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Construction of B-splines for generalized spline spaces generated from local ECT-systems

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Abstract

Certain spaces of generalized splines are considered which are constructed by pasting together smoothly linear combinations of local ECT-systems. For them a basis of splines having minimal compact supports is constructed. These functions that are called B-splines are obtained by solving certain interpolation problems. They can be normalized either to form a partition of unity or to have integral over the real line equal to one each.

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1. Introduction

This paper is concerned with the construction of local support bases for generalized spline spaces. Such spaces are defined by the following data. Given a strictly increasing finite sequence of real *knots*

$$a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b,$$

where we allow $a = -\infty$ or $b = \infty$, given on each *knot interval*

$$I_i := [x_i, x_{i+1}), \quad i = 0, \ldots, k-1, \quad I_k := [x_k, x_{k+1}],$$

an ECT-space of dimension d_i in the sense of Karlin and Studden [6]

$$S_i := \text{span}\{s_1^{(i)}, \ldots, s_{d_i}^{(i)}\}, \quad s_j^{(i)} \in C^{d_i-1}(\overline{I}_i; \mathbb{R})\},$$

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given a sequence μ of nonnegative integers μ_i (μ_i counts the smoothness conditions at the knot x_i for $i=1,\dots,k$)

$$\mu=(\mu_1,\dots,\mu_k),\quad \mu_i<\min(d_i,d_{i-1}),\quad \mu_0:=0,\quad \mu_{k+1}:=0,$$

these data generate the *generalized spline space*

$$\begin{aligned} S^{\mu}_{[a,b]} := \{s | s:[a,b]\mapsto \mathbb{R}, s|_{I_j}\in S_j,\ j=0,\dots,k,\\ D^l_- s(x_i)=D^l_+ s(x_i),\ l=0,\dots,\mu_i-1,\ i=1,\dots,k\}. \end{aligned} \tag{1}$$

Here D^l_- resp. D^l_+ denotes the left resp. right derivative of order l. We recall [14] that $(u_1,\dots,u_n)$ is an *ECT-system* in *canonical form with respect to* $c\in[\alpha,\beta]$ generated by positive weight functions $w_j\in C^{n-j}[\alpha,\beta]$ provided that

$$\begin{aligned} u_1(x)&=w_1(x),\\ u_2(x)&=w_1(x)\int_c^x w_2(t_2)\,\mathrm{d}t_2,\\ u_3(x)&=w_1(x)\int_c^x w_2(t_2)\int_c^{t_2} w_3(t_3)\,\mathrm{d}t_3\,\mathrm{d}t_2,\\ &\vdots\\ u_n(x)&=w_1(x)\int_c^x w_2(t_2)\int_c^{t_2} w_3(t_3)\int_c^{t_3}\dots\int_c^{t_{n-1}} w_n(t_n)\,\mathrm{d}t_n\dots\mathrm{d}t_2. \end{aligned} \tag{2}$$

If S is any ECT-space of dimension n on $[\alpha,\beta]$, then for every $c\in[\alpha,\beta]$ there is an ECT-system in canonical form with respect to c that is a basis of S. Naturally associated with an ECT-system (2) are linear differential operators L^j defined by

$$\begin{aligned} D_0u&:=u,\\ D_ju&:=D\left(\frac{u}{w_j}\right),\quad j=1,\dots,n,\\ L^j&:=D_j\dots D_0,\quad j=0,\dots,n, \end{aligned} \tag{3}$$

L^ju is called the *jth ECT-derivative of u*. The system of functions

$$u_{j,i-j}:=L^ju_i,\quad i=j+1,\dots,n$$

is called the *jth reduced system* of $(u_1,\dots,u_n)$. It is again an ECT-system on $[\alpha,\beta]$ generated by the weight functions $w_{j+1},\dots,w_n$.

Theorem 1.1. *$S^{\mu}_{[a,b]}$ is a linear space of dimension*

$$\delta=\dim S^{\mu}_{[a,b]}=d_0+\sum_{i=1}^{k}(d_i-\mu_i)=d_k+\sum_{i=1}^{k}(d_{i-1}-\mu_i). \tag{4}$$

Proof. Clearly, $S^{\mu}_{[a,b]}$ is a linear space. Let

$$V := \begin{pmatrix} V^1_- & -V^1_+ & 0 & 0 & & \cdots & 0 \\ 0 & V^2_- & -V^2_+ & 0 & & & \\ & & & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & \\ & & & 0 & V^{k-1}_- & -V^{k-1}_+ & 0 \\ 0 & & \cdots & & 0 & V^k_- & -V^k_+ \end{pmatrix},$$

where for $i=1,\dots,k$ the entries are certain *Wronski matrices*

$$V^i_- := (D^l_- s^{(i-1)}_m(x_i))^{m=1,\dots,d_{i-1}}_{l=0,\dots,\mu_i-1}, \quad V^i_+ := (D^l_+ s^{(i)}_m(x_i))^{m=1,\dots,d_i}_{l=0,\dots,\mu_i-1}.$$

The matrices V^i_- and V^i_+ have rank μ_i since by assumption for every i $(s^{(i)}_1,\dots,s^{(i)}_{d_i})$ is an ECT-system on $[x_i,x_{i+1}]$. By definition, every $s \in S^{\mu}_{[a,b]}$ has a representation

$$s = \sum_{m=0}^{k} \sum_{n=1}^{d_m} c^{(m)}_n \cdot s^{(m)}_n$$

meaning that $s|_{I_m} = \sum_{n=1}^{d_m} c^{(m)}_n \cdot s^{(m)}_n$ with coefficients $c^{(m)}_n$ that are related by the connection equations of (1). Denote by $\boldsymbol{c}$ the stack vector of the coefficients of s. Then $s \in S^{\mu}_{[a,b]}$ iff $V \cdot c = 0$. Since V has full row rank the dimension of the null space of V is δ. This proves (4). □

Clearly, there are $N = \sum_{i=0}^{k} d_i$ unknowns and $M = \sum_{j=1}^{k} \mu_i < N$ homogeneous linear equations which are linearly independent. Thus, we can choose $\delta = N - M$ unknowns arbitrarily. Then the remaining M unknowns are determined uniquely.

Next, we will give a second proof of (4) in which we construct a “left-sided basis” of $S^{\mu}_{[a,b]}$ and which will be taken up again later. For $i=0$ and $j=1,\dots,d_0$ let $b_j|_{I_0} := s^{(0)}_j$ and extend b_j to $[x_1,b]$ such that $b_j \in S^{\mu}_{[a,b]}$. Due to the assumptions this is possible in various ways. For

$$i \geqslant 1 \text{ and } j = \sum_{l=0}^{i-1}(d_l - \mu_l) + 1, \dots, \sum_{l=0}^{i-1}(d_l - \mu_l) + d_i - \mu_i, \quad \mu_0 := 0$$

take

$$b_j|_{I_i} := \sum_{l=1}^{d_i} c^{(j)}_l \cdot s^{(i)}_l,$$

where $c^j = (c^{(j)}_1,\dots,c^{(j)}_{d_i})$ is a nontrivial solution of the linear system $V^i_+ \cdot c = 0$. There are exactly $d_i - \mu_i$ linearly independent solutions c^j of this system. Extend $b_j|_{I_i}$ by zero to the left of I_i and as a function $b_j \in S^{\mu}_{[a,b]}$ to the rest of $[a,b]$. It is an easy exercise to prove

Theorem 1.2. $B = \{b_1,\dots,b_\delta\}$ *is a basis of* $S^{\mu}_{[a,b]}$.

