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THE CONTRIBUTIONS OF CARL FRIEDRICH GAUSS TO GEOMAGNETISM

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SUMMARIES

The three areas of geomagnetism in which Gauss made great contributions were those related to the absolute measurement of the field, the analysis in terms of spherical harmonics, and the organization and equipping of magnetic observatories. His approaches and accomplishments in these areas are examined in some detail. Because of the insufficiency of observations over the globe at the time he worked, many of the investigations which he proposed on the basis of the spherical harmonic analysis had to await later workers. These include the quantitative separation of internal and external sources, the effect of the earth's ellipsoidal shape, the possible non-vanishing of the constant term in the expression for the potential and the possible existence of a non-potential portion of the field. The results of these later investigations are outlined.

Les contributions de Gauss à l'étude du geomagnêtisme furent primordiales dans trois domaines: le mesurement absolu du champ, son analyse en termes d'harmoniques sphêriques et l'organisation ainsi que l'équipement d'observatoires magnêtiques. Ses approches ainsi que ses accomplissements dans ces domaines sont examinês de façon quelque peu dêtaillée. Parce que le nombre d'observations magnêtiques à la surface du globe était insuffisant à son époque, l'investigation de nombreux problèmes proposés par Gauss en termes d'analyse en harmoniques sphêriques n'a pu être menée à bien que plus tard. Ce fût le cas notamment de la séparation quantitative du champ en sources internes et externes, de l'effet de la forme

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ellipsoidale de la terre, de la possiblité d'avoir à conserver un terme constant dans l'expression du potentiel ainsi que de l'existence possible d'une contribution non-potentielle au champ total. Les résultats de ces investigations obtenus postérieurement à Gauss sont mentionnés.

INTRODUCTION

The study of the earth's magnetic field has a long history before the time of Gauss. Although it was his principal occupation for only about fourteen years, Gauss's work marks the beginning of the understanding of the quantitative aspects of the field and of its time variations. In order to see the significance of his contributions, it is useful to trace the development of the subject both before and after his time. The earliest history is lost in antiquity, but includes (in the western world) the discovery of lodestone, probably by the Greeks, and the discovery of the directional property of the field (in China) at least as early as the eleventh century [Needham 1962]. Subsequent developments, described by Chapman and Bartels [1940], include discovery in the 15th century of the declination of the compass from true north, and of the inclination or dip of the field by 1576. The geometry of the lines of force around the spherical earth was summarized by William Gilbert in his famous "De Magnete" of 1600. Edmund Halley's voyage in the naval ship "Paramour Pink", undertaken to measure the declination, was probably the first expedition made for purely geophysical purposes. This resulted in the first global map of a geophysical quantity, published in 1700. Toward the end of the eighteenth century, the intensity of the field, as well as well as its direction, was being measured by explorers. The technique used to obtain relative values of the intensity involved comparison of the periods of oscillation of a suspended magnetic needle at different stations, as described in more detail in this paper. Such work was intensified in the early years of the nineteenth century, particularly by Alexander von Humboldt of Germany on his expeditions to South America and Asia, and by several British explorers in the Arctic. But there was no clear understanding of the distribution of the field over the globe, and an erroneous suggestion by Halley, that the field had a quadrupole rather than dipole symmetry, persisted until the time of Gauss.

During the seventeenth and eighteenth centuries, the time variations of the geomagnetic field were recognized also. Gellibrand first recognized the long-period or secular charge in 1634, and Graham established the daily or diurnal variation in 1772. Humboldt recognized the presence of irregular disturbances and gave the name "magnetic storms" to them. But by the year 1830, there was still no understanding of the relationship between these time changes, of their distribution over the globe, nor, of course, of their causes.

The context from which Gauss, in 1829, began to devote so much of his effort to geomagnetism has been discussed by May [1972]; precisely why Gauss's interest was kindled will probably never be known. His three great contributions in geomagnetism deal with the absolute measurement of the field, its mathematical description and the systematic study of the time-variations.

THE ABSOLUTE MEASUREMENT OF THE FIELD [1832]

By the late eighteenth century it had been recognized that the period of oscillation of a suspended needle is inversely proportional to the square root of the field. Most observations upon which the early charts of field intensity were based were made with such needles, where periods were compared with those at a standard station such as Paris or London. The unit of intensity in common use was introduced by Humboldt, and was based on a value of unity for the field at Micuipampa, Peru, which was a point on the magnetic equator. Humboldt assumed this would be minimum value for the total intensity over the earth (a reasonable assumption, but incorrect in detail because of local perturbations in the field). Needles could be suspended as dipping needles, to give relative values of the total intensity, or in the horizontal plane, to give relative values of the horizontal component (H). In the latter case, the period T is given by

$$T = 2\pi \sqrt{\frac{I}{MH+c}} ,$$

where I is the moment of inertia of the suspended system about its axis, M is the magnetic moment of the magnet, and c is the restoring couple due to the elastic suspension. If the suspending fibre is sufficiently fine, c may be made arbitrarily small. In this case

$$MH = \frac{4\pi^2 I}{T^2} . \tag{1}$$

All of the measurements made with such needles, before 1832, were relative, rather than absolute, since the quantity H could not be isolated from the potentially variable magnetic moment M.

Gauss's great contribution ("Intensitas vis magneticae terrestris ad mensuram absolutam revocata", read to the Royal Society of Göttingen on December 15, 1832) was the introduction of a procedure whereby the magnet of moment M is employed in a second experiment, immediately after the period has been measured. In the second experiment it is used to deflect an auxilliary magnet mounted as a compass needle (Fig. 1). If the

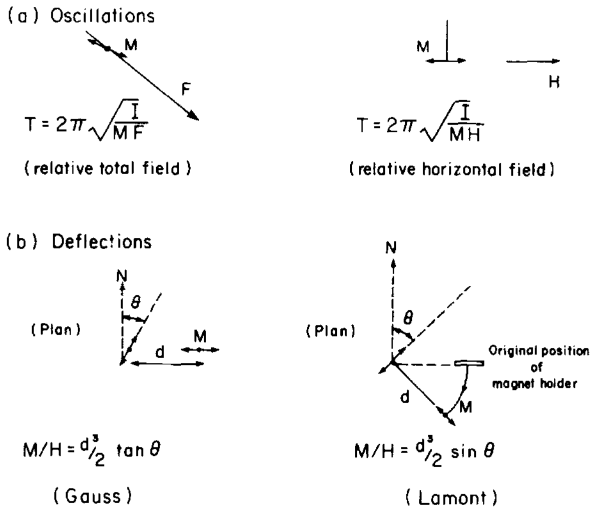

Figure 1: The principle employed by Gauss for the absolute determination of H by means of oscillations (upper right) and deflections (lower left). Older relative measurements included oscillations about the total vector (upper left). Lamont's modification of the deflection experiment is shown at the lower right.

