}{\partial \sigma} - \varepsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\varepsilon} \mathbf{H}(\mathbf{A}_\varepsilon, \sigma, \varepsilon) \right) ds \tag{4.8}$$

and

$$\mathbf{U}(\mathbf{A}_\varepsilon, \tau, \sigma, \varepsilon) = \int_0^{\tau} \left(\mathbf{F}(\mathbf{A}_\varepsilon + \varepsilon \mathbf{U}, s, \sigma, \varepsilon) - \frac{\varepsilon}{\omega(\sigma)} \frac{\partial \mathbf{U}}{\partial \sigma} - \left(\mathbf{I} + \varepsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\varepsilon} \right) \mathbf{H}(\mathbf{A}_\varepsilon, \sigma, \varepsilon) \right) ds. \tag{4.9}$$

Again, $\mathbf{U}$ will be bounded since $\mathbf{H}$ is the average of

$$\mathbf{F} - \frac{\varepsilon}{\omega} \frac{\partial \mathbf{U}}{\partial \sigma} - \varepsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\varepsilon} \mathbf{H}.$$

Lastly, we insert power series for $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{H}$ into (4.8) and (4.9). Equating coefficients, the leading-order terms provide

$$\mathbf{H}_0(\mathbf{A}_\varepsilon, \sigma) = \langle \mathscr{F}_0(\mathbf{A}_\varepsilon, \sigma) \rangle, \tag{4.10}$$

$$\mathbf{U}_0(\mathbf{A}_\varepsilon, \tau, \sigma) = \int_0^{\tau} (\mathscr{F}_0(\mathbf{A}_\varepsilon, s, \sigma) - \mathbf{H}_0(\mathbf{A}_\varepsilon, \sigma))\, ds. \tag{4.11}$$

Equating terms of $\mathcal{O}(\varepsilon^n)$ for $n \geqslant 1$ again yields recursion formulae

$$\mathbf{H}_n(\mathbf{A}_\varepsilon, \sigma) = \langle \mathscr{F}_n(\mathbf{A}_\varepsilon, \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}, \sigma)\rangle - \frac{1}{\omega(\sigma)}\left\langle \frac{\partial \mathbf{U}_{n-1}}{\partial \sigma}\right\rangle - \sum_{k=0}^{n-1}\left\langle \frac{\partial \mathbf{U}_k}{\partial \mathbf{A}_\varepsilon}\right\rangle \mathbf{H}_{n-k-1}(\mathbf{A}_\varepsilon, \sigma), \tag{4.12}$$

$$\mathbf{U}_n(\mathbf{A}_\varepsilon, \tau, \sigma) = \int_0^\tau \left(\mathscr{F}_n(\mathbf{A}_\varepsilon, \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}, s, \sigma) - \mathbf{H}_n(\mathbf{A}_\varepsilon, \sigma) - \frac{1}{\omega(\sigma)}\frac{\partial \mathbf{U}_{n-1}}{\partial \sigma} - \sum_{k=0}^{n-1} \frac{\partial \mathbf{U}_k}{\partial \mathbf{A}_\varepsilon}\mathbf{H}_{n-k-1}(\mathbf{A}_\varepsilon, \sigma)\right) \mathrm{d}s, \tag{4.13}$$

which asymptotically specify the amplitude equation and the solution $\mathbf{x}_\varepsilon(\tau, \sigma)$ to any order in ε.

As an example, consider the linear oscillator with a variable spring constant

$$\ddot{y} + y - \sigma y = 0. \tag{4.14}$$

In this case $\omega(\sigma) = \sqrt{1-\sigma}$, so the new fast scale is $\tau = \frac{2}{3\varepsilon}(1 - (1-\sigma)^{3/2})$, provided $\sigma < 1$. In terms of τ, the differential equation becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + y - \varepsilon \frac{1}{2(1-\sigma)^{3/2}}\frac{\mathrm{d}y}{\mathrm{d}\tau} = 0. \tag{4.15}$$

After introducing polar coordinates and applying (4.12) and (4.13), the amplitude, $\mathbf{A}_\varepsilon = \binom{R_\varepsilon}{\Phi_\varepsilon}$, evolves according to

$$\frac{\mathrm{d}R_\varepsilon}{\mathrm{d}\sigma} = \frac{R_\varepsilon}{4(1-\sigma)} + \mathcal{O}(\varepsilon^2), \tag{4.16}$$

$$\frac{\mathrm{d}\Phi_\varepsilon}{\mathrm{d}\sigma} = -\varepsilon \frac{7}{32(1-\sigma)^{5/2}} + \mathcal{O}(\varepsilon^2). \tag{4.17}$$