In particular, when each ECT-system $(s_1^{(i)},\dots,s_{d_i}^{(i)})$ is given in canonical form with respect to $c=x_i$, we can get B as a *left-sided basis* which in case of polynomial splines is the basis of truncated powers. Starting as above, for

$$i \geqslant 1 \quad \text{and} \quad j=\sum_{l=0}^{i-1}(d_l-\mu_l)+1,\dots,\sum_{l=0}^{i}(d_l-\mu_l)$$

take $b_j|_{I_i}=s^{(i)}_{\sum_{l=0}^{i-1}(d_l-\mu_l)+d_i+1-j}$ and extend b_j to the left of I_i by zero and to the rest of $[a,b]$ arbitrarily such that $b_j\in S^{\mu}_{[a,b]}$.

Similarly, by using the dimension count from right to left and ECT-systems in canonical form with respect to $c=x_{i+1}$ we can construct a *right sided basis* $\tilde{B}$ of $S^{\mu}_{[a,b]}$. The B-splines that we are going to construct below will constitute a basis which is left and right sided.

There exists a vast literature on B-splines for spaces of polynomial splines, Chebyshevian splines, generalized Chebyshevian splines and spaces of generalized splines. In particular, for Chebyshevian splines which are contained in our considerations as the special case that

$$S_i=\operatorname{span}\{s_1,\dots,s_d\}|_{I_i} \quad i=0,\dots,k,$$

where $(s_1,\dots,s_d)$ is an ECT-system on $[a,b]$, Tom Lyche [9] has constructed B-splines recursively. In [13] for generalized Chebyshevian splines based upon the local spaces

$$S_i=\operatorname{span}\{s_1,\dots,s_{d_i}\}|_{I_i}, \quad i=0,\dots,k,$$

where $(s_1,\dots,s_d)$ is an ECT-system on $[a,b]$ and $d\geqslant \max_i d_i$ B-splines are constructed normalized to have integral one over the real line. Sommer and Strauß [15] have constructed B-splines recursively even for generalized spline spaces as defined by (1) where the local spaces $S_i=\operatorname{span}(u_{i,1},\dots,u_{i,d_i})$ on I_i are generated from certain local Descartes systems $V_i=(v_{i,1},\dots,v_{i,n_i})$ on $[x_i,x_{i+1}]$ with $0\leqslant n_i\leqslant d_i$ and certain global weight functions $w_1,\dots,w_p\in C^{p-i}[a,b]$ by integration:

$$u_{i,1}(x)=w_1(x),$$
$$u_{i,j}(x)=\begin{cases} w_1(x)h_{j-1}(x,x_i;w_2,\dots,w_j), & j=2,\dots,p_i:=d_i-n_i,\\ w_1(x)h_{p_i}(x,x_i;w_2,\dots,w_{p_i},v_{i,j-p_i}), & j=p_i+1,\dots,d_i. \end{cases} \tag{5}$$

Here,

$$h_0(x,c):=1,$$
$$h_m(x,c;w_1,\dots,w_m):=\int_c^x w_1(t)h_{m-1}(t,c;w_2,\dots,w_m)\,\mathrm{d}t,$$
$$p:=\max_i p_i \quad \text{and} \quad p_i:=\max\{\mu_i,\mu_{i+1}\}.$$

Moreover, Sommer and Strauß [15] have proved that their B-splines form a weak Descartes system.

There seems to be no obvious relations between the generalized spline spaces generated according to (1) by local ECT-systems on one side or by integrated local Descartes systems on the other side, apart from the following trivial two:

(i) If $n_i = 0$ for all i then the generalized splines introduced by Sommer and Strauß reduce to generalized Chebyshevian splines just mentioned.
(ii) If additionally it is assumed that every local ECT-system $(s_1^{(i)}, \dots, s_{d_i}^{(i)})$ on $[x_i, x_{i+1}]$ forms a Descartes system then with $p_i = 0$, i.e., $d_i = n_i$ the generalized splines according to (1) are particular splines as constructed by Sommer and Strauß [15]. But then necessarily $\mu_i = 0$ for all i, i.e., we are in the case that there are no smoothness conditions at all.

Following work done by Schumaker [14], Nürnberger et al. [12,13] and Sommer and Strauß [15], Freyburger in his thesis [4] has considered generalized spline spaces as defined by (1) where the local spaces $S_i = \operatorname{span}(u_{i,1}, \dots, u_{i,d_i})$ on $\bar{I}_i$ are generated from certain local Chebyshev systems $V_i = (v_{i,1}, \dots, v_{i,n_i}) \subset C(\bar{I}_i, \mathbb{R})$ according to (5) where the global weight functions $w_1, \dots, w_p$ are replaced by local ones $w_{i,1}, \dots, w_{i,p_i}$. For systems $(u_{i,1}, \dots, u_{i,d_i})$ generated this way he has proved a generalization of the Budan-Fourier Theorem for ECT-systems (cf. [14], Theorem 9.12) and used it to derive a zero count for generalized splines. This, in turn, leads immediately to interpolation properties of the generalized spline spaces with respect to Lagrange interpolation, i.e., interpolation at simple nodes. For $n_i = 0$ for all i here our generalized splines are subsumed.

In this paper we discuss interpolation properties of generalized splines as in (1) with respect to Hermite-type interpolation at suitable nodes. In our approach we extend the method due to Jetter et al. [5] to count the zeros of polynomial splines to generalized splines, and we use it to find generalized spline spaces allowing Hermite-type interpolation at suitable nodes. Such spaces are called *interpolation spaces* or *IP-spaces* for short. We are going to show that for them construction of local support bases is possible. By suitably modifying the latter also a basis of $S_{[a,b]}^{\mu}$ consisting of nonnegative functions having minimal compact supports normalized to yield a partition of unity is possible.

2. A zero count for generalized splines

If

$$s = \sum_{i=0}^{k} \sum_{j=1}^{d_i} c_j^{(i)} \cdot s_j^{(i)} \in S_{[a,b]}^{\mu} \tag{6}$$

then an interval $[x_p, x_q] \subset [a, b]$ whose endpoints are knots is called a *support interval* of s iff

(i) $[x_p, x_q] \subset \operatorname{supp} s$,
(ii) $s|_{I_{p-1}} \equiv 0$ if $p > 0$ and $s|_{I_q} \equiv 0$ if $q < k + 1$.

If $[x_p, x_q]$ is a support interval of s and if $\xi \in (x_p, x_q)$, then there are integers $l \geqslant 0$ and i, $p < i \leqslant q$ such that precisely one of the following conditions holds:

(Na) $\xi \in (x_{i-1}, x_i)$ is not a knot and

$$D^0 s(\xi) = D^1 s(\xi) = \cdots = D^{l-1} s(\xi) = 0 \neq D^l s(\xi).$$

(Nb) $\xi = x_i \neq x_q$ is a knot and

$$D^0 s(\xi) = \cdots = D^{l-1} s(\xi) = 0 \neq D^l_- s(\xi)$$

or

$$D^0 s(\xi) = \cdots = D^{l-1} s(\xi) = 0 \neq D^l_+ s(\xi)$$

and

$$D^l_- s(\xi) \cdot D^l_+ s(\xi) \geqslant 0.$$

(Nc) $\xi = x_i \neq x_q$ is a knot and

$$D^0 s(\xi) = \cdots = D^{l-2} s(\xi) = 0 \quad \text{and} \quad D^{l-1}_- s(\xi) \cdot D^{l-1}_+ s(\xi) < 0.$$

Here, by convention conditions $D^0 s, \ldots, D^v s$ with a negative integer v are void. If $l \geqslant 1$ then ξ is called a *zero* of s of *multiplicity* l. A zero ξ of s of type (Nc) of multiplicity $l = 1$ is called *discontinuous*, all others are called *continuous*.

