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Non-isomorphic Solutions of Some Balanced Incomplete Block Designs. I

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ABSTRACT

In this paper we develop a method for generating non-isomorphic solutions of balanced incomplete block designs belonging to the series of symmetric designs with parameters $(4t + 3, 2t + 1, t)$ and to the series with parameters $(4t + 4, 8t + 6, 4t + 3, 2t + 2, 2t + 1)$. We also prove a result about the number of non-isomorphic solutions of these designs as the parameter t tends to infinity.

1. INTRODUCTION

A balanced incomplete block design (BIBD) is an arrangement of v symbols called treatments, in b subsets called blocks of size $k < v$ such that any two treatments occur together in λ blocks. Then each treatment occurs in r blocks and the following relations are satisfied

$$vr = bk,$$

$$\lambda(v - 1) = r(k - 1).$$

Besides these necessary conditions we also have the inequality

$$b \geqslant v,$$

which is due to Fisher. We shall use the term "design" generally to indicate a BIBD. By a (v, b, r, k, λ) design we will mean a BIBD with these parameters. By a symmetric BIBD we mean a BIBD with $v = b$ and hence $r = k$. We shall call such a design a (v, k, λ) design.

Two BIBD's D_1 and D_2 with the same parameters are said to be isomorphic if there exists a bijection of the set of treatments of D_1 into that of D_2 such that under this bijection the set of blocks of D_1 goes into the set of blocks of D_2. Otherwise they are said to be non-isomorphic.

In this paper we develop a technique for generating non-isomorphic solutions of:

(i) a $(4t + 3, 2t + 1, t)$ design, and

(ii) a $(4t + 4, 8t + 6, 4t + 3, 2t + 2, 2t + 1)$ design.

We also prove that if for any t a solution exists for a $(4t + 3, 2t + 1, t)$ design, then the number of non-isomorphic solutions to a

$$(2^{\alpha}(4t + 4) - 1, 2^{\alpha}(2t + 2) - 1, 2^{\alpha}(t + 1) - 1)$$

design tends to infinity as α tends to infinity. Under the same conditions the number of non-isomorphic solutions to a

$$(2^{\alpha}(4t + 4), 2^{\alpha+1}(4t + 4) - 2, 2^{\alpha}(4t + 4) - 1, 2^{\alpha}(2t + 2), 2^{\alpha}(2t + 2) - 1)$$

design also tends to infinity. We illustrate the use of this technique by giving a number of non-isomorphic solutions to some balanced incomplete block designs.

2. Preliminary Results

Number the treatments and blocks of a design by $1, 2, \ldots, v$ and $B_1, B_2, \ldots, B_b$. We define the usual incidence matrix $N = (n_{ij})$ of a (v, b, r, k, λ) design by

$$n_{ij} = 1 \text{ or } 0$$

according as block B_i contains or does not contain the treatment j. Obviously N is a $(0, 1)$ matrix and if N' is the transpose of N, then

$$N'N = (r - \lambda) I + \lambda J, \tag{2.1}$$

where I is the identity matrix of order v and J is the square matrix of order v with all elements 1.

Two BIBD's D_1 and D_2 on the same set $1, 2, \ldots, v$ of treatments and with the same parameters will then be isomorphic if and only if the corresponding incidence matrices N_1 and N_2 are such that each can be obtained from the other by a suitable permutation of its rows and columns.

Corresponding to any design D with incidence matrix N, there exists the complementary design $\bar{D}$ with incidence matrix $\bar{N}$ which is obtained from N by interchanging 0 and 1 in N. If D is a (v, b, r, k, λ) design then obviously $\bar{D}$ is a $(v, b, b - r, v - k, b - 2r + \lambda)$ design. In case $v = 2k$, it is obvious that D and $\bar{D}$ have the same set of parameters.

For a (v, b, r, k, λ) design D with incidence matrix N consider the configuration of b treatments in v blocks with incidence matrix N'. This configuration D' is called the dual of D and in general D' is not a BIBD. However in the case of a (v, k, λ) design it is known that any two blocks intersect in exactly λ treatments and hence the dual of a (v, k, λ) design is again a (v, k, λ) design. These two (v, k, λ) designs in general are non-isomorphic.

It is well known that the existence of a (v, k, λ) design implies the existence of the residual design which is a $(v - k, v - 1, k, k - \lambda, \lambda)$ design and the derived design which is a $(k, v - 1, k - 1, \lambda, \lambda - 1)$ design. They are obtained by omitting an initial block of the (v, k, λ) design and retaining, respectively, in the remaining blocks only those treatments which do not (do) occur in the initial block. The parameters of a (v, b, r, k, λ) design which is the residual of a symmetric BIBD satisfy the following relations

$$r = k + \lambda, \qquad \lambda v = k(k + \lambda - 1), \qquad b\lambda = (k + \lambda)(k + \lambda - 1). \tag{2.2}$$

For $\lambda = 1$ or 2 a (v, b, r, k, λ) design satisfying (2.2) can always be embedded as a residual design in the corresponding symmetric BIBD [3]. For higher values of λ the embedding is possible if and only if certain conditions on the structure of the design with parameters given by (2.2) are satisfied [3].