Moreover, the first correction, $\mathbf{U}_0(\mathbf{A}_\varepsilon, \tau)$, defines the $\mathcal{O}(\varepsilon)$ approximation

$$\rho = R_\varepsilon + \varepsilon\left(-\frac{R_\varepsilon}{8(1-\sigma)^{3/2}}\sin(2(\tau + \Phi_\varepsilon))\right) + \mathcal{O}(\varepsilon^2), \tag{4.18}$$

$$\phi = \Phi_\varepsilon + \varepsilon\left(\frac{R_\varepsilon}{8(1-\sigma)^{3/2}}(1 - \cos(2(\tau + \Phi_\varepsilon)))\right) + \mathcal{O}(\varepsilon^2). \tag{4.19}$$

Using $R_\varepsilon(0) = 1$, we find

$$R_\varepsilon(\sigma) = \frac{1}{(1-\sigma)^{1/4}} + \mathcal{O}(\varepsilon^3\sigma).$$

A direct integration shows that $\Phi_\varepsilon(\sigma) = \mathcal{O}(\varepsilon\sigma)$, so the leading-order approximate solution is

$$y(t;\varepsilon) = \frac{1}{(1-\varepsilon t)^{1/4}}\cos\left(\frac{2}{3\varepsilon}(1 - (1-\varepsilon t)^{3/2})\right) + \mathcal{O}(\varepsilon). \tag{4.20}$$

This result is valid for all $\sigma = \varepsilon t < 1$, in exact agreement with that derived either with the WKBJ method or multiple scales. The exact solution,

$$y(t) = \pi \left[\mathrm{Bi}'(-\varepsilon^{-2/3})\mathrm{Ai}\left(\frac{\varepsilon t - 1}{\varepsilon^{2/3}}\right) - \mathrm{Ai}'(-\varepsilon^{-2/3})\mathrm{Bi}\left(\frac{\varepsilon t - 1}{\varepsilon^{2/3}}\right)\right],$$

involves the Airy functions, $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$. Using their well-known asymptotic expansions for large arguments, one recovers (4.20).

Chen et al. discussed this example in [4, Section II.C]. Using their renormalization group technique, they derive the less satisfactory leading order amplitude equations $\mathrm{d}R_\varepsilon/\mathrm{d}\sigma = \frac{1}{4}R_\varepsilon + \mathcal{O}(\varepsilon)$ and $\mathrm{d}\Phi_\varepsilon/\mathrm{d}\sigma = -\frac{1}{2\varepsilon}\sigma + \mathcal{O}(\varepsilon)$, in our notation, corresponding to

$$y(t;\varepsilon) = \mathrm{e}^{\varepsilon t/4}\cos(t - \tfrac{1}{4}\varepsilon t^2) + \mathcal{O}(\varepsilon). \tag{4.21}$$

They find that no finite number of terms in a perturbation expansion can fully capture the large argument behavior of the Airy functions. They are correct, when one uses the fast time t, but by using the fast time

$$\tau = \frac{2}{3\varepsilon}(1 - (1 - \varepsilon t)^{3/2}) = t - \frac{1}{4}\varepsilon t^2 - \frac{1}{24}\varepsilon^2 t^3 + \mathcal{O}(\varepsilon^3 t^4),$$

we avoid the difficulty that their renormalization procedure only determines the power series for τ termwise. Though the renormalization group procedure is able to determine any number of terms desired, it is more labor intensive.

The exact solution is plotted against the two approximations (4.20) and (4.21) in Fig. 1 on the following page. Notice that the amplitude equation solution remains an excellent approximation to the exact solution until it approaches the turning point $t = \frac{1}{\varepsilon}$. In this region the approximation necessarily breaks down because the turning point is a branch point of the amplitude. The renormalization group solution of [4], however, fails to approximate the solution well after only an $\mathcal{O}(\varepsilon^{-1/2})$ time, when the truncated expansion for τ becomes ambiguously ordered.

5. Systems with fast and slow dynamics

Consider the initial value problem for a system of ordinary differential equations that exhibits combined fast and slow dynamics, i.e.,

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{x}, \mathbf{y}, t, \varepsilon), \tag{5.1}$$

$$\dot{\mathbf{x}} = \varepsilon\mathbf{F}(\mathbf{x}, \mathbf{y}, t, \varepsilon). \tag{5.2}$$

Such a system may arise when a nonlinear system is only partially reduced to the standard form (2.1). Note that the small parameter could be moved to the other side of the differential system by introducing the rescaled time $T = t/\varepsilon$.