Lemma 2.1. *Let* $(u_1, \ldots, u_n)$ *be an ECT-system on* $[\alpha, \beta]$ *in canonical form* (2) *with associated linear differential operators* (3). *Then for every* $u \in \operatorname{span}\{u_1, \ldots, u_n\}$ *and every* $\xi \in [\alpha, \beta]$ *the following assertions hold*:

(i) $D^0 u(\xi) = D^1 u(\xi) = \cdots = D^j u(\xi) = 0 \Leftrightarrow L^0 u(\xi) = L^1 u(\xi) = \cdots = L^j u(\xi) = 0.$

(ii) *If* ξ *is a zero of* u *of multiplicity* l *then there exists a neighbourhood* $N(\xi)$ *of* ξ (*one sided if* $\xi = \alpha$ *or* β), *such that* $L^l u$ *and* $D^l u$ *have the same sign in* $N(\xi)$.

(iii) ξ *is a zero of* u *of multiplicity* l *iff*

$$L^0 u(\xi) = L^1 u(\xi) = \cdots = L^{l-1} u(\xi) = 0 \neq L^l u(\xi).$$

Proof. Lemma 2.1 is easily proved by induction using Leibniz' rule, cf. [14, p. 365]. □

On a generalized spline space $S^\mu_{[a,b]}$ we define $d := \max_{i=0,\ldots,k} d_i$ and differential operators L^j by

$$L^j S^\mu_{[a,b]} := \{L^j s\colon\ s \in S^\mu_{[a,b]} \text{ and } L^j s|_{I_i} = L^j_i s \quad j = 0, \ldots, d\}. \tag{7}$$

Here $L^1_i, \ldots, L^{d_i}_i$ are the linear differential operators associated with a S_i underlying ECT-system $\Gamma^{(i)} := (s^{(i)}_1, \ldots, s^{(i)}_{d_i})$ on I_i and $L^j_i s :\equiv 0$ for $d_i \leqslant j$. Then as an immediate consequence of Lemma 2.1 we have

Lemma 2.2. ξ *is a zero of* $s \in S^\mu_{[a,b]}$ *of multiplicity* l *iff precisely one of the conditions* (Na), (Nb) *or* (Nc) *holds where the operators* D^j *are replaced by the operators* L^j.

Lemma 2.3. *Suppose* $s \in S^\mu_{[a,b]} \backslash \{0\}$.

(i) *If* $\xi \in [x_i, x_{i+1}]$ *is a zero of* s *of multiplicity* $l+1$ *at least, then* $L^l_i s \not\equiv 0$.

(ii) *Let* $x_p \leqslant c < d \leqslant x_q$ *and* $\lim_{x\downarrow c} L^l s(x) = \lim_{x\uparrow d} L^l s(x) = 0$. *Then there exists* $\xi \in (c,d)$, *such that either*
   (a) ξ *is a zero of* $L^{l+1}s$ *with sign change and not a zero of* $L^l s$ *or*
   (b) ξ *is a discontinuous zero of* $L^l s$, *and* $L^{l+1}s$ *has no sign change at* ξ.

Proof. (i) If on I_i $L^l s = L_i^l(\sum_{j=1}^{d_i} c_j^{(i)} s_j^{(i)}) \equiv 0$, *then* $c_j^{(i)} = 0$ for $j = l+1,\ldots,d_i$ since the lth reduced system of the ECT-system $(s_1^{(i)},\ldots,s_{d_i}^{(i)})$ again is an ECT-system. Hence, $L_i^{l-1}s = c_l^{(i)} L_i^{l-1} s_l^{(i)} = c_l^{(i)} s_{l-1,1}^{(i)} = c_l^{(i)} w_l^{(i)}$. Since in any of the cases (Na), (Nb), (Nc) ξ also is a continuous zero of $L_i^{l-1}s$ we must have $c_l^{(i)} = 0$, that is $L^{l-1}s \equiv 0$. Proceeding this way we finally get $s \equiv 0$, a contradiction.

(ii) It is sufficient to prove the assertion for $l = 0$ since $L^l s$ again is a spline. We may suppose that the interval (c,d) is sufficiently small such that it does not contain a continuous zero of s. For sufficiently small $h > 0$ by Lemma 2.1 $s(c+h)Ls(c+h) > 0$ and $s(d-h)Ls(d-h) < 0$. Therefore, s and Ls do not have the same number of sign changes in (c,d). Hence, there is a first point $\xi \in (c,d)$ where either s or Ls changes sign. Since ξ is not a continuous zero of s, we must have either (a) or (b). □

Remark. In other words, Lemma 2.3 (i) tells that a zero ξ of multiplicity $l+1$ of a spline s belongs to the supports of all ECT-derivatives of s up to order l. Moreover, if ξ is a continuous zero of $L^j s$ then $\xi \in \operatorname{supp} L^{j+1}s$, by definition.

Let $d := \max_{i=p,\ldots,q-1} d_i$ and for $j=0,\ldots,d-1$ let l_j be the number of support intervals $I_1^{(j)},\ldots,I_{l_j}^{(j)}$ of $L^j s$ in a support interval $[x_p,x_q]$ of s. Clearly, $l_0 = 1$. Define

$$\boldsymbol{\phi} := \{(x,y) \in \mathbb{R}^2\colon \text{ there exists } j \in \{0,\ldots,d-1\}, \text{ such that } x \in \operatorname{supp} L^j s \text{ and } 0 \leqslant y \leqslant j\}$$

and

$$L(\boldsymbol{\phi}) := \sum_{j=1}^{d-1} l_j.$$

Denote by $\tilde{d}_i = \max\{j : c_j^{(i)} \neq 0,\ j = 1,\ldots,d_i\}$ the order of the spline (6) in $[x_i,x_{i+1}]$. As usual, y_+ denotes the positive part of a real y.

Lemma 2.4. *If* $[x_p,x_q]$ *is a support interval of a spline* (6) *then*

$$L(\boldsymbol{\phi}) = \tilde{d}_p - 1 + \sum_{i=p}^{q-2} (\tilde{d}_{i+1} - \tilde{d}_i)_+ = \tilde{d}_{q-1} - 1 + \sum_{i=p}^{q-2} (\tilde{d}_i - \tilde{d}_{i+1})_+.$$

Proof. Setting for $i = p,\ldots,q-2$

$$\sigma_{i,j} = \begin{cases} 1 & \text{if } (i = p \text{ and } \tilde{d}_p > j) \text{ or } (\tilde{d}_i \leqslant j \text{ and } \tilde{d}_{i+1} > j), \\ 0 & \text{else} \end{cases}$$

then $L(\phi) = \sum_{j=1}^{d-1} \sum_{i=p}^{q-2} \sigma_{i,j}$. There are two ways to count the support intervals of the ECT derivatives of s. Counting them from left to right yields the first formula and counting them from right to left the second. □

Remark. Addition of the two expressions of $L(\phi)$ given in Lemma 2.4 yields

$$2L(\phi) = \tilde{d}_p + \tilde{d}_{q-1} - 2 + \sum_{i=p}^{q-2} |\tilde{d}_{i+1} - \tilde{d}_i|$$

which is the length of the y-boundary of the set ϕ.

Definition. Let $[x_p, x_q]$ be a support interval of a spline (6). If $p \leqslant i \leqslant q$ and $j \in \mathbb{N}_0$, a point $(x_i, j) \in \phi$ is called *singular* iff $x_i \in \operatorname{supp} L^j s$ and there exists k, $0 \leqslant k \leqslant j$, such that $\operatorname{sign} L_{i-1}^k s(x_i-) \neq \operatorname{sign} L_i^k s(x_i+)$. By σ we denote the number of singular points of s in $[x_p, x_q]$.