A design is called resolvable if the set of its blocks can be partitioned into subsets such that each subset is a complete replication of all its treatments. For a resolvable BIBD Fisher's inequality $b \geqslant v$ can be sharpened [1] to

$$b \geqslant v + r - 1.$$

A resolvable BIBD is called an affine resolvable BIBD (ARBIBD) if any two blocks coming from different replications intersect in the same number of treatments. Bose [1] has shown that, if in a resolvable BIBD $b = v + r - 1$, then the design is an ARBIBD and any two blocks of different replications intersect in k^2/v treatments. It then follows that, if in a design with $b = v + r - 1$, the number k^2/v is not an integer, the design cannot be resolvable. He has also shown that the parameters of an ARBIBD can be expressed in terms of two integers $n \geqslant 2$, and $t \geqslant 0$ by

$$v = nk = n^2[(n - 1)\,t + 1], \qquad b = nr = n(n^2 t + n + 1), \qquad \lambda = nt + 1,$$

where the number of treatments common to any two blocks of different replications is $(n - 1)\,t + 1$.

Consider the series of

$$(2\lambda + 2,\ 4\lambda + 2,\ 2\lambda + 1,\ \lambda + 1,\ \lambda)$$

designs. They satisfy the condition $b = v + r - 1$. Since $k^2/v = (\lambda + 1)/2$ such a design can be resolvable and hence affine resolvable only if λ is odd. Bose [2] has given two non-isomorphic solutions of such a design when $\lambda > 1$ is odd and $2\lambda + 1$ is a prime power. One of these is non-resolvable while the other is resolvable and hence affine resolvable.

In the rest of this section we prove a number of lemmas that will be needed later on.

LEMMA 2.1. *The existence of a* $(4t + 3, 2t + 1, t)$ *design is equivalent to the existence of an* $(4t + 4, 8t + 6, 4t + 3, 2t + 2, 2t + 1)$ *ARBIBD.*

PROOF: Let N be the incidence matrix of a $(4t + 3, 2t + 1, t)$ design. Then $\bar{N}$ is the incidence matrix of a $(4t + 3, 2t + 2, t + 1)$ design. It is then easily verified that

$$\begin{pmatrix} N & \underline{1} \\ \bar{N} & \underline{0} \end{pmatrix},$$

where $\underline{1}$ and $\underline{0}$ are column vectors with all elements 1 and 0, respectively, is the incidence matrix of an ARBIBD with the required parameters. Conversely the incidence matrix of an ARBIBD with the above parameters can be put in the above form by suitable permutations of its rows and columns where N is a square matrix of order $4t + 3$ having $2t + 1$ elements 1 in each row and column. If x is the scalar product of any two columns of N, then the scalar product of the corresponding columns of $\bar{N}$ is $4t + 3 - 2(2t + 1) + x = x + 1$ and hence $2x + 1 = 2t + 1$ or $x = t$. It now follows that N is the incidence matrix of a $(4t + 3, 2t + 1, t)$ design.

LEMMA 2.2. *If N and M are the incidence matrices of two* $(4t + 3, 2t + 1, t)$ *designs, then*

$$\begin{pmatrix} N & \underline{1} \\ \bar{M} & \underline{0} \end{pmatrix}$$

is the incidence matrix of a $(4t + 4, 8t + 6, 4t + 3, 2t + 2, 2t + 1)$ *design.*

Proof is obvious.

LEMMA 2.3. *If (without loss of generality)*

$$\begin{pmatrix} N & \underline{1} \\ \bar{N} & \underline{0} \end{pmatrix}$$

is the incidence matrix of an $(4t+4,\ 8t+6,\ 4t+3,\ 2t+2,\ 2t+1)$ *ARBIBD and* M *is the incidence matrix of a* $(4t+3, 2t+1, t)$ *design, then*

$$P = \begin{pmatrix} N & \underline{1} & M \\ \bar{N} & \underline{0} & M \\ \underline{0}' & 0 & \underline{1}' \end{pmatrix}$$

is the incidence matrix of a $(8t+7, 4t+3, 2t+1)$ *design.*

PROOF: We need only check the value $\lambda = 2t+1$ for two treatments, one from the set of first $4t+3$ and the other from the set of last $4t+3$ treatments. From the structure of P it is obvious that this is equal to the number of unities in each column of M which is equal to $2t+1$.

COROLLARY. *If* N *is the incidence matrix of a* $(4t+3, 2t+1, t)$ *design, then*

$$\begin{pmatrix} N & \underline{1} & N \\ \bar{N} & \underline{0} & N \\ \underline{0}' & 0 & \underline{1}' \end{pmatrix}$$

is the incidence matrix of a $(8t+7, 4t+3, 2t+1)$ *design.*

We note that the above corollary implies that an $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$ ARBIBD is always embeddable as a residual in a $(8t+7, 4t+3, 2t+1)$ design. Using the fact that any two blocks of the ARBIBD coming from different replications intersect in $t+1$ treatments, it is obvious that Lemma 2.3 gives the most general method of embedding such a design in a corresponding $(8t+7, 4t+3, 2t+1)$ design.

We have seen that the dual of a (v, k, λ) design is again a (v, k, λ) design. In general these two designs are non-isomorphic. If, however, there is a unique (up to isomorphism) solution of a (v, k, λ) design, then the dual of such a design is isomorphic to the design. We now prove

LEMMA 2.4. *Let* G *be an additive Abelian group with elements* $\alpha_0 = 0, \alpha_1, \ldots, \alpha_{v-1}$ *and let (without loss of generality)* $(\alpha_0, \alpha_{i_1}, \ldots, \alpha_{i_{k-1}})$ *be a difference set generating a* (v, k, λ) *design* D. *Then the dual* D' *is isomorphic to* D.