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magnet M can be treated as a dipole, then

$$\frac{M}{H} = \frac{d^3}{2} \tan\theta. \tag{2}$$

All of the quantities on the right of equations (1) and (2) can be measured in terms of mass, length and time, and from the equations both M and H can be determined. The moment of inertia I can be computed or measured by adding a non-magnetic body of known moment of inertia to the suspension, and then observing the period again. If the magnet M is too large to be treated as a dipole, Gauss showed that a geometrical factor could be introduced in (2), but this does not alter the dimensional analysis. In particular,

$$H = \left(\frac{8\pi^2 I}{d^3 \sin\theta T^2} \right)^{\frac{1}{2}}.$$

The dimensions of magnetic field H therefore follow directly as $[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}]$. Gauss measured M in milligrams, L in millimeters, and T in seconds. His absolute unit therefore corresponds to 0.1 of a centimetre-gram-second (c.g.s.) unit of magnetic field (i.e. $\mathrm{gm}^{\frac{1}{2}}\ \mathrm{cm}^{-\frac{1}{2}}\ \mathrm{sec}^{-1}$). The magnetic field thus became the first

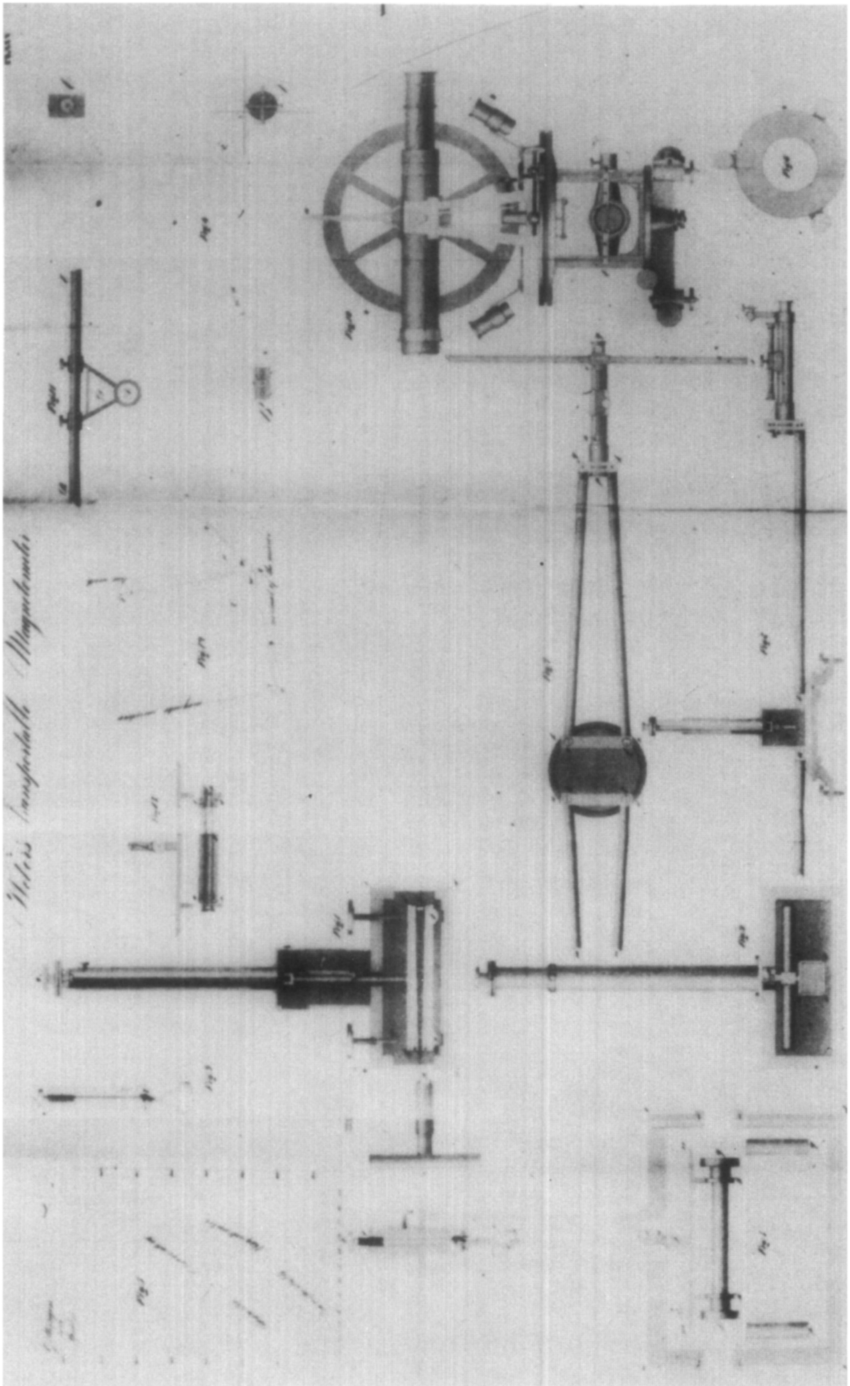

Figure 2: Portable absolute instruments designed by Gauss and Weber. The meticulous detail is typical of the instructional papers written for the Magnetische Verein.

Figure 3: The non-magnetic building erected at Göttingen for Gauss's experiments. The building still stands, although not on its original site. (Courtesy of Prof. U. Schmucker, Göttingen).

non-mechanical quantity to be expressed in terms of mass, length and time. Gauss, however, never heard of the "Gauss" as a magnetic unit. After being proposed for different electromagnetic quantities, it was adopted by the International Electrical Congress in 1900 as the c.g.s. unit of magnetic field (see Glossary).

With one slight modification, the method of Gauss became virtually the standard for magnetic field determinations until almost the middle of the twentieth century. The modification, introduced by Lamont in 1849, relates to the position of the magnet M in the second experiment, and is illustrated in Fig. 1. Lamont's method has the advantage of eliminating any possible effect of torsion in the fibre on the deflection of the auxilliary needle.

It is probable that Gauss at first visualized his absolute method as being used at only a small number of few base stations, where relative instruments would be calibrated. However, he and his collaborators soon developed portable absolute instruments and the successors of these were used for measurements at land stations over the entire earth (Fig. 2).

THE GÖTTINGEN MAGNETIC UNION (MAGNETISCHE VEREIN), 1834

The collaboration of Gauss with the much younger Wilhelm Weber began in 1831, and lasted for many years. Their initial efforts were to make systematic observations of the magnetic elements, at prescribed times of day, in a new non-magnetic building at Göttingen (Fig. 3). At the urging of the von Humboldt, they eventually organized a network of observatories, at first in Europe, where measurements could be made simultaneously with those in Göttingen.