Using these concepts it is not hard to see that the zero count for polynomial splines due to Jetter et al. [5], see also [8, p. 157ff] carries over to generalized splines:

Theorem 2.5. *Let $[x_p, x_q]$ be a support interval of a generalized spline $s \in S_{[a,b]}^{\mu}$. Then the number of zeros of s in $[x_p, x_q]$ counting multiplicities is*

$$Z(s, (x_p, x_q)) \leqslant \sigma - L(\phi) - 2.$$

For polynomial splines the estimate of Theorem 2.5 has the immediate consequence

$$Z(s, (x_p, x_q)) \leqslant \sum_{i=p}^{q-1} d_i - \sum_{i=p}^{q} \mu_i - 1. \tag{8}$$

Does (8) carry over to generalized splines? In general, the answer is: no! A counterexample can be found in [2, p. 25]. Only under certain conditions on the weight functions of the ECT-systems in neighbouring knot intervals we can prove the estimate (8) to hold for generalized splines.

Theorem 2.6. *Let $[x_p, x_q]$ be a support interval of a generalized spline $s \in S_{[a,b]}^{\mu}$. Under the assumption*

$$\operatorname{sign} L_{i-1}^j s(x_i-) = \operatorname{sign} L_i^j s(x_i+) \quad \text{for } i = p+1, \ldots, q-1 \text{ and } j = 0, \ldots, \mu_i - 1 \tag{9}$$

there holds (8). Here $\mu_0 := 0$ if $x_p = x_0 = a$ and $\mu_{k+1} := 0$ if $x_q = x_{k+1} = b$.

Proof. In view of (9) since $L^j s(x_i) = 0$ for $j \geqslant \max\{\tilde{d}_{i-1}, \tilde{d}_i\}$ only the points $(x_i, \mu_i), \ldots, (x_i, \max\{\tilde{d}_{i-1}, \tilde{d}_i\} - 1)$ or the points $(x_p, \mu_p), \ldots, (x_p, \tilde{d}_p - 1)$, $(x_q, \mu_q), \ldots, (x_q, \tilde{d}_{q-1} - 1)$ might be singular. Since $[x_p, x_q]$ is a support interval of s we have $L^j s(x_p) = 0$ for $j = 0, \ldots, \mu_p - 1$, but $L^{\tilde{d}_p} s(x_p) \neq 0$ whence $\tilde{d}_p \geqslant \mu_p$. Similarly, $\tilde{d}_{q-1} \geqslant \mu_q$ follows. (It should be noticed that a similar estimate for the inner knots does not hold necessarily.) Therefore,

$$\sigma \leqslant \tilde{d}_p - \mu_p + \tilde{d}_{q-1} - \mu_q + \underbrace{\sum_{i=p}^{q-2} (\max\{\tilde{d}_i, \tilde{d}_{i+1}\} - \mu_{i+1})_+}_{=:K}. \qquad (10)$$

Now

$$K = \sum_{i=p}^{q-2} \max\{\tilde{d}_i, \tilde{d}_{i+1}\} - \sum_{i=p}^{q-2} \mu_{i+1} + \sum_{i=p}^{q-2} (\mu_{i+1} - \max\{\tilde{d}_i, \tilde{d}_{i+1}\})_+$$

$$= \sum_{i=p}^{q-2} \tilde{d}_i + \sum_{i=p}^{q-2} (\tilde{d}_{i+1} - \tilde{d}_i)_+ - \sum_{i=p+1}^{q-1} \mu_i + \sum_{i=p}^{q-2} (\mu_{i+1} - \max\{\tilde{d}_i, \tilde{d}_{i+1}\})_+.$$

By inserting this into (10)

$$\sigma \leqslant \tilde{d}_p + \sum_{i=p}^{q-1} \tilde{d}_i + \sum_{i=p}^{q-2} (\tilde{d}_{i+1} - \tilde{d}_i)_+ - \sum_{i=p}^{q} \mu_i + \sum_{i=p}^{q-2} (\mu_{i+1} - \max\{\tilde{d}_i, \tilde{d}_{i+1}\})_+$$

is obtained. It remains to show that

$$\tilde{d}_i + (\mu_{i+1} - \max\{\tilde{d}_i, \tilde{d}_{i+1}\})_+ \leqslant d_i, \quad i = p, \ldots, q-2.$$

When for some i $\mu_{i+1} \leqslant \max\{\tilde{d}_i, \tilde{d}_{i+1}\}$ this holds trivially true. When $\mu_{i+1} > \max\{\tilde{d}_i, \tilde{d}_{i+1}\}$ then $\tilde{d}_i + (\mu_{i+1} - \max\{\tilde{d}_i, \tilde{d}_{i+1}\})_+ = \tilde{d}_i + \mu_{i+1} - \max\{\tilde{d}_i, \tilde{d}_{i+1}\} \leqslant \tilde{d}_i + \mu_{i+1} - \tilde{d}_i = \mu_{i+1} < d_i$. Now (8) follows by inserting this bound for σ and the first equation for $L(\boldsymbol{\phi})$ of Lemma 2.4 into the estimate of Theorem 2.5. □

For (9) to hold for all $s \in S^{\mu}_{[a,b]}$ Freyburger [4, pp. 39–43] has given an equivalent condition in terms of the weight functions of the ECT-systems in neighboured knot intervals.

Lemma 2.7. *The following assertions are equivalent*:

(i) *For all* $s \in S^{\mu}_{[a,b]}$

$$\operatorname{sign} L^l_{i-1} s(x_i-) = \operatorname{sign} L^l_i s(x_i+) \quad \textit{for } i = 1, \ldots, k \textit{ and } l = 0, \ldots, \mu_i - 1.$$

(ii) *For* $i = 1, \ldots, k$ *and* $j = 1, \ldots, \mu_i - 1$ *there exist positive constants* $K_{i,j}$, *such that*

$$D^l_- w^{(i-1)}_j (x_i) = K_{i,j} D^l_+ w^{(i)}_j (x_i), \quad l = 0, \ldots, \mu_i - j.$$

Remarks.

(i) The assertions of Lemma 2.7 hold trivially if one ECT-system on $[a,b]$ is “cut into pieces”, i.e., if the global weight functions w_j on $[a,b]$ define the “local weight functions” $w^{(i)}_j$ for $i = 0, \ldots, k$ by restriction

$$w^{(i)}_j := w_j|_{I_i}, \quad j = 1, \ldots, d_i.$$

Generalized splines generated this way in [13] are called *generalized Chebyshevian splines*. In particular, this holds true for polynomial splines where the global weight functions w_j are positive constants.

(ii) The assertions of Lemma 2.7 hold true also in case that the ECT-systems in neighboured knot intervals are constructed from the same weight functions where only the last two weight functions may be chosen differently for the two systems. For instance, if

$$S_i = \text{span}(1, x, \dots, x^{d-3}, \mathrm{e}^{\alpha_i x}, \mathrm{e}^{\beta_i x})$$

with real and distinct α_i, β_i and $\mu_i = d-1$ for all i then Lemma 2.7 holds, for S_i is an ECT-space on $\bar{I}_i$ with weights $w_1^{(i)} = \dots = w_{d-2}^{(i)} = 1, w_{d-1}^{(i)}(x) = \mathrm{e}^{\alpha_i x}, w_d^{(i)}(x) = \mathrm{e}^{(\beta_i - \alpha_i)x}$. Choosing $d = 4, \beta_i = -\alpha_i \neq 0$ and $\mu_i = 3$ for all i $S^{\mu}_{[a,b]}$ is the space of *splines in tension* (cf. [1, p. 264]) where the tension parameter α_i may be chosen differently in each knot interval. We remark that also the local spaces

$$S_i = \text{span}\ (1, x, \dots, x^{r_0}, \mathrm{e}^{\gamma_1 x}, x\mathrm{e}^{\gamma_1 x}, \dots, x^{r_1}\mathrm{e}^{\gamma_1 x}, \dots \mathrm{e}^{\gamma_m x}, \dots, x^{r_m}\mathrm{e}^{\gamma_m x}, \mathrm{e}^{\alpha_i x}, \mathrm{e}^{\beta_i x}), \quad x \in \bar{I}_i$$

with $r_0 \geqslant 0$, $\sum_{l=0}^{m}(r_l + 1) = d - 2$, and $\gamma_1, \dots, \gamma_m, \alpha_i, \beta_i$ real and pairwise distinct and $\gamma_1, \dots, \gamma_m$ independent of i fit into our approach.