PROOF: Let the treatments and blocks of D be numbered $\alpha_0, \alpha_1, \ldots, \alpha_{v-1}$. Then the block α_j of D contains the treatments

$$(\alpha_j, \alpha_j + \alpha_{i_1}, \ldots, \alpha_j + \alpha_{i_{k-1}});\ j = 0, 1, \ldots, v-1.$$

The design D' is seen to be generated by the difference set

$$(\alpha_0, -\alpha_{i_1}, \ldots, -\alpha_{i_{k-1}}).$$

It is easily seen that the permutation $\alpha_i \to -\alpha_i$, takes the block numbered α_j of D into the block numbered $-\alpha_j$ of D'. Thus D and D' are isomorphic.

Now consider a $(4\lambda + 3, 2\lambda + 1, \lambda)$ design, say, D. Any two blocks of D intersect in λ treatments. We note that there cannot exist 4 blocks in D containing the same λ-tuple. For suppose there exist 3 blocks of D containing the same λ-tuple S, then we can write these 3 blocks as $B_i = (S, S_i)$, $i = 1, 2, 3$, where each S_i is a $(\lambda + 1)$-tuple and S, S_1, S_2, S_3 are mutually disjoint and account for all the treatments of D. It there is any other block of D containing S then it is obvious that this block cannot contain any other treatment, violating the condition $k = 2\lambda + 1$. The number α of such λ-tuples occurring in 3 blocks of D is then a characteristic of D and we shall call α the characteristic number of D. We shall call such λ-tuples special λ-tuples and the corresponding blocks in which they occur as B-triples.

It is easily verified that the characteristic number α is always zero for a $(4\lambda + 3, 2\lambda + 1, \lambda)$ design D when λ is even. If $\lambda = 1$, then D is the unique (7, 3, 1) design for which $\alpha = 7$.

It is obvious that two solutions of a $(4\lambda + 3, 2\lambda + 1, \lambda)$ design having different characteristic numbers are non-isomorphic. Even if the two solutions have the same characteristic number α, it may still be possible to prove that they are non-isomorphic. For example, this will be so if we can prove that the 2 sets of α special λ-tuples are non-isomorphic.

We have already seen that any $(4t + 4, 8t + 6, 4t + 3, 2t + 2, 2t + 1)$ ARBIBD has the incidence matrix

$$\begin{pmatrix} N & \underline{1} \\ \bar{N} & \underline{0} \end{pmatrix}$$

where N is the incidence matrix of a $(4t + 3, 2t + 1, t)$ design. It is easy to verify that in the ARBIBD there is no $(t + 1)$-tuple occurring in more than 3 blocks of the design. We will call a $(t + 1)$-tuple occurring in 3 blocks of the design a special $(t + 1)$-tuple and the blocks in which they occur as the corresponding B-triples. The number β of special $(t + 1)$-tuples will be called the characteristic number of the design.

We now consider how these special $(t + 1)$-tuples arise. Obviously any special t-tuple in N gives rise to a special $(t + 1)$-tuple in $(N\ \underline{1})$. Again no special $(t + 1)$-tuple can arise in $(\bar{N}\ \underline{0})$, since there are no special $(t + 1)$-tuples in $\bar{N}$. Again a special $(t + 1)$-tuple cannot arise from 2 blocks of $(N\ \underline{1})$ and one block of $(\bar{N}\ \underline{0})$. The only case left is when such a $(t + 1)$-tuple arises from 2 blocks of $(\bar{N}\ \underline{0})$ and one block of $(N\ \underline{1})$, i.e., from 2 blocks of $\bar{N}$ and 1 block of N. It is easily seen that such a situation arises only in the following case, where B_i and $\bar{B}_i$ are the corresponding blocks of N and $\bar{N}$, $i = 1, 2, 3$:

$$\begin{aligned} &B_1 = (S, S_1), \qquad &B_2 = (S, S_2), \qquad &B_3 = (S, S_3), \\ &\bar{B}_1 = (S_2, S_3), \qquad &\bar{B}_2 = (S_3, S_1), \qquad &\bar{B}_3 = (S_1, S_2), \end{aligned}$$

where S is a t-tuple and S_1, S_2, S_3 are $(t + 1)$-tuples such that all these tuples are mutually disjoint and account for all the treatments of N. If we denote the last treatment in the ARBIBD by ∞, then it is obvious that the only $(t + 1)$-tuples of the ARBIBD are of the form (S, ∞), S_1, S_2, and S_3 where S is a special t-tuple in N. We have thus proved

LEMMA 2.5. *If the characteristic number of a* $(4t + 3,\ 2t + 1,\ t)$ *design is* α, *then the characteristic number* β *of the corresponding* $(4t + 4, 8t + 6, 4t + 3, 2t + 2, 2t + 1)$ *ARBIBD is given by* $\beta = 4\alpha$.

LEMMA 2.6. *If the characteristic number of a* $(4t + 3, 2t + 1, t)$ *design with incidence matrix* N *is* α, *then the characteristic number* γ *of the* $(8t + 7, 4t + 3, 2t + 1)$ *design with incidence matrix*

$$P = \begin{pmatrix} N & \underline{1} & N \\ \bar{N} & \underline{0} & N \\ \underline{0}' & 0 & \underline{1}' \end{pmatrix}$$

is given by $\gamma = 4t + 3 + 4\alpha$.