The correspondence of Gauss during this period [Schering 1887] shows that, while von Humboldt approached foreign academies, Gauss maintained the closest possible contact with scientists at the magnetic stations. The Magnetic Union, in fact, grew to include 35 observatories in Europe, and this was increased by the addition of the British Colonial Observatories, and by observatories in Siberia and other parts of Asia. During the years 1836-1841, many of the original papers by Gauss and Weber were published in a series known as the *Annual Reports* of the Magnetic Union. In a very real sense, the Union was one of the first examples of international scientific cooperation and it was the forerunner of the International Association of Geomagnetism and Aeronomy (a constituent association of the International Union of Geodesy and Geophysics).

Great effort was devoted by Gauss and Weber in designing observatory instruments (Fig. 4) suitable for the measurement of time-changes in the magnetic field. The measurement of short-period changes in the direction of the field presented no great difficulty, for these could be followed by means of suspended or pivoted needles supplemented by an optical system comprising scale, mirror and telescope. But the intensity (see Glossary) could not be measured by the usual method of timing oscillations, since the computation of a mean period would obscure the time variations. The simplest static technique used to record changes in H employs a needle suspended by means of a torsion fibre. By twisting the fibre until the needle is at right angles to the meridian, it is possible to record small changes in the needle's orientation. But in Gauss's day, to obtain adequate sensitivity with the fibres then available, a large magnetic moment was required, but this made it difficult to twist the needle and maintain stability. Gauss overcame this problem with the bifilar suspension, whose characteristics he discussed in detail [1837]. The sensitivity of the needle to small changes in H increases with magnetic moment and with the length of the fibres relative to their distance apart (Fig. 5). In his original apparatus, Gauss used a magnet of weight 25 pounds, apparently almost one metre long. It was suspended by two portions of a steel wire, the suspension being 17 feet long, with the wires 1.5 inches apart. With the suspension head rotated until the magnet was perpendicular to the meridian, the angle θ between the azimuth

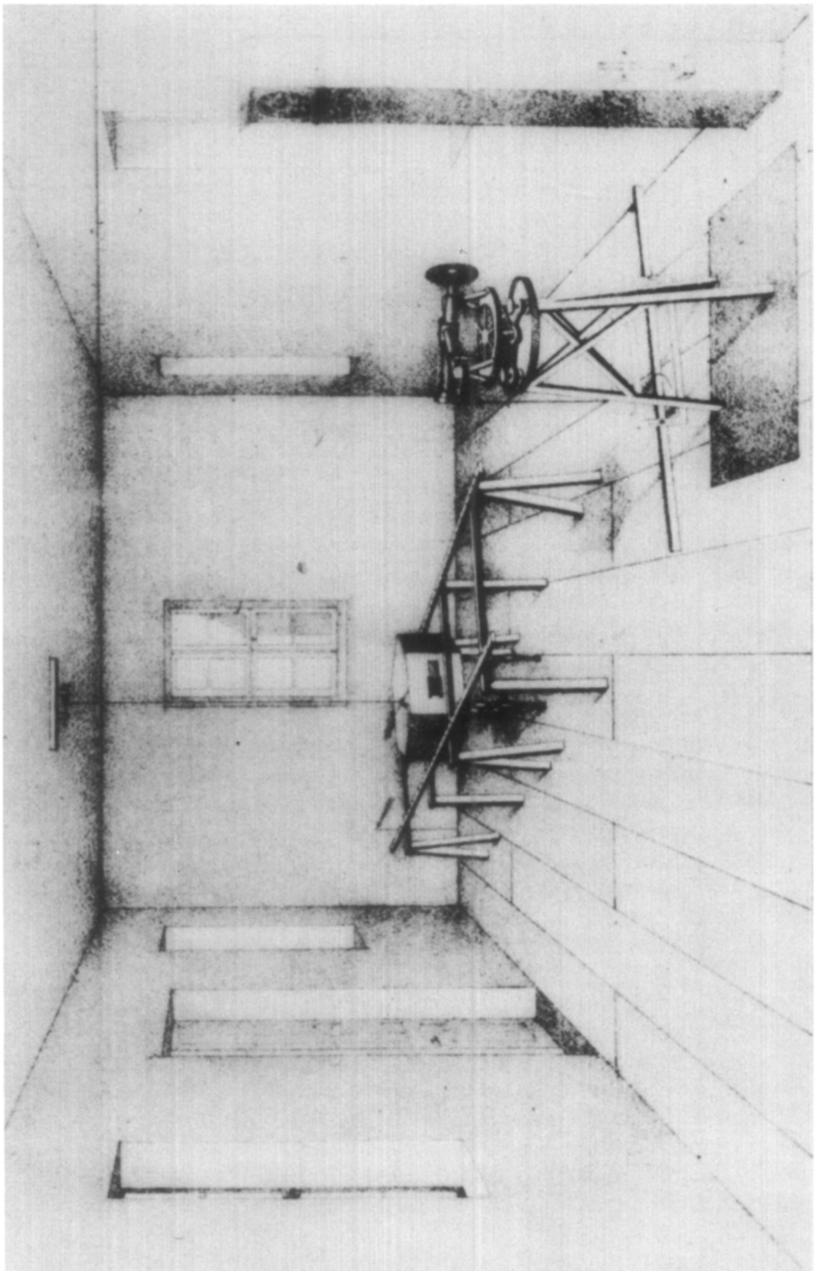

Figure 4: The interior of the Göttingen magnetic observatory, viewed from the south. A suspended magnet observed with the theodolite in the foreground was used for both intensity and declination measurements. A long-case clock kept time for the observatory.

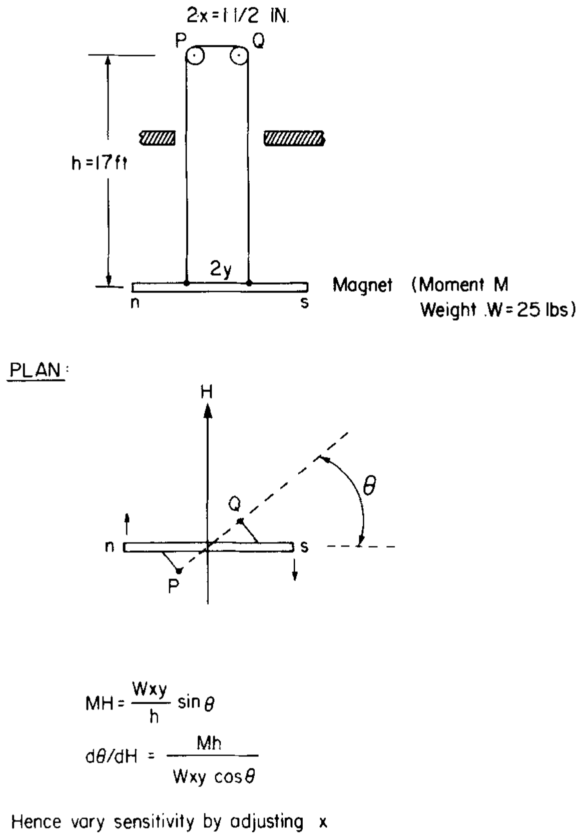

Figure 5: Principal of the bifilar magnetometer for the detection of changes in H. The dimension shown are those of Gauss's original instrument. As shown in the plan (lower), the quantity recorded is the angle θ between the vertical plane through the needle and that through the supports, P and Q.