Another generalized spline space with Lemma 2.7 valid is generated by local Cauchy–Vandermonde spaces

$$S_i = \text{span}\left(1, x, \dots, x^{d-3}, \frac{1}{x - x_i + \varepsilon}, \frac{1}{x - x_{i+1} - \varepsilon}\right)$$

with $\varepsilon > 0, d \geqslant 3$ and $\mu_i = d - 1$ for all i. It will be shown below that all these spaces and even the more general local spaces

$$S_i = \text{span}\left(1, x, \dots, x^{r_0 - 1}, \frac{1}{x - p_1}, \dots, \frac{1}{(x - p_1)^{r_1}}, \dots, \right.$$
$$\left. \frac{1}{x - p_m}, \dots, \frac{1}{(x - p_m)^{r_m}}, \frac{1}{x - x_i + \varepsilon}, \frac{1}{x - x_{i+1} - \varepsilon}\right), \quad x \in \bar{I}_i$$

where $r_0 \geqslant 1$, $\sum_{l=0}^{m} r_l = d - 2$ and $p_1, \dots, p_m$ are real, pairwise distinct, independent of i and outside $[a, b]$ admit a basis of B-splines normalized to form a nonnegative partition of unity. Rational B-splines with prescribed poles are investigated in some detail in [3].

(iii) The validity of the assertions of Lemma 2.7 is sufficient for Theorem 2.6 to hold but not necessary. This is shown by an example, see [2, p. 28] and also [10].

(iv) If the assertions of Lemma 2.7 hold then $S^{\mu}_{[a,b]}$ is a weak Chebyshev space. In particular, all examples of generalized spline spaces mentioned in remark (ii) are weak Chebyshev spaces and the B-splines to be constructed in Section 4 form a weak Chebyshev system.

3. Interpolation by generalized splines

It is well known that Chebyshevian splines [7] and also generalized Chebyshevian splines [12] can be used to solve interpolation problems of Hermite type provided the interpolation points ("*nodes*") and the knots interlace properly. There is a simple description of this interlacing property. Modify the knot sequence $(x_i)_{i=0}^{k+1}$ to knot sequences $y := (y_j)_{j=1}^{\delta}$ and $z := (z_j)_{j=1}^{\delta}$ which are weakly increasing both and where in the sequence y the knot x_0 has multiplicity d_0 and for $i = 1,\dots,k$ each knot x_i is repeated precisely $(d_i - \mu_i)$times. Similarly, in the sequence z the knot x_{k+1} has multiplicity d_k and for $i = 1,\dots,k$ each knot x_i, is repeated precisely $(d_{i-1} - \mu_i)$times. The sequences y and z correspond to the two ways to calculate the dimension δ of the space $S^{\mu}_{[a,b]}$ as done in the second proof of Theorem 1.1. They are used to define intervals

$$M_i := \begin{cases} [a, z_i) & \text{for } i = 1,\dots,d_0, \\ (y_i, z_i) & \text{for } i = d_0 + 1,\dots,\delta - d_k \\ (y_i, b] & \text{for } i = \delta - d_k + 1,\dots,\delta. \end{cases}$$

Consider nodes

$$\tau := (\tau_1,\dots,\tau_\delta),$$

where

$$a \leqslant \tau_1 \leqslant \cdots \leqslant \tau_\delta \leqslant b \tag{11}$$

which are not necessarily distinct from the knots x_j but are restricted by the natural *accumulation condition* that

$$v_i \leqslant \mu_j \quad \text{if } \tau_i = x_j, \; j = 1,\dots,k \tag{12}$$

v_i denoting the multiplicity of τ_i in τ. We say that the nodes (3) have the *interlacing property* with respect to the knots $(x_i)_{i=0}^{k+1}$ provided

$$\tau_i \in M_i, \quad i = 1,\dots,\delta. \tag{13}$$

With these concepts the proof of Karlin and Ziegler [7] carries over to generalized spline spaces in the form

Theorem 3.1. *Let $S^{\mu}_{[a,b]}$ be a space of generalized splines such that the zero count (8) holds. If τ is a system of nodes satisfying the accumulation condition (12) then every interpolation problem for τ*

given a sufficiently smooth function f, find $s \in S^{\mu}_{[a,b]}$ such that

$$D^{\lambda_i} s(\tau_i) = D^{\lambda_i} f(\tau_i), \quad i = 1,\dots,\delta \text{ where } \lambda_i := \max\{l : \tau_i = \tau_{i-1} = \cdots = \tau_{i-l}\}$$

has a unique solution in $S^{\mu}_{[a,b]}$ iff τ has the interlacing property (13) with respect to the knots of $S^{\mu}_{[a,b]}$.

Generalized spline spaces $S^{\mu}_{[a,b]}$ are called *interpolation spaces* or *IP-spaces* provided for them Theorem 3.1 holds. For such spaces construction of local support bases is possible via interpolation.

4. Construction of local support bases for generalized spline spaces

In order to construct B-splines we need an extension of the knot sequence $\Delta = (x_i)_{i=0}^{k+1}$ and we need auxiliary ECT-spaces. Let us assume $x_0 = a \in \mathbb{R}$ and $x_{k+1} = b \in \mathbb{R}$. Then we extend Δ to a knot sequence $\tilde{\Delta} = (x_i)_{-d_0+1}^{k+d_k}$ with $-\infty < x_{-d_0+1} < \cdots < x_{-1} < x_0 < \cdots < x_k < x_{k+1} < \cdots < x_{k+d_k} < \infty$.

For simplicity, we assume that $\Gamma^{(0)} = (s_1^{(0)}, \ldots, s_{d_0}^{(0)})$ is an ECT-system on $[x_{-d_0+1}, x_1]$ with corresponding weight functions $w_j^{(0)}$ $(j = 1, \ldots, d_0)$ defined on $[x_{-d_0+1}, x_1]$ and that $\Gamma^{(k)} = (s_1^{(k)}, \ldots, s_{d_k}^{(k)})$ is an ECT-system on $[x_k, x_{k+d_k}]$ with weight functions $w_j^{(k)}$ $(j = 1, \ldots, k)$ defined on that interval. We are going to use the ECT-spaces

$$S_i := \mathrm{span}\{s_1^{(i)}, \ldots, s_{d_i}^{(i)}\}, \quad s_j^{(i)} \in C^{d_i-1}(\bar{I}_i; \mathbb{R}),$$

where

$$s_j^{(i)} := \begin{cases} s_j^{(0)}|_{\bar{I}_i}, & i = -d_0 + 1, \ldots, 0, \ d_i = d_0, \ j = 1, \ldots, d_0, \\ s_j^{(i)}, & i = 1, \ldots, k-1, \ j = 1, \ldots, d_i, \\ s_j^{(k)}|_{\bar{I}_i} & i = k, \ldots, k + d_k - 1, \ d_i = d_k, \ j = 1, \ldots, d_k. \end{cases}$$

Assuming the conditions of remark (ii) to Lemma 2.7 then $S_{[a,b]}^{\mu}$ is an IP-space as well as $S_{[x_{-d_0+1}, x_{k+d_k}]}^{\mu}$. We do need

Lemma 4.1. *Let B be a basis of the generalized spline space $S_{[a,b]}^{\mu}$ having the property that for some $i \in \{0, \ldots, k\}$ precisely d_i elements of B are nonzero on I_i where*

$$B|_{(x_i, x_{i+1})} := \{b \in B\colon \ \mathrm{supp}\ b \cap (x_i, x_{i+1}) \neq \emptyset\} =: \{s_{i,1}, \ldots, s_{i,d_i}\}.$$

Then $\{s_{i,1}|_{I_i}, \ldots, s_{i,d_i}|_{I_i}\}$ is a basis of S_i.