PROOF: We note that the special $(2t + 1)$-tuples of P are of one of the two types. In the first type the corresponding B-triples do not contain the last block of P and the number of such tuples is obviously 4α. In the special $(2t + 1)$-tuples of the second type, the corresponding B-triples contain the last block of P. The number of such tuples is obviously $4t + 3$ and these special $(2t + 1)$-tuples are the subsets of the first $4t + 3$ blocks of P corresponding to the last $4t + 3$ treatments.

LEMMA 2.7. *If the characteristic number of a* $(4t + 3, 2t + 1, t)$ *design with incidence matrix* N *is* α, *and* M *is the incidence matrix of any* $(4t + 3, 2t + 1, t)$ *design, then the characteristic number* x *of the* $(8t + 7, 4t + 3, 2t + 1)$ *design with incidence matrix*

$$P = \begin{pmatrix} N & \underline{1} & M \\ \bar{N} & \underline{0} & M \\ \underline{0}' & 0 & \underline{1}' \end{pmatrix}$$

satisfies the inequality

$$4t + 3 \leqslant x \leqslant 4t + 3 + 4\alpha.$$

Proof is obvious.

REMARK. We note that the maximum value of x is attained if and only if the characteristic number corresponding to M is $\geqslant \alpha$ and if for each special t-tuple of N occurring in blocks numbered i_1, i_2, i_3 there is a special t-tuples in M occurring in the same 3 blocks. In particular, if $M = N$ the number x attains the maximum value $4t + 3 + 4\alpha$. In general, if δ is the number of common B-triples in M and N then the characteristic number of the design corresponding to P is given by $x = 4t + 3 + 4\delta$. By suitable permutation of the rows of M it is possible to make δ assume different values between 0 and α. This helps in obtaining a large number of non-isomorphic solutions to a $(8t + 7, 4t + 3, 2t + 1)$ design.

Now consider the $(4t + 4, 8t + 6, 4t + 3, 2t + 2, 2t + 1)$ design with incidence matrix

$$\begin{pmatrix} N & \underline{1} \\ \overline{M} & \underline{0} \end{pmatrix}$$

when N and M, $N \neq M$, are incidence matrices of two $(4t + 3, 2t + 1, t)$ designs. This design cannot have any $(t + 1)$-tuple occurring in more than 3 blocks. Any special $(t + 1)$-tuple of this design arises in two distinct ways. Any special t-tuple of N obviously gives a special $(t + 1)$-tuple of this design. Any other special $(t + 1)$-tuple arises from 2 blocks of $\overline{M}$ and 1 block of N and this happens when the intersection of any two blocks of $\overline{M}$ is contained in a block of N. The number of special $(t + 1)$-tuples in the design will be called its characteristic number.

3. SOLUTIONS OF (15, 7, 3) DESIGNS

Nandi [4] proved that there exist, in all, 5 non-isomorphic solutions of a (15, 7, 3) design which he denoted by $(\alpha_1\alpha_1')_1$, $(\alpha_1\alpha_1')_2$, $(\alpha_2\alpha_2')$, $(\beta_1\beta_1')$, and $(\gamma\gamma')$. We will denote these designs by C_1, C_2, C_3, C_4, and C_5, respectively, where the blocks in C_i, $i = 1, 2, 3, 4$ are identical with the corresponding blocks in $(\alpha_1\alpha_1')_1$, $(\alpha_1\alpha_1')_2$, $(\alpha_2\alpha_2')$, $(\beta_1\beta_1')$ whereas C_5 is isomorphic to $(\gamma\gamma')$ and is given by the difference set (1, 2, 4, 5, 8, 10, 15) mod 15. This will be the first block of C_5 and the block numbered i is obtained by adding $(i - 1)$ to each element of the difference set and reducing mod 15. For the sake of future reference we write (Table 3.1) down the designs in which the blocks are numbered from 1 to 15. We will denote the incidence matrix of C_i by N_i; $i = 1, 2, \ldots, 5$.

It is easily verified that these designs have respectively 7, 11, 7, 19, and 35 special 3-tuples. We show these special 3-tuples and the corresponding B-triples in Table 3.2.

TABLE 3.1

Block No.	C_1							C_2						
1	1,	2,	3,	4,	9,	10,	11	1,	2,	3,	4,	9,	10,	11
2	1,	2,	3,	5,	12,	13,	14	1,	2,	3,	5,	12,	13,	14
3	1,	2,	6,	7,	10,	13,	15	1,	2,	6,	7,	10,	13,	15
4	1,	3,	7,	8,	9,	12,	15	1,	3,	7,	8,	11,	14,	15
5	1,	4,	5,	6,	11,	12,	15	1,	4,	5,	6,	11,	12,	15
6	1,	4,	6,	8,	9,	13,	14	1,	4,	6,	8,	9,	13,	14
7	1,	5,	7,	8,	10,	11,	14	1,	5,	7,	8,	9,	10,	12
8	2,	3,	6,	8,	11,	14,	15	2,	3,	6,	8,	9,	12,	15
9	2,	4,	5,	7,	9,	14,	15	2,	4,	5,	7,	9,	14,	15
10	2,	4,	7,	8,	11,	12,	13	2,	4,	7,	8,	11,	12,	13
11	2,	5,	6,	8,	9,	10,	12	2,	5,	6,	8,	10,	11,	14
12	3,	4,	5,	8,	10,	13,	15	3,	4,	5,	8,	10,	13,	15
13	3,	4,	6,	7,	10,	12,	14	3,	4,	6,	7,	10,	12,	14
14	3,	5,	6,	7,	9,	11,	13	3,	5,	6,	7,	9,	11,	13
15	9,	10,	11,	12,	13,	14,	15	9,	10,	11,	12,	13,	14,	15