We defer a more complete discussion of asymptotic solutions for (5.1), (5.2) to a later paper, but note that such problems may be separated into several important subclasses of immediate interest, two of which are mentioned here. If $\mathbf{G}$ is strictly nonlinear in $\mathbf{y}$ and expected to have periodic solutions, then one may use a multiple scales procedure [2,3,11]. If the fast dynamics $\mathbf{G}$ is linear or weakly nonlinear in $\mathbf{y}$ and either periodic or decaying solutions are expected, we may solve system (5.1), (5.2) relatively

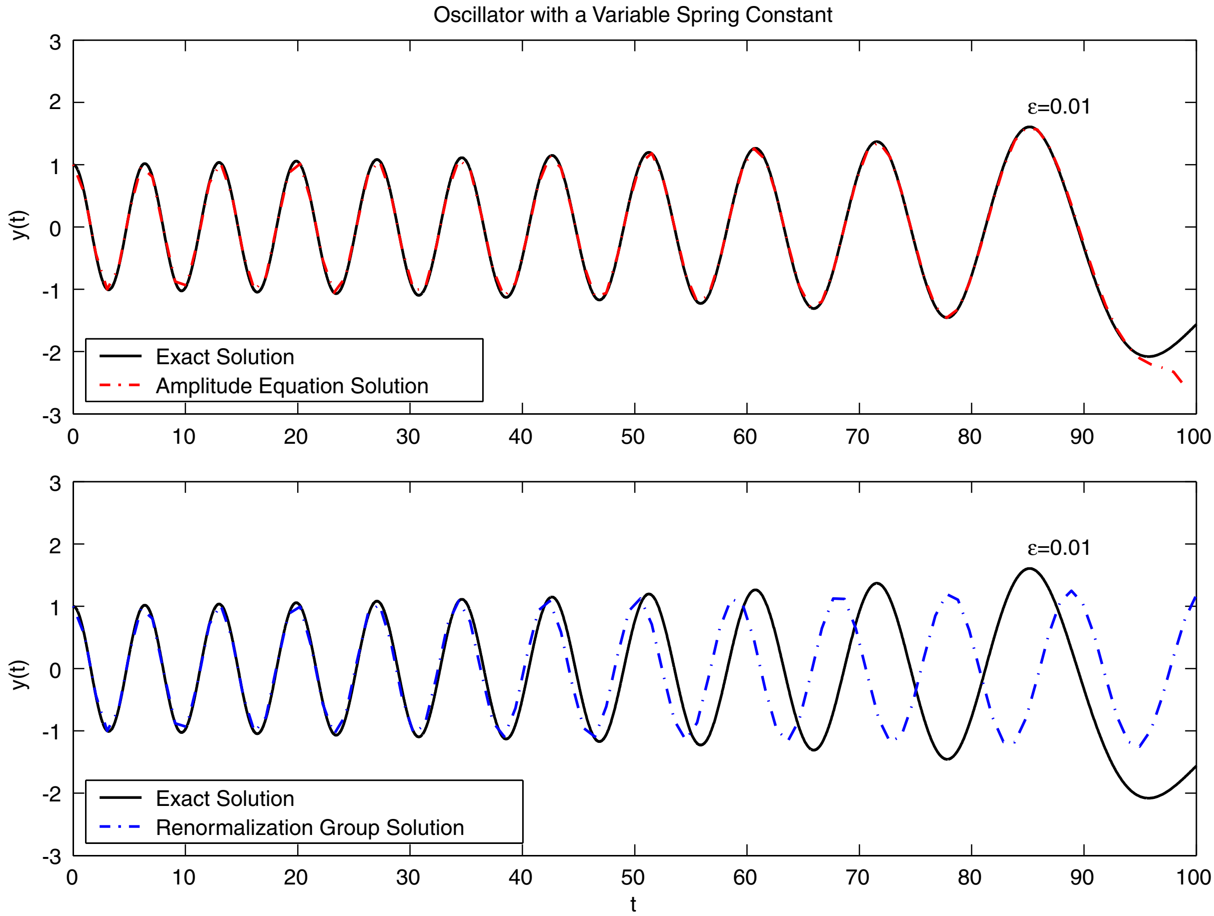

Fig. 1. A comparison of the amplitude equation solution and the renormalization group solution of [4] for the oscillator with a variable spring constant ($\varepsilon = 0.01$).

easily with a hybrid version of the amplitude equation method. In this case the fast dynamics is initially parameterized by the slow dynamics, so one can use the amplitude equation method in conjunction with a multiple scales ansatz to solve the problem quite efficiently. We will show how this method proceeds in the case of decaying solutions for a scalar example.