Proof. Let $f \in S_i$ be arbitrary. We may extend f to $[a,b]$ as a function $\hat{f} \in S_{[a,b]}^{\mu}$ as is clear from the first proof of Theorem 1.1. Represent $\hat{f}$ in the basis B as $\hat{f} = \sum_{b \in B} c_b \cdot b$. Restriction of this representation to I_i gives

$$\hat{f}|_{I_i} = f = \sum_{j=1}^{d_i} c_{i,j} \cdot s_{i,j}\ , \quad c_{i,j} := c_{s_{i,j}},$$

where $(c_{i,j})_{j=1}^{d_i}$ is uniquely determined by $\hat{f}$. Thus, $s_{i,1}, \ldots, s_{i,d_i}$ generate S_i. Therefore they must be a basis of this space. □

For simplicity reasons in this and the next section we assume that all local spaces S_i have the same dimension $d_i = d$, $i = -d+1, \ldots, k+d-1$, and that $\mu_i = d-1$, $i = -d+2, \ldots, k+d-1$, i.e., we are considering generalized splines having maximal smoothness order C^{d-2} at all knots. By $S_{[a,b]}^{\max}$ we denote the space $S_{[a,b]}^{\mu}$ with $d_i = d \geqslant 2$, $i = 0, \ldots, k$, $\mu = (d-1, \ldots, d-1)$ and by $S_{[x_{-d+1}, x_{k+d}]}^{\mu}$ we denote its extension described above.

Theorem 4.2. *Under the above assumptions suppose that* $p \in \{-d+1,\dots,k\}$. *Then the following assertions hold*:

(i) *There exists a spline* $s_p \in S^{\max}_{[x_{-d+1},x_{k+d}]}$ *with* $\operatorname{supp} s_p = [x_p, x_{p+d}]$, *which can be extended onto the real line as a* C^{d-2}-*function.*
(ii) s_p *has no zero in* (x_p, x_{p+d}).
(iii) s_p *is determined uniquely up to a nonzero constant factor by properties* (i) *and* (ii).
(iv) $B = \{s_{-d+1}|_{[a,b]},\dots,s_k|_{[a,b]}\}$ *is a basis of* $S^{\max}_{[a,b]}$ *of splines having compact supports.*
(v) *There is no spline* $s \in S^{\max}_{[x_{-d+1},x_{k+d}]}$ *with a support interval being the union of only* $d-1$ *knot intervals.*
(vi) B *is left and right sided.*

Proof. By assumption, all spline spaces involved are IP-spaces. Consider the spline space $S^* := S^{\max}_{[x_{-d+1},x_{d+k}]}$. Its dimension is $\delta^* = 3(d-1)+k+1$. For any $p \in \{-d+1,\dots,k\}$ let $x' \in (x_p, x_{p+d})$ be arbitrary. Then the system of nodes

$$\tau^* := \left(x_{-d+1}, x_{-d+2},\dots,x_{p-1}, \underbrace{x_p,\dots,x_p}_{d-1}, x', \underbrace{x_{p+d},\dots,x_{p+d}}_{d-1}, x_{p+d+1},\dots,x_{k+d} \right)$$

has the accumulation and interlacing properties with respect to the knot system of S^*. Hence the interpolation problem

find $s_p \in S^*$ such that

$$s_p(x_i) = 0 \quad \text{for } i = -d+1,\dots,p-1 \text{ and } i = p+d+1,\dots,d+k,$$

$$s_p(x') = 1,$$

$$D^j s_p(x_p) = 0 \quad \text{for } j = 0,\dots,d-2,$$

$$D^j s_p(x_{p+d}) = 0 \quad \text{for } j = 0,\dots,d-2$$

has a unique solution. Since s_p has the maximal number δ^*-1 of zeros a nontrivial spline from S^* can have, s_p has no further zeros in (x_p, x_{p+d}). Moreover, $\operatorname{supp} s_p = [x_p, x_{p+d}]$ since otherwise there would be in S^* a nontrivial solution of the homogeneous interpolation problem $s|_{\tau^*} = 0$, a contradiction.

Now, for any nonzero constant factor α the spline αs_p has the properties (i) and (ii).

To prove (iii) assume that $s \in S^*$ is any spline having the properties (i) and (ii). Then there exist $\xi \in (x_p, x_{p+d})$ and $\alpha \neq 0$ such that $s(\xi) = a s_p(\xi)$. This implies that the spline $s - \alpha \cdot s_p \in S^*$ would have the zero ξ. According to Theorem 3.1 $s - \alpha s_p$ must be the zero function.

To prove (iv) we will show that $\{s_{-d+1}|_{[a,b]},\dots,s_k|_{[a,b]}\}$ is a basis of $S^{\max}_{[a,b]}$. By assumption, all knots $x_1,\dots,x_k$ are simple and $D^l s_i(x_i) = 0$ for $l = 0,\dots,d-2$. Therefore, $s_i|_{[x_i,x_{i+1}]} \not\equiv 0$ is a basis of the one-dimensional subspace of S_i as constructed in the second proof of Theorem 1.1 where a left-sided basis of $S^{\mu}_{[a,b]}$ has been constructed in analogy to the truncated powers.

By construction $s_i|_{[a,b]} \in S^{\max}_{[a,b]}$ for $i=-d+1,\dots,k$. It remains to show that $\{s_{-d+1}|_{[x_0,x_1]},\dots,s_0|_{[x_0,x_1]}\}$ is a basis of S_0. Extend $\hat{\Delta}$ once more to $\tilde{\Delta}=(x_i)_{i=-d}^{k+d}$ with $x_{-d}<x_{-d+1}$ and $S^{\mu}_{[x_{-d+1},x_{k+d}]}$ to $S^{\mu}_{[x_{-d},x_{k+d}]}$ where we assume that also $S_{-d}=\operatorname{span}\{s_1^{(-d)},\dots,s_d^{(-d)}\}\subset C^{d-1}(\bar{I}_{-d};\mathbb{R})$ with $s_j^{(-d)}=s_j^{(0)}|_{\bar{I}_{-d}}$ $(j=1,\dots,d)$ $(s_1^{(0)},\dots,s_d^{(0)})$ being an ECT-system on $[x_{-d},x_1]$. Then $S^{\max}_{[x_{-d},x_1]}$ is an IP-space of dimension $2d-1$. Take any point $x'\in(x_{-d},x_0)$, then the nodes

$$\underbrace{x_{-d},\dots,x_{-d}}_{d-1},x',\underbrace{x_0,\dots,x_0}_{d-1}$$

have the accumulation and the interlacing properties with respect to the knots of $S^{\max}_{[x_{-d},x_0]}$. Hence, for $l=1,\dots,d$ the interpolation problem

find $F_l\in S^{\max}_{[x_{-d},x_0]}$ such that

$$D^jF_l(x_{-d})=D^js_l^{(-d)}(x_{-d})\quad\text{for } j=0,\dots,d-2,$$

$$F_l(x')=1,$$

$$D^jF_l(x_0)=0\quad\text{for } j=0,\dots,d-2$$

has a unique solution. For $m=1,\dots,d$ let

$$G_m:[x_{-d},x_1]\to\mathbb{R},\quad x\mapsto\begin{cases}F_m(x) & \text{for } x_{-d}\leqslant x<x_0,\\ 0 & \text{for } x_0\leqslant x\leqslant x_1\end{cases}$$

and for $m=-d+1,\dots,0$ let

$$G_m:[x_{-d},x_1]\to\mathbb{R},\quad x\mapsto\begin{cases}0 & \text{for } x_{-d}\leqslant x<x_m,\\ s_m(x) & \text{for } x_m\leqslant x\leqslant x_1.\end{cases}$$