Block No.	C_3							C_4						
1	1,	2,	3,	4,	9,	10,	11	1,	2,	3,	4,	9,	10,	11
2	1,	2,	3,	5,	12,	13,	14	1,	2,	3,	5,	12,	13,	14
3	1,	2,	6,	7,	9,	13,	15	1,	2,	6,	7,	10,	13,	15
4	1,	3,	7,	8,	11,	14,	15	1,	3,	6,	8,	11,	12,	15
5	1,	4,	5,	6,	11,	12,	15	1,	4,	5,	6,	9,	14,	15
6	1,	4,	6,	8,	10,	13,	14	1,	4,	7,	8,	9,	12,	13
7	1,	5,	7,	8,	9,	10,	12	1,	5,	7,	8,	10,	11,	14
8	2,	3,	6,	8,	10,	12,	15	2,	3,	7,	8,	9,	14,	15
9	2,	4,	5,	8,	9,	14,	15	2,	4,	5,	7,	11,	12,	15
10	2,	4,	7,	8,	11,	12,	13	2,	4,	6,	8,	11,	13,	14
11	2,	5,	6,	7,	10,	11,	14	2,	5,	6,	8,	9,	10,	12
12	3,	4,	5,	7,	10,	13,	15	3,	4,	5,	8,	10,	13,	15
13	3,	4,	6,	7,	9,	12,	14	3,	4,	6,	7,	10,	12,	14
14	3,	5,	6,	8,	9,	11,	13	3,	5,	6,	7,	9,	11,	13
15	9,	10,	11,	12,	13,	14,	15	9,	10,	11,	12,	13,	14,	15

From the characteristic numbers of the above designs it is obvious that C_2, C_4, C_5 are mutually non-isomorphic as also non-isomorphic with C_1 and C_3. We note that the special tuples in C_1 form a (7, 3, 1) design, whereas those in C_3 contain treatment numbered 5, 7 times and the rest exactly once. C_1 and C_3 are, therefore, non-isomorphic. It is easy to verify that, if D is a (15, 7, 3) design, then D and D' both have the same characteristic numbers. Thus C_2, C_4, C_5 are isomorphic to their

TABLE 3.2

Special 3-tuples				B-triples		
			C_1			
3	4	10		1	13	12
3	5	13		2	14	12
10	13	15		3	15	12
3	8	15		4	8	12
4	5	15		5	9	12
4	8	13		6	10	12
5	8	10		7	11	12
			C_2			
3	4	10		1	12	13
3	5	13		2	12	14
1	5	12		2	5	7
2	5	14		2	9	11
10	13	15		3	12	15
3	8	15		4	8	12
4	5	15		5	9	12
5	6	11		5	11	14
4	8	13		6	10	12
5	7	9		7	9	14
5	8	10		7	11	12
			C_3			
1	12	5		2	5	7
2	14	5		2	9	11
3	13	5		2	12	14
4	15	5		5	9	12
6	11	5		5	11	14
8	9	5		7	9	14
7	10	5		7	11	12
			C_4			
4	5	15		5	9	12
1	4	9		5	1	6
1	5	14		5	2	7
1	6	15		5	3	4
9	14	15		5	8	15
4	6	14		5	10	13
5	6	9		5	11	14
2	4	11		9	1	10
2	5	12		9	2	11
2	7	15		9	3	8

(Table continued)

TABLE 3.2 (*continued*)

Special 3-tuples				B-triples		
11	12	15		9	4	15
4	7	12		9	6	13
5	7	11		9	7	14
3	4	10		12	1	13
3	5	13		12	2	14
10	13	15		12	3	15
4	8	13		12	6	10
5	8	10		12	7	11
3	8	15		12	4	8
			C_5			
1	6	11		2	7	12
2	7	12		3	8	13
3	8	13		4	9	14
4	9	14		5	10	15
5	10	15		1	6	11
	and				and	
15 each generated by				15 each generated by		
1	4	15		1	12	15
	and				and	
2	8	15		1	8	14
	mod 15				mod 15	

own duals, whereas it is easy to verify that C_1 is isomorphic to the dual of C_3.

Starting from the unique solution of a (7, 3, 1) design given by the difference set (1, 2, 4) mod 7, we apply Lemma 2.6 to obtain the solution of a (15, 7, 3) design with characteristic number $7 + 4(7) = 35$. This solution is then isomorphic to C_5. We now illustrate the remark at the end of Lemma 2.7. The special 1-tuples and the corresponding B-triples of the (7, 3, 1) design are given below.

Special 1-tuple		B-triples	
1	1	5	7
2	2	6	1
3	3	7	2
4	4	1	3
5	5	2	4
6	6	3	5
7	7	4	6

Now consider the design with incidence matrix P of Lemma 2.6 by taking $M = N$, when N is the incidence matrix of the above (7, 3, 1) design. The characteristic number of this design as already seen is 35. The value of δ under the permutations (1, 2), (2, 6, 5, 4, 3), and (1, 2) (3, 4) (5, 6) is easily seen to be 3, 1, and 0, respectively, and hence the corresponding solutions have 19, 11, and 7 as the characteristic numbers. The first two of these solutions are isomorphic to C_4 and C_2, respectively. In the last solution the special 3-tuples form a (7, 3, 1) design and hence this solution is isomorphic to C_1. The solution C_3 cannot be obtained by this method. However, it is obtained by taking the dual of C_1.