Consider the system

$$\dot{y} = -y + x^2, \quad \dot{x} = \varepsilon(y + x\cos^2(t)), \tag{5.3}$$

subject to the initial conditions $x(0) = -1/4$, $y(0) = 1$. This relates to the system $\dot{y} = -y + g(x)$, $\dot{x} = \varepsilon f(x, y)$, discussed in [6]. The $\varepsilon = 0$ problem,

$$\dot{y_0} = -y_0 + x_0^2, \quad \dot{x_0} = 0, \tag{5.4}$$

with the given initial conditions, clearly has the asymptotically stable rest point $(x_0(\infty), y_0(\infty)) = (-1/4, 1/16)$ in the phase plane. The existence of such a rest point leads us to anticipate that we will be able to successfully apply the amplitude equation method [7]. The general solution of (5.4),

$$y_0(t) = x(0)^2 + C\mathrm{e}^{-t}, \quad x_0(t) = x(0),$$

shows how the slow dynamics parameterize the fast dynamics. Thus if we replace $x(0)$ with a slowly varying function, the fast dynamics will also vary on the slow timescale. Hence, we shall replace $x(t)$ using a near-identity transformation

$$x_\varepsilon(t, \sigma) = A_\varepsilon(\sigma) + \varepsilon U(A_\varepsilon(\sigma), t, \varepsilon) \tag{5.5}$$

and assume that $y(t)$ follows the generalized multiple scales ansatz

$$y_\varepsilon(t, \sigma) \equiv B(A_\varepsilon(\sigma), t, \varepsilon). \tag{5.6}$$

We further introduce the amplitude equation

$$\frac{\mathrm{d}A_\varepsilon}{\mathrm{d}\sigma} \equiv H(A_\varepsilon, \varepsilon) \tag{5.7}$$

and suppose H and U have the power series (2.8) and (2.9), respectively, and that

$$B(A_\varepsilon(\sigma), t, \varepsilon) = B_0(A_\varepsilon, t) + \varepsilon B_1(A_\varepsilon, t) + \varepsilon^2 B_2(A_\varepsilon, t) + \cdots. \tag{5.8}$$

Substituting (5.5) and (5.6) into (5.3), we obtain the system of PDEs

$$\frac{\partial B}{\partial t} + \varepsilon \frac{\partial B}{\partial A_\varepsilon} \frac{\mathrm{d}A_\varepsilon}{\mathrm{d}\sigma} = -B + A_\varepsilon^2 + \varepsilon 2 A_\varepsilon U + \varepsilon^2 U^2, \tag{5.9}$$

$$\frac{\mathrm{d}A_\varepsilon}{\mathrm{d}\sigma} + \varepsilon \frac{\partial U}{\partial A_\varepsilon} \frac{\mathrm{d}A_\varepsilon}{\mathrm{d}\sigma} + \frac{\partial U}{\partial t} = B + \frac{1}{2} A_\varepsilon + \frac{1}{2} A_\varepsilon \cos(2t) + \frac{1}{2} \varepsilon U + \frac{1}{2} \varepsilon U \cos(2t). \tag{5.10}$$

Thus, the leading-order fast dynamics, B_0, must satisfy

$$\frac{\partial B_0}{\partial t} = -B_0 + A_\varepsilon^2. \tag{5.11}$$

The general solution of (5.11) is

$$B_0(A_\varepsilon, t) = A_\varepsilon^2 + C_0(A_\varepsilon)\mathrm{e}^{-t}, \tag{5.12}$$

where $C_0(A_\varepsilon)$ is an "integration constant". This solution could have been easily anticipated from the reduced problem's solution. Since $x(0) = -1/4$ and $y(0) = 1$ imply that $A_\varepsilon(0) = -1/4$ and $B_0(-1/4, 0) = 1$, we find $C_0(-1/4) = 15/16$. To specify the evolution of $C_0(A_\varepsilon)$, we must eliminate secular terms from the fast dynamics at the next order in ε.