Clearly, $G_m\in S^{\max}_{[x_{-d},x_1]}$ for $m=-d+1,\dots,d$ since by construction $G_m\in C^{d-2}$ at all inner knots of $\tilde{\Delta}$. Moreover, $(G_m)_{m=-d+1}^{d}$ is a basis of the space $S^{\max}_{[x_{-d},x_1]}$. According to the second proof of Theorem 1.1 adapted to $S^{\max}_{[x_{-d},x_1]}$ it remains to show that $G_1,\dots,G_m$ are linearly independent on $\bar{I}_{-d}=[x_{-d},x_{-d+1}]$. Assume that

$$\sum_{l=0}^{d}c_mF_m(x)\equiv 0\quad\text{in } \bar{I}_{-d}. \tag{14}$$

We have to show $c_m=0$ for all m. Taking the ECT-system $s_1^{(-d)},\dots,s_d^{(-d)}$ in canonical form with respect to $c=x_{-d}$ by inserting $x=x_{-d}$ into (14) we find $c_1=0$. By applying L^1_{-d} to (14) and inserting again $x=x_{-d}$ we find $c_2=0$. Continuing this way we find from the interpolation conditions of F_l at x_{-d} that $c_1=c_2=\cdots=c_{d-1}=0$, hence $c_dF_d(x)\equiv 0$ in $\bar{I}_{-d}$. To complete the proof observe that F_d cannot be the zero function in $\bar{I}_d$. If it were then $[x_{-d+1},x_0]$ would be a support interval of the spline F_d contradicting (v) which we are going to prove next. Since precisely the d functions $G_{-d+1},\dots,G_0$ are nonzero on $[x_0,x_1]$ being there identical with $s_{-d+1},\dots,s_0$, correspondingly, it follows from Lemma 4.1 that $\{s_{-d+1}|_{[x_0,x_1]},\dots,s_0|_{[x_0,x_1]}\}$ is a basis of S_0.

To prove (v) observe that any spline $s\in S^{\max}_{[a,b]}$ with $\operatorname{supp} s$ smaller than $[x_i,x_{i+d}]$, say with $\operatorname{supp} s=[x_i,x_{i+d-1}]$ for some $i\in\{-d+2,\dots,k\}$ must be the zero function. Indeed, it satisfies the homogeneous interpolation conditions with nodes

$$x_{-d+1},\ldots,x_{i-1},\underbrace{x_i,\ldots,x_i}_{d-1},\underbrace{x_{i+d-1},\ldots,x_{i+d-1}}_{d-1},x_{i+d},\ldots,x_{k+d}$$

having the accumulation and the interlacing properties with respect to the knots of $S^{\max}_{[x_{-d+1},x_{k+d}]}$.

Assertion (vi) holds by construction of B. □

We call the basic functions $s_p,\ p=-d+1,\ldots,k$ of Theorem 4.2 each being uniquely determined up to a nonzero constant factor *B-splines*.

Corollary 4.3. *Under the assumptions of Theorem* 4.2 *if* $B=\{s_i\colon\ i=-d+1,\ldots,k\}$ *is a system of functions having supports* $\operatorname{supp} s_i=[x_i,x_{i+d}]$ *then* $B|_{[a,b]}$ *is a basis of* $S^{\max}_{[a,b]}$ *provided* $B\subset S^{\max}_{[x_{-d+1},x_{k+d}]}$.

Next we want to normalize the basic functions by one or the other of the two conditions:

(i) every B-spline has integral equal to one over the real line;
(ii) the B-splines form a partition of unity on $[a,b]$.

For condition (ii) it is necessary that the constant function 1 belongs to every ECT-system. This is guaranteed by the assumption that the first weight function of every ECT-system is the constant function 1.

Lemma 4.3. *For* $i=-d+1,\ldots,k+d-1$ *let all ECT-systems* $\Gamma^{(i)}$ *on* $[x_i,x_{i+1}]$ *have dimension* d *and the first weight function* $w_1^{(i)}\equiv 1$. *Suppose that the assumptions of remark* (ii) *to Lemma* 2.7 *hold such that* $S^{\max}_{[a,b]}$ *is an IP-space. If* $B=\{s_{-d+1},\ldots,s_k\}$ *is a B-spline basis as described in Theorem* 4.2, *then the spline* $s^0\equiv 1\in S^{\max}_{[a,b]}$ *belongs to no subspace generated by a proper subset of* B.

Proof. It is easily seen that the assertion holds true for $d=1,2$ and every $k\geqslant 0$. Therefore let $d\geqslant 3$. Suppose now that $i_0\in\{-d+1,\ldots,k\}$ and that

$$\sum_{\substack{i=-d+1\\ i\neq i_0}}^{k}\lambda_i s_i(x)\equiv 1\quad\text{for } x_0\leqslant x\leqslant x_{k+1}. \tag{15}$$

We are going to show that this assumption leads to a contradiction. By applying the differential operator $L^1=D$ to both sides of this equation we get the identity

$$\sum_{\substack{i=-d+1\\ i\neq i_0}}^{k}\lambda_i L^1 s_i(x)\equiv 0\quad\text{for } x_0\leqslant x\leqslant x_{k+1}.$$

Consider first the case $i_0\geqslant 0$ and the interval I_{i_0}. Then

$$z_{i_0}(x):=\sum_{i=i_0-d+1}^{i_0-1}\lambda_i L^1 s_i(x)\equiv 0\quad\text{for } x_{i_0}\leqslant x\leqslant x_{i_0+1} \tag{16}$$

since all other B-splines vanish on I_{i_0}. If not all coefficients in (16) are zero then $z_{i_0} \in L^1 S^{\max}_{[a,b]}$ has support contained in $[x_{i_0-d+1}, x_{i_0}] \cup [x_{i_0+1}, x_{i_0+d-1}]$. Since $L^1 S^{\max}_{[a,b]}$ again is an IP-space whose local ECT-spaces have dimension $d-1$ each, z_{i_0} must vanish on $[x_{i_0+1}, x_{i_0+d-1}]$ since according to Theorem 4.2 (v) this interval is too short to be a support interval of a spline from $L^1 S^{\max}_{[a,b]}$. On the other side, z_{i_0} must be a constant multiple of the B-spline of the space $L^1 S^{\max}_{[a,b]}$ having support $[x_{i_0-d+1}, x_{i_0}]$. Obviously, this constant factor must be 0. Consequently, z_{i_0} is the zero function on $\mathbb{R}$. From the fundamental Theorem of calculus we get

$$\int_{-\infty}^{x} z_{i_0}(t)\,\mathrm{d}t = \sum_{i=i_0-d+1}^{i_0-1} \lambda_i \int_{-\infty}^{x} L^1 s_i(t)\,\mathrm{d}t = \sum_{i=i_0-d+1}^{i_0-1} \lambda_i s_i(x) = \sum_{\substack{i=-d+1 \\ i \neq i_0}} \lambda_i s_i(x) \equiv 0$$

for $x_{i_0} \leqslant x \leqslant x_{i_0+1}$ contradicting (15).