4. Solutions of (16, 30, 15, 8, 7) Designs

The 5 solutions of (15, 7, 3) designs with the use of Lemma 2.2 give rise to 25 designs with parameters (16, 30, 15, 8, 7).

Consider a block of a (v, b, r, k, λ) design. If f_i is the number of other blocks having i treatments in common with this block, $0 \leqslant i \leqslant k$, we express the pattern of intersection of this block with the remaining blocks by the notation $(0^{f_0}1^{f_1} \cdots k^{f_k})$. If b_1 is the number of blocks with the pattern of block intersection $(0^{f_0}1^{f_1} \cdots k^{f_k})$, b_2 is the number of blocks with the pattern $(0^{g_0}1^{g_1} \cdots k^{g_k})$ etc., we express the block intersections for the design by the notation

$$(0^{f_0}1^{f_1} \cdots k^{f_k})^{b_1} (0^{g_0}1^{g_1} \cdots k^{g_k})^{b_2} \cdots .$$

Whenever $f_i = 0$ we omit the term i^{f_i}. We use the following notation for some of the possible patterns of block intersections of a (16, 30, 15, 8, 7) design:

$$a = (0^1 4^{28}), \qquad b = (2^3 4^{25} 6^1), \qquad c = (1^1 3^4 4^{21} 5^3), \qquad d = (2^2 3^4 4^{19} 5^4),$$
$$e = (2^1 3^6 4^{19} 5^2 6^1), \qquad f = (2^3 3^1 4^{22} 5^3), \qquad g = (2^2 3^3 4^{22} 5^1 6^1),$$
$$h = (2^3 3^1 4^{22} 5^3).$$

Table 4.1 gives the pattern of block intersections as well as the characteristic numbers of these 25 designs. If even one of them is different for two solutions then these solutions are non-isomorphic. The converse, however, is not necessarily true.

The four pairs of solutions corresponding to characteristic numbers 33, 34, 56, 60 have the same block intersections and hence cannot be distinguished by our method. We have not made any attempt further to analyze these solutions for non-isomorphism. The solutions E_1, E_7, E_{13}, E_{19}, E_{25} are affine resolvable. We distinguish the two solutions with

TABLE 4.1

Design	N	M	Characteristic number	Block intersections
E_1	N_1	N_1	28	a^{30}
E_2	N_1	N_2	44	$a^{22}b^8$
E_3	N_1	N_3	33	$a^{10}b^2c^{12}deg^3h$
E_4	N_1	N_4	56	$a^{18}b^{12}$
E_5	N_1	N_5	48	$b^2c^{10}d^7e^7fg^3$
E_6	N_2	N_1	42	$a^{22}b^8$
E_7	N_2	N_2	44	a^{30}
E_8	N_2	N_3	34	$a^{14}c^{16}$
E_9	N_2	N_4	60	$a^{14}b^{16}$
E_{10}	N_2	N_5	50	$bc^6d^{11}e^{10}fg$
E_{11}	N_3	N_1	33	$a^{10}b^2c^{12}deg^3h$
E_{12}	N_3	N_2	34	$a^{14}c^{16}$
E_{13}	N_3	N_3	28	a^{30}
E_{14}	N_3	N_4	46	$a^6b^4c^8d^2e^2f^2g^6$
E_{15}	N_3	N_5	44	$bc^4d^{14}e^9f^2$
E_{16}	N_4	N_1	56	$a^{18}b^{12}$
E_{17}	N_4	N_2	60	$a^{14}b^{16}$
E_{18}	N_4	N_3	49	$a^6b^4c^8d^2e^2f^2g^6$
E_{19}	N_4	N_4	76	a^{30}
E_{20}	N_4	N_5	59	$b^3c^6d^{10}e^9g^2$
E_{21}	N_5	N_1	70	$b^2c^{10}d^7e^7fg^3$
E_{22}	N_5	N_2	73	$bc^6d^{11}e^{10}fg$
E_{23}	N_5	N_3	68	$bc^4d^{14}e^9f^2$
E_{24}	N_5	N_4	76	$b^3c^6d^{10}e^9g^2$
E_{25}	N_5	N_5	140	a^{30}

characteristic number 28 as follows. From the nature of special 3-tuples in C_1 it is obvious that the pair $(3, \infty)$ occurs thrice in the corresponding 4-tuples of the ARBIBD corresponding to N_1. The 4-tuples of the ARBIBD corresponding to N_3 are given by the 4-tuples (S_i, S_j), $i < j = 1, 2, \ldots, 8$, where $S_1 = (1, 12)$, $S_2 = (2, 14)$, $S_3 = (3, 13)$, $S_4 = (4, 15)$, $S_5 = (6, 11)$, $S_6 = (8, 9)$, $S_7 = (7, 10)$, $S_8 = (5, \infty)$. It is obvious that these 4-tuples form a group divisible design with parameters

$$v = 16, \quad b = 28, \quad r = 7, \quad k = 4, \quad \lambda_1 = 7, \quad \lambda_2 = 1,$$

where any two treatments belonging to the same S_i occur together 7 times, whereas those coming from different S_i's occur together only once. It is now obvious that the two ARBIBD's with characteristic number 28 are non-isomorphic.