Now the leading-order slow dynamics from (5.10) will be

$$H_0(A_\varepsilon) + \frac{\partial U_0}{\partial t} = A_\varepsilon^2 + C_0(A_\varepsilon)\mathrm{e}^{-t} + \frac{1}{2} A_\varepsilon + \frac{1}{2} A_\varepsilon \cos(2t). \tag{5.13}$$

We shall proceed as before to equate its average and non-average terms. However, since the forcing in (5.13) is no longer periodic, it is necessary to replace the periodic average (2.11) with the more general KBM vector field average

$$\langle g \rangle = \lim_{r \to \infty} \frac{1}{r} \int_0^r g(s)\, \mathrm{d}s, \tag{5.14}$$

(see [19, Chapter 3]) whenever this limit exists. Applying (5.14) to (5.13) we find the first term of the amplitude equation

$$H_0(A_\varepsilon) = A_\varepsilon^2 + \tfrac{1}{2}A_\varepsilon,$$

since $\langle \frac{\partial U_0}{\partial t} \rangle = \langle \mathrm{e}^{-t} \rangle = \langle \cos(2t) \rangle = 0$. Integrating the remaining terms of (5.13) from 0 to t,

$$U_0(A_\varepsilon, t) = C_0(A_\varepsilon)(1 - \mathrm{e}^{-t}) + \tfrac{1}{4}A_\varepsilon \sin(2t),$$

since U_0 is zero at $t = 0$.

Using the known forms of B_0 and U_0, the $\mathcal{O}(\varepsilon)$ terms in (5.9) are

$$\frac{\partial B_1}{\partial t} + B_1 = \left(-2A_\varepsilon C_0(A_\varepsilon) - \frac{\mathrm{d}C_0}{\mathrm{d}A_\varepsilon} H_0(A_\varepsilon)\right)\mathrm{e}^{-t} + 2A_\varepsilon\left(C_0(A_\varepsilon) - H_0(A_\varepsilon) + \frac{1}{4}\sin(2t)\right). \tag{5.15}$$

Since e^{-t} satisfies the homogeneous equation, B_1 will contain secular terms unless we eliminate all multiples of e^{-t} from the forcing. Thus we require $C_0(A_\varepsilon)$ to satisfy the differential equation

$$\frac{\mathrm{d}C_0}{\mathrm{d}A_\varepsilon} H_0(A_\varepsilon) + 2A_\varepsilon C_0(A_\varepsilon) = 0. \tag{5.16}$$

Using the initial condition $C_0(-1/4) = 15/16$, we obtain

$$C_0(A_\varepsilon) = \frac{15}{64(2A_\varepsilon + 1)^2}.$$

Integrating what remains of (5.15), we then get

$$B_1(A_\varepsilon, t) = 2A_\varepsilon(C_0(A_\varepsilon) - H_0(A_\varepsilon)) + \tfrac{1}{10}A_\varepsilon^2(\sin(2t) - 2\cos(2t)) + C_1(A_\varepsilon)\mathrm{e}^{-t}.$$

Now the initial condition $B_1(-1/4, 0) = 0$ implies $C_1(-1/4) = 41/80$. Having calculated B_1 we may proceed to calculate U_1 and H_1 from the $\mathcal{O}(\varepsilon)$ terms in (5.10). For brevity the solution U_1 is omitted, however the second term of the amplitude equation is found to be

$$H_1(A_\varepsilon) = 2A_\varepsilon(C_0(A_\varepsilon) - H_0(A_\varepsilon)) + \frac{\mathrm{d}H_0}{\mathrm{d}A_\varepsilon} C_0(A_\varepsilon).$$

Next, we eliminate secular terms from the $\mathcal{O}(\varepsilon^2)$ fast dynamics to determine $C_1(A_\varepsilon)$, completing the solution of (5.3) to $\mathcal{O}(\varepsilon)$. All that remains is to solve the $\mathcal{O}(\varepsilon)$ amplitude equation

$$\frac{\mathrm{d}A_\varepsilon}{\mathrm{d}\sigma} = H_0(A_\varepsilon) + \varepsilon\left(2A_\varepsilon(C_0(A_\varepsilon) - H_0(A_\varepsilon)) + \frac{\mathrm{d}H_0}{\mathrm{d}A_\varepsilon} C_0(A_\varepsilon)\right) + \mathcal{O}(\varepsilon^2). \tag{5.17}$$

Such calculations may readily be carried to higher order.

6. Summary

We have shown that the renormalization procedure of Chen et al. can be further simplified. The resulting amplitude equation procedure can be applied to weakly nonlinear oscillators, oscillators with slowly varying frequency and systems with both fast and slow dynamics.

Though the underlying idea that integration constants should be replaced by slowly varying functions is basic to other methods (e.g. the method of multiple scales), our amplitude equation procedure makes a most efficient use of this assumption. With some additional extensions, our methodology can be applied to two-point boundary-value problems and, we believe, to partial differential equations.

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