Consider next the case $i_0 < 0$ and the interval I_0. This case similarly leads to a contradiction. Since all cases for i_0 are covered, the proof is complete. □

The following notation will be useful. For $i = -d+1, \ldots, k$ and $j > i$ let

$$V_i^j := \begin{pmatrix} V_+^i & 0 & & \cdots & & 0 \\ V_-^{i+1} & -V_+^{i+1} & 0 & & & \\ 0 & V_-^{i+2} & -V_+^{i+2} & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & 0 & V_-^{j-2} & -V_+^{j-2} & 0 \\ & & & 0 & V_-^{j-1} & -V_+^{j-1} \\ 0 & & \cdots & & 0 & V_-^j \end{pmatrix}$$

and

$$J_i := \left(\int_{x_i}^{x_{i+1}} s_1^{(i)}(x)\,\mathrm{d}x, \ldots, \int_{x_i}^{x_{i+1}} s_d^{(i)}(x)\,\mathrm{d}x \right).$$

Here, V_+^l resp. V_-^l are the Wronskian matrices introduced in the proof of Theorem 1.1 and $s_j^{(i)}$ $(j = 1, \ldots, d)$ is the basic ECT-system on $[x_i, x_{i+1}]$. By $V_{J,i}^j$ we denote the matrix obtained by rendering V_i^j below by the row vector $(J_i, J_{i+1}, \ldots, J_{j-1}) \in \mathbb{R}^{d(j-i)}$. Observe that $V_{J,i}^{i+d} \in \mathbb{R}^{d^2 \times d^2}$.

Theorem 4.4. *For $i = -d+1, \ldots, k+d-1$ let all ECT-systems $\Gamma^{(i)}$ on $[x_i, x_{i+1}]$ have dimension d. Let $S^{\max}_{[a,b]}$ be an IP-space such that its extension $S^* := S^{\max}_{[x_{-d+1}, x_{k+d}]}$ constructed in the proof of Theorem 4.2 is an IP-space too.*

(i) *There are splines* $\{B_{-d+1},\dots,B_k\}\subset S^*$ *having supports* $\operatorname{supp} B_p=[x_p,x_{p+d}]$ *such that*
   (a) $\{B_{-d+1}|_{[a,b]},\dots,B_k|_{[a,b]}\}$ *is a basis of* $S^{\max}_{[a,b]}$,
   (b) B_p *has no zero in* (x_p,x_{p+d}), $p=-d+1,\dots,k$,
   (c) $\int_{x_{-d+1}}^{x_{d+k}} B_p(x)\,\mathrm{d}x=1$, $p=-d+1,\dots,k$,
   (d)

$$B_p(x)=\sum_{i=0}^{d-1}\sum_{j=1}^{d} c_{p+i,j}s_j^{(p+i)},$$

where the stack vector of this representation

$$c_p=(c_{p,1},\dots,c_{p,d},c_{p+1,1},\dots,c_{p+1,d},\dots,c_{p+d-1,1},\dots,c_{p+d-1,d})^{\mathrm{T}}$$

is the unique solution of the linear system

$$V_{J,p}^{p+d}x=(0,\dots,0,1)^{\mathrm{T}}. \tag{17}$$

(ii) *If all ECT-systems* $\Gamma^{(i)}$ on $[x_i,x_{i+1}]$ *have the first weight function* $w_1^{(i)}\equiv 1$, *then there are splines* $\{N_{-d+1},\dots,N_k\}\subset S^*$ *having supports* $\operatorname{supp} N_p=[x_p,x_{p+d}]$ *such that*
   (a) $\{N_{-d+1}|_{[a,b]},\dots,N_k|_{[a,b]}\}$ *is a basis of* $S^{\max}_{[a,b]}$,
   (b) N_p *has no zero in* (x_p,x_{p+d}), $p=-d+1,\dots,k$,
   (c) $\sum_{i=-d+1}^{k} N_i(x)=1$, *for* $a\leqslant x\leqslant b$,
   (d) *By choosing any system of nodes* $\tau=(\tau_1,\dots,\tau_\delta)$ *in* $[a,b]$ *having the accumulation and the interlacing properties with respect to the knots of* $S^{\max}_{[a,b]}$ *the interpolation problem*

$$s(\tau_l)=\sum_{i=-d+1}^{k}\lambda_i B_i(\tau_l)=1,\quad l=1,\dots,\delta$$

has a unique solution. Here, $\lambda_i\neq 0$ *for all* $i=-d+1,\dots,k$. *Then*

$$N_i:=\lambda_i B_i,\quad i=-d+1,\dots,k.$$

Proof. (i) follows from Theorem 4.2. Accordingly, for every $p=-d+1,\dots,k$ there exists a spline s_p which in (x_p,x_{p+d}) is positive. Then $B_p:=(\int_{x_p}^{x_{p+d}} s_p(x)\,\mathrm{d}x)^{-1}\cdot s_p$ $(p=-d+1,\dots,k)$ are the splines satisfying (a), (b) and (c) of Theorem 4.4. Indeed, B_p is uniquely determined by these conditions. For if there were another spline in S^* different from B_p having these properties their nontrivial difference must have a zero in (x_p,x_{p+d}). According to Theorem 3.1 the difference is the zero function. Clearly, B_p has coefficient vector c_p with respect to the local ECT-systems on the knot intervals belonging to the support of B_p that solves (17). c_p is uniquely determined since B_p is unique.

To prove (ii) notice that by assumption $s=s^0\equiv 1$ belongs to $S^{\max}_{[a,b]}$. Hence uniquely

$$s^0=\sum_{i=-d+1}^{k}\lambda_i B_i.$$

By Lemma 4.3 no coefficient λ_i is zero, hence

$$N_i:=\lambda_i B_i,\quad i=-d+1,\dots,k$$

are nontrivial and $\{N_{-d+1}|_{[a,b]}, \ldots, N_k|_{[a,b]}\}$ is a basis of $S^{\max}_{[a,b]}$ normalized to be a partition of unity on $[a,b]$. Since $S^{\max}_{[a,b]}$ is an IP-space the coefficients can be determined by interpolation. This proves (ii). □

Remarks.

(i) Theorem 4.4 may be generalized (cf. [2, Section 2.4.2]). A similar result holds also in the more general case that the dimensions d_i of the basic ECT-spaces are different and the smoothness count μ is prescribed arbitrarily.
(ii) For a different approach to B-splines for spaces of generalized splines generated from different local ECT-systems via connection matrices where the natural differential operators of the local ECT-systems are involved we refer to [11].
(iii) As already mentioned the B-splines constructed form a weak Chebyshev system. Indeed, each local ECT-system $\Gamma^{(i)}$ on $[x_i, x_{i+1}]$ depends continuously on the knots x_i, x_{i+1} for the local weight functions may be extended smoothly. Since also generalized B-splines are continuous functions of their knots, the same arguments as used for the proof of Theorem 4.64 in [14] show that also the generalized B-splines form a weak Chebyshev system. As a consequence, from this it follows that the B-splines constructed also form a weak Descartes system (cf. [14, p. 169]).
(iv) It is well known (cf. [9]) that in the particular case of Chebyshevian splines the B-splines can be computed recursively. In the more general case considered in Theorem 4.4 existence of B-splines has been proved by showing that the linear system (17) has a unique solution. It remains an open problem if also in the general case there exists a recurrence relation for the B-splines. Since the structure of (17) is a particular one at least for particular ECT-systems existence of a recurrence relation may be expected.

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