Therefore we have at least 21 mutually non-isomorphic solutions for a (16, 30, 15, 8, 7) design among which 5 are affine resolvable. The block intersections show that these 21 solutions are different from the 30 solutions given by Preece [5].

5. Solutions of (31, 15, 7) Designs

Starting from the 5 solutions of a (15, 7, 3) design and using Lemma 2.7 one can obtain a large number of non-isomorphic solutions of a (31, 15, 7) design. We indicate some of these solutions in Table 5.1, where we take

TABLE 5.1

Design	N	M	Characteristic number
F_1	N_1	$(1, 2,..., 15)N_1$	15
F_2	N_1	$(1, 2,..., 7)N_1$	15
F_3	N_1	$(2, 3,..., 7)N_1$	19
F_4	N_1	$(3, 4,..., 7)N_1$	23
F_5	N_1	$(4, 5,..., 7)N_1$	27
F_6	N_1	$(5, 6, 7)N_1$	31
F_7	N_1	$(6, 7)N_1$	35
F_8	N_1	N_1	43
F_9	N_3	N_3	43
F_{10}	N_2	$(1, 3, 4)N_2$	47
F_{11}	N_2	$(1, 3)N_2$	51
F_{12}	N_2	N_2	59
F_{13}	N_5	$(2, 7, 12, 13)N_5$	63
F_{14}	N_4	$(1, 2)N_4$	67
F_{15}	N_4	$(4, 8)N_4$	75
F_{16}	N_5	$(5, 10, 15)N_5$	83
F_{17}	N_4	N_4	91
F_{18}	N_5	$(1, 2)N_5$	107
F_{19}	N_5	N_5	155

$N = N_i$ and M to be either N_i or the incidence matrix obtained by taking suitable permutations of rows of N_i. Thus, in the design F_1, we take the incidence matrix P obtained by taking $N = N_1$ and taking M to be the matrix obtained by applying the cyclic permutation $(1, 2,..., 15)$ on the rows of N_1. The designs with different characteristic numbers are obviously non-isomorphic.

We now show that designs F_8 and F_9 are non-isomorphic. If the treatments of these designs are ∞ and $1, 2,..., 15$ corresponding to N

and 16, 17,..., 30 corresponding to M, then it is easily verified from the nature of special 3-tuples of N_1 that, in the 43 7-tuples of F_8, treatments numbered 18, 19, 20, 23, 25, 28, and 30 occur 19 times whereas the remaining 24 treatments occur each 7 times. On the other hand, it is easily verified that, in the corresponding 43 7-tuples of F_9, treatment numbered 20 occurs 35 times. This proves our assertion.

It is evident that corresponding to a special t-tuple occurring 3 times in a $(4t + 3, 2t + 1, t)$ design there is a 3-tuple occurring t times in the dual design and conversely. Therefore, the number of 3-tuples occurring t times in a given $(4t + 3, 2t + 1, t)$ design is precisely the characteristic number of its dual design. We use this fact to prove that F_1' and F_2' are non-isomorphic which, in turn, will prove that F_1 and F_2 are non-isomorphic. We note that, in the design P given by Lemma 2.7, a 3-tuple of M occurring in its t blocks occurs in $2t + 1$ blocks of P. The number of such 3-tuples is the characteristic number of the design corresponding to M'. Besides these, the other possible 3-tuples occurring in $2t + 1$ blocks of P contain either two treatments of N and one treatment of M or else contain ∞ and one treatment each from N and M. It can be verified that F_1 has in all 13 such 3-tuples. They are (1, 10, 29), (4, 10, 25), (5, 13, 25), (6, 8, 26), (8, 15, 25), (3, 25, ∞) besides the 7 special 3-tuples of M. But F_2 has besides these 7 special 3-tuples of M only the 3-tuple (1, 16, ∞). Thus F_1' and F_2' have the characteristic numbers 13 and 8, respectively, showing that F_1', F_2' and hence F_1, F_2 are non-isomorphic.

Besides these 21 non-isomorphic solutions we have another solution to a (31, 15, 7) design which is obtained from the difference set containing the 15 quadratic residues mod 31. This solution has characteristic number 0. We thus have at least 22 non-isomorphic solutions.

We note that C_1 and C_2 are isomorphic to duals of each other whereas C_2, C_4, and C_5 are isomorphic to their own duals. Hence F_8', F_9', F_{12}', F_{17}', and F_{19}' do not provide any new solutions. Further, the quadratic residue solution, being a difference set solution, is isomorphic to its own dual. The duals of the remaining solutions may be tried for non-isomorphisms by calculating the 3-tuples occurring 7 times in these designs.

One can conceivably obtain a much larger number of non-isomorphic solutions by taking $N = N_i$ and $M = N_j$, $i \neq j$, or the matrix obtained from N_j by taking a suitable permutation of its rows. We have, however, not attempted this. The number δ in these cases can be calculated by making use of the B-triples of the Table 3.2.

We note that 20 of the above solutions for a (31, 15, 7) design have different characteristic numbers. From Lemmas 2.5 and 2.6 it then follows that the number of non-isomorphic solutions to a (32, 62, 31, 16, 15) ARBIBD or a (63, 31, 15) design is at least 20.

6. An Asymptotic Result

Let D_0 be an existing $(4t+3, 2t+1, t)$ design. Then, from Lemma 2.6 for each $\alpha \geqslant 1$, there exists a solution to a design D_α with parameters $(2^\alpha(4t+4)-1, 2^\alpha(2t+2)-1, 2^\alpha(t+1)-1)$. We actually show that the number of non-isomorphic solutions for D_α tends to ∞ as α tends to ∞.