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defined by suspension points *PQ* and the needle became very sensitive to small changes in *H*. Particularly important is the fact that the sensitivity can be set to any desired value, to compensate for changes in *M*, for example, by altering the distance *x*, without disturbing the suspension itself. The bifilar suspension (not usually of such great length) was in general use in magnetic observatories for many years. The introduction of sensitive, stable quartz fibres and improved optical systems eliminated the need for such large magnets and it is today rarely seen.

It must be remembered that until 1848 there was no such thing as photographic recording at an observatory. All readings of time variations were taken manually. It was therefore important that schedules be adopted for the taking of observations and vital that these schedules be adhered to. Much of the correspondence of Gauss during the early years of the Magnetische Verein was devoted to arranging periods of simultaneous observations and periods of more frequent observations to detect shorter period variations. Outside of these special intervals, the usual routine was to observe the three magnetic elements each hour, throughout the twenty-four. The staffs of magnetic observatories were indeed devoted.

ALLGEMEINE THEORIE DES ERDMAGNETISMUS [1838]

Modern analysis of the magnetic field over the earth's surface dates from this great paper, in which Gauss expressed the potential of the field as a sum of spherical harmonics. Although others, in particular Legendre, had used spherical harmonics before, Gauss's application of the theory to the very sparse observational data then available was most courageous, and his insight into what could be deduced from the representation was remarkable.

In his analysis, the earth was taken to be spherical and the source of the field to be completely internal. The potential V was expressed, in slightly revised notation, as

$$V = a \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} \left[g_n^m \cos m\lambda + h_n^m \sin m\lambda \right] P_n^m (\cos\theta),$$

where a is the earth's radius, r, θ and λ are the spherical coordinates of a point (θ the co-latitude), and $P_n^m(\cos\theta)$ is an associated Legendre function. Coordinates of the magnetic field X (northward), Y (eastward) and Z (downward) are given by

$$X = \frac{\partial V}{r\,\partial\theta}, \quad Y = \frac{1}{r\sin\theta}, \quad Z = \frac{\partial V}{or}.$$

The problem was to determine the unknown coefficients g_n^m and h_n^m (called by Gauss the "elements" of the field.) The coefficients g_n^m and h_n^m have the units of magnetic field. Because he worked with data expressed in the older or von Humboldt's units, Gauss expressed the coefficients in these terms, despite the fact that he had previously demonstrated that the field could be measured absolutely. (The old unit is equivalent to 0.3494 Gauss.) Writing in 1838, Gauss remarked that for many years he had wished to attempt the analysis, but that is was necessary for him to await the publication of sufficient measurements. He attributed his immediate incentive to produce finally such an analysis to the publication by Sabine [1837] of a global chart for the total intensity.

Sabine's chart shows the actual stations at which the magnetic elements had been measured. Many of these are of considerable interest in themselves. For example, stations in what is now British Columbia were those of the botanist David Douglas (after whom the fir is named). Douglas died during his expedition (1829-1834), but his notebooks were returned to London. Gauss combined Sabine's chart with older charts for the declination and inclination (see Glossary), reading off the three quantities at "12 points on 7 parallels of latitude". He realized the inherent weakness of combining observations from different epochs (Sabine's intensity chart itself was not reduced to a common epoch (see Glossary), and contained observations made over a period of several years before 1837), but he had no other choice.

For the numerical determination, Gauss expressed each field component along a given parallel in the form

$$X = k + k_1 \cos \lambda + K_1 \sin \lambda + k_2 \cos 2\lambda + \ldots$$

$$Y = \ell + \ell_1 \cos \lambda + L_1 \sin \lambda + \ell_2 \cos 2\lambda + \ldots$$

$$Z = m + m_1 \cos \lambda + M_1 \sin \lambda + m_2 \cos 2\lambda + \ldots \quad .$$

Then by equating coefficients of sin $j\lambda$ and cos $j\lambda$,

$$-k_j = \sum_{n=j}^{\infty} g_n^j \frac{\partial P_n^j (\cos \theta)}{\partial \theta} \quad ,$$

$$L_j = 2 \sum_{n=j}^{\infty} g_n^j \frac{P_n^j (\cos \theta)}{\sin \theta} \quad ,$$

$$m_j = \sum_{n=j}^{\infty} (n+1)\, g_n^j \, P_n^j (\cos \theta) \quad ,$$

$$-K_j = \sum_{n=j}^{\infty} h_n^j \frac{d\, P_n^j (\cos \theta)}{d\theta} \quad ,$$

$$-\ell_j = \sum_{n=j}^{\infty} h_n^j \frac{P_n^j (\cos \theta)}{\sin \theta} \quad .$$

$$M_j = 2 \sum_{n=j}^{\infty} (n+1)\, h_n^j \, P_n^j (\cos \theta) \quad .$$

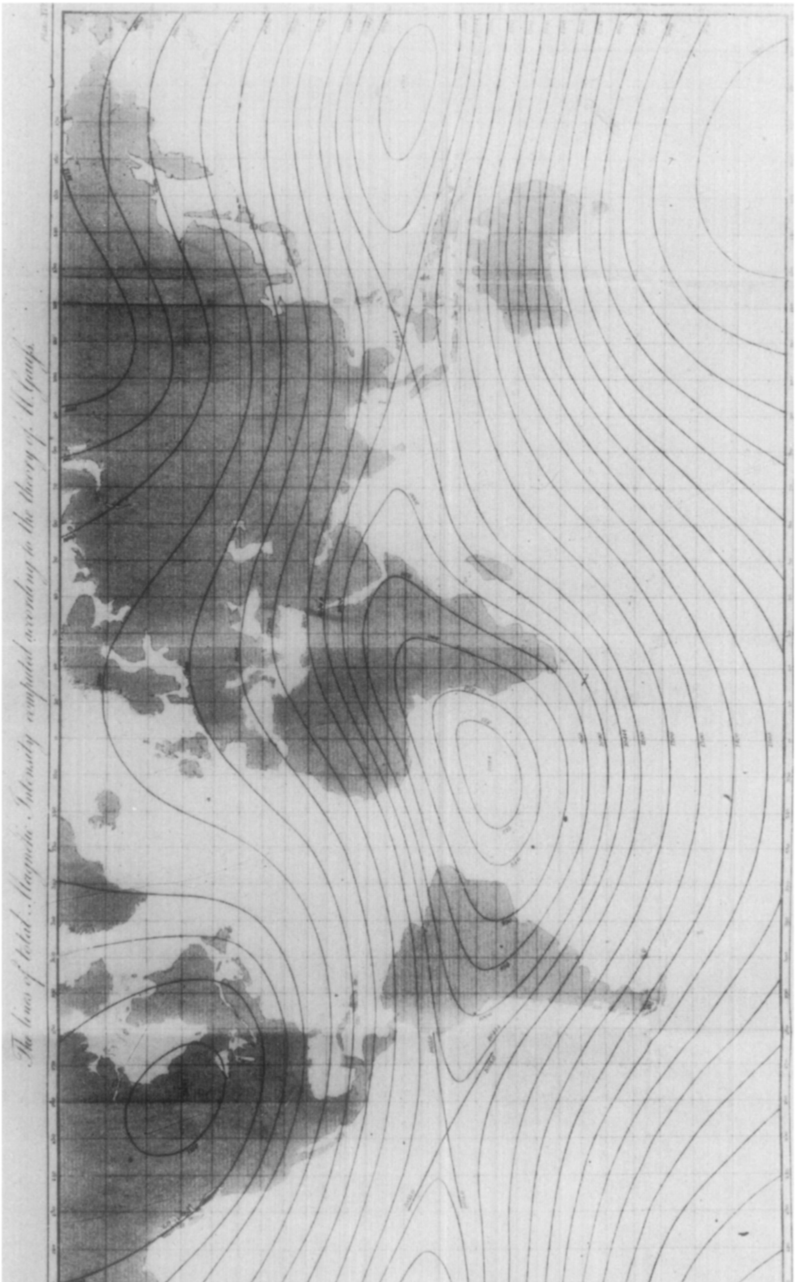

Figure 6: (a) The total intensity field of the earth, as synthesized by Gauss from his spherical harmonic expansion to the fourth degree. It may be compared with Figure 6 (b), a modern chart (the International Geomagnetic Reference Field) of the total intensity.

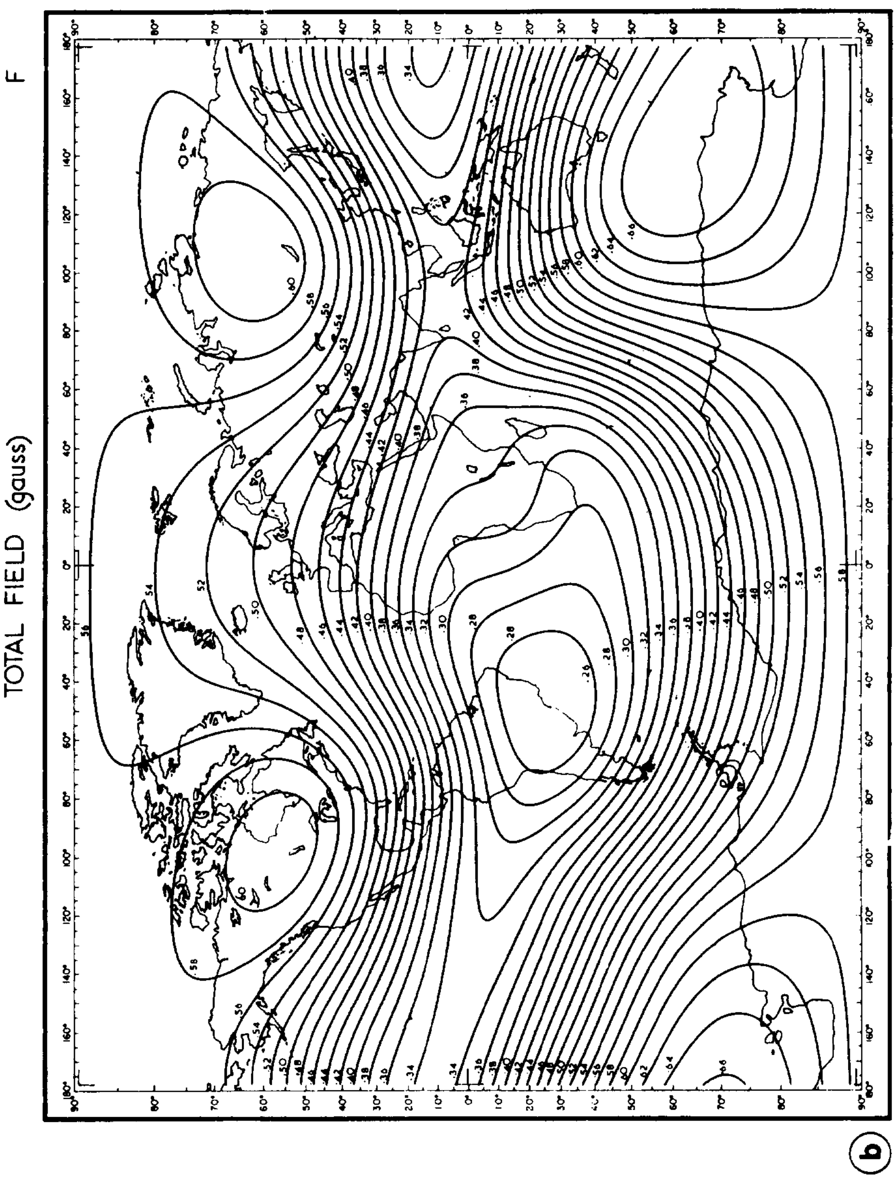

Figure 6: (continued)

Thus, for each order j, there were 6 equations for each of the 7 parallels of latitude along which the field was expanded, or 42 equations. Gauss used the method of least squares, which involved 168 equations to solve for the 24 coefficients g_4^4 and h_4^4.

Having obtained the coefficients, it was then possible to compute the total force, declination or inclination at any point on earth. In addition to producing charts (Fig. 6), Gauss computed the field at 86 observation points (Fig. 7), and obtained values whose mean difference from the observed, without regard to sign, were 0.046 units for the intensity, 1°6' for the inclination and 1°30' for the declination. Of course, many of the comparison points shown on Fig. 7 were computed for the same stations whose values went into the production of the charts. However, Gauss felt that the agreement between the observed values and those obtained by the lengthy computations based on chart-gridded values, was a good test of the analysis. By contrast, discrepancies between different observers at the same site were often much larger than the differences between observed and computed values. The memoir notes the case of Otaheite, where the difference in intensity between measurements made in 1830 and 1835 was 0.155 units.