Assume that for $\alpha \geqslant 1$ there exist solutions $D_{\alpha,1}, D_{\alpha,2}, \ldots, D_{\alpha,\alpha}$ of a design D_α with characteristic numbers $x_{\alpha,1} > x_{\alpha,2} > \cdots > x_{\alpha,\alpha} > 0$. Using these solutions and Lemma 2.6 we obtain solutions $D_{\alpha+1,1}, \ldots, D_{\alpha+1,\alpha}$ having characteristic numbers $x_{\alpha+1,1} > x_{\alpha+1,2} > \cdots > x_{\alpha+1,\alpha} > 0$. We now use Lemma 2.7 with N as the incidence matrix of $D_{\alpha,\alpha}$ and M as the matrix obtained from N by permutation (i_1, j_1), on its blocks where (i_1, i_2, i_3) is a B-triple of N and $j_1 \neq i_1, i_2, i_3$. Since (i_1, i_2, i_3) is a B-triple of N, (j_1, i_2, i_3) is not a B-triple of N, for otherwise there will be a special tuple of a $(2^\alpha(4t+4)-1, 2^\alpha(2t+2)-1, 2^\alpha(t+1)-1)$ design occurring in 4 of its blocks. This is, however, a contradiction. The number δ of common B-triples of N and M is then strictly less than $x_{\alpha,\alpha}$. The solution $D_{\alpha+1,\alpha+1}$ thus obtained has the characteristic number $x_{\alpha+1,\alpha+1}$ which is strictly less than $x_{\alpha+1,\alpha}$. We thus obtain $\alpha+1$ non-isomorphic solutions to a design $D_{\alpha+1}$ corresponding to characteristic numbers

$$x_{\alpha+1,1} > x_{\alpha+1,2} > \cdots > x_{\alpha+1,\alpha+1} > 0.$$

We note that the existing solution of D_0 with the help of Lemma 2.6 gives a solution $D_{1,1}$ of D_1 with characteristic number $\geqslant 4t+3$. It then follows by induction that there exist at least α non-isomorphic solutions for D_α and hence the number of non-isomorphic solutions of D_α tends to ∞ with α.

It also follows from Lemma 2.5 that the number of non-isomorphic solutions to a $(2^\alpha(4t+4), 2^{\alpha+1}(4t+4)-2, 2^\alpha(4t+4)-1, 2^\alpha(2t+2), 2^\alpha(2t+2)-1)$ ARBIBD also tends to ∞ with α.

7. Solutions of (23, 11, 5) Designs

There exists a unique solution of a (11, 5, 2) design given by the difference set (1, 3, 4, 5, 9) mod 11. We write this design in Table 7.1. We note that the characteristic number of this design is 0.

Let N be the incidence matrix of this design. We now apply Lemma 2.7 to obtain solutions G_1, G_2, G_3, G_4 of a (23, 11, 5) design. In G_1 we take $M = N$ and in G_2, G_3, G_4, respectively, we take for M the matrix obtained by applying permutations (6, 7)(9, 10), (6, 7)(9, 10, 11), and (6, 8, 11, 9, 7) to the rows of N. All 4 designs have the characteristic

TABLE 7.1

Block number	Block				
1	1,	3,	4,	5,	9
2	1,	4,	6,	7,	8
3	1,	5,	8,	10,	11
4	1,	2,	6,	9,	11
5	1,	2,	3,	7,	10
6	2,	4,	5,	6,	10
7	2,	5,	7,	8,	9
8	2,	3,	4,	8,	11
9	3,	5,	6,	7,	11
10	3,	6,	8,	9,	10
11	4,	7,	9,	10,	11

number 11. The dual designs G_1', G_2', G_3', G_4' are easily seen to have the characteristic numbers 11, 3, 2, and 1 and hence are non-isomorphic. The designs G_1, G_2, G_3, G_4 are then also non-isomorphic. It is obvious from the nature of G_1 that G_1' is isomorphic to G_1. We also have the quadratic residue solution mod 23 which is easily seen to have the characteristic number 0. Thus there exist at least 8 non-isomorphic solutions of a (23, 11, 5) design.

8. Concluding Remarks

The concept of a special t-tuple of a $(4t + 3, 2t + 1, t)$ design can be generalized to the series S_n of $(n(nt + 1) + 1, nt + 1, t)$ designs. It can easily be verified that any t-tuple occurs at most in $(n + 1)$ blocks of such a design. We call such a t-tuple a special t-tuple and the number of such t-tuples the characteristic number of this design. Correspondingly we may consider $(n + 1)$-tuples occurring in t blocks of the design. The number of such $(n + 1)$-tuples will be the characteristic number of the dual design. For $n = 2$, the series S_n reduces to the class of $(4t + 3, 2t + 1, t)$ designs.

We note that, if a design S_n contains a special t-tuple, then any other block besides the $(n + 1)$ blocks of the t-tuple can contain only $t + in$ treatments where i is a non-negative integer. If n and t are both even, then $t + in$ and the block size $nt + 1$ are of opposite parity, which is impossible. Thus the characteristic number of S_n is always 0 when n and t are both even.

The characteristic number of designs of the series S_n can be utilized to distinguish two solutions for non-isomorphism.

Some additional techniques to generate non-isomorphic solutions of designs will be given in a forthcoming sequel to this paper.

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