Gauss recognized that when further observations become available, extensions of his method of analysis and the detailed investigation of certain characteristics of the field would be very desirable. In particular, he outlined the modification required to adapt the analysis to the ellipsoidal figure of the earth, but for the observations that were then available, Gauss did not believe that the modification was justified.

His values for the coefficients showed immediately that there was a preponderance of the dipole terms. In his units, g_1^0 was 925,782, much greater than any of the remaining coefficients, the next in value being h_1^1 at -178,744. This was in itself an important result, as there were still scientists who, following Halley, believed in a quadrupole symmetry of the field over the earth. From the expansion, Gauss was also able to obtain the location of the field source (internal as opposed to external), to determine whether or not P_0 vanished, and the question of whether the field is entirely derivable from a potential.

Gauss firmly believed that the field source was completely internal and that this view was supported by the agreement between observations and his synthesized field components computed from the internal-source expression. He wrote:

> *Another part of our theory, which can resolve an uncertainty, is the proof that the source of the*

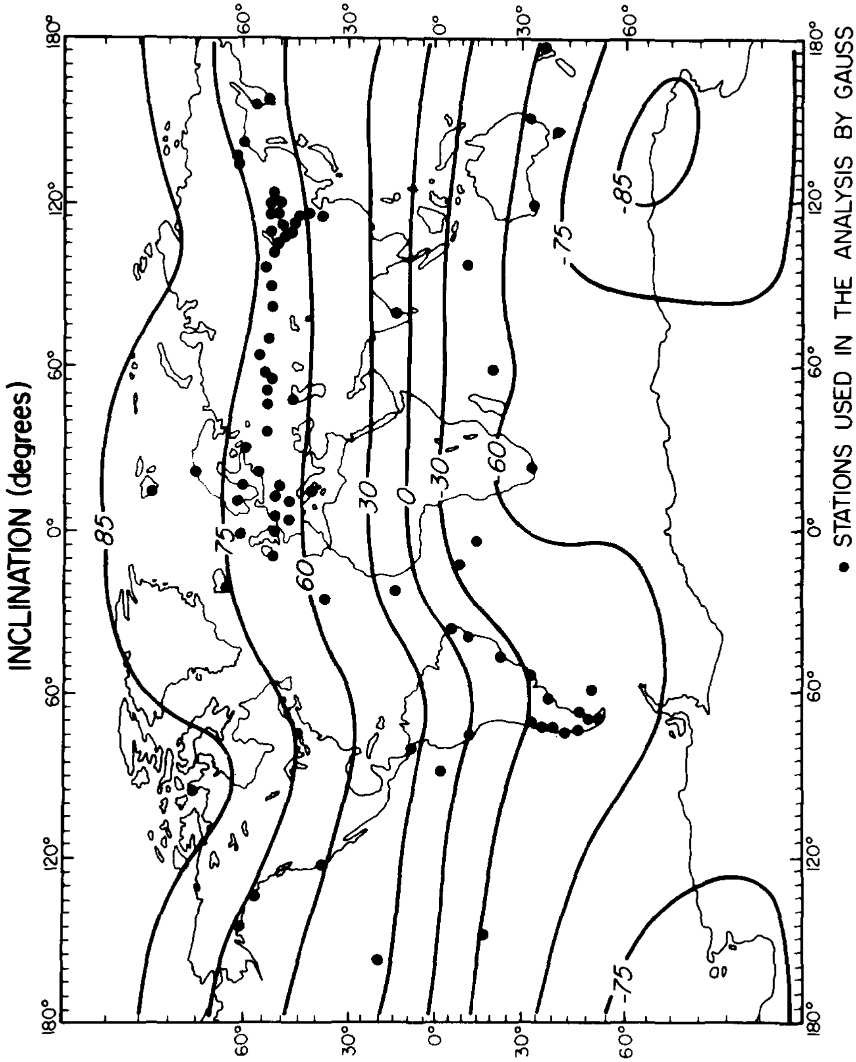

Figure 7: The distribution of observing stations for which Gauss computed the field given by his expansion, to compare with measurements.

> *magnetic field is internal to the earth. If the source of all or part of the field is sought outside the earth, we can, insofar as we avoid fantasy and stick to known facts, only attribute it to a galvanic current. The ordinary air is not a conductor, nor is empty space; this leaves only the upper atmosphere as a possible conductor. Among those theories of which electricity in motion plays a role, only the mysterious displays of the aurora keep us from disavowing the possibility of such a current...* [Gauss, *Werke* 5, 169]

The formal separation of internal and external contributions to the field depends upon the fact that the dependence of the potential upon r is different in the two cases. If only horizontal components (x and y) are known over the earth, then the spherical harmonic coefficients yield the sum of the contributions of internal and external sources. On the other hand, if the vertical component is known, then these contributions are obtained separately.

When there are both internal and external sources, the potential V may be conveniently written:

$$V = a \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[\left\{ C_n^m \left(\frac{r}{a}\right)^n + (1-C_n^m)\left(\frac{a}{r}\right)^{n+1} \right\} g_n^m \cos m\lambda + \left\{ S_n^m \left(\frac{r}{a}\right)^n + (1-S_n^m)\left(\frac{a}{r}\right)^{n+1} \right\} h_n^m \sin m\lambda \right] P_n^m (\cos\theta).$$

Here C_n^m and S_n^m are dimensionless quantities lying between 0 and 1; they represent the fraction of each term which arises from *external* sources.

On the earth's surface, $r = a$ and the components x and y of the field do not involve C_n^m and S_n^m. In this case, the coefficients g_n^m and h_n^m may be derived as before. However, the vertical component z is given by

$$z = \frac{\partial V}{\partial r} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[\{nC_n^m - (n+1)(1-C_n^m)\} g_n^m \cos m\lambda + \{n S_n^m - (n+1)(1-S_n^m)\} h_n^m \sin m\lambda \right] P_n^m (\cos\theta).$$

The analysis of z over the globe yields values for the quantities in the caliper brackets from which C_n^m and S_n^m are determined. In this way, the relative importance of internal and external sources for each harmonic term is known.

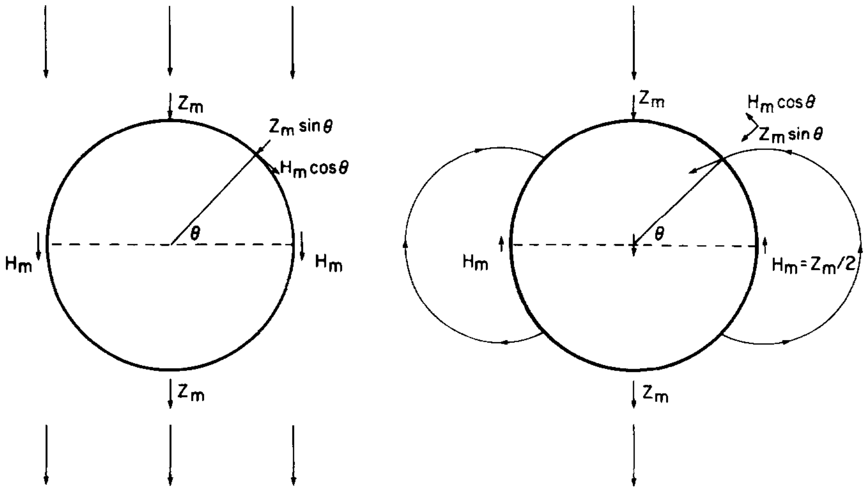

Figure 8: Relations between components in the case of an external, uniform source (left) and an internal, dipole source (right).

As an example, we may consider the variation of X and Z corresponding to the first zonal harmonic,

$$X = g_1^0 \sin\theta$$

$$Z = \{c_1^0 - 2(1-c_1^0)\}\, g_1^0 \cos\theta.$$

If $c_1^0 = 0$, the field is completely internal in origin and is due to a dipole (Fig. 8); if $c_1^0 = 1$, the field is external in origin and is uniform. Although the distribution of the component X is identical in the two cases, Z is very different.

The method outlined above was proposed by Gauss, but not carried out. Similarly, he noted that if the field is derivable from a potential, the variations of X and Y are not independent. In particular, the coefficients g_n^m and h_n^m obtained from either X or Y alone should be identical. Equivalently, observed field increments between neighbouring points on the surface should be independent of the path joining those points. The existence of a potential in turn is equivalent to the vanishing of the line integral of the field around any path on the surface, or to the vanishing of the curl of the field. Failure for this condition to be observed would in principle indicate a non-potential portion of the field (that is a non-conservative field (see Glossary)) such as would be due to current flowing across the earth's surface. Gauss stressed the importance of testing this, but felt that the observations available to him, and especially their reduction to a common epoch, were insufficient.

Finally, Gauss noted the desirability of investigating the term P_0. For, if it were shown that $P_0 \neq 0$, this would mean non-equality of north and south poles within the earth, hence the existence of magnetic monopoles. He wrote, "the possibility of inequality should not yet be dismissed", and pointed out the advantage of investigating the term P_0 for a body the size of the earth. Regretfully, he admitted that with the available data, no conclusion would be reached if a non-zero value of P_0 was taken.

Many more observations were required before his proposals for further investigations of the field could be carried out, and Gauss was not to live to see this happen. A second analysis of the field was published 30 years after his death [Schmidt 1889], and it has only been within recent decades that satisfactory answers to some of the questions have been found.

THE BRITISH MAGNETIC OBSERVATORIES

The temporal recording of the magnetic elements at a fixed observatory serves two purposes: it provides the long-period secular change (see Glossary) for the reduction of observations to a common epoch and yields information on the shorter-period, externally-produced variations in the field. Gauss and Weber certainly recognized the importance of both functions and the desirability of expanding the original Magnetische Verein to a more global distribution. To accomplish this, it was necessary to seek the cooperation of countries outside of Europe.

In April 1836, von Humboldt wrote to the Duke of Sussex, President of the Royal Society of London. In this remarkable letter, Humboldt summarized the developments in geomagnetism, describing his own observations and the work of Gauss. He pointed to Britain's unique opportunity to establish magnetic observatories in its colonies around the world. The term "magnetic storm" was probably introduced for the first time in this letter, which also notes the long association of British scientists with geomagnetism.

This letter, from which we quote below, pre-dates the publication of the *Allgemeine Theorie*. The references to Gauss relate to his work on instrumentation and observation:

> *England, from the former works of William Gilbert, Graham and Halley to the modern work of Gilpin, Beaufoy, Barlow and Christie has made available a rich collection of material suitable for the discovery of physical laws which govern the variation of the magnetic declination.......*
>
> *The great geometer, Mr. Gauss, to whom we owe this method of observation, as well as the method of*

> *obtaining in absolute terms the magnetic intensity anywhere on earth and the ingenious investion of a magnetometer--published in the years 1834 and 1835, series of observations.....*
>
> *I have had to recall the beautiful works of Mr. Gauss so that members of the Royal Society could take (them) into consideration (in establishing new stations).* [Translated from Schering 1887, 15-20]
> (Original in French)

The President and Council of the Royal Society submitted the letter to S. Hunter Christie and G. B. Airy (the Astronomer Royal) for a detailed analysis [Hunter Christie and Airy 1837]. Christie and Airy responded enthusiastically, and in a report dated 9 June 1836, they made the following recommendation:

> *to His Royal Highness, the President and to the Council that such a representation be made to the Government in order that means may be ensured for the establishment, in the first instance, of magnetical observatories... .*

By 1839, the British Government had agreed to fund the establishment and operation of magnetic observatories in Canada, Hobart, South Africa and St. Helena, as well as in Greenwich and Dublin (the last supported in part by the University of Dublin). All of these observatories, whose origin can be ascribed, through the intervention of von Humboldt, to the concurrent activity of Gauss, had long and distinguished records of contributing significantly to geomagnetism.

LATER DEVELOPMENTS BASED UPON THE IDEAS OF GAUSS

There have been many spherical harmonic analyses of the field in the years since 1838, each in turn benefitting from the more complete distribution of observations, better reduction to a common epoch, and, in recent years, from high-speed computers. Individual efforts [e.g., Schmidt 1889, Bauer 1922, Vestine et al 1947, Cain et al 1965, Leaton et al 1965, Hurwitz et al 1966] led to the establishment of International Geomagnetic Reference Field [Zmuda 1971], and the adoption of an internationally-uniform set of magnetic coefficients for the harmonics up to degree 8. The first few coefficients are compared with those obtained by Gauss in Table 1. (The recalculation of coefficients is not simply a matter of different units; the functions P_n^m are now normalized in such a way that they differ, by factors involving m, from those used by Gauss). Also shown is the secular rate of change of each coefficient, as determined in 1965. Obviously, a portion of the discrepancies could be real, simply due to accumulated secular change over approximately 130 years.

TABLE 1

	g_1^0	g_1^1	h_1^1	g_2^0	g_2^1	h_2^1	g_2^2	h_2^2
Gauss (γ)	-32350	-3110	+6250	+510	+2920	+120	-20	+1570
IGRF 1965.0 (γ)	-30339	-2123	+5758	-1654	+2994	-2006	+1567	+130
Secular Change (γ/yr)	+15.3	+8.7	-2.3	-24.4	+0.3	-11.8	-1.6	-16.7

$$1\ \gamma = 10^{-5}\ \text{G.}$$

$$V = a \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} \left[g_m^n \cos m\lambda + h_m^n \sin m\lambda \right] P_n^m (\cos \theta)$$

Points of special physical interest, raised by Gauss in the *Allgemeine Theorie* and investigated by later workers, include the necessity to use spheroidal functions, the reality of the external field, the existence of a non-conservative part of the field, and the possible non-vanishing of P_0. Small, but significant, contributions to the coefficients from external sources have been observed in most of the analyses since that of Schmidt. Winch [1967] showed that the magnitude obtained for the external contributions is extremely dependent upon whether or not spheroidal geometry is used. He developed a very straightforward way of expanding the potential in oblate spheroidal coordinates, still using the conventional associated Legendre polynomials, and was able to show that to the first order in the square of the ellipticity (ε^2),

$$g^{*0}_{1,i} = (1 + \frac{2}{5}\varepsilon^2)\, g^0_{1,i}$$

and

$$g^{*0}_{1,e} = g^0_{1,e} - \frac{2}{5}\varepsilon^2\, g^0_{1,i} \; .$$

The asterisk indicates the value given by analysis in spheroidal coordinates and the subscripts e and i indicate external and internal contributions respectively. For the earth, ε^2 is 0.006723, and $g^0_{1,i}$ is always found to be of the order of -30,000 γ ($\gamma = 10^{-5}$ Gauss).

The improvement on the estimate of $g^0_{1,i}$, by using spheroidal coordinates, is relatively very small, but the absolute value of $g^0_{1,e}$ is reduced by some 80 γ. Winch found that, when the analysis was applied to the data of Schmidt for 1885, 18 coefficients for the external field remained significant at the 95 per cent confidence level, $g^{*0}_{1,e}$ being -139 γ. By contrast, applied to the data of Hurwitz et al for the epoch 1965, the equation indicates $g^{*0}_{1,e}$ to be 0 γ.

The most probable cause of an external contribution to g^0_1 is a steady ring current of charged particles above the equator, since shorter period ring currents are known to have an important effect on the magnetic disturbance field. Since 1965 was close to a minimum of the solar cycle (1964), it is not unreasonable that any steady ring current should be very small at that epoch. It is important, in principle, that the full expression for the potential, including external sources, be used in analyses. However, a number of analyses, including the International

Geomagnetic Reference Field and the original one made by Gauss, have been based on a totally internal source.

Most analyses have assumed that the field is entirely derivable from a potential, basing the determination of coefficients upon a combination of X and Y. It is obviously not possible to test for a non-potential portion of the field in those cases. From time to time, the suggestion has been made that the magnetic charts themselves show the presence of a non-potential portion; that is, that $\oint H \cdot ds \neq 0$.

A very simple test to determine if such a non-potential portion exists is to construct a path on a global chart of the declination which is everywhere perpendicular to the horizontal component, and to see if the path closes on itself. Some tests of this type have indicated the presence of earth-air current in densities as great as 3×10^{-11} amp/cm^2 [Schmidt 1924], which would be 10^5 times as great as that indicated by atmospheric electricity measurements. Bartels [1939] concluded that these estimates simply reflected errors in the charts, in particular, systematic errors in reduction to epoch by correction for the secular variation. He believed "that the computed values for the currents are within the limits of the observational errors and that the onus of proof rests with those who hold the opposite opinion". While this was undoubtedly the sensible point of view, it is worth pointing out that magnetic instrumentation has greatly improved in recent years, and that a direct measurement of $\oint H \cdot ds$ around suitably chosen circuits could be a useful exercise.

Gauss's suggestion that the value of P_0 be investigated (equivalent to the search for magnetic monopoles) does not appear to have been followed up. All investigators have fitted the field to series beginning with $n = 1$. A simple preliminary test based on observations would be the integration over the surface of the vertical component; non-vanishing of the total radial flux would be equivalent to a non-vanishing P_0 term. However, it is questionable that the vertical component is sufficiently well mapped, even today, to make this worthwhile.

Apparently Gauss spent little time in speculating on the origin of the field. In connection with his mathematical analysis, he wrote, "... the terrestrial magnetic force is the collective action of all the magnetized particles of the earth's mass.....No alteration in the results would be caused by changing this mode of representation ... to consist in constant galvanic currents" [Gauss, *Werke* 5, 126]. He was clearly unwilling to decide between permanent magnetization and electric currents. After Gauss, theories based on permanent magnetization and the rotation of separated electric charges have gradually been discarded in favour of galvanic currents deep within the earth. Larmor [1919] is generally credited with the suggestion of a dynamo in the earth's fluid core. The most recent theoretical work [Gubbins 1974] suggests that small scale and even turbulent motions of the conducting fluid may support the dipole field.

CONCLUSIONS

Many observations in the area of geomagnetism were made long before Gauss's time, and there have been considerable advance since then. However, Gauss's contributions are of first-rank importance. His method of spherical harmonic analysis is still considered to be the best way to describe the field, and only the computational details have been altered. His absolute method of measuring the horizontal intensity remained the standard for many years, and his organization of the *Magnetische Verein* led to the present world-wide network of magnetic observatories. His contribution is not diminished by acknowledging that he did not work alone. As May [1972] pointed out, all of Gauss's publications in geomagnetism date from the period of his collaboration with Weber. The interaction with Humboldt, however, is more difficult to evaluate. Humboldt's interest in geomagnetism dates back to 1799 and he consistently supported and encouraged Gauss in his magnetic research. It was Humboldt who approached the Royal Society of London on Gauss's behalf. Yet Gauss denied that Humboldt initiated his own interest in geomagnetism [May 1972, 305]. The precise reason for Gauss's great preoccupation with the subject between 1830 and 1844 remains a mystery, but the result was of enormous benefit to geophysics.

GLOSSARY

Total intensity:
The strength of the earth's magnetic field at a point on earth, measured in the direction of the field vector. It may be resolved into the *horizontal intensity* in the direction of the compass, and the *vertical intensity*. The angle between the field vector and the horizontal plane is the *inclination*; that between the horizontal intensity and geographic north is the *declination*.

Gauss (unit):
The centimetre-gram-second unit of magnetic field strength. It may be defined in various ways, but an appreciation of the magnitude is given by the fact that the field is 1 Gauss at a distance of 1 cm. from a long wire carrying a current of 5 amperes. The maximum value of the earth's magnetic field is about 0.6 Gauss.

Secular change:
The time-variation of the earth's magnetic field which may be observed over a span of a few decades to a few centuries.

Epoch:
A point in time (e.g., 1978.0) to which magnetic observations made on different dates are reduced, by removal of the effect of secular change.

Conservative field:
A field of force in which the work done in moving a particle between two points is independent of the path. The magnetic field is conservative except in a region through which electric current is flowing.

Magnetic monopole:
A hypothetical particle consisting of a simple magnetic pole. Its existence has been predicted in some theories, but it has not been observed.

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